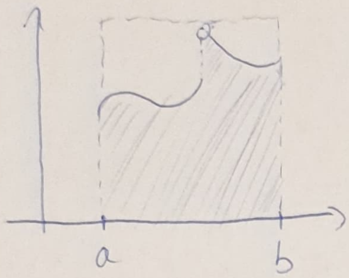
Лекция 1

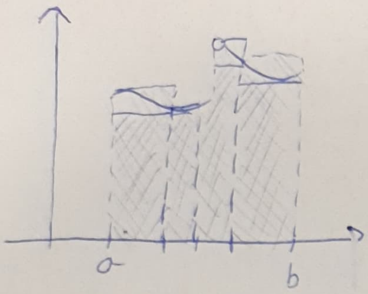


$f:[a,b] \longrightarrow \mathbb{R}$

ограничена

$\sup\{f(x) : x \in [a,b]\} . (b-a)$ - горни оценки

$\inf\{f(x) : x \in [a,b]\} . (b-a)$ - долни оценки



$\tau : a = x_0 < x_1 < \dots < x_n = b$

$\tau$ - подразбиване на интервала $[a,b]$

Разглеждаме $[x_{i-1}, x_i]$

$M_i := \sup\{f(x) : x \in [x_{i-1}, x_i]\}$

$S_f(\tau) := \sum_{i=1}^{n} M_i (x_i - x_{i-1})$ - голяма сума на Дарбу за $f$ при подр. $\tau$

$s_f(\tau) : \sum_{i=1}^{n} m_i (x_i - x_{i-1})$ - малка сума на Дарбу

$m_i := \inf\{f(x) : x \in [x_{i-1}, x_i]\}$

Лема 1 Ако $\tau^* \geq \tau$, то

$S_f(\tau^*) \leq S_f(\tau)$ и

$s_f(\tau^*) \geq s_f(\tau)$

$\tau^*, \tau$ - подразбивания на $[a,b]$

$\tau^* \geq \tau$

($\tau^*$ по-подробно и по-фино от $\tau$) ако $\tau^*$ съдържа всички делящи точки от $\tau$

Д-во

Б.о.о. $\tau^*$ се получава от $\tau$ прибавяне на една точка

$\tau : a = x_0 < x_1 < \dots < x_n = b$

$\tau^* : a = \underbrace{x_0 < x_1 < \dots <} x_{\underbrace{i-1} < x^*} \underbrace{< x_i} \underbrace{< \dots < x_n} = b$

$S_f(\tau) - S_f(\tau^*) = \sum_{j=1}^{n} \sup_{[x_{j-1},x_j]} f \,(x_j - x_{j-1}) - \sum_{j=1}^{i-1} \sup_{[x_{j-1},x_j]} f \,(x_j - x_{j-1}) +$

$+ \sup_{[x_{i-1},x^*]} f \,(x^* - x_{i-1}) + \sup_{[x^*,x_i]} f \,(x_i - x^*) + \sum_{j=i+1}^{n} \sup_{[x_{j-1},x_j]} f \,(x_j - x_{j-1}) =$

$= \sup_{[x_{i-1},x_i]} f \,(x_i - x_{i-1}) - \sup_{[x_{i-1},x^*]} (x^* - x_{i-1}) - \sup_{[x^*,x_i]} f \,(x_i - x^*) \geq$

$\geq \sup_{[x_{i-1},x_i]} f \,(x_i - x_{i-1}) - \sup_{[x_{i-1},x_i]} f \,(x^* - x_{i-1}) - \sup_{[x_{i-1},x_i]} f \,(x_i - x^*) = 0$

