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Zero-inflated Poisson and negative binomial integer-valued GARCH models

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ABSTRACT

Zero inflation means that the proportion of 0's of a model is greater than the proportion of 0's of the corresponding Poisson model, which is a common phenomenon in count data. To model the zero-inflated characteristic of time series of counts, we propose zero-inflated Poisson and negative binomial INGARCH models, which are useful and flexible generalizations of the Poisson and negative binomial INGARCH models, respectively. The stationarity conditions and the autocorrelation function are given. Based on the EM algorithm, the estimating procedure is simple and easy to be implemented. A simulation study shows that the estimation method is accurate and reliable as long as the sample size is reasonably large. A real data example leads to superior performance of the proposed models compared with other competitive models in the literature.

1. Introduction

In the probability model the Poisson distribution is usually assumed for count data; however, in many real applications it is likely to observe that the number of zeroes is greater than what would be expected for the Poisson model, which is called zero inflation. The zero inflation is of interest because zero counts frequently have special status, e.g., in counting disease lesions on plants, a plant may have no lesions either because it is resistant to the disease, or simply because no disease spores have landed on it. This is the distinction between *structural zeros*, which are inevitable, and *sampling zeros*, which occur by chance (Ridout et al., 1998). Ignoring zero inflation can have at least two consequences; first, the estimated parameters and standard errors may be biased, and second, the excessive number of zeros can cause overdispersion (Zuur et al., 2009, p. 269).

In recent years there has been considerable and growing interest in modeling zero-inflated count data, and many models have been proposed, e.g., the hurdle model (Mullahy, 1986), the zero-inflated Poisson (ZIP) model (Lambert, 1992), and the two-part model (Heilbron, 1994, also known as the zero-altered model). Ridout et al. (1998) reviewed this literature and cited examples from econometrics, manufacturing defects, patent applications, road safety, species abundance, medical consultations, use of recreational facilities, and sexual behavior. For the ZIP model, Böhning (1998) also reviewed the related literature and provided a variety of examples from different disciplines. As a generalization of the ZIP model, the zero-inflated negative binomial (ZINB) model has been discussed by many authors, such as Ridout et al. (2001) considered the score test for testing the ZIP model against the ZINB model. Zeileis et al. (2008) gave a nice overview and comparison of Poisson, negative binomial, and zero-inflated models in the software R. For a recent review and applications to ecology, see Zuur et al. (2009, Chapter 11).

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In general, zero-inflated model can be viewed as a mixture of a degenerate distribution with mass at zero and a nondegenerate distribution such as the Poisson or negative binomial distribution. Now many researchers are still studying that how to extend and test these models (see, e.g., Xie et al., 2009; Yang et al., 2009; Min and Czado, 2010; Hall and Shen, 2010; Garay et al., 2011). To our best knowledge, all the zero-inflated models are considered in regression context, not yet in time series context, except that Bakouch and Ristić (2010) considered a zero-truncated Poisson INAR(1) model. But zero inflation is also common in time series analysis, see the example given in Section 6.

In addition to the zero-inflated characteristic, many time series count datasets also display overdispersion, which means that the variance is greater than the mean. Overdispersion has been well modeled in the literature, such as the integer-valued generalized autoregressive conditional heteroscedastic (INGARCH) model proposed by Ferland et al. (2006) and its various generalizations. The INGARCH model is defined as follows:

$$\begin{cases} X_t|\mathcal{F}_{t-1} : \mathcal{P}(\lambda_t), \quad \forall t \in \mathbb{Z}, \\ \lambda_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i} + \sum_{j=1}^{q} \beta_j \lambda_{t-j}, \end{cases} \tag{1.1}$$

where $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, $p \geq 1$, $q \geq 0$, and $\mathcal{F}_{t-1}$ is the σ-field generated by $\{X_{t-1}, X_{t-2} \ldots\}$. This model has been studied by many authors. Zhu et al. (2008), Zhu and Li (2009) and Zhu and Wang (2010, 2011) considered various estimation and testing methods. Specially, Zhu and Wang (2011) gave a necessary and sufficient condition for the existence of higher-order moments. Fokianos et al. (2009) considered geometric ergodicity and likelihood-based inference, and Fokianos and Fried (2010) transferred the concept of intervention effects to model (1.1). Weiß (2009) derived a set of equations from which the variance and the autocorrelation function of the general case can be obtained. Weiß (2010a) derived the unconditional distributions via the Poisson–Charlier expansion, while Weiß (2010b) considered higher-order moments and jumps. Zhu et al. (2010) extended model (1.1) to the mixture model context, while Zhu (2011) extended the Poisson deviate to the negative binomial one, which are useful generalizations. For more generalizations, see Fokianos and Tjøstheim (2011) and Matteson et al. (2011). Fokianos (2011) reviewed some recent progress in INGARCH models.

To model overdispersion and zero inflation in the same framework, we will generalize the Poisson model (1.1) and the negative binomial model proposed in Zhu (2011) and show the usefulness of these generalizations. The paper is organized as follows. In Sections 2 and 3 we describe the zero-inflated Poisson INGARCH (ZIP-INGARCH) model and the zero-inflated negative binomial INGARCH (ZINB-INGARCH) model, respectively. The stationarity conditions and the autocorrelation functions are given. We discuss the estimation procedure in Section 4 via the EM algorithm. Section 5 presents a simulation study. In Section 6 we apply the proposed models to a real data example. Section 7 gives some discussions.

2. The zero-inflated Poisson INGARCH model

First, recall the definition of ZIP distribution (see Johnson et al., 2005, Section 4.10.3). A distribution is said to be ZIP (λ, ω) if its probability mass function (pmf) can be written in the form

$$P(X = k) = \omega \delta_{k,0} + (1-\omega) \frac{\lambda^k e^{-\lambda}}{k!}, \quad k = 0, 1, 2 \ldots,$$

where $0 < \omega < 1$, $\delta_{k,0}$ is the Kronecker delta, i.e., $\delta_{k,0}$ is 1 when $k=0$ and is zero when $k \neq 0$. The probability generating function (pgf) is $G(z) = \omega + (1-\omega) e^{\lambda(z-1)}$, then from Lemma 1 in Ferland et al. (2006) we know that the uncentered moments of X satisfy

$$E(X^m) = (1-\omega) \sum_{j=0}^{m} \mathfrak{S}_m^{(j)} \lambda^j, \tag{2.1}$$

where $\mathfrak{S}_m^{(j)}$ is the Stirling number of the second kind (for details, see Gradshteyn and Ryzhik, 2007, p. 1046). Specially, we have

$$E(X) = (1-\omega)\lambda, \quad \mathrm{Var}(X) = (1-\omega)\lambda(1+\omega\lambda) > E(X).$$

