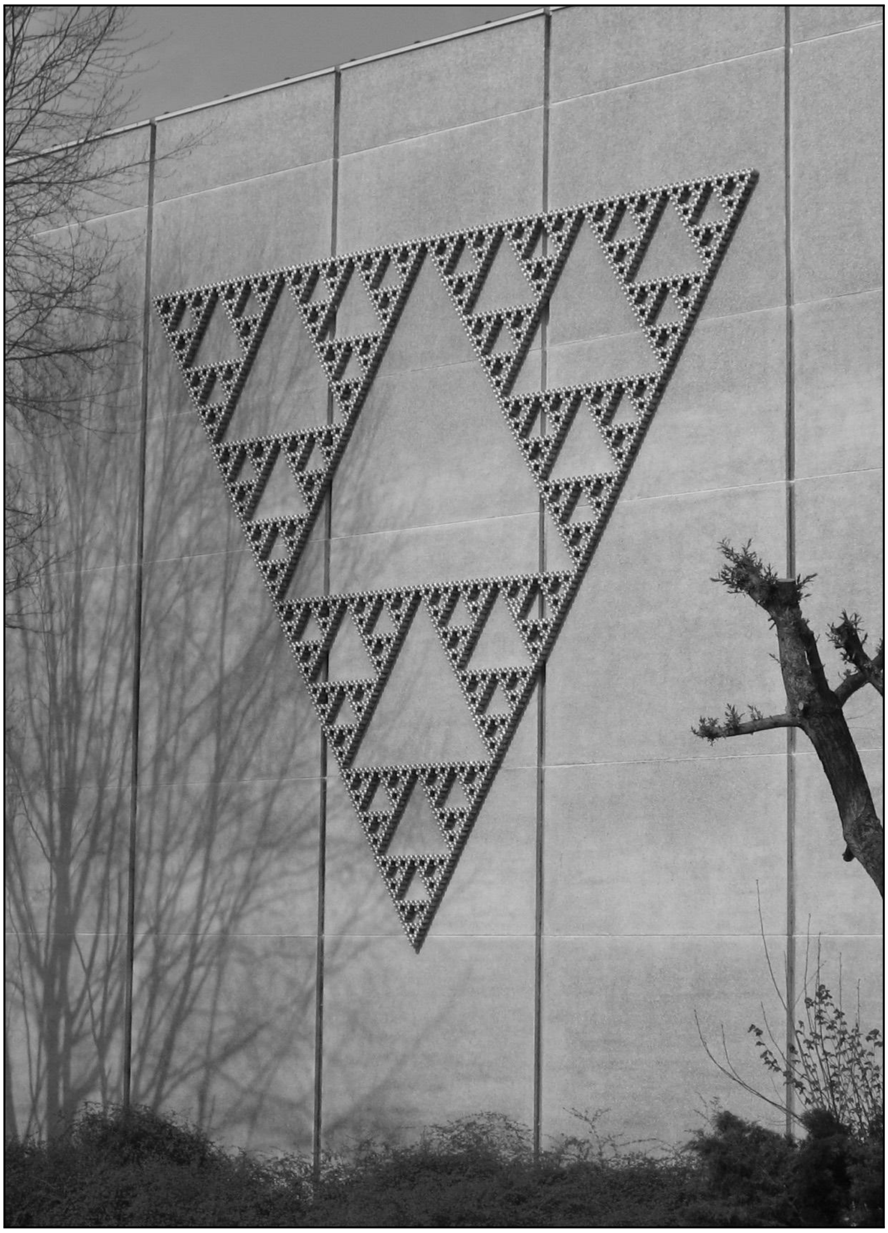
Chapter 3 · Trigonometry



Computer animation often deals with variables representing position, distance, angle, ... The major mathematical concepts that cover these measures are sine and cosine. All the remaining concepts in this chapter recur frequently in the subsequent chapters.

Trigonometry originated from measuring our surrounding space and definitely leads to a multitude of applications such as coordinate systems, rotation, periodical phenomena (light and sound), ... We initially define what an angle is in order to turn to triangles. These triangles allow us to define the trigonometric concepts which we then extend to the unit circle. We finalise this chapter by outlining the inverse trigonometric functions.

3.1 Angles

We consider the cone that connects the pupil of our eye to a golf ball at arm's length. Cutting this 3D-cone vertically, we obtain a plane angle formed by its two rays or sides, sharing the common endpoint or vertex O (at our pupil). We now try to measure the size of this spacial quantity 'angle' as the inner sector bordered by both intersecting sides.

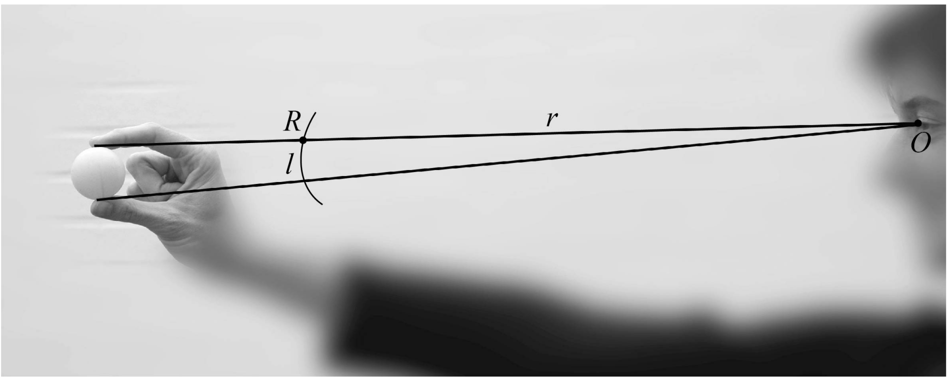


Figure 3.1: Definition of an angle

We catch this sector using a pair of compasses and define an **angle** as the circular arc length l to its radius $|OR| = r$:

$$\alpha = \frac{l}{r}. \tag{3.1}$$

As the above fraction is insensitive to scaling, it indeed guarantees a reliable measurement for any angle. For instance, when we measure the size of an angle as a circular arc of 6 cm to a radius of 5 cm, then it will remain as a circular arc of 24 cm to its proportionally adjusted radius of 20 cm. We then calculate $\alpha = \frac{6\,\text{cm}}{5\,\text{cm}} = \frac{24\,\text{cm}}{20\,\text{cm}} = 1.2$. The unit for this angle α, based upon the dimensionless fraction $\frac{\text{circular arc length}}{\text{radius}}$, is called **radians** and commonly typeset as rad. Hence the above discussion led to an angle of 1.2 rad.

