AMP(1) - Lab09 – Cross Product

1. Content

2. Learning objectives

2.1. Exam objectives

By the end of this lab you should be able to (pen and paper):

- Apply the cross product of vectors
- Be mindful of the cross product's anticommutativity
- Apply the cross product's geometric properties to determine surface normal (inwards or outwards) and area of its subtended triangle (or parallelogram)
- Apply the cross product's criterion for (anti)parallelism

We advise you to **make your own summary of topics** which are new to you.

2.2. Supportive objectives

Specifically related to the above you should in GeoGebra Classic**5.0** be able to:

- Apply the cross product of vectors
- Apply the cross product to determine and visualize a surface normal in the View/Graphics

3. Exercises

3.1. Basic exercises

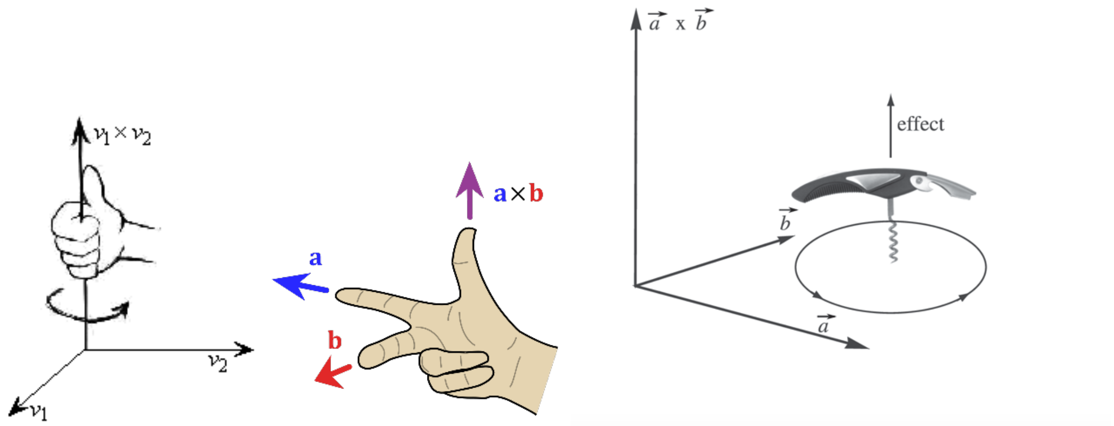
3.1.1. Calculate & visualize Cross product using unit vectors

- Given vector $\vec{\boldsymbol{u}} = \begin{pmatrix}1\\0\\0\end{pmatrix}$ and the vector $\vec{\boldsymbol{v}} = \begin{pmatrix}0\\1\\0\end{pmatrix}$
- Calculate the cross product $\vec{\boldsymbol{w}}\mathbf{1} = \vec{\boldsymbol{u}}\ \mathbf{X}\,\vec{\boldsymbol{v}}$ $\begin{pmatrix}0\\0\\1\end{pmatrix}$

- Visualize $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{v}}$ in Geogebra (View – Graphics3D)
- Visualize $\vec{\boldsymbol{w}}\mathbf{1}$ using the Cross-command

- Calculate the cross product $\vec{\boldsymbol{w}}\mathbf{2} = \vec{\boldsymbol{v}}\,\mathbf{X}\,\vec{\boldsymbol{u}}$ $\begin{pmatrix}0\\0\\-1\end{pmatrix}$
- Visualize $\vec{\boldsymbol{w}}\mathbf{2}$ using the Cross-command

- Check the directions of $\vec{\boldsymbol{w}}\mathbf{1}$ and $\vec{\boldsymbol{w}}\mathbf{2}$ with the right hand-rule. Make use of one of the mnemonics visualized below (Opposite direction)



3.1.2. Calculate & Visualize Cross product using other vectors

- Given vector $\vec{\boldsymbol{d}}$ $\begin{pmatrix}2\\-2\\2\end{pmatrix}$and the vector $\vec{\boldsymbol{e}}$ $\begin{pmatrix}2\\2\\2\end{pmatrix}$
- Calculate the cross product $\vec{\boldsymbol{f}}\mathbf{1} = \vec{\boldsymbol{d}}\ \mathbf{X}\,\vec{\boldsymbol{e}}$ $\begin{pmatrix}-8\\0\\8\end{pmatrix}$
- Visualize $\vec{\boldsymbol{d}}$ and $\vec{\boldsymbol{e}}$ in Geogebra (View – Graphics3D)