**Лема 2** $\tau_1, \tau_2$ призволни подр. на $[a,b]$

Тогава $s_f(\tau_1) \le S_f(\tau_2)$

Сгу $\tau^* \ge \tau_1$, $\tau^* \ge \tau_2$

$A \subset B$
$\sup A \le \sup B$

$$s_f(\tau_1) \le s_f(\tau^*) \le S_f(\tau^*) \le S_f(\tau_2)$$

$\Lambda_1$: $\tau^* \ge \tau_1$ ; $\Lambda_1$: $\tau^* \ge \tau_2$

$$[s_f(\tau_1), S_f(\tau_1)] \cap [s_f(\tau_2), S_f(\tau_2)] \neq \emptyset$$

$f : [a,b] \to \mathbb{R}$
ограничена

$$\overline{\int_a^b} f := \inf\{S_f(\tau) : \tau \text{ подразбиване на } [a,b]\}$$

горен интеграл на $f$ в $[a,b]$

$$\underline{\int_a^b} := \sup\{s_f(\tau) : \tau \text{ подразбиване на } [a,b]\}$$

долен интеграл

От Лема 2

$$\underbrace{s_f(\tau_1) \le S_f(\tau_2)}_{\tau_2 \text{ фикс.}} \quad \forall \tau_1, \tau_2 \text{ - разбиване на } [a,b]$$

$\Rightarrow \underline{\int_a^b} f \le S_f(\tau_2) \ \forall \tau_2$ подр. $[a,b]$

$\Rightarrow \underline{\int_a^b} f \le \overline{\int_a^b} f$

**Дефиниция** $f : [a,b] \to \mathbb{R}$ се нарича интегруема по Риман, ако е ограничена и $\underline{\int_a^b} f = \overline{\int_a^b} f$. В този случай числото $\underline{\int_a^b} f = \overline{\int_a^b} f$ се нарича риманов интеграл по $f$ в $[a,b]$ и се бележи

$\int_a^b f$ или $\int_a^b f(x)dx$

Пример. функция на Дирихле (от упражнение)

Критерий за интегруемост: $f:[a,b]\to\mathbb{R}$, ограничена

Твърдим, че

$f$ е интегруема по Риман $\overset{1}{\Leftrightarrow}$ $\forall \varepsilon>0\ \exists \tau_1,\tau_2$ подр. $[a,b]$: $S_f(\tau_1)-s_f(\tau_2)<\varepsilon$ $\overset{2}{\Leftrightarrow}$ $\forall \varepsilon>0\ \exists \tau$-подр. $[a,b]$: $S_f(\tau)-s_f(\tau)<\varepsilon$

$\Leftrightarrow$ $\forall \varepsilon>0\ \exists \tau$ подр на $[a,b]$: $\sum_{i=1}^{n} w(f,[x_{i-1},x_i])$

(1⇒) $\varepsilon>0$

$\int_a^b f+\frac{\varepsilon}{2} = \overline{\int_a^b} f-\frac{\varepsilon}{2} > \overline{\int_a^b} f \Rightarrow \exists\ \tau_1$ подр. на $[a,b]$

$S_f(\tau_1)<\int_a^b f+\frac{\varepsilon}{2}$

$\int_a^b f-\frac{\varepsilon}{2} = \underline{\int_a^b} f-\frac{\varepsilon}{2} < \underline{\int_a^b} f \Rightarrow \exists\ \tau_2$ подр на $[a,b]$

$s_f(\tau_2)>\int_a^b f-\frac{\varepsilon}{2}$

$S_f(\tau_1)-s_f(\tau_2) < \left(\int_a^b f+\frac{\varepsilon}{2}\right)-\left(\int_a^b f-\frac{\varepsilon}{2}\right) = \frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon$

(1⇐) ~~доп. противоречие~~ доп. прот.

$\Rightarrow \forall\ \tau_1,\tau_2$ : $S_f(\tau_1)-s_f(\tau_2)\geq \overline{\int_a^b} f-\underline{\int_a^b} f>0$