We may be more familiar with **degrees** as a common unit for angles. This ancient Sumerian measure refers to the approximately 360 days for the Earth to complete its 'circular' orbit around the Sun, stating $\alpha_{\max} = 360°$.

Converting angles from radians to degrees and the other way round can be done easily by applying the **Rule Of Three**. We obtain the maximal plane angle by revolving one ray until $\alpha_{\max} = \frac{l_{\max}}{r} = \frac{\text{circle circumference}}{r} = \frac{2\pi r}{r} = 2\pi$ rad. Hence we can rely on the **full angle** equality 2π rad $= 360°$ given $\pi \approx 3.14$.

Example: Conversion between radians and degrees.

from degrees to radians	from radians to degrees
$360° = 2\pi$ rad	2π rad $= 360°$
$1° = \frac{2\pi \text{ rad}}{360}$	$1 \text{ rad} = \frac{360°}{2\pi}$
$30° = \frac{2\pi \text{ rad}}{360} \cdot 30 = \frac{\pi}{6}$ rad	$\frac{\pi}{6}$ rad $= \frac{360°}{2\pi} \cdot \frac{\pi}{6} = 30°$

The fractional part of an angle represented in degrees can be expressed in two different base forms. On the one hand, we can use the popular DD-form (Duo Decimal) for it, writing the fraction in the decimal number base 10. On the other hand, we might use the ancient DMS-form (Degrees Minutes Seconds) for it, writing the fraction in the sexagesimal number base 60 using accents for separating minutes from seconds. Our contemporary time keeping still uses minutes and seconds as parts of one hour. We might for instance express an angle of $180.5°$ as $180°30'00''$ in the latter system. Converting fractional parts of angles from DD to DMS and the other way round goes the same way as converting numbers from decimal base to number base $B = 60$.

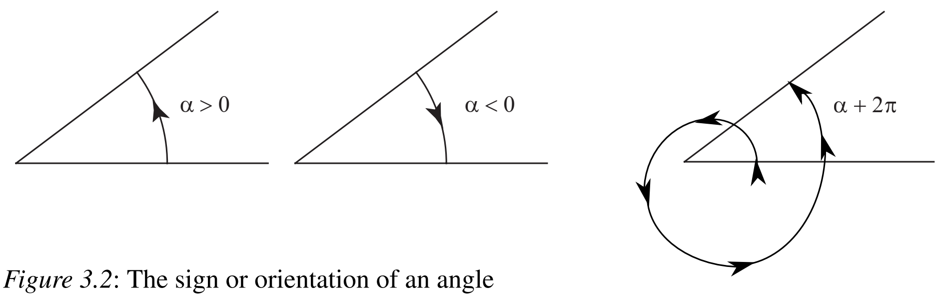


Figure 3.2: The sign or orientation of an angle

We can represent the same angle in different ways. We even may attribute a sign to an angle. We define **positive angles** by revolving a ray counter clockwise. Alternatively, we define **negative angles** by revolving a ray clockwise. Furthermore, all angles α and α' for which $\alpha' = \alpha + k\,2\pi$ with parameter $k \in \mathbb{Z}$ (or alternatively $\alpha' = \alpha + k\,360°$) are equivalent due to the full angle being their **elementary period**.

We differentiate the types of angles in some more detail. A **zero** angle equals exactly $0°$. An **acute** angle is larger than $0°$ and smaller than $90°$. A **right** angle equals exactly $90°$ or the quarter of a circle. Both sides of a right angle are said to be orthogonal or perpendicular. An **obtuse** angle is larger than $90°$ and smaller than $180°$. A **straight** angle equals exactly $180°$ or the half of a circle. A **reflex** angle is larger than $180°$ and smaller than $360°$ (full circle).

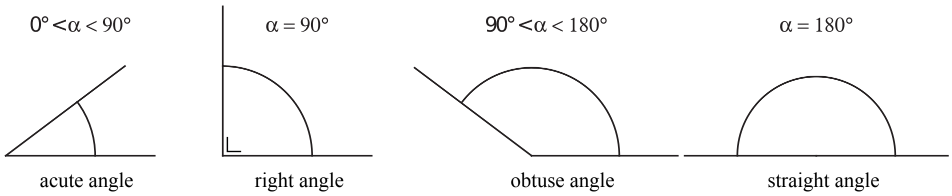


Figure 3.3: Types of angles

3.2 Triangles

Every three points or vertices A, B and C that do not lie on the same straight line make up a **scalene triangle** ABC when it has no two sides of equal length (see figure 3.10). The measures of the interior angles of a triangle ABC always add up to the straight angle, $\hat{A} + \hat{B} + \hat{C} = 180° = \pi$ rad.

A **right triangle** is a triangle having one of its interior angles equal to $90° = \frac{\pi}{2}$ rad. Consequently, the sum of both acute angles of any right triangle equals $90° = \frac{\pi}{2}$ rad. Its largest side is the edge opposite the right angle and is called the **hypotenuse**.

An **isosceles triangle** is a triangle having two equal sides through the apex and their corresponding base angles having the same measure. An **equilateral triangle** is a triangle having three equal sides, their equal interior angles measuring $60°$.

