AMP(1) – Lab10 – Matrices

1. Content

Contents

2. Learning objectives

2.1. Exam objectives

By the end of this lab you should be able to (pen and paper):

- Master specific vocabulary about matrices like row matrix, column matrix, square matrix, zero matrix and identity matrix
- Given its conditions are met, apply the addition of matrices
- Apply the scalar multiplication of matrices
- Given its conditions are met, apply the subtraction of matrices
- Apply the transpose of matrices and understand symmetric matrices

- Given its conditions are met, apply the dot product of matrices
- Given its conditions are met, apply the matrix powers with natural exponents
- Understand and **use** the inverse of regular matrices
- Understand and apply the transpose or inverse of a matrix product (Socks ‘n Boots rule)

We advise you to **make your own summary of topics** which are new to you.

2.2. Supportive objectives

Specifically related to the above you should in GeoGebra Classic**5.0** be able to:

- Create specific matrices like row matrix, column matrix, square matrix, zero matrix and identity matrix
- Perform the addition, scalar multiplication and substraction of matrices
- Perform the transpose of matrices
- Perform the dot product of matrices and matrix powers with natural exponent
- Distinguish between invertible and singular matrices via their determinant
- Calculate the inverse of an invertible square matrix

3. Exercises

Dependent of the lab session you may work individually or teamed (organized by the lab attendant). In either case make sure that throughout the course of this lab, you re-save sufficiently your solution file on your local machine as

1DAExx-0y-name1(+name2+name3).GGB given **xx**=groupcode, **0y**=labindex

3.1. Matrices and Geogebra

Defining matrices in GeoGebra can be done in different ways. The easiest way is the following:

- Open the spreadsheet view
- Type in a matrix in the spreadsheet view
- Select the matrix element (use the shift key to allow multiple selection)
- Right click the selected matrix, choose create.Matrix

The generated matrix can be manipulated by double clicking (eg: change its name)

You can use this procedure when working on the exercises below, to check your calculated results.

3.2. Basic exercises

3.2.1. Matrix addition and scalar multiplication

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} + \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = 2 * \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

What conditions must be met to be able to add 2 matrices? Same dimensions

$$\begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} - \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} =?$$

0 (notation for zero matrix, p215)

$$\begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix}$$

What conditions must be met to perform the scalar multiplication? none

$$0 * \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} =?$$

0 (zero matrix)

$$1 * \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$7 * \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 7 & 14 & 21 \\ 28 & 35 & 42 \\ 49 & 56 & 63 \end{bmatrix}$$

3.2.2. Dot product and (natural) matrix powers

Write down the dimensions of both matrices. Indicate whether the multiplication is compatible, and if so, do the calculation:

$\begin{pmatrix} 1 & 2 \\ 8 & 6 \end{pmatrix} . \begin{pmatrix} 5 & 6 \\ 1 & 2 \end{pmatrix}$	2x2 . 2x2	OK	$\begin{pmatrix} 7 & 10 \\ 46 & 60 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 \end{pmatrix} . \begin{pmatrix} 1 & 2 \\ 8 & 6 \end{pmatrix}$	1X2 . 2X1	OK	$\begin{pmatrix} 17 & 14 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 8 \end{pmatrix} . \begin{pmatrix} 1 & 2 \\ 8 & 6 \end{pmatrix}$	2X1 . 2X2	NOK	
$\begin{pmatrix} 1 & 2 \\ 8 & 6 \end{pmatrix} . \begin{pmatrix} 5 \\ 6 \end{pmatrix}$	2X2 . 2X1	OK	$\begin{pmatrix} 17 & 56 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 \\ 8 & 6 \end{pmatrix} . \begin{pmatrix} 5 & 6 \end{pmatrix}$	2X2 . 1X2	NOK	
$\begin{pmatrix} 1 & 2 \end{pmatrix} . \begin{pmatrix} 5 & 6 \end{pmatrix}$	1X2 . 1X2	NOK	
$\begin{pmatrix} 1 & 2 \end{pmatrix} . \begin{pmatrix} 5 \\ 1 \end{pmatrix}$	1X2 . 2X1	OK	7
$\begin{pmatrix} 1 \\ 8 \end{pmatrix} . \begin{pmatrix} 5 & 6 \end{pmatrix}$	2X1 . 1X2	OK	$\begin{pmatrix} 5 & 6 \\ 40 & 48 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 8 \end{pmatrix} . \begin{pmatrix} 5 \\ 1 \end{pmatrix}$	2X1 . 2X1	NOK	