Let $\{X_t\}$ be a time series of counts. We assume that, conditional on $\mathcal{F}_{t-1}$, the random variables $X_1, \ldots, X_n$ are independent, and the conditional distribution of X_t is specified by a ZIP distribution. To be specific, we consider the following model:

$$X_t|\mathcal{F}_{t-1} : \mathcal{ZIP}(\lambda_t, \omega), \quad \lambda_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i X_{t-i} + \sum_{j=1}^{q} \beta_j \lambda_{t-j}, \tag{2.2}$$

where $0 < \omega < 1$, $\alpha_0 > 0$, $\alpha_i \geq 0$, $\beta_j \geq 0$, $i = 1, \ldots, p$, $j = 1, \ldots, q$, $p \geq 1$, $q \geq 0$, $\mathcal{F}_{t-1}$ is the σ-field generated by $\{X_{t-1}, X_{t-2} \ldots\}$. The above model is denoted by ZIP-INGARCH(p, q). The conditional mean and conditional variance of X_t are given by

$$E(X_t|\mathcal{F}_{t-1}) = (1-\omega)\lambda_t, \quad \mathrm{Var}(X_t|\mathcal{F}_{t-1}) = (1-\omega)\lambda_t(1+\omega\lambda_t), \tag{2.3}$$

then $\mathrm{Var}(X_t|\mathcal{F}_{t-1}) > E(X_t|\mathcal{F}_{t-1})$. Furthermore,

$$\begin{aligned}\mathrm{Var}(X_t) &= E(\mathrm{Var}(X_t|\mathcal{F}_{t-1}))+\mathrm{Var}(E(X_t|\mathcal{F}_{t-1})) = E((1-\omega)\lambda_t(1+\omega\lambda_t))+\mathrm{Var}((1-\omega)\lambda_t)\\ &= (1-\omega)E(\lambda_t)+\omega(1-\omega)(E(\lambda_t))^2+(1-\omega)\mathrm{Var}(\lambda_t) > (1-\omega)E(\lambda_t) = E(X_t),\end{aligned} \tag{2.4}$$

which indicates that the model (2.2) can handle integer-valued time series with overdispersion.

Remark 1. If $\omega=0$, then the ZIP-INGARCH model (2.2) reduces to the Poisson INGARCH model considered in Ferland et al. (2006).

The first-order stationarity conditions for the ZIP-INGARCH model (2.2) can be given in a different way. To notational simplicity, let $p \ge q$.

Theorem 1. *A necessary and sufficient condition for the ZIP-INGARCH(p, q) process to be stationary in the mean is that the roots of the equation*

$$1-\sum_{i=1}^{q}((1-\omega)\alpha_i+\beta_i)z^{-i}-\sum_{i=q+1}^{p}(1-\omega)\alpha_i z^{-i}=0 \tag{2.5}$$

all lie inside the unit circle.

Proof. Let $\mu_t = E(X_t)$. Then

$$\begin{aligned}\mu_t &= E(X_t) = E(E(X_t|\mathcal{F}_{t-1})) = (1-\omega)E(\lambda_t) = (1-\omega)\alpha_0+\sum_{i=1}^{p}(1-\omega)\alpha_i E(X_{t-i})+\sum_{j=1}^{q}(1-\omega)\beta_j E(\lambda_{t-j})\\ &= (1-\omega)\alpha_0+\sum_{i=1}^{p}(1-\omega)\alpha_i\mu_{t-i}+\sum_{j=1}^{q}\beta_j\mu_{t-j}.\end{aligned}$$

The necessary and sufficient condition for a nonhomogeneous difference equation to have a stable solution, which is finite and independent of t, is that the roots $z_1,\ldots,z_p$ of Eq. (2.5) all lie inside the unit circle (Goldberg, 1958). □

Remark 2. Suppose that the process $\{X_t\}$ following the ZIP-INGARCH(p,q) model is first-order stationary, then we have $\mu =: E(X_t) = (1-\omega)\alpha_0/(1-(1-\omega)\sum_{i=1}^{p}\alpha_i-\sum_{j=1}^{q}\beta_j)$.

The second-order stationarity conditions for the ZIP-INGARCH model (2.2) are given in the following theorem. To illustrate the main idea of the proof, we focus on the simpler ZIP-INGARCH(p, q) model for the second-order stationary condition, i.e., $q=0$, which is denoted by ZIP-INARCH (p).

Theorem 2. *Suppose that the process $\{X_t\}$ following the ZIP-INARCH (p) model is first-order stationary. A necessary and sufficient condition for the process to be second-order stationary is that $1-C_1z^{-1}-\cdots-C_pz^{-p}=0$ has all roots lie inside the unit circle, where for $u,l=1,\ldots,p-1$,*

$$C_u=(1-\omega)\left(\alpha_u^2-\sum_{v=1}^{p-1}\sum_{|i-j|=v}\alpha_i\alpha_j b_{vu}\beta_{u0}\right),\quad C_p=(1-\omega)\alpha_p^2,$$

$$\beta_{l0}=(1-\omega)\alpha_l,\quad \beta_{ll}=(1-\omega)\sum_{|i-l|=l}\alpha_i-1 \quad \textit{and} \quad \beta_{lu}=(1-\omega)\sum_{|i-l|=u}\alpha_i,\ u\neq l,$$

where B and B^{-1} are $(p-1)\times(p-1)$ matrices such that $B=(\beta_{ij})_{i,j=1}^{p-1}$ and $B^{-1}=(b_{ij})_{i,j=1}^{p-1}$.

