algbra 0 aluffi

note: I write function / morphism composition as $\mathbf{f(g(x)) = gf = f \circ g}$.

2.5: f is an epimorphism means $\forall a, b \quad fa = fb \Rightarrow a = b$.

Let $f : X \to Y$ be an epimorphism, and suppose it is not surjective, i.e. $\exists\ y \in Y$ with an empty fiber. Consider two maps $a,\ b$ from Y to $\{0, 1\}$: a sends all of Y to 0, and b is the same, except for sending $y \mapsto 1$. Then $fa = fb$ but $a \neq b$, so epimorphisms must be surjections.

Now let f be surjective, i.e. every y has a nonempty fiber f_y (let $f(f_y)$ mean to pick an arbitrary $x \in f_y$ to put in f). Suppose for $a, b : Y \to Z$, $a \neq b$ but $fa = fb$.
Then $\exists y\ a(y) \neq b(y) \Rightarrow a \circ f(f_y) \neq b \circ f(f_y)$, but we supposed $a \circ f = b \circ f$. The only way out was if f_y was empty for some y, but f is surjective. Hence f surjective $\implies (fa = fb \Rightarrow a = b)$, i.e. f is an epimorphism.

2.9: Sets are isomorphic when they have the same cardinality.
$A' \cap B' = \emptyset = A'' \cap B'' \iff |A' \cup B'| = |A'| + |B'| = |A''| + |B''| = |A'' \cup B''| \iff A' \cup B' \cong A'' \cup B''$.

2.10: Each $a \in A$ has a choice of mapping to each $b \in B$. Since these choices are independent between different a's, we multiply by $|B|$ for each $a \in A$, i.e. $|B^A| = |B|^{|A|}$

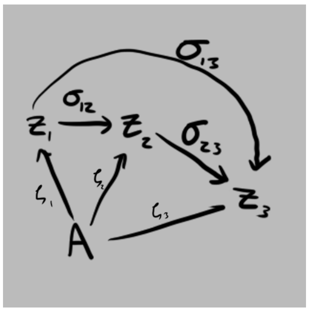
2.11: A bijection is: $\forall p \in \mathcal{P}(A)$, $\forall e \in A \quad e \mapsto 1$ if $e \in P$, otherwise $e \mapsto 0$. Since a unique subset determines a unique map and vice-versa (construct the inverse), this is indeed a bijection.

3.1: For $f \in \mathrm{Hom}_{\mathsf{C}^{op}}(B, A)$ and $g \in \mathrm{Hom}_{\mathsf{C}^{op}}(C, B)$, we define composition as $fg \in \mathrm{Hom}_{\mathsf{C}^{op}}(C, A)$. This is well defined through the parent category: $f \in \mathrm{Hom}_{\mathsf{C}}(A, B)$ and $g \in \mathrm{Hom}_{\mathsf{C}}(B, C) \Rightarrow fg \in \mathrm{Hom}_{\mathsf{C}}(A, C)$. Associativity and existence of the identity morphism also follow from the parent category.

3.4: No, there is no identity.

3.5: The $\subseteq$ relation is reflexive and transitive.

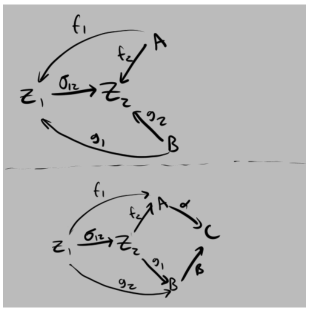
3.6: finite dimensional vector spaces, the morphisms are maps between these spaces. A matrix with 0 rows/columns is a map involving $\vec{0}$.

3.7: 

Elements of $\mathrm{Hom}(\zeta_1, \zeta_2)$ are morphisms σ_{12} such that $\zeta_1\sigma_{12} = \zeta_2$.
Composition is well-defined: $\zeta_1\sigma_{12}\sigma_{23} = \zeta_3 \Rightarrow \sigma_{12}\sigma_{23} = \sigma_{13} \in \mathrm{Hom}(\zeta_1, \zeta_3)$

3.9: I think an isomorphism between msets should map equivalent elements to equivalent elements, with the objects of mset being a set equipped with an equivalence relation $\sim$. Generalizing, the morphisms are just functions where $a \sim b \Rightarrow f(a) \sim f(b)$, that way any morphism which is a bijection is an isomorphism. Set is a full subcategory as it is true for all functions that $a = b \Rightarrow f(a) = f(b)$.

3.10: Any set with two elements is a subobject classifier in Set, with the subobjects being subsets.

3.11: 

Note that the bottom category $\mathsf{C}_{\alpha,\beta}$ "looks like" a full subcategory of $\mathsf{C}_{A,B}$, as in there is a one-to-one and onto mapping of objects and morphisms, respecting domain and composition, between $\mathsf{C}_{\alpha,\beta}$ and a full subcategory of $\mathsf{C}_{A,B}$.

For the top category $\mathsf{C}^{A,B}$, we have $f_1\sigma_{12} = f_2$ and $g_1\sigma_{12} = g_2$.

4.1: