Modern Sampling Methods

Class 6: Covariate-Adaptive Randomization

January 11, 2022

Outline

- Randomized Treatment Assignment
  - Simple Randomization
  - Efron's Biased Coin Design
  - Forced Balance
- Covariate-Adaptive Randomization / Stratification
  - Two-sample t-test
  - Regression with Strata Effects
  - Strata Saturated Regression

Setup

Units $i = 1, \ldots, n$.

Treatment: $T_i = 0, 1$.

Potential Outcomes: $Y_i(0)$, $Y_i(1)$

Observed outcome:

$$Y_i = (1 - T_i) Y_i(0) + T_i Y_i(1).$$

May also observe covariates X_i (invariant to treatment).

Average treatment effect:

$$\begin{aligned} ATE &= E\left[Y_i(1) - Y_i(0)\right] \\ &= E\left[Y_i(1)\right] - E\left[Y_i(0)\right]. \end{aligned}$$

Assignment

Recall: The *assignment mechanism* is the procedure that determines T_i given $Y_i(0)$ and $Y_i(1)$ (and given X_i if we also have covariates).

The assignment mechanism is key to understanding how observed outcomes (and treatments) are generated by potential outcomes.

$$\begin{array}{lcc} \underline{\text{``Model''}} & & \underline{\text{Data}} \\ \left.\begin{array}{l} (a)\ \{(Y_i(0), Y_i(1), X_i)), i = 1, \ldots, n\} \\ (b)\ Y_i = (1 - T_i)Y_i(0) + T_i Y_i(1) \\ (c)\ \textbf{Assignment Mechanism} \end{array}\right\} & \longrightarrow & \begin{array}{l} \{(Y_i, T_i, X_i)), \\ \quad i = 1, \ldots, n\} \end{array} \end{array}$$

Assignment Mechanisms

Let $W^{(n)} = \{(Y_i(0), Y_i(1), X_i)), i = 1, \dots, n\}$, $T^{(i-1)} = (T_{i-1}, \dots, T_1)$, and $T^{(0)} = \emptyset$.

1. Simple/Complete/Pure Randomization ("Coin Flipping")
   - $T_1, \dots, T_n$ i.i.d.
     - $\Pr(T_i = 1 | T^{(i-1)}, W^{(n)}) = \Pr(T_i = 1) = p$
     - $Cov(T_i, T_j) = 0, \quad i \neq j$
2. Restricted Randomization
   - Allows for dependence across units in randomization assignment
     - $\Pr(T_i = 1 | T^{(i-1)}, W^{(n)}) = \Pr(T_i = 1 | T^{(i-1)})$
     - $Cov(T_i, T_j) \neq 0, \quad i \neq j$
3. Covariate-Adaptive Randomization (Stratification)
   - Allows assignment to depend on baseline covariates
     - $\Pr(T_i = 1 | T^{(i-1)}, W^{(n)}) = \Pr(T_i = 1 | X^{(i)}, T^{(i-1)})$
4. Response-Adaptive Randomization
   - Allows assignment to depend on "earlier" outcomes
     - $\Pr(T_i = 1 | T^{(i-1)}, W^{(n)}) \Pr(T_i = 1 | X^{(i)}, Y^{(i-1)}, T^{(i-1)})$

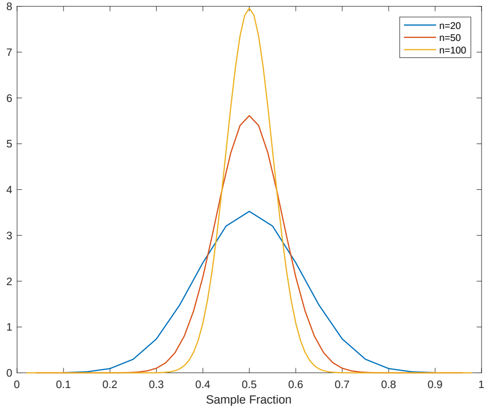
Simple Randomization

- Assignment Rule: $\Pr(T_i = 1) = \frac{1}{2}$
- Independence: $Var(T_i) = \frac{1}{4}$, $Cov(T_i, T_j) = 0$
- Balance: Large Sample vs. Small Sample
- Degree of Imbalance $=$ (#Treated) - (Target #Treated)