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^2 = ?$$ Impossible, because not square

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}^2 = ?$$ Impossible, because not square

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^2 = \begin{bmatrix} 7 & 10 \\ 15 & 22 \end{bmatrix}$$

What condition must be met if you want to calculate the power of a matrix? It must be square

3.2.3. Understanding the dot- product

Given are the integer matrices A_1 and A_2:

$$A_1 = \begin{bmatrix} 11 \\ 21 \\ 1 \end{bmatrix} \qquad A_2 = \begin{bmatrix} 11 & 12 & 13 \\ 21 & 22 & 23 \\ 1 & 1 & 1 \end{bmatrix}$$

Find the **integer** matrix B such that

$$B.A_1 = \begin{bmatrix} 11+5 \\ 21+6 \\ 1 \end{bmatrix} \quad \text{and} \quad B.A_2 = \begin{bmatrix} 11+5 & 12+5 & 13+5 \\ 21+6 & 22+6 & 23+6 \\ 1 & 1 & 1 \end{bmatrix}$$

$$\underbrace{\underset{3\times 1}{B.A_1} = \underbrace{\begin{pmatrix} 11+5 \\ 21+6 \\ 1 \end{pmatrix}}_{3\times 1}}$$

$\Leftarrow B = 3\times 3$

$$\begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix} \begin{pmatrix} 11 \\ 21 \\ 1 \end{pmatrix} = \begin{pmatrix} 11+5 \\ 21+6 \\ 1 \end{pmatrix}$$

2e product matricial

$$\begin{pmatrix} \text{first order} + 5 \\ \text{second order} + 6 \\ 1 \end{pmatrix}$$

$$\underbrace{b_{11}}_{1} \cdot 11 + \underbrace{b_{12}}_{\emptyset} \cdot 21 + \underbrace{b_{13}}_{5} \cdot 1 = 11 + 5$$

$$b_{21} \cdot 11 + b_{22} \cdot 21 + 1 \cdot b_{23} = 21 + 6$$
$$b_{21} = 0 \qquad b_{22} = 1 \qquad b_{23} = 6$$

$$b_{31} \cdot 11 + b_{32} \cdot 21 + b_{33} \cdot 1 = 1$$

$$b_{31} \cdot 11 + b_{32} \cdot 21 + b_{33} \cdot 1 = 1$$
$$b_{31} = 0 \qquad b_{32} = 0 \qquad b_{33} = 1$$

$$\begin{pmatrix} 1 & 0 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix} = B$$

Find the matrix C such that

$$C.A_1 = \begin{bmatrix} 11\pi \\ 21\pi \\ 1 \end{bmatrix} \quad \text{and} \quad C.A_2 = \begin{bmatrix} 11\pi & 12\pi & 13\pi \\ 21\pi & 22\pi & 23\pi \\ 1 & 1 & 1 \end{bmatrix}$$

$$\underbrace{C}_{3\times3} . \underbrace{A_1}_{3\times1} = \underbrace{\begin{matrix} 11\pi \\ 21\pi \\ 1 \end{matrix}}_{3\times1}$$

$$C . A_1 = \begin{pmatrix} \pi & a_{11} \\ \pi & a_{21} \\ 1 & \end{pmatrix}$$

$$\begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix} = \begin{pmatrix} \pi a_{11} \\ \pi a_{21} \\ \pi a_{31} \end{pmatrix}$$

$$\underset{\pi}{\underset{\shortparallel}{c_{11}}} a_{11} + \underset{0}{\underset{\shortparallel}{c_{12}}} a_{21} + \underset{0}{\underset{\shortparallel}{c_{13}}} a_{31} = \pi a_{11}$$

$$\underset{0}{\underset{\shortparallel}{c_{21}}} a_{11} + \underset{\pi}{\underset{\shortparallel}{c_{22}}} a_{21} + \underset{0}{\underset{\shortparallel}{c_{13}}} a_{31} = \pi a_{21}$$

$$\underset{0}{\underset{\shortparallel}{c_{31}}} a_{11} + \underset{0}{\underset{\shortparallel}{c_{32}}} a_{21} + \underset{1}{\underset{\shortparallel}{c_{33}}} a_{31} = 1$$

$$C = \begin{pmatrix} \pi & 0 & 0 \\ 0 & \pi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Find the matrix D such that

$$D.A_1 = \begin{bmatrix} 11\pi + 5 \\ 21\pi + 6 \\ 1 \end{bmatrix} \quad \text{and} \quad D.A_2 = \begin{bmatrix} 11\pi + 5 & 12\pi + 5 & 13\pi + 5 \\ 21\pi + 6 & 22\pi + 6 & 23\pi + 6 \\ 1 & 1 & 1 \end{bmatrix}$$