We recall some geometric concepts as they are defined and used in triangles. An **angle bisector** of a triangle is the straight line through a vertex which cuts the corresponding angle in half. A **median** or **side bisector** of a triangle is the straight line through a vertex and the midpoint of the opposite side. An **altitude** of a triangle is the straight line through a vertex and perpendicular to the opposite side. This opposite side is called the **base** of the altitude and its intersection point is called the **foot**. A **perpendicular bisector** of a triangle is the straight line through the midpoint of a side and being perpendicular to it. In a scalene triangle (see figure 3.4) these four geometric lines differ clearly. In isosceles triangles the four geometric lines through the apex coincide. In equilateral triangles the four geometric lines always coincide.

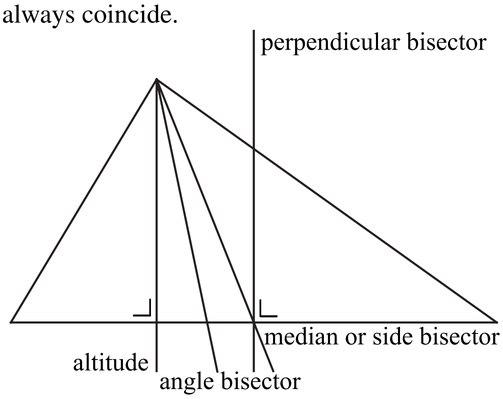


Figure 3.4: Geometric lines in a triangle

Two plane polygons are **similar** in case

- ▷ the ratio of their corresponding edges is constant, and
- ▷ their corresponding angles measure the same size.

We recall more specifically a similarity criterion for triangles: *two triangles are similar whenever two of their corresponding angles measure the same size.*

Before proceeding with the discussion on triangles we explain the ruler-and-compass method to construct a perpendicular bisector on a given line segment, as it will be useful in subsequent chapters.

- ▷ We construct the perpendicular bisector m of a segment $[AB]$ by drawing two circles $C(A,r)$ and $C(B,r)$ with the same radius r. These circles intersect, when applying a sufficiently large radius $r > 0$, in two points spanning the perpendicular bisector of the segment $[AB]$. Consequently we acquire the midpoint M of the segment $[AB]$ as a bonus.
- ▷ We construct the altitude l from apex D to a base k by drawing a circle $C(D,s)$ which intersects the base k in two distinguished points P and Q. Finally we apply the above construction for the perpendicular bisector m of the segment $[PQ]$. Consequently we acquire the foot as the midpoint D' of the segment $[PQ]$.

We refresh the **Pythagorean theorem**, named after the Greek **Pythagoras of Samos** (582–507 Before Christ), which states in any right triangle: the square of the length of the

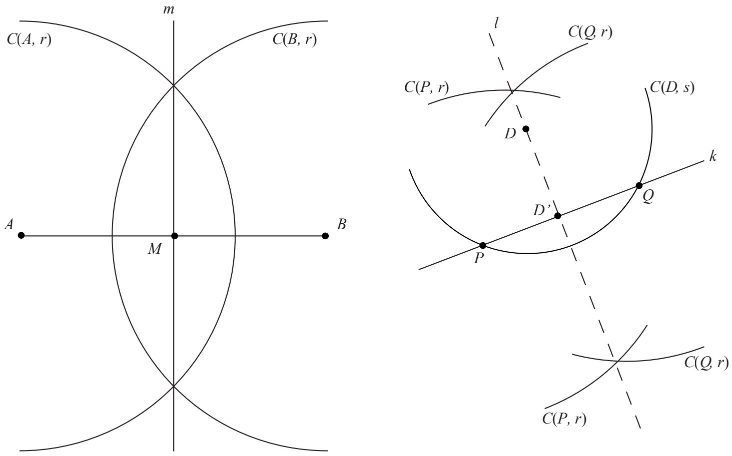


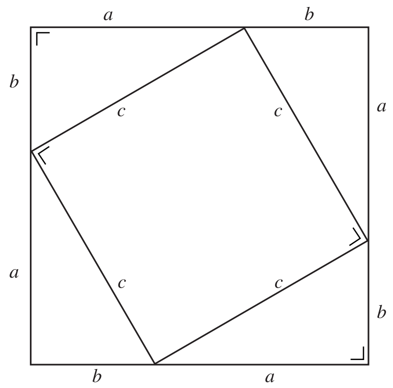
Figure 3.5: Constructions of the perpendicular bisector m and the altitude l

hypotenuse equals the sum of the squares of the lengths of the two other sides.

$$(\text{side}_1)^2 + (\text{side}_2)^2 = (\text{hypotenuse})^2 \tag{3.2}$$

Figure 3.6: Statue of Pythagoras on the isle of Samos

Proof: One way to prove this famous theorem goes via the area of this tiled square. In this square we count four identical right triangles featuring hypotenuse c and other sides a and b.



large square area = inner square area + 4 times right triangle area

$$(a+b)^2 = c^2 + 4\frac{ab}{2}$$

$$a^2 + 2ab + b^2 = c^2 + 2ab$$

$$a^2 + b^2 = c^2$$

■

Example: A staircase consists of 17 steps of height 19 cm and depth 15 cm. Find the length of the entire banisters.

The Pythagorean theorem yields $\sqrt{19^2+15^2}$ for the length of the hypotenuse of a single step. In conclusion, for the length of the banisters we multiply $17 \cdot \sqrt{19^2+15^2} = 411.53$ cm.

Game programming often involves calculating the distance between two points on the screen, e.g. between the anchor points of two colliding objects or two interacting personages. The programmed game may for instance respond as soon as its player moves sufficiently near the enemy. We realise now that various screen situations require a fast calculation of the distance between two points. Applying the Pythagorean theorem is the straightforward way to do so.

Assuming we know two points $P(x_1,y_1)$ and $Q(x_2,y_2)$ allows us to draw a right triangle with hypotenuse $[PQ]$. Figure 3.7 reveals (x_2-x_1) and (y_2-y_1) for the lengths of the other sides. In conclusion, the Pythagorean theorem states for the distance between P and Q, typeset as $d(P,Q)$, the formula

$$d(P,Q) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}. \tag{3.3}$$

Example: Find the distance between the points $A(1,2)$ and $B(5,6)$.
Their distance yields $d(A,B) = \sqrt{(5-1)^2+(6-2)^2} = \sqrt{32}$.

We can extend the plane distance formula to three dimensions. We just need to take the z-coordinate into account to realise it. Consequently, the distance between two spacial points $P(x_1,y_1,z_1)$ and $Q(x_2,y_2,z_2)$ then equals

$$d(P,Q) = |PQ| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}. \tag{3.4}$$

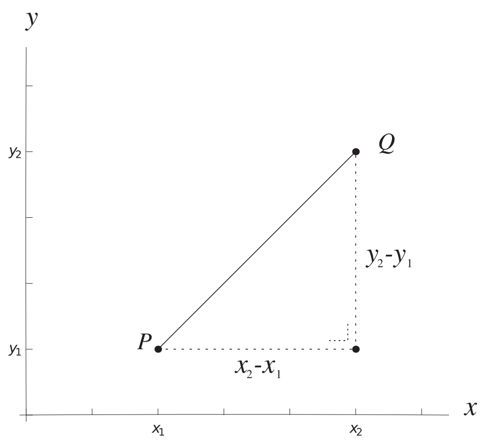


Figure 3.7: Distance between two points

3.3 Right triangle

We recall measuring the size of angles in radians as defined by the ratio $\alpha = \frac{l}{r}$. Alternatively we define more such ratios as based upon a right triangle. We replace the drawing of a circular arc on radius r (using a pair of compasses) by the drawing of an opposite side perpendicular to r (using a square). We realise the elegance of using a square over the use of a pair of compasses, as the length of a side is far easier to determine than the length of a circular arc.

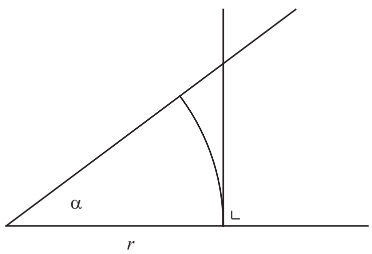


Figure 3.8: Definition of trigonometric ratios

This way we acquire a new measure for the interior angle α as the ratio $\frac{\text{opposite side}}{r}$. Such a measure is a 'trigonometric ratio'. We define the above ratio of sides as the **tangent** of the acute angle α. As the above fraction is insensitive to scaling, it again guarantees a reliable measurement for any angle.

The total number of trigonometric ratios is six and they are called **sine**, **cosine**, tangent, **cotangent**, cosecant and secant. They are of fundamental importance to study and practice the more advanced trigonometry.

We hereby list all correct definitions for the ratios sine (sin), cosine (cos) and tangent (tan), as for their reciprocals cosecant (csc), secant (sec) and cotangent (cot).

$$\sin\alpha = \frac{\text{opposite side}}{\text{hypotenuse}} \qquad \csc\,\alpha = \frac{1}{\sin\alpha}$$
$$\cos\alpha = \frac{\text{adjacent side}}{\text{hypotenuse}} \qquad \sec\alpha = \frac{1}{\cos\alpha}$$
$$\tan\alpha = \frac{\text{opposite side}}{\text{adjacent side}} \qquad \cot\,\alpha = \frac{1}{\tan\alpha}$$

As a useful mnemonic for the above ratios we make acronyms using their starting characters: 'SOH, CAH, TOA'. We realise that these definitions, as based upon right triangles, are limited to acute angles α only. We recall that each of both acute angles of a right triangle measures a size smaller than 90°.

Based upon the above trigonometric ratios, we immediately discover our first trigonometric formula as

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}. \tag{3.5}$$

3.4 Unit circle

The **unit circle** is a circle within the orthonormal (see paragraph 6.1) frame. The centre of the unit circle is the origin O and its radius equals one. We consider the unit circle as being our trigonometric 'dashboard' because for any angle α revolved from the horizontal x-axis (reference side), we find a corresponding point E_α on it (see figure 3.9).

We subdivide this unit circle in four equal parts that we index counter clockwise, starting from the zero angle (on the positive x-axis). Consequently, all acute angles between 0° and 90° lie in the first quadrant and all obtuse angles are in the second quadrant bordered by 90° and 180°. The reflex angles fall in the third quadrant as bordered by 180° and 270° or eventually in the fourth quadrant between 270° and 360°.

We simplify the former trigonometric ratios sine, cosine and tangent for a hypotenuse corresponding to the radius of the unit circle, in other words for the hypotenuse set to one. Note that we redefine the trigonometric tangent via a larger right triangle bordering the vertical tangent line to the unit circle at the point $E_0(1,0)$.

$$\cos\alpha = \frac{\text{adjacent side}}{1}$$
$$\sin\alpha = \frac{\text{opposite side}}{1}$$
$$\tan\alpha = \frac{\text{opposite side at the vertical tangent line } x = 1}{1}$$

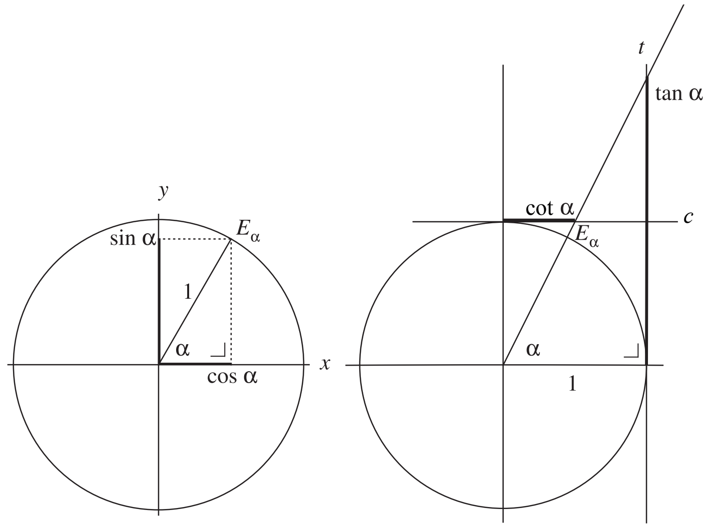


Figure 3.9: Trigonometric ratios in the unit circle

In the unit circle we interpret $\cos\alpha$ as the horizontal shadow on the x-axis caused by the revolving side of angle α. Likewise, we meet $\sin\alpha$ as the vertical shadow on the y-axis caused by the revolving side of angle α. Finally, we encounter $\tan\alpha$ as the opposite side to the angle α on the vertical tangent line $x = 1$. In other words, for the revolving side of an angle α intersecting the unit circle in the corresponding point E_α we define the x-coordinate of E_α as the **cosine** of α and the y-coordinate of E_α as the **sine** of α. Any angle α therefore determines unambiguously its cosine and sine. But for the other way round, every couple of cosine and sine values determines its angle α only to an integer multiple of 2π rad or 360°. This is due to the periodicity of each angle with its **elementary period** of 2π rad or 360°.

The redefined sine, cosine and tangent in a unit circle overcome their previous limitation to acute angles in such a way that they now are defined for any angle size.

Applying the Pythagorean theorem (3.2) in the unit circle, yields

$$(\sin\alpha)^2 + (\cos\alpha)^2 = 1. \tag{3.6}$$

This important formula linking sine and cosine is known as the trigonometric **Pythagorean Identity**.

The Pythagorean theorem applies to right triangles only. Scalene triangles are ruled by the **Law of Cosines** and the **Law of Sines** to link their interior angles to the length of their sides. For a scalene triangle ABC featuring interior angles α, β and γ at corresponding vertices A, B and C and opposite sides a, b and c, we state both laws (omitting their proofs):

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} \qquad \text{Law of Sines,} \tag{3.7}$$

$$a^2 = b^2 + c^2 - 2bc\cos\alpha \qquad \text{Law of Cosines.} \tag{3.8}$$

The Law of Cosines for a right angle α implies $\cos\alpha = 0$ and thus simplifies to the Pythagorean theorem. In addition, we refresh the general formula to calculate the area of a scalene triangle as

$$\text{area}_{triangle} = \frac{1}{2} a\, b \sin\gamma \qquad \text{Area of triangles.} \tag{3.9}$$

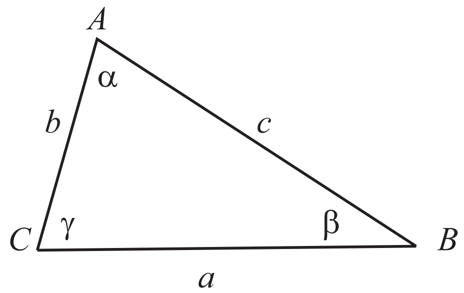


Figure 3.10: Scalene triangle formulas

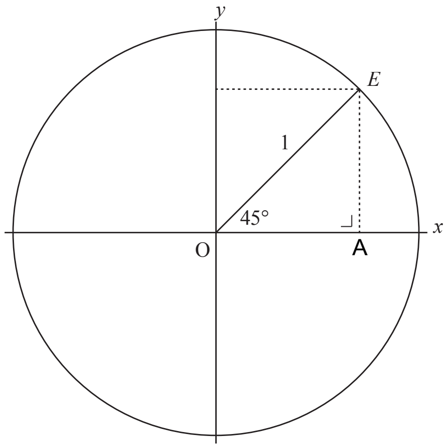
3.5 Special angles

Given their definitions, the trigonometric ratios are limited to these intervals:

$\sin\alpha \in [-1, 1]$ $\qquad$ $\cos\alpha \in [-1, 1]$

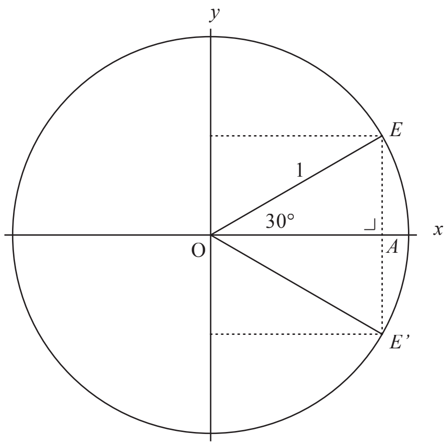
$\tan\alpha \in]-\infty, +\infty[$ $\qquad$ $\cot\alpha \in]-\infty, +\infty[$

Trigonometric ratios for an angle of $45° = \frac{\pi}{4}$ rad



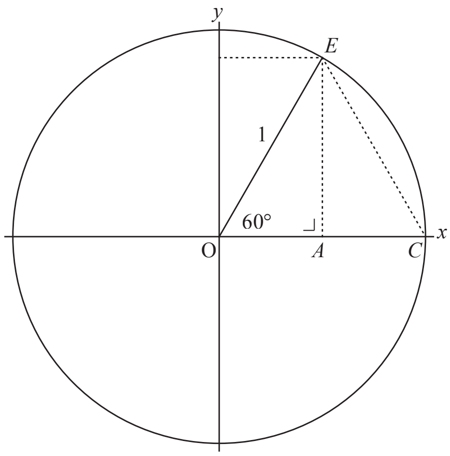
As the interior angles of the triangle OAE add up to $180°$, we find the angle $\hat{E}$ equals $45°$. As this triangle features equal angles $\hat{O} = \hat{E} = 45°$ we conclude the triangle OAE to be isosceles with apex A and for its corresponding length of sides $|OA| = |AE|$. This geometric reasoning leads finally to $\cos 45° = \sin 45°$. Substituting the above geometric conclusion in the Pythagorean Identity $(\sin 45°)^2 + (\cos 45°)^2 = 1$ yields $(\sin 45°)^2 + (\sin 45°)^2 = 1$. Solving this equality for $\sin 45°$ leads to $\sin 45° = \pm\sqrt{\frac{1}{2}}$. Since the angle of $45°$ resides in the first quadrant, it limits $\sin 45° \geqslant 0$, and so we conclude $\sin 45° = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$. As an immediate consequence we conclude $\cos 45° = \frac{\sqrt{2}}{2}$.

Trigonometric ratios for an angle of $30° = \frac{\pi}{6}$ rad



Firstly we **reflect** the point $E_{30°}$ over the x-axis to its image point E'. Reflections are isometric mappings: they leave distances and angle sizes unchanged. Therefore we conclude the triangle EOE' to have an apex angle $\hat{O} = 60°$ and two equal sides $|OE| = |OE'|$. As the interior angles of the triangle EOE' add up to $180°$, we find equal angles $\hat{E} = \hat{E}' = 60°$. Since $\hat{E} = \hat{O} = \hat{E}' = 60°$ we conclude the triangle EOE' to be equilateral with length of sides $|OE| = |OE'| = |EE'| = 1$. Based on the coinciding of all geometric lines in equilateral triangles, we consider the altitude OA on the base $[EE']$ also to be the median that bisects the segment $[EE']$. Therefore we conclude $|EA| = \sin 30° = \frac{1}{2}$. Substituting this geometric conclusion in the Pythagorean Identity straightforwardly yields the corresponding $\cos 30° = \frac{\sqrt{3}}{2}$.

Trigonometric ratios for an angle of $60° = \frac{\pi}{3}$ rad



For the special angle of 60°, we consider the triangle OEC as the start of a geometric reasoning. Given its vertices E and C lying on the unit circle, we know that $|OE| = |OC| = 1$ is leading us to an isosceles triangle OEC with apex O and consequently equal base angles $\hat{E} = \hat{C}$. As the interior angles of the triangle OEC add up to 180°, we find equal angles $\hat{E} = \hat{C} = 60°$. Based on the coinciding of all geometric lines in equilateral triangles, we consider the altitude EA on the base $[OC]$ also to be the median that bisects the segment $[OC]$. Therefore we conclude that $|OA| = \cos 60° = \frac{1}{2}$. Substituting this geometric conclusion in the Pythagorean Identity yields the corresponding $\sin 60° = \frac{\sqrt{3}}{2}$.

Overview

Given the above sine and cosine values, we can easily calculate the corresponding tangents via their quotient (see formula (3.5)). We hereby draw for some special angles α their corresponding values for sine and cosine within the unit circle and their tangents on the vertical tangent line $x = 1$.

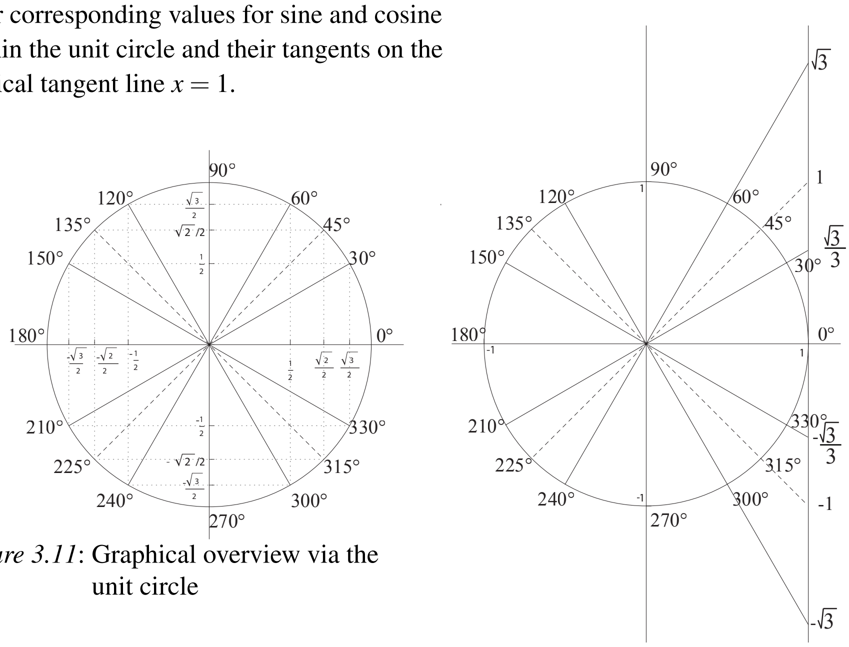


Figure 3.11: Graphical overview via the unit circle

3.6 Pairs of angles

We briefly explain the properties of two pairs of angles that are useful in this book.

Coterminal angles or oppositely signed angles; their measurements add up to 0°. In other words, if α and β are coterminal then $\alpha+\beta=0^\circ$ or $\beta=-\alpha$. The corresponding figure shows how the cosines of coterminal angles remain invariant, while their sines receive opposite signs. This leads to the trigonometric formulas $\cos(-\alpha)=\cos\alpha$ and $\sin(-\alpha)=-\sin\alpha$.

Complementary angles; their measurements add up to 90°. In other words, if α and β are complementary then $\alpha+\beta=90^\circ$ or $\beta=90^\circ-\alpha$. The corresponding figure shows how the sine of α equals the cosine of $90^\circ-\alpha$ and the cosine of α equals the sine of $90^\circ-\alpha$. This leads to the trigonometric formulas $\cos(90^\circ-\alpha)=\sin\alpha$ and $\sin(90^\circ-\alpha)=\cos\alpha$.