- Visualize $\vec{f}1$ using the Cross-command. Check that the vector is perpendicular to $\vec{d}$ and $\vec{e}$ **(**See the previous lab on dot-products)
- Repeat for $\vec{f}2 = \vec{e}\,\mathbf{X}\,\vec{d}$ $\begin{pmatrix} 8 \\ 0 \\ -8 \end{pmatrix}$
- Visualize $\vec{f}2$ using the Cross-command and check the direction with the right hand-rule.
- Normalize the vectors f1 and f2 (UnitVector) $\begin{pmatrix} -0.71 \\ 0 \\ 0.71 \end{pmatrix}$ **and** $\begin{pmatrix} 0.71 \\ 0 \\ -0.71 \end{pmatrix}$

3.2. Bridging exercises

3.2.1. Determine a surface normal via the cross product

- Visualize the triangle determined by the vertices A = (-3,0,2), B = (2,0,3) and C = (2,-4,3)
- Calculate and visualize the normalized normal in **A**
  Hint :
  - Determine vector $\overrightarrow{AB}$ B – A = $\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$
  - Determine vector $\overrightarrow{AC}$ C – A = $\begin{pmatrix} 5 \\ -4 \\ 1 \end{pmatrix}$
  - Visualize the vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$
  - Calculate $\overrightarrow{AB}$ x $\overrightarrow{AC}$ and $\overrightarrow{AC}$ x $\overrightarrow{AB}$ + visualize in geogebra

    $\overrightarrow{AB}$ x $\overrightarrow{AC}$ **=** $\begin{pmatrix} 4 \\ 0 \\ -20 \end{pmatrix}$

    $\overrightarrow{AC}$ x $\overrightarrow{AB}$ **=** $\begin{pmatrix} -4 \\ 0 \\ 20 \end{pmatrix}$

  => What is your conclusion? Opposite direction – values are identical

  Calculate and visualise the normalized normal in **B**
  Hint :
  - Determine vector $\overrightarrow{BA}$ A – B = $\begin{pmatrix} -5 \\ 0 \\ -1 \end{pmatrix}$
  - Determine vector $\overrightarrow{BC}$ C – B = $\begin{pmatrix} 0 \\ -4 \\ 0 \end{pmatrix}$
  - Visualize the vectors $\overrightarrow{BA}$ and BC
  - Calculate $\overrightarrow{BA}$ x $\overrightarrow{BC}$ and $\overrightarrow{BC}$ x $\overrightarrow{BA}$ + visualize in geogebra

    $\overrightarrow{BA}$ x $\overrightarrow{BC}$ **=** $\begin{pmatrix} -4 \\ 0 \\ 20 \end{pmatrix}$

    $\overrightarrow{BC}$ x $\overrightarrow{BA}$ **=** $\begin{pmatrix} 4 \\ 0 \\ -20 \end{pmatrix}$

  ⇨ Compare with the normal in vertex A. Wat is your conclusion?
  Normal on a plane is identical in all vertices

3.2.2. Area of parallelogram via the cross product

The length of the cross product-vector is a measure of the Area of the parallelogram determined by the 2 side-vectors of the parallelogram.

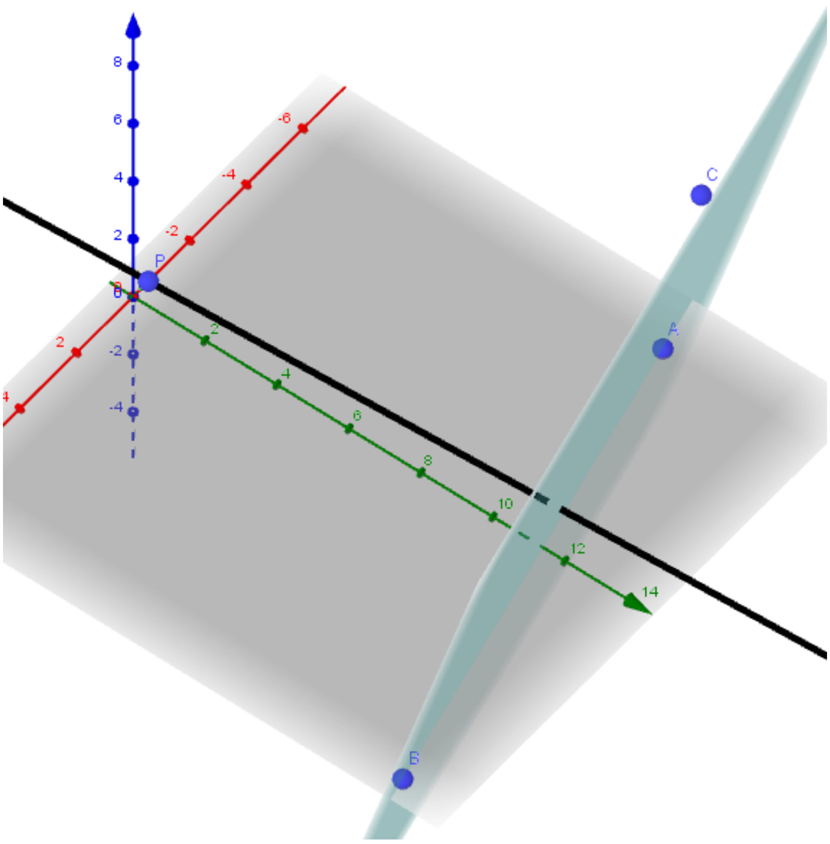
Find the area of the parallelogram spanned by the location vectors $\vec{a}$ $\begin{pmatrix} 4 \\ -10 \\ 5 \end{pmatrix}$ and $\vec{b}$ $\begin{pmatrix} -3 \\ -1 \\ -3 \end{pmatrix}$

Check your answer in geogebra.

$\vec{a} \times \vec{b} = \begin{pmatrix} 35 \\ -3 \\ -34 \end{pmatrix}$ Length = Area of parallelogram = 48.89

3.2.3. Distance from point to a plane

Given a point P (2,2,4) and a plane defined by A (-6, 10, 0), B (7, 13, 0) and C (-1, 15, 14). Calculate the perpendicular distance from the point P tot the plane ABC.



Plane defined by Vector ac (5,5,14) and vector ab(13,3,0)

Normal of plane Cross(ab, ac) = (42,-182,50)

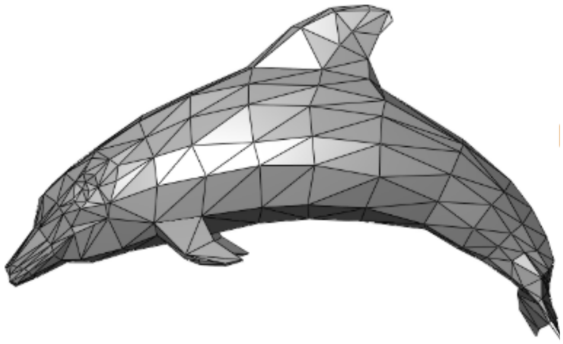
Normal normalized = (0.22,-0.94,0.26)