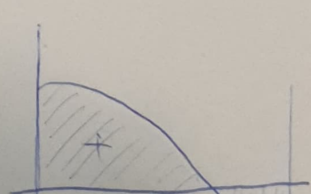
Д(⇐2) $\tau_1:=\tau \quad \tau_2:=\tau$

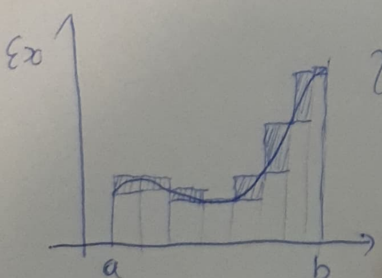
(⇒2) $\varepsilon>0$

$\tau_1,\tau_2 \Rightarrow S_f(\tau_1)-s_f(\tau_2)<\varepsilon$

$\tau\supseteq\tau_1, \tau\supseteq\tau_2$

$\Rightarrow S_f(\tau)-s_f(\tau)\leq S_f(\tau_1)-s_f(\tau_2)<\varepsilon$




$\tau: a=x_0<x_1<\dots<x_n=b$

$S_f(\tau)-s_f(\tau)=\sum_{i=1}^{n}\left(\sup_{[x_{i-1},x_i]} f-\inf_{[x_{i-1},x_i]} f\right)(x_i-x_{i-1})$

* осцилация (провери)

Лема: $\omega(f,[a,b])=\sup_{[a,b]} f-\inf_{[a,b]} f$

$x,y\in[a,b] \longrightarrow f(x)\leq \sup_{[a,b]} f$, $f(y)\geq \inf_{[a,b]} f$

$$|f(x)-f(y)| - \left| \begin{array}{l} f(x)-f(y) \le \sup\limits_{[a,b]} f - \inf\limits_{[a,b]} f \\ f(y)-f(x) \le \sup\limits_{[a,b]} f - \inf\limits_{[a,b]} f \end{array} \right.$$

$$\Rightarrow w(f,[a,b]) \le \sup f - \inf f$$

($\Leftarrow$) $\varepsilon > 0$

$\sup f - \inf f - \varepsilon$

$x_0 \in [a,b],\ f(x_0) > \sup f - \frac{\varepsilon}{2}$

$x_0 \in [a,b],\ f(y_0) < \inf f + \frac{\varepsilon}{2}$

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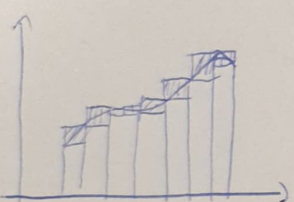
$\forall \varepsilon > 0\ \exists \tau$ подр. на $[a,b]$: $\sum\limits_{i=1}^{n} w(f,[x_{i-1},x])$

**Твърдение:** Непрекъснатите функции са интегруеми

$f:[a,b] \to R$ непрекъснати     Вайерщрас $\longrightarrow$ $f$ ограничена



$\varepsilon > 0$

Кантор $\Rightarrow \exists \delta > 0\ \forall x', x'' \in [a,b],\ |x'-x''| < \delta : |f(x')-f(x'')| < \frac{\varepsilon}{2(b-a)}$

Взимаме $\tau : x_0 = a < x_1 < \dots < x_n = b$

такова, че $\underbrace{\max\{x_i - x_{i-1},\ i \in \{1,\dots,n\}\}}_{d(\tau) \text{ - диаметър}} < \delta$

Тогава

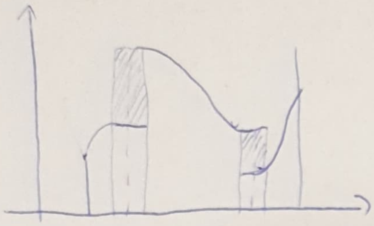
$$S_f(\tau) - s_f(\tau) = \sum_{i=1}^{n} w(f,[x_{i-1},x_i])(x_i - x_{i-1}) \overset{*}{\le}$$

$[x_{i-1},x_i]$, $x_i - x_{i-1} < \delta$   $x', x'' \in [x_{i-1},x_i] \Rightarrow |x'-x''| < \delta$

$\Rightarrow |f(x')-f(x'')| < \frac{\varepsilon}{2(b-a)} \Rightarrow w(f,[x_{i-1},x_i]) \le \frac{\varepsilon}{2(b-a)}$

$$\overset{*}{\le} \sum_{i=1}^{n} \frac{\varepsilon}{2(b-a)}(x_i - x_{i-1}) = \frac{\varepsilon}{2(b-a)} \underbrace{\sum_{i=1}^{n}(x_i - x_{i-1})}_{b-a} = \frac{\varepsilon}{2} < \varepsilon$$