Proof. The proof of the theorem follows that in Fong et al. (2007). Let $\gamma_{it}=E(X_tX_{t-i})$ for $i=0,1,\ldots,p$ and C be a constant independent of t. If the process is second-order stationary, we have $\gamma_{*,t}=\gamma_{*,t-i}$ for $i=0,1,\ldots,p$. Consider the conditional second moment,

$$\begin{aligned}E(X_t^2|\mathcal{F}_{t-1}) &= \mathrm{Var}(X_t|\mathcal{F}_{t-1})+[E(X_t|\mathcal{F}_{t-1})]^2\\ &= (1-\omega)(\lambda_t+\lambda_t^2) = (1-\omega)\left(\alpha_0+\alpha_0^2+(1+2\alpha_0)\sum_{i=1}^{p}\alpha_iX_{t-i}+\sum_{i=1}^{p}\alpha_i^2X_{t-i}^2+\sum_{\substack{i,j=1\\ i\neq j}}^{p}\alpha_i\alpha_jX_{t-i}X_{t-j}\right).\end{aligned}$$

For $l=1,\ldots,p-1$, the covariance between X_t and X_{t-l} are

$$\begin{aligned}\gamma_{lt} &= E(E(X_t|\mathcal{F}_{t-1})X_{t-l}) = (1-\omega)E\left(\alpha_0X_{t-l}+\sum_{i=1}^{p}\alpha_iX_{t-i}X_{t-l}\right) = (1-\omega)\left(\alpha_0\mu+\alpha_l\gamma_{0,t-l}+\sum_{\substack{i=1\\ i\neq l}}^{p}\alpha_i\gamma_{|i-l|,t}\right)\\ &= (1-\omega)\left(\alpha_0\mu+\alpha_l\gamma_{0,t-l}+\sum_{|i-l|=1}\alpha_i\gamma_{1t}+\cdots+\sum_{|i-l|=l}\alpha_i\gamma_{lt}+\cdots+\sum_{|i-l|=p-1}\alpha_i\gamma_{p-1,t}\right),\end{aligned}$$

where $\gamma_{*,t-i}$ are replaced by $\gamma_{*,t}$ for $i=1,\dots,p-1$. Hence for $l=1,\dots,p-1$,

$$(1-\omega)\alpha_0\mu+\beta_{l0}\gamma_{0,t-l}+\sum_{u=1}^{p-1}\beta_{lu}\gamma_{ut}=0.$$

Therefore,

$$B(\gamma_{1t},\dots,\gamma_{p-1,t})^\top=-((1-\omega)\alpha_0\mu+\beta_{10}\gamma_{0,t-1},\dots,(1-\omega)\alpha_0\mu+\beta_{p-1,0}\gamma_{0,t-p+1})^\top,$$

then

$$\gamma_{lt}=-(1-\omega)\alpha_0\mu\sum_{u=1}^{p-1}b_{lu}-\sum_{u=1}^{p-1}b_{lu}\beta_{u0}\gamma_{0,t-u},\quad l=1,\dots,p-1.$$

The unconditional second moment can be rewritten as

$$\begin{aligned}\gamma_{0t}&=C+(1-\omega)\left[\sum_{u=1}^{p}\alpha_u^2\gamma_{0,t-u}+\sum_{\substack{i,j=1\\ i\neq j}}^{p}\alpha_i\alpha_j\gamma_{|i-j|,t}\right]=C+(1-\omega)\left[\sum_{u=1}^{p}\alpha_u^2\gamma_{0,t-u}+\sum_{v=1}^{p-1}\sum_{|i-j|=v}\alpha_i\alpha_j\gamma_{vt}\right]\\ &=C_0+(1-\omega)\left[\sum_{u=1}^{p}\alpha_u^2\gamma_{0,t-u}+\sum_{v=1}^{p-1}\sum_{|i-j|=v}\alpha_i\alpha_j\left(-\sum_{u=1}^{p-1}b_{vu}\beta_{u0}\gamma_{0,t-u}\right)\right]\\ &=C_0+(1-\omega)\left[\sum_{u=1}^{p}\alpha_u^2\gamma_{0,t-u}-\sum_{u=1}^{p-1}\left(\sum_{v=1}^{p-1}\sum_{|i-j|=v}\alpha_i\alpha_j b_{vu}\beta_{u0}\right)\gamma_{0,t-u}\right]\\ &=C_0+(1-\omega)\left[\sum_{u=1}^{p-1}\left(\alpha_u^2-\sum_{v=1}^{p-1}\sum_{|i-j|=v}\alpha_i\alpha_j b_{vu}\beta_{u0}\right)\gamma_{0,t-u}+\alpha_p^2\gamma_{0,t-p}\right],\end{aligned}$$

or equivalently,

$$\gamma_{0t}=C_0+\sum_{u=1}^{p}C_u\gamma_{0,t-u},$$

where $C_0=C-(1-\omega)^2\alpha_0\mu\sum_{v=1}^{p-1}\sum_{|i-j|=v}\alpha_i\alpha_j\sum_{u=1}^{p-1}b_{vu}$. Therefore, the nonhomogeneous difference equation has a stable solution if the equation $1-C_1z^{-1}-\cdots-C_pz^{-p}=0$ has all roots lie inside the unit circle. □

In the following, we give some special cases of Theorems 1 and 2.

Corollary 1. *Suppose that the process $\{X_t\}$ following the ZIP-INARCH (p) model, then for $p=1$ and2, the necessary and sufficient conditions for X_t is first-order stationary are $(1-\omega)\alpha_1<1$ and $(1-\omega)(\alpha_1+\alpha_2)<1$, respectively.*

Now suppose that X_t is first-order stationary. Then for $p=1$ and2, the second-order stationarity conditions are $(1-\omega)\alpha_1^2<1$ and $\delta_1+\delta_2<1$, respectively, where

$$\delta_1=(1-\omega)\left(\alpha_1^2+\frac{2(1-\omega)\alpha_1^2\alpha_2}{1-(1-\omega)\alpha_2}\right),\quad \delta_2=(1-\omega)\alpha_2^2.$$

Proof. The validation for the case $p=1$ is simple, we only give the proof for the case $p=2$. From Theorem 1 we know that the first-order stationarity condition is that $1-(1-\omega)\alpha_1z^{-1}-(1-\omega)\alpha_2z^{-2}=0$ has all roots lie inside the unit circle, which is equivalent to the following condition:

$$(1-\omega)(\alpha_2+\alpha_1)<1,\quad (1-\omega)(\alpha_2-\alpha_1)<1,\quad \left|(1-\omega)\alpha_2\right|<1. \tag{2.6}$$

The condition given in (2.6) is also equivalent to $(1-\omega)(\alpha_1+\alpha_2)<1$. Similarly, from Theorem 2 we know that the second-order stationarity condition is equivalent to the following condition:

$$\delta_2+\delta_1<1,\quad \delta_2-\delta_1<1,\quad \left|\delta_2\right|<1. \tag{2.7}$$

Then $\delta_1>0$ holds under the assumption of the first-order stationarity, thus the condition in (2.7) is equivalent to $\delta_1+\delta_2<1$. □

The necessary and sufficient condition for the process X_t following the ZIP-INARCH (p) model to be mth order stationary is difficult to be given. Next, we will derive a necessary condition for the process X_t that is mth order stationary.