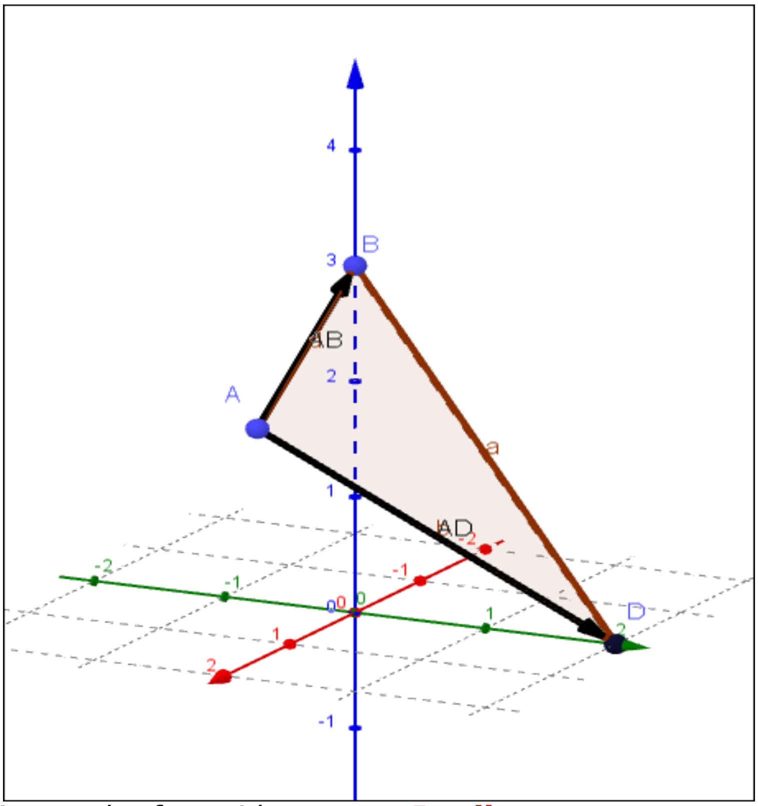
Vector from P to a point on the plane ap = (8,-8,4)