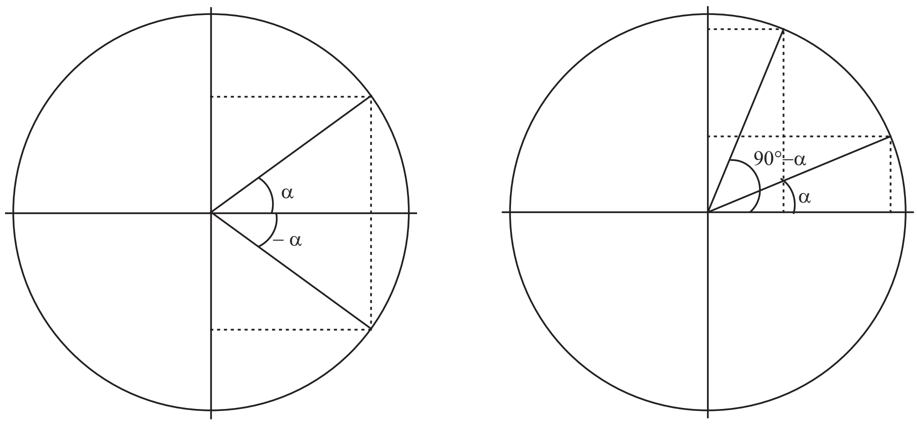


Figure 3.12: Coterminal and complementary angles

3.7 Sum identities

In this paragraph we state and prove all trigonometric ratios of a sum of two angles. We firstly emphasise the non-linearity of all trigonometric ratios: e.g. for the sine we encounter $\sin(\alpha+\beta)\neq\sin\alpha+\sin\beta$. Indeed, e.g. for angles $\alpha=60^\circ$ and $\beta=30^\circ$ the value $\sin 90^\circ=1$ does not equal the sum $\sin 60^\circ+\sin 30^\circ=\frac{\sqrt{3}+1}{2}$. Given the above inequality, we realise the need for the correct formulas which are stated below.

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \qquad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

$$\tan(\alpha+\beta) = \tfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta} \qquad \tan(\alpha-\beta) = \tfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

Proof: First of all we prove the formula for the cosine of a subtraction of angles, using the included right triangles and the Law of Cosines. Thereafter we will prove all other Sum Identities based upon this formula for the cosine of a subtraction.

▷ $\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

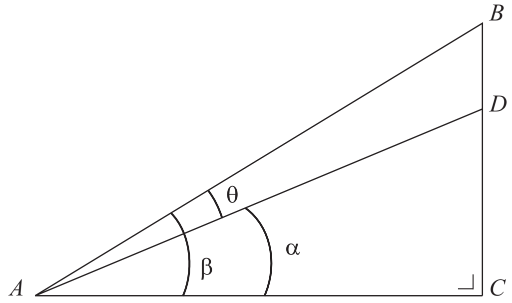


Figure 3.13: Proving the formula for the cosine of a subtraction

The right triangle ACD yields $\cos\alpha = \frac{|AC|}{|AD|}$ and $\sin\alpha = \frac{|DC|}{|AD|}$ and the right triangle ABC yields $\cos\beta = \frac{|AC|}{|AB|}$ and $\sin\beta = \frac{|BC|}{|AB|}$. Substituting the sines and cosines in the right hand side of the formula leads to

$$\begin{aligned}\cos\alpha\cos\beta + \sin\alpha\sin\beta &= \frac{|AC|}{|AD|}\cdot\frac{|AC|}{|AB|} + \frac{|DC|}{|AD|}\cdot\frac{|BC|}{|AB|}\\ &= \frac{|AC|^2 + |DC|\cdot|BC|}{|AD|\cdot|AB|}.\end{aligned}$$

Secondly, we interpret the left hand side $\cos(\alpha-\beta) = \cos(\beta-\alpha)$ as $\cos\theta$ in the scalene triangle BAD. We determine $\cos\theta$ using the Law of Cosines (3.8) for the angle θ and its opposite side $|BD|$ as $|BD|^2 = |AB|^2 + |AD|^2 - 2|AB|\cdot|AD|\cdot\cos\theta$.

To finalise the proof, eliminating $\cos\theta$ from both steps should lead to an equality. Substituting the factor $\cos\theta$ in the second step results in:

$$\begin{aligned}|BD|^2 &= |AB|^2 + |AD|^2 - 2|AB|\cdot|AD|\cdot\left(\frac{|AC|^2 + |DC|\cdot|BC|}{|AD|\cdot|AB|}\right)\\ &= |AB|^2 + |AD|^2 - 2(|AC|^2 + |DC|\cdot|BC|)\end{aligned}$$

We may replace the length $|BD|$ by the difference $|BC|-|DC|$ and apply the Perfect Square:

$$\begin{aligned}|BD|^2 &= |AB|^2+|AD|^2-2(|AC|^2+|DC|\cdot|BC|)\\ (|BC|-|DC|)^2 &= |AB|^2+|AD|^2-2(|AC|^2+|DC|\cdot|BC|)\\ |BC|^2-2|BC|\cdot|DC|+|DC|^2 &= |AB|^2+|AD|^2-2|AC|^2-2|DC|\cdot|BC|\\ |BC|^2+|DC|^2 &= |AB|^2+|AD|^2-2|AC|^2.\end{aligned}$$

We finally obtain the equality $(|BC|^2+|AC|^2)+(|DC|^2+|AC|^2)=|AB|^2+|AD|^2$ after grouping terms and applying the Pythagorean Theorem twice.

▷ Based upon $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$, we have less difficulties in proving the five remaining Sum Identities

$$\begin{aligned}\cos(\alpha+\beta) &= \cos(\alpha-(-\beta))\\ &= \cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)\\ &= \cos\alpha\cos\beta-\sin\alpha\sin\beta \qquad \text{(coterminal angles)}\end{aligned}$$

$$\begin{aligned}\sin(\alpha-\beta) &= \cos(90^\circ-(\alpha-\beta)) \qquad \text{(complementary angles)}\\ &= \cos((90^\circ-\alpha)+\beta)\\ &= \cos(90^\circ-\alpha)\cos\beta-\sin(90^\circ-\alpha)\sin\beta\\ &= \sin\alpha\cos\beta-\cos\alpha\sin\beta \qquad \text{(complementary angles)}\end{aligned}$$

$$\begin{aligned}\sin(\alpha+\beta) &= \sin(\alpha-(-\beta))\\ &= \sin\alpha\cos(-\beta)-\cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta+\cos\alpha\sin\beta \qquad \text{(coterminal angles)}\end{aligned}$$

$$\begin{aligned}\tan(\alpha\pm\beta) &= \frac{\sin(\alpha\pm\beta)}{\cos(\alpha\pm\beta)}\\ &= \frac{\sin\alpha\cos\beta\pm\cos\alpha\sin\beta}{\cos\alpha\cos\beta\mp\sin\alpha\sin\beta}\\ &= \frac{\frac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}\pm\frac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}\mp\frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\tan\alpha\pm\tan\beta}{1\mp\tan\alpha\tan\beta} \qquad \blacksquare\end{aligned}$$

When applying the quotient formula of tangent we should never divide by zero! This would occur in case $\tan\alpha=\pm\frac{1}{\tan\beta}$ or equivalently whenever $\alpha\pm\beta=90^\circ+k180^\circ$ given $k\in\mathbb{Z}$. These angular values would cause $\tan(\alpha\pm\beta)=\pm\infty$.

