Idealization Axiom:
$\forall^{sf} F \;\; \exists x \; R(x,F) \iff \exists x \; \forall^{s} y \; R(x,y)$

Standardization Axiom:
$\forall P(x) \;\; \exists^{s} A \subseteq E \quad \forall^{s} x \; \big(x \in A \iff x \in E \text{ and } P(x)\big)$
(E is some parent set)

Transfer Axiom:
For any standard parameters $A, B, ..., L$ of a **classical** formula F:
$\forall^{s} x \; F(x, A, B, ..., L) \iff \forall x \; F(x, A, B, ..., L)$

letting $F' = \neg F$ gives the equivalent $\exists x \; F'(x, A, B, ..., L) \iff \exists^{s} x \; F'(x, A, B, ..., L)$

By the dual form, all objects uniquely defined by classical formulas are both unique and standard.

Note: in general not all properties $P(x)$ are set-forming, but we can always create a standard set ${}^{s}\{x \in E \;:\; P(x)\}$

Example to think about: for v illimited, the interval in $\mathbb{N}$ of $[0,v] \subset \mathbb{N}$ but ${}^{s}[0,v] = \mathbb{N}$...

Well-Ordering Principle of the Natural Numbers:
"Every nonempty subset of $\mathbb{N}$ has a smallest element"

Induction comes from this well-ordering principle:

$P(x)$ is true for $x = 0$ and $P(n) \Rightarrow P(n+1)$ means $P(x)$ holds for all $x \in \mathbb{N}$.

Convert to a set so we can justify by the well-ordering principle: $T = \{x \in \mathbb{N} \;:\; P(x)\}, \; F = \mathbb{N}/A$. Assume the premise and that $F \neq \emptyset$, so F must have a least element f. However, since $N = T + F$, $f - 1 \in T$ and $P(f-1) \Rightarrow P(f)$, a contradiction. So F must be empty.

So as long as we can *properly* form the set T (see below), we can apply induction without further thought (see exercise 2.8.4).

P being **set-forming** in E implies (means?): letting $F = \{x \in E \;:\; P(x)\}$
$\forall x \in F \;\; P(x)$ and $\forall x \in E/F \;\; \neg P(x)$

Set-forming nonexample: $P(x) =$ "x is standard", form $\{x \in \mathbb{N} \;:\; P(x)\}$
Suppose $P(x)$ set forming and that this set is finite; then the set of illimited integers has a smallest element b for which $b - 1 = n$, where n is standard. However, transfer applies to the formula $b = n + 1$, so b must be standard too, a contradiction. Hence the set of standard integers must be infinite. However, every infinite set of integers contains illimited integers by idealization, so $P(x)$ does not hold for some elements in the set, hence $P(x)$ is not set-forming in $\mathbb{N}$.

Note the converse is not true: given an illimted n, $[0,n]$ is nonstandard (invert theorem 2.4.2), but we are still forbidden from collecting all the standard elements here.

nonexample 2: From a standard and infinite E and same $P(x)$, form $T = \{x \in E \;:\; P(x)\}$
Suppose $P(x)$ set forming and T finite, so T must be standard (theorem 2.4.2). Then, due to sharing exactly the same standard elements, transfer applies and we have $T = E$, a contradiction. So T must be infinite. Since any infinite set contains nonstandard elements (apply idealization to the $\neq$ relation), T contains nonstandard elements, hence $P(x)$ is not set forming in standard infinite sets.

This complements theorem 2.4.2:
For infinite standard sets, "x is standard" must not be set-forming (just proved).
A finite set is standard $\iff$ all elements are standard (2.4.2).

Exercises 1.9 (didn't know how to solve #3 without axioms from the next chapter???)

1.9.1: To do this without axioms from chapt 2 requires an inductive argument, which I did type up before, but I deleted it for some reason and am too lazy to type up again because annoying :)

Injective and surjective are classical properties, hence we can form the classical statement $P(x) =$ "an injective map of x to itself is surjective".
This is true for all standard intervals of $\mathbb{N}$: an injective map from $[0, n-1]$ implies the image must have cardinality n. Since the codomain also has size n, the map is surjective.
Since $P(x)$ is classical, transfer applies and $P(x)$ holds for all intervals $[0, n] \subset \mathbb{N}$, including illimited n.

1.9.2: Let n be illimited and form the infinite set $S = \{1\} \cap \{n,\ n+1,\ n+2,\ ...\}$ Then the property forming $T = \{x \in S\ :\ x \text{ is standard}\}$ is set forming: 1 is standard, and all elements of S/T are nonstandard (this shows S is nonstandard, see nonexample 2).

Let n be illimited and form the set $\{1, n\}$. By the same reasoning above, the property "x is standard" is set-forming here. (also note this set is nonstandard by the inversion of theorem 2.4.2)

1.9.3: All 3 propositions are false. Counterexample: for every standard integer n, $2^n > n$, so transfer applies: for any illimited integer n, $n < 2^n$, so 2^n is also illimited.

For positive x illimited and a limited, $x - a$ is illimited (and thus also nonstandard). Suppose the contrary: $x - a = b$, for some b limited. Then $x = a + b$, which by F', would imply x is standard, contradictory to x being illimited.

Exercises 2.8

2.8.1: Suppose we had a standard nonempty set with all nonstandard elements. Then the property "x is standard" would be set-forming here, and nonexample 2 implies this set must be finite, but inversion of theorem 2.4.2 says this set is then nonstandard, a contradiction. So any standard nonempty set must have at least one standard element.

2.8.2: A standard infinite set must have infinite nonstandard elements, otherwise by listing the nonstandard elements in a set and taking the difference we'd obtain that "x is standard" is set-forming, a contradiction by nonexample 2. By the same example, any infinite set of all nonstandard elements must be nonstandard, as "x is standard" is set-forming in this set.

2.8.3: The projections (two standard functions) uniquely define the sets E and F from $E \times F$, so E and F are both standard.

2.8.4: Continuing the earlier discussion on induction, for any (classical or not) property P, standardization says we can always form $T = {}^s\{x \in \mathbb{N}\ :\ P(x)\}$. Then $F = \mathbb{N}/T$ is also standard. Should $P(0)$ hold and $\forall^s n\ P(n) \Rightarrow P(n+1)$, then F must be empty (see the earlier discussion, it fits right in). So the conclusion that $P(n)$ holds for all standard $n \in \mathbb{N}$ holds.

2.8.5: B and C are obviously standard. Take $D = C/A = \{x \in \mathbb{Q}\ :\ 0 \leq x < \varepsilon\}$. D must be nonstandard as "x is standard" is set forming. So supposing A standard immediately leads to contradiction as this would uniquely define D via a standard formula, hence A is nonstandard.

2.8.6: Form