Dotproduct(ap, normalizednormal) = 10.3

3.3. Contextual practice

3.3.1. Backface culling revisited

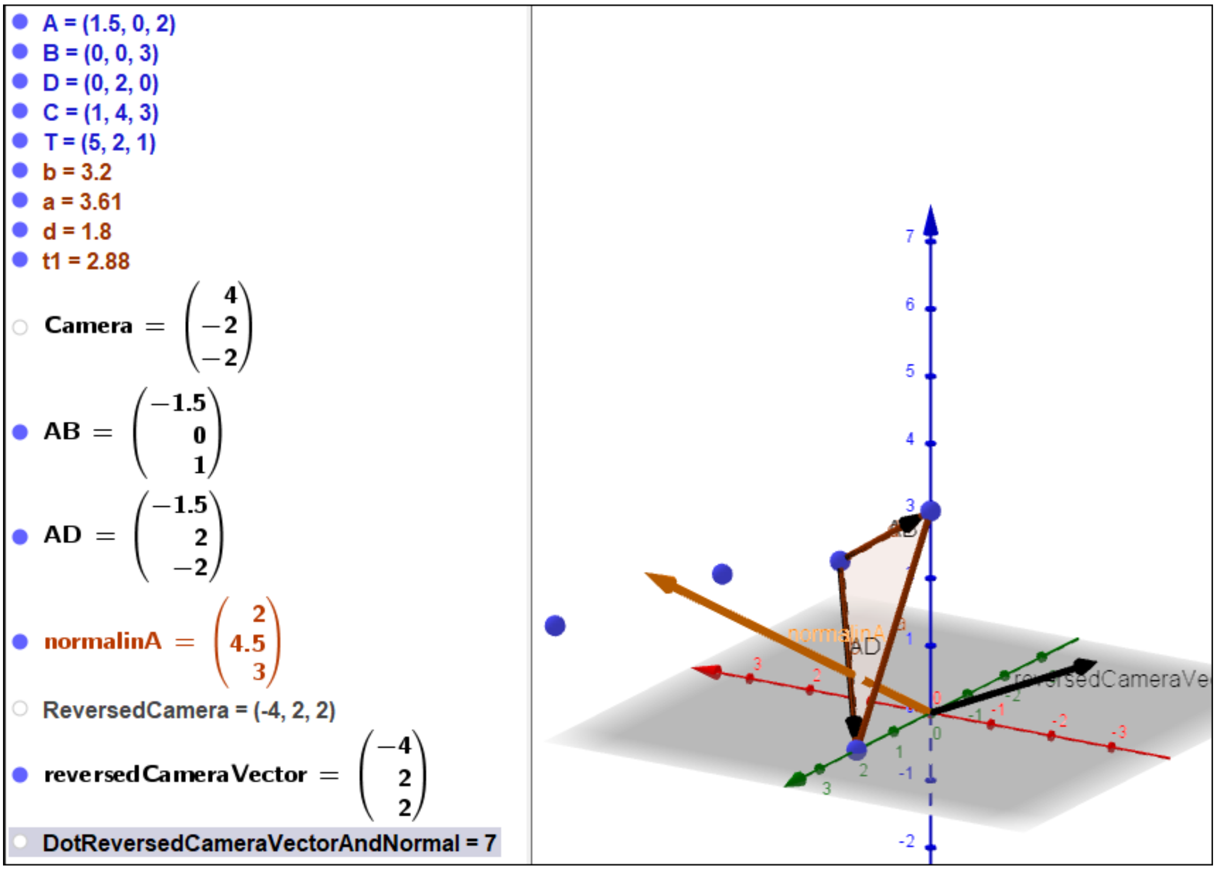




Assume a face determined by the vertices A(1.5,0,2), B(0,0,3) and D(0,2,0) . The frontside of the triangle is determined by the counterclockwisely crossed produced normal (in A, B or D)

The camera is positioned in C(1,4,3) band point towards the target in T(5,2,1). Use the cross-product to calculate the normal and the dot-product to determine if the camera is looking at the frontside or not. **Realize:** determining e.g. the frontside of a triangle in 3D is an inherent ambiguous setup. This is due to a lack of reference to define the winding 'counterclockwise' in 3D.

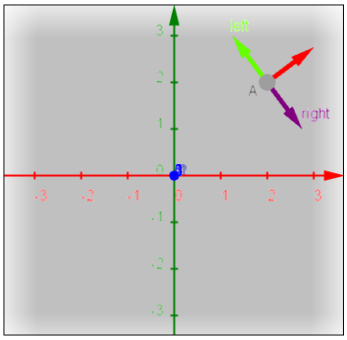
Solution :



Dot-product = 7 => > 0 = face is visible

3.3.2. Relative Right and Left

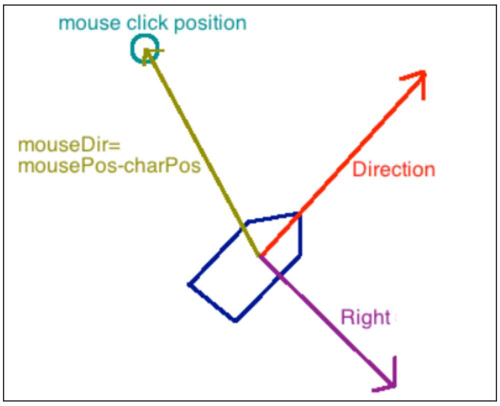
In a topdown game, the movement is always relative to the front facing direction of the avatar. The avatar is in position (2,2,0). The front vector is described by $\vec{f} = \begin{pmatrix} 1 \\ 0.75 \\ 0 \end{pmatrix}$. It's up direction is the same as that of the z-axis.



- Calculate the vector that indicates the direction 90 degrees to the right and 90 degrees to the left using a cross-product with the z-axis.

  Right = $\begin{pmatrix} 0.75 \\ -1 \\ 0 \end{pmatrix}$ and Left = $\begin{pmatrix} -0.75 \\ 1 \\ 0 \end{pmatrix}$

- When the player clicks in (2,4,0), how can you calculate that is to the avatars left side or right side?



Hint : Make use of the dot-product to find the angle between Right and mouseDir.

mouseDir = $\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$

The dot product will return a positive value if the angle is less than 90, a negative value if the angle is more than 90.