3.8 Inverse trigonometric functions

When an angle α returns $\sin\alpha = \frac{1}{2}$ then we may conclude, apart from the obvious $\alpha = \frac{\pi}{6}$, for infinitely more angular sizes α to return $\sin\alpha = \frac{1}{2}$. This is due to the periodicity of the function sine and the sine values (except for -1 and 1) to appear twice within one period. We have $\sin\alpha = \frac{1}{2}$ if $\alpha = \frac{\pi}{6}$ but also if $\alpha = \frac{5\pi}{6}$ and we may add multiples of 2π to each of them (due to the periodicity of the plane angle α). We avoid that an infinite number of angles α are the solution to $\sin\alpha = \frac{1}{2}$ when we restrict the angle α to the radian interval $[\frac{-\pi}{2}, \frac{\pi}{2}]$. A similar issue also applies for the cosine and for the tangent, solved in a similar way. We agree on the restricted interval $[0, \pi]$ for cosine and on the restricted interval $]\frac{-\pi}{2}, \frac{\pi}{2}[$ for tangent.

For any x being the result of a sine, cosine or tangent, we find one unique angle α in their restricted interval giving $\sin\alpha = x$, $\cos\alpha = x$ or $\tan\alpha = x$ respectively. The corresponding functions to trace this unique angle α are called the **arcsine**, the **arccosine** and the **arctangent**. We can typeset the arcsine either as arcsin, asin or $\sin^{-1}$.

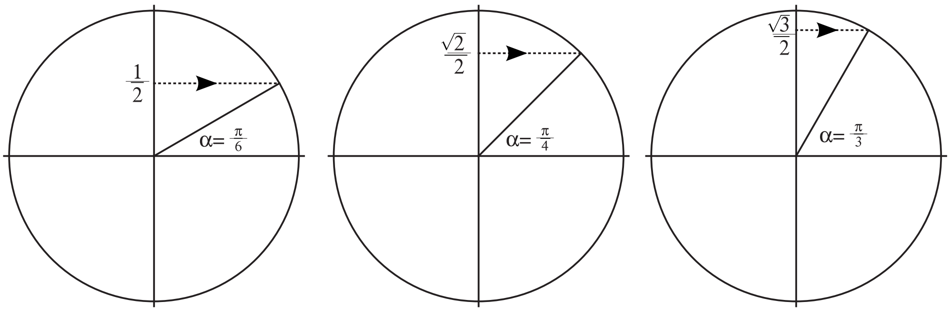


Figure 3.14: The arcsine

Arcsine returns for a given sine value its corresponding circular arc (in radians restricted to quadrants I and IV). For instance $\arcsin\frac{\sqrt{2}}{2} = \frac{\pi}{4}$ because $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

$$\alpha = \arcsin x \Leftrightarrow x = \sin\alpha \text{ with } \frac{-\pi}{2} \leqslant \alpha \leqslant \frac{\pi}{2}$$

Similarly, arccosine returns for any given cosine value its corresponding circular arc (in radians restricted to quadrants I and II).

$$\alpha = \arccos x \Leftrightarrow x = \cos \alpha \text{ with } 0 \leqslant \alpha \leqslant \pi$$

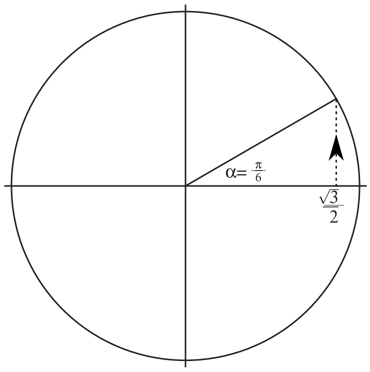


Figure 3.15: The arccosine

Finally, arctangent returns for a given tangent value its corresponding circular arc (in radians restricted to quadrants I and IV).

$$\alpha = \arctan x \Leftrightarrow x = \tan \alpha \text{ with } \frac{-\pi}{2} < \alpha < \frac{\pi}{2}$$

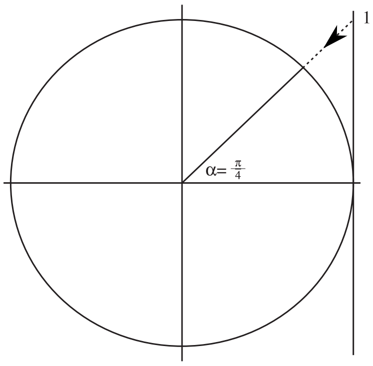


Figure 3.16: The arctangent

Example: What size is the (smallest) angle made by the straight line through the points $O(0,0)$ and $P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and the x-axis?

Drawing the lines shows an opposite side of length $\frac{\sqrt{3}}{2}$ and an adjacent side of length $\frac{1}{2}$. This allows us to calculate the size of angle α made by the line OP and the x-axis via the tangent: $\tan \alpha = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$. We need to restrict the size of angle α to the radian interval $]\frac{-\pi}{2}, \frac{\pi}{2}[$ for the arctangent to return the corresponding size of the targeted angle as $\alpha = \arctan \sqrt{3} = \frac{\pi}{3}$.