What is the relationship between B, C and D? Prove this relationship

B * C = D (see geogebra file 3.2.3 understanding the dot product.ggb)

3.3. Bridging exercises

3.3.1. Inverse matrices

Calculate the determinant of the following matrices if any:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$$

Only for square matrices, so only the two last

Third one: 1*4 -2*3 = -2: has an inverse

Fourth one: 3*4 – 2*6 = 0: no inverse

For the above matrices, which of the matrices below is its inverse?

$$\begin{bmatrix} 2 & -1 \\ -3/2 & 1/2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/4 & 1/5 & 1/6 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ 3/2 & -1/2 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1/4 \\ 1/2 & 1/5 \\ 1/3 & 1/6 \end{bmatrix}$$

$$\begin{bmatrix} -2 & 1 \\ -3/2 & 1/2 \end{bmatrix}$$

Its the second one in the first column (calculate by hand or use geogebra)

3.3.2. Noncommutative dot product of matrices

Calculate the following matrix (dot) products:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

See uncummutative.ggb

3.4. Terminology

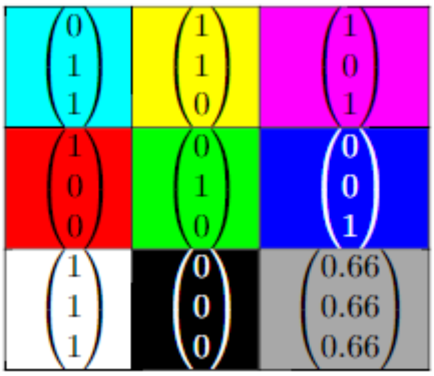
- What is a square matrix? Colcount = rowcount
- What is a zero matrix? What is the notation of a zero matrix? 0 (all zeros)
- What is an identity matrix? What is the notation of an identity matrix? all zeros, except main diagonal which contains 1, square only
- What is the minor of the element $a_{i,j}$ in a matrix?p216, top paragraph, square only
- What is the cofactor of the element $a_{i,j}$ in a matrix? p216, top paragraph, square only
- Write down the definition of the determinant of a matrix for 3x3: p 216, square only
- Why is it important to know the determinant of a matrix? If not 0, invertible, square only
- What is an opposite matrix? P218: opposite of B = -1B
- What is the transpose of a matrix?p220: rewriting rows as columns and vice versa (dimension does change if not squared)
- What is the notation of a transposed matrix?superscript T
- What is a zero divisor? A non zero matrix A for which exists a non zero matrix B such that A.B=0
- What is the inverse of a matrix? The inverse of A is the matrix B such that A.B = I = B.A (square only)
- What is a singular matrix? Matrix with determinant 0 (no inverse), square only
- What is an invertible matrix? Matrix with determinant !=0 (has inverse): square only
- Explain the 'Socks-and-Boots' formulation: while dressing: first socks then boots, while undressing: first boots, then socks

For the above definitions, which apply to square matrices only?

3.5. Contextual practice

3.5.1. Matrix operations to perform colour transformations

The following matrix represents RGB colours using floats, a very common format in 3D environments:



Apply the following matrixes to all the colours and write down the new colour. The matrices that are given are all 4x4 matrices which allows for some special operations (for example the negative operation). To be able to perform the matrix multiplications it is necessary to add an extra row with value 1 to each color:

An example:

$$\begin{pmatrix}0\\1\\1\end{pmatrix} \text{ becomes } \begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

Do this for all the colors in the above table.
It is now possible to carry out the matrix multiplication:

$$\begin{pmatrix}1&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}0\\1\\1\\1\end{pmatrix}$$

a. Monochrome – 'yellowish

Apply the following matrix to the above colours and explain what is happening and how it is happening. Try to look up the colours

$$\begin{pmatrix}1&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&0&1\end{pmatrix}$$

$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\end{pmatrix}$
$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$
$\begin{pmatrix}1\\1\\0\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0.66\\0.66\\0\end{pmatrix}$

b. Saturation

Start by calculating the following matrix and apply it to the colours. What are the new colours you get?

