LAGA random notes

Hi, Peter here to explain the joke.

Euler's formula (well, one of them...) says $e^{\theta i} = \cos(\theta) + i\sin(\theta)$. So $e^{\theta i}$ is just shorthand for a vector of sorts, spinning in a circle according to θ. We add vectors by putting the head of one on the tail of the other, hence why graphing $e^{\theta i} + e^{\pi\theta i}$ looks like that.

Suppose while graphing $e^{\theta i} + e^{\pi\theta i}$, we find it "covers its own tracks", i.e. there exists $\theta_0,\ \theta_1$ such that

$(\theta_0 = \theta_1 \mod 2\pi)$ and $(\theta_0\pi = \theta_1\pi \mod 2\pi)$.

I.e. we just want to see if there is an initial condition θ_0 which comes with a final point θ_1 that puts the two vectors defined in each half of the function in the same "configuration" that it started with.

Letting $\phi = \theta/2$, this is the same as saying:

$(\phi_0/\pi = \phi_1/\pi \mod 1)$ and $(\phi_0 = \phi_1 \mod 1)$

The second condition says that ϕ_0 and ϕ_1 are rational "with respect to each other":

$$(\phi_0 = \phi_1 \mod 1) \iff (\phi_0 - \phi_1 = 0 \mod 1) \iff \phi_0 - \phi_1 \text{ is an integer.}$$

The first condition says similar:

$$(\phi_0 - \phi_1)/\pi \text{ is some integer } d \iff \pi = \frac{\phi_0 - \phi_1}{d}$$

So if the graph ever began tracing over itself, this is equivalent to saying we've found a rational expression for π.