Theorem 3. *Suppose that the process X_t following the ZIP-INARCH (p) model is mth order stationary, then $(1-\omega)Q^m<1$, where $Q=\sum_{k=1}^{p}\alpha_k$.*

Proof. The theorem is proven by using similar techniques to the proof of Theorem 2 in Zhu and Wang (2011). From (2.1) we have

$$E(X_t^m|\mathcal{F}_{t-1})=(1-\omega)\sum_{j=0}^{m}\mathfrak{S}_m^{(j)}\lambda_t^j,$$

then

$$E(X_t^m)=(1-\omega)\sum_{j=0}^{m}\mathfrak{S}_m^{(j)}E(\lambda_t^j)=(1-\omega)\sum_{j=0}^{m}\mathfrak{S}_m^{(j)}\sum_{i=0}^{j}\binom{j}{i}\alpha_0^{j-i}E\left(\sum_{k=1}^{p}\alpha_k X_{t-k}\right)^i. \tag{2.8}$$

Let $q_k=\alpha_k/Q$ and note that x^i is convex for $x\geq 0$. Then by Jensen's inequality we have

$$E\left(\sum_{k=1}^{p}\alpha_k X_{t-k}\right)^i=Q^iE\left(\sum_{k=1}^{p}q_k X_{t-k}\right)^i\leq Q^i\sum_{k=1}^{p}q_kE(X_{t-k}^i)=Q^iE(X_t^i), \tag{2.9}$$

then from (2.8), (2.9) and $\mathfrak{S}_m^{(m)}=1$ we have

$$\begin{aligned}E(X_t^m)&\leq(1-\omega)\sum_{j=0}^{m}\mathfrak{S}_m^{(j)}\sum_{i=0}^{j}\binom{j}{i}\alpha_0^{j-i}Q^iE(X_t^i)\\&=(1-\omega)\sum_{j=0}^{m-1}\mathfrak{S}_m^{(j)}\sum_{i=0}^{j}\binom{j}{i}\alpha_0^{j-i}Q^iE(X_t^i)+(1-\omega)\sum_{i=0}^{m-1}\binom{m}{i}\alpha_0^{m-i}Q^iE(X_t^i)+(1-\omega)Q^mE(X_t^m).\end{aligned}$$

Thus

$$E(X_t^m)\leq(1-\omega)\frac{\sum_{j=0}^{m-1}\mathfrak{S}_m^{(j)}\sum_{i=0}^{j}\binom{j}{i}\alpha_0^{j-i}Q^iE(X_t^i)+\sum_{i=0}^{m-1}\binom{m}{i}\alpha_0^{m-i}Q^iE(X_t^i)}{1-(1-\omega)Q^m}. \tag{2.10}$$

The numerator in (2.10) involves the moments of X_t of order $\leq m-1$ and is finite, thus $E(X_t^m)<\infty$ holds under the condition $(1-\omega)Q^m<1$. □

We can extend the results in Weiß (2009) to the model (2.2). The following theorem gives a set of equations from which the variance and autocorrelation function can be obtained.

Theorem 4. *Suppose that $\{X_t\}$ following the ZIP-INGARCH(p, q) process is second-order stationary. Let the autocovariances $\gamma_X(k)=\mathrm{Cov}(X_t,X_{t-k}),\gamma_\lambda(k)=\mathrm{Cov}(\lambda_t,\lambda_{t-k})$, then they satisfy the equations*

$$\gamma_X(k)=(1-\omega)\sum_{i=1}^{p}\alpha_i\gamma_X(|k-i|)+\sum_{j=1}^{\min(k-1,q)}\beta_j\gamma_X(k-j)+(1-\omega)^2\sum_{j=k}^{q}\beta_j\gamma_\lambda(j-k),\quad k\geq 1,$$

$$\gamma_\lambda(k)=(1-\omega)\sum_{i=1}^{\min(k,p)}\alpha_i\gamma_\lambda(k-i)+\frac{1}{1-\omega}\sum_{i=k+1}^{p}\alpha_i\gamma_X(i-k)+\sum_{j=1}^{q}\beta_j\gamma_\lambda(|k-j|),\quad k\geq 0.$$

Proof. Let $\mathcal{I}_t$ be the σ-field generated by $\{\lambda_t,\lambda_{t-1}\ldots\}$, then we have

$$E(X_t|\mathcal{F}_{t-1},\mathcal{I}_t)=E(X_t|\mathcal{F}_{t-1})=(1-\omega)\lambda_t. \tag{2.11}$$

For $k\geq 0$, from (2.3) and (2.11) we have

$$\begin{aligned}\mathrm{Cov}(X_t-(1-\omega)\lambda_t,(1-\omega)\lambda_{t-k})&=E[(X_t-(1-\omega)\lambda_t)((1-\omega)\lambda_{t-k}-\mu)]=E[((1-\omega)\lambda_{t-k}-\mu)E((X_t-(1-\omega)\lambda_t)|\mathcal{I}_t)]\\&=E[((1-\omega)\lambda_{t-k}-\mu)[E(E(X_t|\mathcal{F}_{t-1},\mathcal{I}_t)|\mathcal{I}_t)-(1-\omega)\lambda_t]]=E[((1-\omega)\lambda_{t-k}-\mu)[E((1-\omega)\lambda_t|\mathcal{I}_t)-(1-\omega)\lambda_t]]=0.\end{aligned} \tag{2.12}$$

Similarly, for $k<0$, from (2.2) we have

$$\begin{aligned}\mathrm{Cov}(X_t,X_{t-k}-(1-\omega)\lambda_{t-k})&=E[(X_t-\mu)(X_{t-k}-(1-\omega)\lambda_{t-k})]=E[(X_t-\mu)E((X_{t-k}-(1-\omega)\lambda_{t-k})|\mathcal{F}_{t-k-1})]\\&=E[(X_t-\mu)[(1-\omega)\lambda_{t-k}-E((1-\omega)\lambda_{t-k}|\mathcal{F}_{t-k-1})]]=0.\end{aligned} \tag{2.13}$$

Then from (2.12) and (2.13) we have

$$\mathrm{Cov}(X_t,(1-\omega)\lambda_{t-k})=\begin{cases}\mathrm{Cov}((1-\omega)\lambda_t,(1-\omega)\lambda_{t-k}), & k\geq 0,\\ \mathrm{Cov}(X_t,X_{t-k}), & k<0.\end{cases} \tag{2.14}$$