Right . mouseDir = -2 => at the left of the avatar

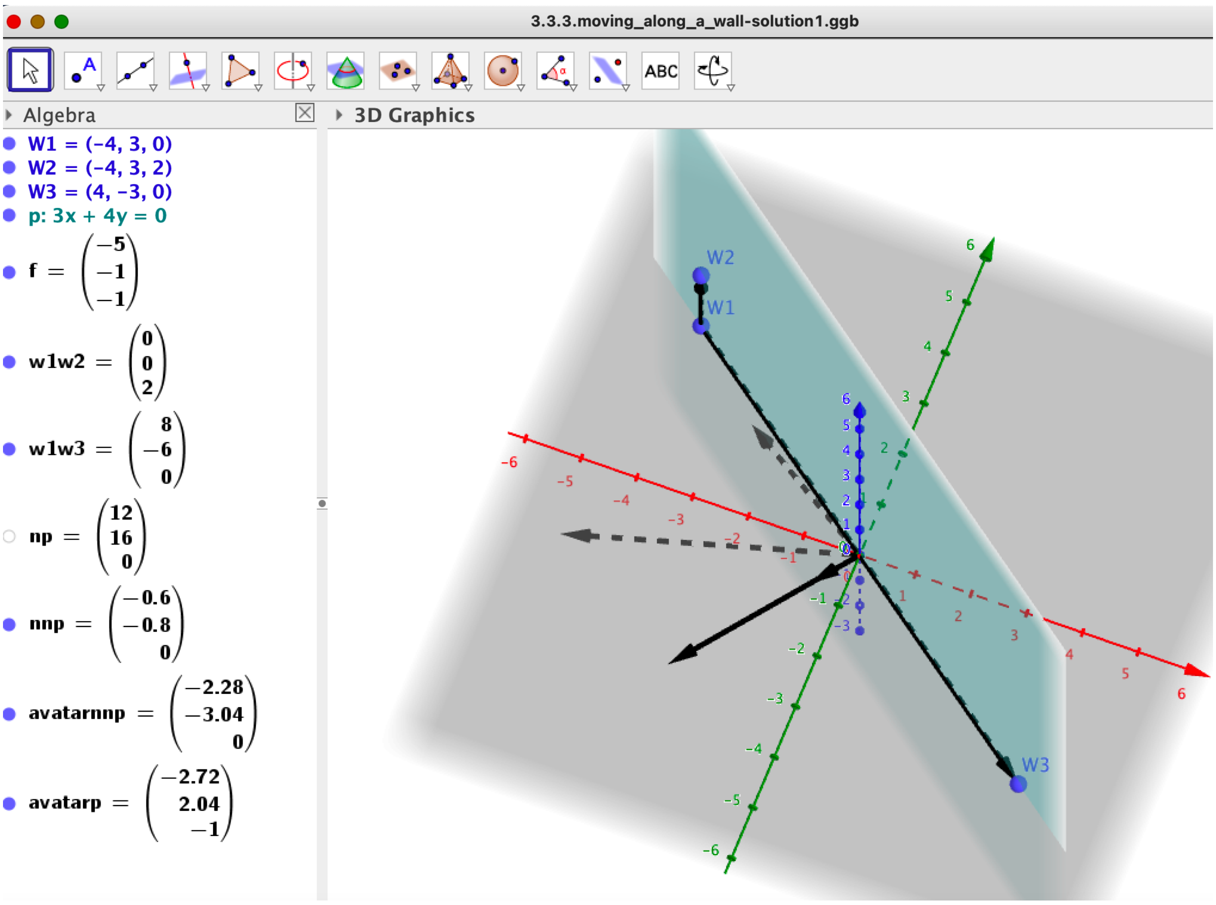
3.3.3. Moving along the wall

A player is directing his avatar into a wall. The direction in which the avatar moves is $\vec{f}\begin{pmatrix}-5\\-1\\-1\end{pmatrix}$. The wall is determined by the vertices W1(-4,3,0), W2(-4,3,2) and W3(4,-3,0).