$$
\begin{aligned}
D_n &= \left(\sum_{i=1}^{n} T_i\right) - p \cdot n \\
\left(\text{setting } p = \frac{1}{2}\right) \qquad &= \sum_{i=1}^{n}\left(T_i - \frac{1}{2}\right) \\
&= \frac{1}{2}(\#Treated - \#Control)
\end{aligned}
$$

Degree of Imbalance under Simple Randomization

$$\Pr(D_n = d) = \frac{\binom{n}{d+\frac{n}{2}}}{2^n}, \quad d = -\frac{n}{2}, -\frac{n}{2}+1, \ldots, \frac{n}{2}$$



Approximate Distribution for Degree of Imbalance under Simple Randomization

Since degree of imbalance is a sum over i.i.d. random variables, $D_n = \sum_{i=1}^{n} \left(T_i - \frac{1}{2}\right)$, we can approximate its behavior by a normal distribution,

$$\frac{1}{\sqrt{n}} D_n \xrightarrow{d} N\left(0\,, \frac{1}{4}\right)$$

so, for large sample sizes,

$$\Pr(|D_n| \geq r) \approx 2\left[1 - \Phi\left(\frac{2r}{\sqrt{n}}\right)\right]$$

Effect of Imbalance on Power

Recall $H_0 : ATE = 0$ vs. $H_1 : ATE \neq 0$.

where

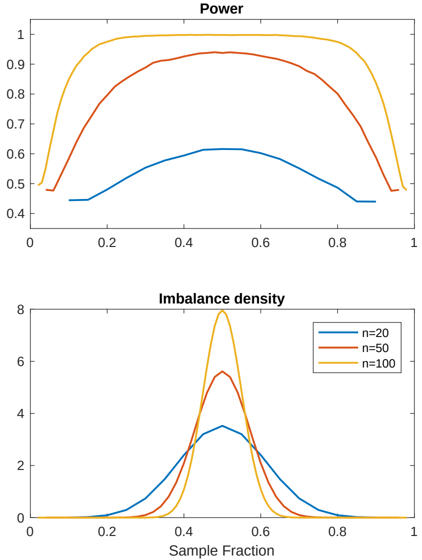
$$\begin{aligned} ATE &= \alpha_1 - \alpha_0 \\ &= E\left[Y_i \mid T_i = 1\right] - E\left[Y_i \mid T_i = 0\right] \end{aligned}$$

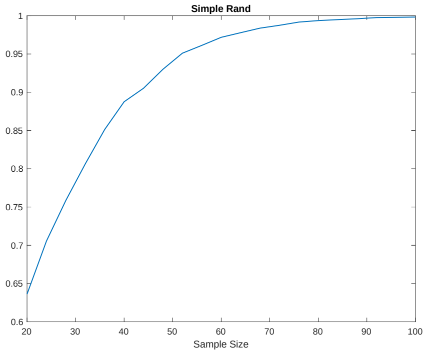
$$\hat{\beta} = \overline{Y}_1 - \overline{Y}_0$$

Two Sample t-test: $$t = \frac{\hat{\beta}}{\sqrt{\frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_0^2}{n_0}}}$$

$\Rightarrow$ Reject if $|t| > 1.96$

Simple Randomization - Power Simulation





▸▸ local power adjusted

Efron's Biased Coin Design

- Assignment Rule:

$$\Pr(T_i = 1 | T^{(i-1)}) = \begin{cases} \frac{1}{2} & \text{if } D_{i-1} = 0 \\ q_e & \text{if } D_{i-1} < 0 \\ 1 - q_e & \text{if } D_{i-1} > 0 \end{cases}$$

  where $q_e \in \left[\frac{1}{2}, 1\right]$, and $D_{i-1} = \left(\sum_{j=1}^{i-1} T_j\right) - \frac{i-1}{2}$