For $k \geq 0$, from (2.2) and (2.14) we have

$$\gamma_\lambda(k) = \text{Cov}(\lambda_t, \lambda_{t-k}) = \sum_{i=1}^{p} \alpha_i \text{Cov}(X_{t-i}, \lambda_{t-k}) + \sum_{j=1}^{q} \beta_j \text{Cov}(\lambda_{t-j}, \lambda_{t-k})$$

$$= (1-\omega) \sum_{i=1}^{\min(k,p)} \alpha_i \text{ Cov}(\lambda_{t-i}, \lambda_{t-k}) + \frac{1}{1-\omega} \sum_{i=k+1}^{p} \alpha_i \text{ Cov}(X_{t-i}, X_{t-k}) + \sum_{j=1}^{q} \beta_j \text{ Cov}(\lambda_{t-j}, \lambda_{t-k}).$$

Similarly, for $k \geq 1$, we have

$$\gamma_X(k) = \text{Cov}(X_t, X_{t-k}) = (1-\omega)\text{Cov}(\lambda_t, X_{t-k}) = (1-\omega) \sum_{i=1}^{p} \alpha_i \text{ Cov}(X_{t-i}, X_{t-k}) + (1-\omega) \sum_{j=1}^{q} \beta_j \text{ Cov}(\lambda_{t-j}, X_{t-k})$$

$$= (1-\omega) \sum_{i=1}^{p} \alpha_i \text{ Cov}(X_{t-i}, X_{t-k}) + \sum_{j=1}^{\min(k-1,q)} \beta_j \text{ Cov}(X_{t-j}, X_{t-k}) + (1-\omega)^2 \sum_{j=k}^{q} \beta_j \text{ Cov}(\lambda_{t-j}, \lambda_{t-k}).$$

This completes the proof. $\square$

Example 1. Consider the ZIP-INGARCH $(1,1)$ model. From Remark 2 we know

$$\mu = E(X_t) = \frac{(1-\omega)\alpha_0}{1-(1-\omega)\alpha_1-\beta_1}.$$

From Theorem 4 we obtain

$$\gamma_X(k) = (1-\omega)\alpha_1\gamma_X(k-1) + \beta_1\gamma_X(k-1) = [(1-\omega)\alpha_1+\beta_1]^{k-1}\gamma_X(1), \quad k \geq 2,$$

$$\gamma_\lambda(k) = (1-\omega)\alpha_1\gamma_\lambda(k-1) + \beta_1\gamma_\lambda(k-1) = [(1-\omega)\alpha_1+\beta_1]^{k}\gamma_\lambda(0), \quad k \geq 1.$$

From Theorem 4 and (2.4) we have

$$\gamma_X(1) = (1-\omega)\alpha_1\gamma_X(0) + (1-\omega)^2\beta_1\gamma_\lambda(0) = (1-\omega)^2(\alpha_1+\beta_1)\gamma_\lambda(0) + \alpha_1(1-\omega)\left(\mu + \frac{\omega\mu^2}{1-\omega}\right),$$

$$\gamma_\lambda(0) = \frac{\alpha_1}{1-\omega}\gamma_X(1) + \beta_1\gamma_\lambda(1) = [(1-\omega)\alpha_1^2 + 2(1-\omega)\alpha_1\beta_1 + \beta_1^2]\gamma_\lambda(0) + \alpha_1^2\left(\mu + \frac{\omega\mu^2}{1-\omega}\right),$$

so $\text{Var}(\lambda_t) = \alpha_1^2(\mu + \omega\mu^2/(1-\omega))/[1-(1-\omega)\alpha_1^2 - 2(1-\omega)\alpha_1\beta_1 - \beta_1^2]$. Then

$$\text{Var}(X_t) = (1-\omega)\text{Var}(\lambda_t) + \mu + \frac{\omega\mu^2}{1-\omega} = \frac{1-2(1-\omega)\alpha_1\beta_1-\beta_1^2}{1-(1-\omega)\alpha_1^2-2(1-\omega)\alpha_1\beta_1-\beta_1^2}\left(\mu + \frac{\omega\mu^2}{1-\omega}\right).$$

The autocorrelations are given by

$$\rho_\lambda(k) = [(1-\omega)\alpha_1+\beta_1]^k, \quad k \geq 0,$$

$$\rho_X(k) = [(1-\omega)\alpha_1+\beta_1]^{k-1}\frac{(1-\omega)\alpha_1[1-(1-\omega)\alpha_1\beta_1-\beta_1^2]}{1-2(1-\omega)\alpha_1\beta_1-\beta_1^2}, \quad k \geq 1.$$

Specially, the mean, variance and autocorrelations of the ZIP-INARCH (1) model are given by

$$E(X_t) = \frac{(1-\omega)\alpha_0}{1-(1-\omega)\alpha_1}, \quad \text{Var}(X_t) = \frac{(1-\omega)\alpha_0[1+\omega\alpha_0-(1-\omega)\alpha_1]}{[1-(1-\omega)\alpha_1^2][1-(1-\omega)\alpha_1]^2},$$

$$\rho_X(k) = [(1-\omega)\alpha_1]^k, \quad k \geq 1.$$

Similarly, the mean and variance of the ZIP-INARCH(2) model are given by

$$E(X_t) = \frac{(1-\omega)\alpha_0}{1-(1-\omega)(\alpha_1+\alpha_2)},$$

$$\text{Var}(X_t) = \frac{(1-\omega)E(X_t)+\omega[E(X_t)]^2}{1-\omega-(1-\omega)^2\left[\frac{1+(1-\omega)\alpha_2}{1-(1-\omega)\alpha_2}\alpha_1^2+\alpha_2^2\right]},$$

which will be used in Section 6.

Remark 3. From the above example we know that the second-order stationary condition for the ZIP-INGARCH(1,1) model is $(1-\omega)\alpha_1^2 + 2(1-\omega)\alpha_1\beta_1 + \beta_1^2 < 1$.

Corollary 2. *Suppose that $\{X_t\}$ following the ZIP-INARCH (p) model is second-order stationary, then the autocovariance function $\gamma_X(k)$ satisfies the equations*

$$\gamma_X(k) = \sum_{i=1}^{p}(1-\omega)\alpha_i\gamma_X(|k-i|), \quad k \geq 1. \tag{2.15}$$

The equations of Corollary 2 are obviously nearly identical to the Yule–Walker equations off the standard AR (p) model. As a consequence, the model order p can be identified with the help of the partial autocorrelation function $\rho_X^*(k)$. It follows from Corollary 2 that $\rho_X^*(k)=0$ for $k>p$, which can be used to identify the model order p.

3. The zero-inflated negative binomial INGARCH model

Analogously to Section 2, we can consider the ZINB-INGARCH model. First, recall the pmf of the ZINB (λ,a,ω) can be written in the form

$$P(X=k) = \omega\delta_{k,0}+(1-\omega)\frac{\Gamma\left(k+\dfrac{\lambda^{1-c}}{a}\right)}{k!\Gamma\left(\dfrac{\lambda^{1-c}}{a}\right)}\left(\frac{1}{1+a\lambda^c}\right)^{\lambda^{1-c}/a}\left(\frac{a\lambda^c}{1+a\lambda^c}\right)^k, \quad k=0,1,2\ldots,$$

where $\lambda>0, 0<\omega<1$, and $a\geq 0$ is the dispersion parameter. This distribution reduces to ZIP (λ,ω) in the limit $a \to 0$. The index $c(=0,1)$ identifies the particular form of the underlying NB distribution (Ridout et al., 2001). For $c=0$, this distribution is denoted by ZINB1 (λ,a,ω) and for $c=1$, it is denoted by ZINB2 (λ,a,ω). The distribution is just denoted by ZINB (λ,a,ω) when c is not emphasized. The mean and variance of this distribution are

$$E(X) = (1-\omega)\lambda, \quad \mathrm{Var}(X) = \begin{cases}(1-\omega)\lambda(1+a+\omega\lambda), & c=0,\\ (1-\omega)\lambda(1+(\omega+a)\lambda), & c=1.\end{cases}$$

The ZINB-INGARCH(p, q) model is defined as

$$X_t|\mathcal{F}_{t-1} : \mathcal{ZINB}(\lambda_t,a,\omega), \quad \lambda_t = \alpha_0+\sum_{i=1}^{p}\alpha_i X_{t-i}+\sum_{j=1}^{q}\beta_j\lambda_{t-j}, \tag{3.1}$$

where $0<\omega<1$, $a>0$, $\alpha_0>0$, $\alpha_i\geq 0$, $\beta_j\geq 0$, $i=1,\ldots,p$, $j=1,\ldots,q$, $p\geq 1$, $q\geq 0$, $\mathcal{F}_{t-1}$ is the σ-field generated by $\{X_{t-1},X_{t-2}\ldots\}$. The conditional mean and conditional variance of X_t are given by

$$E(X_t|\mathcal{F}_{t-1}) = (1-\omega)\lambda_t, \quad \mathrm{Var}(X_t|\mathcal{F}_{t-1}) = (1-\omega)\lambda_t(1+\omega\lambda_t+a\lambda_t^c),$$

then $\mathrm{Var}(X_t|\mathcal{F}_{t-1}) > E(X_t|\mathcal{F}_{t-1})$. Furthermore,

$$\begin{aligned}\mathrm{Var}(X_t) &= E(\mathrm{Var}(X_t|\mathcal{F}_{t-1}))+\mathrm{Var}(E(X_t|\mathcal{F}_{t-1})) = E((1-\omega)\lambda_t(1+\omega\lambda_t+a\lambda_t^c))+\mathrm{Var}((1-\omega)\lambda_t)\\ &= \begin{cases}(1+a)(1-\omega)E(\lambda_t)+\omega(1-\omega)[E(\lambda_t)]^2+(1-\omega)\mathrm{Var}(\lambda_t), & c=0,\\ (1-\omega)E(\lambda_t)+(\omega+a)(1-\omega)[E(\lambda_t)]^2+(1+a)(1-\omega)\mathrm{Var}(\lambda_t), & c=1\end{cases} > (1-\omega)E(\lambda_t) = E(X_t),\end{aligned}$$

which indicates that the model (3.1) can handle integer-valued time series with overdispersion.

Theorems 1, 2, 4 and Example 1 given in Section 2 can be extended to the model (3.1). We only state the results here, the proofs are identical to or similar to those for the model (2.2) and are omitted.

Theorem 5. *A necessary and sufficient condition for the ZINB-INGARCH(p, q) process to be stationary in the mean is identical to that of the ZIP-INGARCH(p, q) model.*

Theorem 6. *Suppose that the process $\{X_t\}$ following the ZINB-INARCH (p) model is first-order stationary. For the ZINB1 process, a necessary and sufficient condition for the process to be second-order stationary is identical to that of the ZIP model; while for the ZINB2 process, a necessary and sufficient condition for the process to be second-order stationary is that $1-C_1^*z^{-1}-\cdots-C_p^*z^{-p}=0$ has all roots lie inside the unit circle, where for $u,l=1,\ldots,p-1$,*

$$C_u^* = (1-\omega)(1+a)\left(\alpha_u^2-\sum_{v=1}^{p-1}\sum_{|i-j|=v}\alpha_i\alpha_j b_{vu}\beta_{u0}\right), \quad C_p^* = (1-\omega)(1+a)\alpha_p^2,$$

$$\beta_{l0}^* = (1-\omega)\alpha_l, \quad \beta_{ll}^* = (1-\omega)\sum_{|i-l|=l}\alpha_i-1 \quad \text{and} \quad \beta_{lu}^* = (1-\omega)\sum_{|i-l|=u}\alpha_i, \; u\neq l,$$

where B^* and $(B^*)^{-1}$ are $(p-1)\times(p-1)$ matrices such that $B^*=(\beta_{ij}^*)_{i,j=1}^{p-1}$ and $(B^*)^{-1}=(b_{ij}^*)_{i,j=1}^{p-1}$.

Theorem 7. *The recurrence relationships for $\gamma_X(k)=\mathrm{Cov}(X_t,X_{t-k})$ and $\gamma_\lambda(k)=\mathrm{Cov}(\lambda_t,\lambda_{t-k})$ for the ZINB-INGARCH(p, q) process are the same as those for the ZIP-INGARCH(p, q) process, respectively.*

Example 2. The mean μ and autocorrelations of the ZINB-INGARCH $(1,1)$ model are the same as those for the ZIP-INGARCH $(1,1)$ process, respectively. But for the variance,

$$\mathrm{Var}(X_t)=\begin{cases}\dfrac{1-2(1-\omega)\alpha_1\beta_1-\beta_1^2}{1-(1-\omega)\alpha_1^2-2(1-\omega)\alpha_1\beta_1-\beta_1^2}\left((1+a)\mu+\dfrac{\omega\mu^2}{1-\omega}\right), & c=0,\\ \dfrac{1-2(1-\omega)\alpha_1\beta_1-\beta_1^2}{1-(1+a)(1-\omega)\alpha_1^2-2(1-\omega)\alpha_1\beta_1-\beta_1^2}\left(\mu+\dfrac{(\omega+a)\mu^2}{1-\omega}\right), & c=1.\end{cases}$$

Specially, the mean, variance and autocorrelations of the ZINB-INARCH (1) model are given by

$$E(X_t)=\frac{(1-\omega)\alpha_0}{1-(1-\omega)\alpha_1},\quad \rho_X(k)=[(1-\omega)\alpha_1]^k,\quad k\ge 1,$$

$$\mathrm{Var}(X_t)=\begin{cases}\dfrac{(1+a)(1-\omega)\alpha_0[1+\omega\alpha_0-(1-\omega)\alpha_1]}{[1-(1-\omega)\alpha_1^2][1-(1-\omega)\alpha_1]^2}, & c=0,\\ \dfrac{(1-\omega)\alpha_0[1+(a+\omega)\alpha_0-(1-\omega)\alpha_1]}{[1-(1+a)(1-\omega)\alpha_1^2][1-(1-\omega)\alpha_1]^2}, & c=1.\end{cases}$$

4. Estimation

In this section, we use the EM algorithm to estimate the parameters.

4.1. The ZIP-INGARCH model

Suppose that the observation $X=(X_1,\dots,X_n)$ is generated from the model (2.2). Suppose we knew which zeros came from the generate zero and which came from the Poisson, that is, suppose we could observe $Z_t=1$ when X_t is from the generate zero and $Z_t=0$ when X_t is from the Poisson. The distribution of Z_t is $P(Z_t=1)=\omega, P(Z_t=0)=1-\omega$. Let $Z=(Z_1,\dots,Z_n)$, $\theta=(\alpha_0,\alpha_1,\dots,\alpha_p,\beta_1,\dots,\beta_q)^\top=(\theta_0,\theta_1,\dots,\theta_{p+q})^\top, \Theta=(\omega,\theta^\top)^\top$. The distribution of $(Z|\Theta)$ is

$$\prod_{t=p+1}^{n}\omega^{Z_t}(1-\omega)^{1-Z_t},$$

and the distribution of $(X|Z,\Theta)$ is

$$\prod_{t=p+1}^{n}\left(Z_t\times 1+(1-Z_t)\frac{\lambda_t^{X_t}e^{-\lambda_t}}{X_t!}\right)=\prod_{t=p+1}^{n}\left(\frac{\lambda_t^{X_t}e^{-\lambda_t}}{X_t!}\right)^{1-Z_t}.$$

Consequently, we are led to the conditional likelihood function for the complete data

$$\prod_{t=p+1}^{n}\omega^{Z_t}\left((1-\omega)\frac{\lambda_t^{X_t}e^{-\lambda_t}}{X_t!}\right)^{1-Z_t},$$

then the conditional log-likelihood is given by

$$l(\Theta)=\sum_{t=p+1}^{n}\{Z_t\log\omega+(1-Z_t)[\log(1-\omega)+X_t\log\lambda_t-\lambda_t-\log(X_t!)]\}. \tag{4.1}$$

The first derivatives of the log-likelihood with respect to θ are given as follows:

$$\frac{\partial l}{\partial\omega}=\sum_{t=p+1}^{n}\left(\frac{Z_t}{\omega}-\frac{1-Z_t}{1-\omega}\right), \tag{4.2}$$

$$\frac{\partial l}{\partial\theta_i}=\sum_{t=p+1}^{n}(1-Z_t)\left(\frac{X_t}{\lambda_t}-1\right)\frac{\partial\lambda_t}{\partial\theta_i},\quad i=0,1,\dots,p+q. \tag{4.3}$$

The iterative EM procedure estimates the parameters by maximizing the log-likelihood function (4.1). It consists of an E step and an M step described as follows:

E step: suppose that Θ is known. The missing data Z_t are replaced by their expectations, conditional on the parameters θ and on the observed data X, which are denoted by τ_t. Then

$$\tau_t=\begin{cases}\dfrac{\omega}{\omega+(1-\omega)e^{-\lambda_t}} & \text{if } X_t=0,\\ 0 & \text{if } X_t=1,2\dots,\end{cases}$$

M step: suppose that the missing data are known. The estimates of Θ can then be obtained by maximizing the log-likelihood function l. This can be done by equating expressions (4.2)–(4.3) to 0. The M step equations become

$$\hat{\omega} = \frac{1}{n-p}\sum_{t=p+1}^{n} \tau_t,$$

$$\sum_{t=p+1}^{n} (1-\tau_t)\left(\frac{X_t}{\lambda_t}-1\right)\frac{\partial \lambda_t}{\partial \theta_i}\bigg|_{\hat{\theta}} = 0, \quad i=0,1,\ldots,p+q. \tag{4.4}$$

Since closed form solutions of Eq. (4.4) are unavailable, we use a standard Newton-Raphson algorithm to obtain estimates. For $i,j=0,1,\ldots,p+q$,

$$-\frac{\partial^2 l}{\partial \theta_i\,\partial \theta_j} = \sum_{t=p+1}^{n} (1-Z_t)\left[\frac{X_t}{\lambda_t^2}\frac{\partial \lambda_t}{\partial \theta_i}\frac{\partial \lambda_t}{\partial \theta_j} - \left(\frac{X_t}{\lambda_t}-1\right)\frac{\partial^2 \lambda_t}{\partial \theta_i \partial \theta_j}\right].$$

This suggests that starting with the initial values $\theta^{(0)}$, the values of θ in the subsequent iteration can be given as

$$\theta^{(i+1)} = \theta^{(i)} - \left\{\frac{\partial^2 l}{\partial\theta\partial\theta^\top}\bigg|_{\theta^{(i)}}\right\}^{-1}\frac{\partial l}{\partial \theta}\bigg|_{\theta^{(i)}},$$

where $\theta^{(i)}$ is the value in the ith iteration. In practice, the Z_t's are set to the τ_t's from the previous E step of the EM procedure.

The estimates of Θ are obtained by iterating these two steps until convergence. The criterion used for checking convergence of the EM procedure is $|(\Theta_j^{(i+1)}-\Theta_j^{(i)})/\Theta_j^{(i)}| \le 10^{-5}$.