$$\begin{aligned} s &= 0.1\\ sr &= (1-s)*0.2125\\ sg &= (1-s)*0.71154\\ sb &= (1-s)*0.0721 \end{aligned}$$

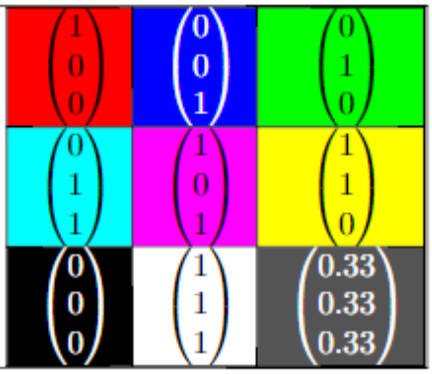
$$\begin{pmatrix}sr+s&sg&sb&0\\sr&sg+s&sb&0\\sr&sg&sb+s&0\\0&0&0&1\end{pmatrix}$$

$\begin{pmatrix}0.70\\0.80\\0.80\end{pmatrix}$	$\begin{pmatrix}0.35\\0.25\\0.35\end{pmatrix}$	$\begin{pmatrix}0.93\\0.93\\0.83\end{pmatrix}$
$\begin{pmatrix}0.29\\0.19\\0.19\end{pmatrix}$	$\begin{pmatrix}0.64\\0.74\\0.64\end{pmatrix}$	$\begin{pmatrix}0.70\\0.80\\0.78\end{pmatrix}$
$\begin{pmatrix}0.99\\0.99\\0.99\end{pmatrix}$	$\begin{pmatrix}0\\0\\0\end{pmatrix}$	$\begin{pmatrix}0.65\\0.65\\0.65\end{pmatrix}$

c. Negative

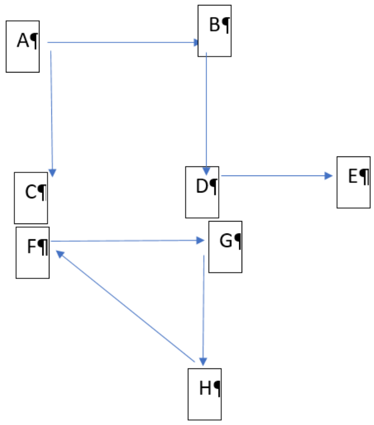
Apply the matrix to the starting colours above and explain what is happening

$$\begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$



Notice that the W-value of the RGB vector needs to be 1 or the translation will not be applied and you will get negative numbers.

3.5.2. Adjacency matrix



Look up the definition of a directed adjacency matrix. Create the adjacency matrix for the following graph describing waypoint navigation in a game

Use powers of this matrix to find how many steps it takes to

- move from A to D: 2
- move from A to H: never

What's the maximum exponent for any given graph, which (the exponent) adds information concerning the number of steps it takes to move from 1 vertex in the graph to another one: number of vertices - 1

Hint: define a matrix which is the power of the adjacency matrix, in which the power is a slider variable

3.5.3. Population models and matrices

Matrices can be used to describe how a population evolves. Look at the video found at population models and matrices.

Consider an initial population of 100 immature, 0 mature and 0 post mature elements:

I	M	P
100	0	0

We can describe the population evolution from this current situation to the next (let's say a breeding cycle) with a 3x3 matrix:

	I	M	P
I	0.5	0.3	0
M	0.8	0.6	0.2
P	0	0	0.4

- Pen and paper: calculate the population after 1 step (see ggb)
- Geogebra: define a power matrix for the breeding matrix, in which the power is a slider variable. Allow this slider to take value in [1,500].
- Will this population flourish or perish? flourish
- Whatever the result, change the breeding factor of the mature individuals such that the result (perish of flourish) changes. Change the factor in steps of 0.1, and find at which new value the change occurs (if any). 0.6
- Put that number to 0.8 again and change the number specifying the evolution from mature to mature such that you switch the result again. Move in steps of 0.1. What's the first number causing the change? 0.5
- What would this 3x3 matrix look like when all individuals were female? 0.8 (row2, col1) would be 0 (no reproduction)

4. References

4.1. Demos in art and programming

- If you want to program some advanced graphical matrix operations, have a look at Kernel (image processing). These matrix operations allow you to enhance digital material.
- The population matrix is a simplified version of Markov chains. A different population approach can be found at Lotka–Volterra equations (predator-prey)