- Special Cases: $q_e = \frac{1}{2}$: Simple Randomization

  $q_e = 1$: "Pairwise" Randomization"

  $q_e = \frac{2}{3}$: Efron's suggested value

- $E(T_i) = \frac{1}{2}$, $Var(T_i) = \frac{1}{4}$
- $\Sigma = Var(T)$ and (Exact) distribution of D_n

  see Markaryan and Rosenberger (2010)

Approximate Distribution for Degree of Imbalance under Efron's BC

- $q_e = \frac{1}{2}$ Independent observations (Simple Randomization)

$$\frac{1}{\sqrt{n}} D_n \xrightarrow{d} N\left(0\ ,\frac{1}{4}\right)$$

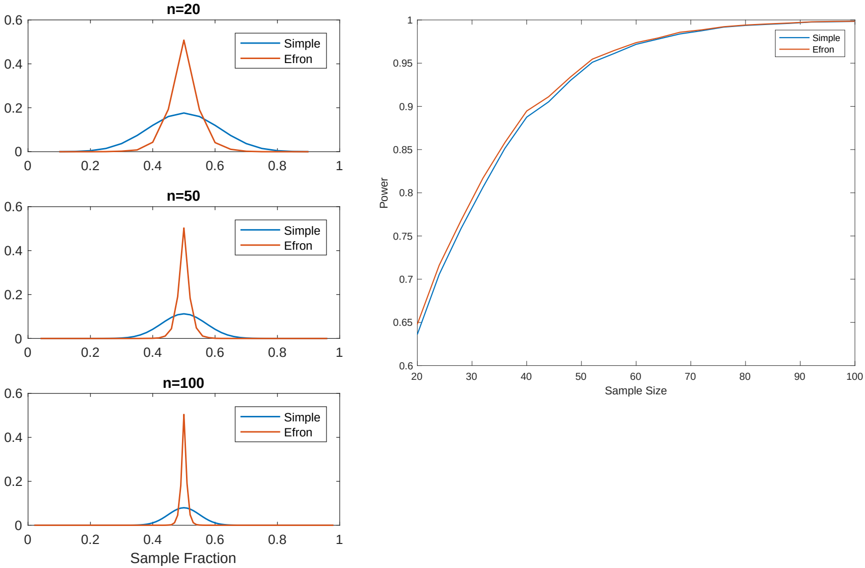
- $q_e > \frac{1}{2}$ Dependent observations

$$\frac{1}{\sqrt{n}} D_n \xrightarrow{p} 0$$

In fact, $D_n = 0$ with positive probability for large (even) sample sizes.

$$\Pr(D_n = 0 | n \text{ even}) \longrightarrow \frac{2q_e - 1}{q_e} \quad = \frac{1}{2} \left(\text{when } q_e = \frac{2}{3}\right)$$

Efron's Biased Coin - Power Simulation



Forced Balance Randomization

- Assignment Rule: $\binom{n}{n/2}$ equally likely arrangements with $n/2$ assigned $T = 1$ and $n/2$ assigned $T = 0$. (n even)

- Example. Sequential allocation rule

  Urn: $n/2$ balls labeled T (or 1) and $n/2$ balls labeled C (or 0)

$$\Pr(T_i = 1 | T^{(i-1)}) = \frac{\frac{n}{2} - \sum_{j=1}^{i-1} T_j}{n - (i-1)}$$