**Твърдение 2** Нека $f[a,b] \longrightarrow \mathbb{R}$ е интегруема и има краен брой точки на прекъсване.



$y_1, y_2, \dots, y_k$ точки на прекъсване на $f$

$\eta > 0$ разгл. $C = [a,b] \setminus \left( \bigcup_{i=1}^{k} (y_i - \eta, y_i + \eta) \right)$

Обединение на краен брой затворени интервали и $f$ е непр. в/у $C$

От Кантор $\Rightarrow \exists\, \delta > 0\ \forall x', x'' \in C, |x' - x''| < \delta : |f(x') - f(x'')| < \frac{\varepsilon}{4(b-a)}$

$\tau : a = x_0 < x_1 < \dots < x_n = b$ такива, че $\left| \begin{array}{l} [x_{i-1}, x_i] \subset C \Rightarrow \text{тогава } x_i - x_{i-1} < \delta \\ [x_{i-1}, x_i] = [y_j - \eta, y_j + \eta] \text{ за} \\ \text{някое } j \in \{1, \dots, k\} \end{array} \right.$

$$S_f(\tau) - s_f(\tau) = \sum_{i=1}^{n} \omega(f;[x_{i-1}, x_i])(x_i - x_{i-1}) \leq \sum_{[x_{i-1}, x_i] \subset C} \omega(f;[x_{i-1}, x_i])(x_i - x_{i-1}) +$$

$$+ \sum_{j=1}^{k} \underbrace{\omega(f;[y_j - \eta, y_j + \eta] \cap [a,b])}_{\leq (M-m)} 2\eta \leq$$

$$\left\{ M = \sup_{[a,b]} f \quad m = \inf_{[a,b]} f \right\}$$

$$\leq \frac{\varepsilon}{4(b-a)} \underbrace{\sum_{[x_{i-1}, x_i] \subset C} (x_i - x_{i-1})}_{\leq b-a} + (M-m) \cdot 2\eta \cdot k$$

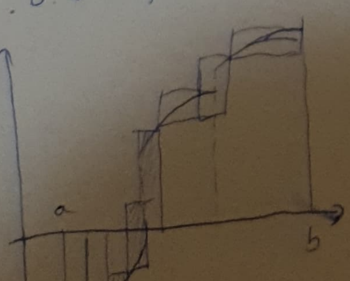
$$S_f(\tau) - s_f(\tau) \leq \frac{\varepsilon}{4(b-a)} (b-a) + (M-m) 2k \cdot \eta$$

Трябва ни $\eta < \frac{\varepsilon}{4k(M-m)}$ $\square$

**Твърдение 3** Монотонните функции са интегруеми

$f : [a,b] \longrightarrow \mathbb{R}$

б.о.о. $f$ е растяща $f(a) \leq f(x) \leq f(b)\ \forall x \in [a,b]$ — $f$ ограничена




$\tau : a = x_0 < x_1 < \dots < x_n = b$

$x \in [x_{i-1}, x_i] \longrightarrow f(x_{i-1}) \leq f(x) \leq f(x_i)$

$\inf_{[x_{i-1}, x_i]} f = f(x_{i-1}) \qquad \sup_{[x_{i-1}, x_i]} f = f(x_i)$

$$S_f(\tau) - s_f(\tau) = \sum_{i=1}^{n} \left( f(x_i) - f(x_{i-1}) \right)(x_i - x_{i-1}) \le$$

$$\le d(\tau) \sum_{i=1}^{n} \left( f(x_i) - f(x_{i-1}) \right) = d(\tau)\left( f(b) - f(a) \right) < \frac{\varepsilon}{f(b) - f(a) + 1} \left( f(b) - f(a) \right) < \varepsilon$$

$\varepsilon > 0 \rightarrow$ избираме $\tau$ с $d(\tau) < \dfrac{\varepsilon}{f(b) - f(a) + 1}$