Like any decent game, the player will not be able to traverse the wall, but the character will start sliding along the wall. Determine the direction in which the character will move along the wall

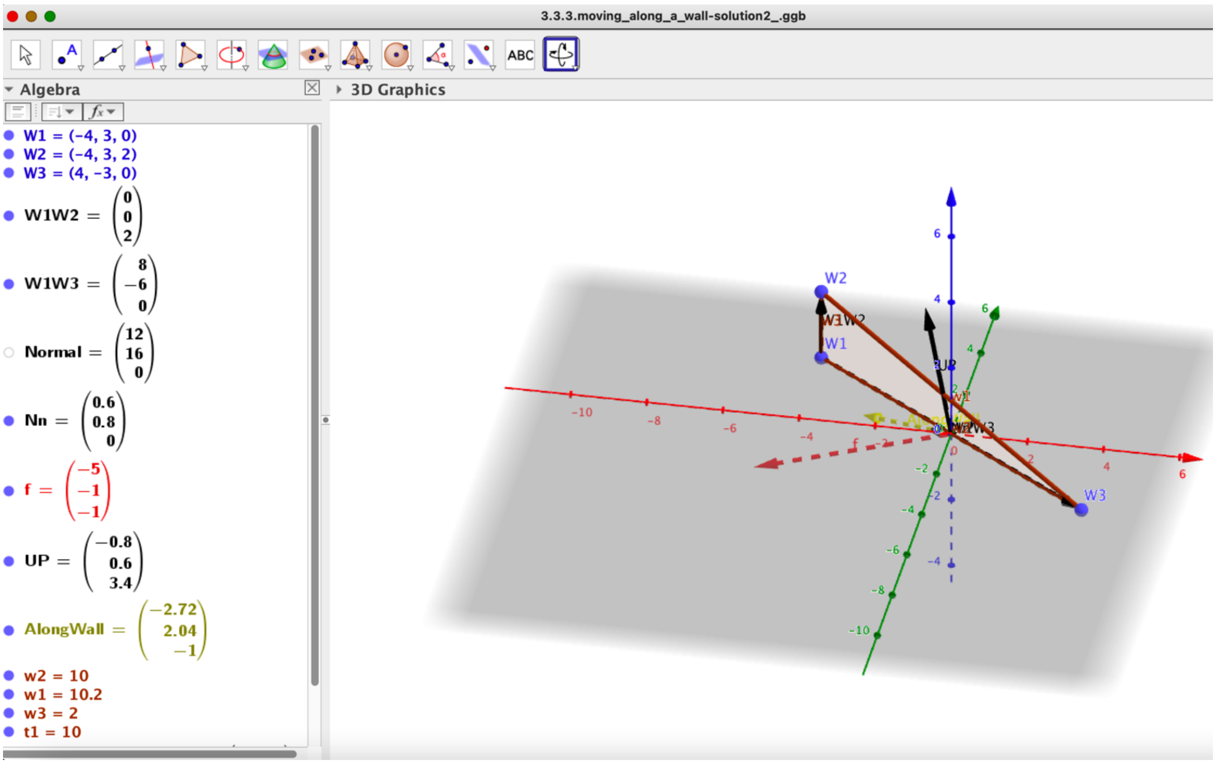
3.3.3.moving_along_a_wall-solution1

- Step 1 : calculate **unit normal** on the wall by $\hat{n} = \dfrac{\overrightarrow{W1W2} \times \overrightarrow{W1W3}}{\left\|\overrightarrow{W1W2} \times \overrightarrow{W1W3}\right\|}$
- Step 2 : calculate dot product $\vec{f} \odot \hat{n}$ yields orthogonal projection of $\vec{f}$
- Step 3 : difference vector $\vec{f} - (\vec{f} \odot \hat{n})\hat{n}$ gives the $\vec{f}$-part within the wall



3.3.3.moving_along_a_wall-solution2

- Step 1 : calculate **unit normal** on the wall by $\hat{n} = \frac{\overrightarrow{W1W2} \times \overrightarrow{W1W3}}{\left\|\overrightarrow{W1W2} \times \overrightarrow{W1W3}\right\|}$
- Step 2 : calculate **up-vector** along the wall (up = f x N)
- Step 3 : calc the **forward-vector** along the wall (alongWall = up x N)



4. References

Normal mapping
https://blog.teamtreehouse.com/understanding-normal-maps

Backface culling
https://en.wikipedia.org/wiki/Back-face_culling

Inherent ambiguity **of normal directions – fix by an algorithm**

https://cgvr.cs.uni-bremen.de/papers/orientation/orientation_electr.pdf