- $E(T_i) = \frac{1}{2}$, $Var(T_i) = \frac{1}{4}$,

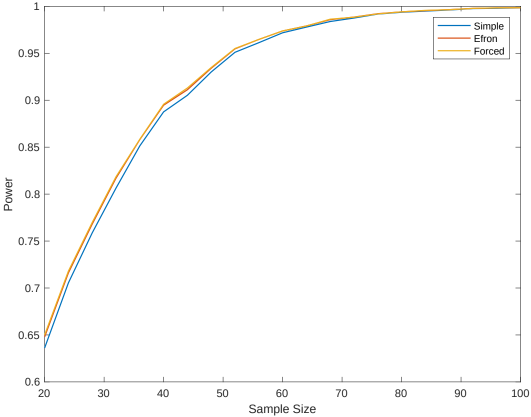
  Dependence: $Cov(T_i, T_j) = -\frac{1}{4(n-1)}$

- D_n ...

Simple: $$\frac{1}{\sqrt{n}} D_n \xrightarrow{d} N\left(0\,,\frac{1}{4}\right)$$

Efron's BC: $$D_n \xrightarrow{p} 0$$

Forced: $$D_n = 0 \quad (n \text{ even})$$



Baseline Covariate Stratification

1. Construct strata
2. Apply randomization within strata

Stratification: Partition covariate space into a finite number of strata.

e.g. $X_{1,i} = \text{region} \in \{1, 2, 3, 4\} \quad \text{(N, S, E, W)}$

$X_{2,i} = \text{income category} \in \{0, 1\} \quad \text{(low, high)}$

Strata $\mathscr{S} = \{(1,0),(1,1),(2,0),(2,1),(3,0),(3,1),(4,0),(4,1)\}$

$|\mathscr{S}| = 4 \times 2 = 8$

Let S denote the map from covariate values into strata values, $S(X_i) = S_i \in \mathscr{S}$

For each strata $s \in \mathscr{S}$, apply randomization ((e.g. Simple, Efron's BC, Forced Balance) with target fraction of treated $p(s)$.

Assignment Rule:

$$\Pr(T_i = 1 \mid S^{(i)}, T^{(i-1)})$$

Degree of Imbalance for each strata s:

$$D_n(s) = \sum_{i=1}^{n} (T_i - p(s))\, \mathbf{1}\{S_i = s\}$$

Suppose $p = p(s)$.

$$\left\{ \frac{1}{\sqrt{n}} D_n(s) : \; s \in \mathscr{S} \right\} \xrightarrow{d} N(0, \; \tau \; \Omega_D)$$

where $\Omega_D = \text{diag}\left(\Pr(S_i = 1), \ldots, \Pr(S_i = |\mathscr{S}|)\right)$

- Simple Randomization: $\quad \tau = p(1-p)$
- Efron's Biased Coin Design: $\quad \tau = 0$
- Forced Balance based on $\begin{pmatrix} n(s) \\ [pn(s)] \end{pmatrix}$ re-arrangments: $\quad \tau = 0$

$\tau = 0$: less imbalance, but more dependence in observations

Behavior of tests for $ATE = 0$ will depend on the value of τ

Inference on ATE with Covariates

Suppose $(Y_i(0), Y_i(1), X_i)$ i.i.d. (across i). Then stratified randomization implies unconfoundedness w.r.t. strata:

$$(Y_i(0), Y_i(1)) \perp T_i \mid S_i$$

$H_0 : ATE = 0$ vs. $H_1 : ATE \neq 0$.

- Two Sample t-test
- Regression with Strata Effects
- Saturated Regression

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see Bugni, Canay, and Shaikh (2018, 2019)

Two Sample t-test

$$t = \frac{\overline{Y}_1 - \overline{Y}_0}{\sqrt{\frac{\hat{\sigma}_1^2}{n_1} + \frac{\hat{\sigma}_0^2}{n_0}}}\ , \qquad \text{Reject if } |t| > 1.96$$

Regression equivalent:

$$Y_i = \gamma + \beta T_i + \varepsilon_i$$

t-test for $\beta = 0$ with heteroskedasticity-robust standard errors

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Result:
(i) Under H_0,

$$t \xrightarrow{d} N(0, \sigma^2_{t-test})$$

where $\sigma^2_{t-test} \leq 1$;

Result: (cont'd)

(ii) Moreover, the test is conservative, $\sigma^2_{t-test} < 1$, unless:

- No stratification

  So, for all previous simple and restricted randomization cases (ignoring covariates), two sample t-test is *not* conservative

  \- or -

- $E[Y_i(t)|S_i] = E[Y_i(t)]$ for $t = 0, 1$

  So, two sample t-test is *not* conservative when, there's no (mean) heterogeneity by strata

  \- or -

- Assignment by simple randomization

  So, stratified block randomization (forced balance by strata), two sample t-test will be too conservative

BCS (2018): fix two sample t-test standard error

Regression with Strata Effects

$$Y_i = \beta T_i + \sum_{s \in \mathscr{S}} \gamma_s \mathbf{1}\{S_i = s\} + \varepsilon_i^{se}$$

- $\beta = ATE$
- t-test for $\beta = 0$ with heteroskedasticity-robust standard errors

$$t^{se} = \frac{\hat{\beta}}{std\ err(\hat{\beta})}$$

- "Within" estimation (deviations from strata means)
  - Same $\hat{\beta}$
  - Different $std\ err(\hat{\beta})$, in general

Result:
(i) Under H_0,

$$t^{se} \xrightarrow{d} N(0, \sigma^2_{t^{se}})$$

where $\sigma^2_{t^{se}} \leq 1$;
(ii) Moreover, the test is conservative, $\sigma^2_{t^{se}} < 1$ unless

- As before, no stratification, no heterogeneity by strata, simple randomization
  - or -
- $p = \frac{1}{2}$
  Strata effects has an additional condition leading to non-conservativeness. This condition $p = \frac{1}{2}$ is common.

Notes:

- If $\tau = 0$, $\sigma^2_{t^{se}} = \sigma^2_{t-test}$. So, two-sample t-test and strata effects regression have same power (asymptotically).
- Imbens-Rubin (2015) consider other stratified block randomization designs and show $\hat{\beta}$ not consistent

Strata Saturated Regression

- $p(s) = p$: Two Sample t-test and Regression with Strata Effects
- $p(s)$ possibly nonconstant: Saturated Regression

$$Y_i = \sum_{s \in \mathscr{S}} \gamma_s \mathbf{1}\{S_i = s\} + \sum_{s \in \mathscr{S}} \theta_s \mathbf{1}\{T_i = 1, S_i = s\} + \varepsilon_i^{sat}$$

- $\theta_s = ATE(s)$ and $\beta = ATE = \sum_{s \in \mathscr{S}} \theta_s \Pr(S_i = s)$
- t-test for β with heteroskedasticity-robust standard errors

$$\hat{\beta} = \sum_{s \in \mathscr{S}} \hat{\theta}_s \left(\frac{\sum_{i=1}^{n} \mathbf{1}\{S_i = s\}}{n} \right) \quad \text{and} \quad t^{se} = \frac{\hat{\beta}}{std\ err(\hat{\beta})}$$

Result:

(i) Under H_0,

$$t^{sat} \xrightarrow{d} N(0, \sigma^2_{t^{sat}})$$

where $\sigma^2_{t^{sat}} \geq 1$;

(ii) Moreover, the test is invalid (does not control size), $\sigma^2_{t^{sat}} > 1$ unless

- No stratification (ignoring covariates)
  - or -
- No heterogeneity by strata:
  $E[Y_i(1) - Y_i(0)|S_i] = E[Y_i(1) - Y_i(0)]$

Notes:

- For two sample t-test and strata effects regression, the usual heteroskedasticity-robust variance estimator is too *large*, in general.
  For saturated regression, the usual heteroskedasticity-robust variance estimator is too *small*, in general.

- When $\tau = 0$ and $p(s) = p$, all three tests have the same asymptotic power. But, when $\tau > 0$, the saturated regression approach is most powerful.

Simple Randomization - Local Power Simulation