Remark 4. We consider the EM algorithm because it is simple and easy to be implemented. We can also directly maximize the log-likelihood function. The likelihood function for the ZIP-INGARCH model is

$$\prod_{X_t=0}[\omega+(1-\omega)e^{-\lambda_t}] \times \prod_{X_t>0}\left[(1-\omega)\frac{\lambda_t^{X_t}e^{-\lambda_t}}{X_t!}\right],$$

then the log-likelihood function is

$$\sum_{X_t=0}\log[\omega+(1-\omega)e^{-\lambda_t}] + \sum_{X_t>0}[\log(1-\omega)+X_t\log\lambda_t-\lambda_t-\log(X_t!)].$$

The first term complicates the maximization of the log-likelihood function, although which can be maximized by using the Newton–Raphson algorithm.

4.2. The ZINB-INGARCH model

The estimation procedure of the ZINB-INGARCH model is analogous to that in Section 4.1. The conditional log-likelihood for the complete data is given by $l(\Theta,a) = \sum_{t=p+1}^{n}(l_t^{*}+l_t^{\star})$, where

$$l_t^{*} = Z_t\log\omega + (1-Z_t)\log(1-\omega),$$

$$l_t^{\star} = (1-Z_t)[X_t\log(a\lambda_t^c)-(X_t+\lambda_t^{1-c}/a)\log(1+a\lambda_t^c)+\log\Gamma(X_t+\lambda_t^{1-c}/a)-\log\Gamma(\lambda_t^{1-c}/a)-\log(X_t!)].$$

We jointly estimated the parameters (Θ,a) by adopting the approach of Lawless (1987) and Benjamin et al. (2003), maximizing the likelihood (Θ,a) with respect to Θ for selected values of a, using the similar EM algorithm outlined in Section 4.1, but here

$$\tau_t = \begin{cases} \dfrac{\omega}{\omega+(1-\omega)(1+a\lambda_t^c)^{-\lambda_t^{1-c}/a}} & \text{if } X_t=0, \\ 0 & \text{if } X_t=1,2\ldots, \end{cases}$$

for $c=0$,

$$\frac{\partial l}{\partial \theta_i} = \sum_{t=p+1}^{n}(1-Z_t)\left[\mathrm{DG}\left(X_t+\frac{\lambda_t}{a}\right)-\mathrm{DG}\left(\frac{\lambda_t}{a}\right)-\log(1+a)\right]\frac{1}{a}\frac{\partial \lambda_t}{\partial \theta_i},$$

$$-\frac{\partial^2 l}{\partial \theta_i\,\partial \theta_j} = \sum_{t=p+1}^{n}(1-Z_t)\left\{\left[\mathrm{DG}'\left(\frac{\lambda_t}{a}\right)-\mathrm{DG}'\left(X_t+\frac{\lambda_t}{a}\right)\right]\frac{1}{a^2}\frac{\partial \lambda_t}{\partial \theta_i}\frac{\partial \lambda_t}{\partial \theta_j} - \left[\mathrm{DG}\left(X_t+\frac{\lambda_t}{a}\right)-\mathrm{DG}\left(\frac{\lambda_t}{a}\right)-\log(1+a)\right]\frac{1}{a}\frac{\partial^2 \lambda_t}{\partial \theta_i \partial \theta_j}\right\},$$

where $\mathrm{DG}(x)=\Gamma'(x)/\Gamma(x)$ is the digamma function; while for $c=1$,

$$\frac{\partial l}{\partial\theta_i}=\sum_{t=p+1}^{n}(1-Z_t)\left(\frac{X_t}{\lambda_t}-\frac{1+aX_t}{1+a\lambda_t}\right)\frac{\partial\lambda_t}{\partial\theta_i},$$

$$-\frac{\partial^2 l}{\partial\theta_i\partial\theta_j}=\sum_{t=p+1}^{n}(1-Z_t)\left[\left(\frac{X_t}{\lambda_t^2}-\frac{a(1+aX_t)}{(1+a\lambda_t)^2}\right)\frac{\partial\lambda_t}{\partial\theta_i}\frac{\partial\lambda_t}{\partial\theta_j}-\left(\frac{X_t}{\lambda_t}-\frac{1+aX_t}{1+a\lambda_t}\right)\frac{\partial^2\lambda_t}{\partial\theta_i\partial\theta_j}\right].$$

This gives a maximum likelihood estimate of $\hat{\Theta}(a)$ and hence the profile likelihood $l(\hat{\Theta}(a),a)$, from which the maximum likelihood estimate $\hat{a}$ can be obtained.

Remark 5. Similar to the ZIP-INGARCH model, we can also directly maximize the log-likelihood function for the ZINB-INGARCH model, which is given as

$$\sum_{X_t=0}\log[\omega+(1-\omega)(1+a\lambda_t^c)^{-\lambda_t^{1-c}/a}]+\sum_{X_t>0}[\log(1-\omega)+X_t\log(a\lambda_t^c)$$
$$-(X_t+\lambda_t^{1-c}/a)\log(1+a\lambda_t^c)+\log\Gamma(X_t+\lambda_t^{1-c}/a)-\log\Gamma(\lambda_t^{1-c}/a)-\log(X_t!)].$$

5. Simulation

A simulation study was conducted to evaluate the finite sample performance of the estimators. The evaluation criterion is the mean absolute deviation error (MADE), i.e.

$$\frac{1}{m}\sum_{i=1}^{m}|\hat{\Theta}_j-\Theta_j|,$$

where m is the number of replications. We consider the following models:

(1) INARCH (1) models:
- ZIP models with $(\omega,\alpha_0,\alpha_1)^\top=$ (A1) $(0.5,2,0.5)^\top$ and (A2) $(0.2,1,0.4)^\top$;
- ZINB1 models with $(\omega,\alpha_0,\alpha_1,a)^\top=$ (A3) $(0.5,2,0.5,0.2)^\top$ and (A4) $(0.2,1,0.4,0.5)^\top$;
- ZINB2 models with $(\omega,\alpha_0,\alpha_1,a)^\top=$ (A5) $(0.5,2,0.5,0.2)^\top$ and (A6) $(0.2,1,0.4,0.5)^\top$;

(2) INARCH (2) models:
- ZIP models with $(\omega,\alpha_0,\alpha_1,\alpha_2)^\top=$ (B1) $(0.4,2,0.3,0.1)^\top$ and (B2) $(0.6,3,0.2,0.3)^\top$;
- ZINB1 models with $(\omega,\alpha_0,\alpha_1,\alpha_2,a)^\top=$ (B3) $(0.4,2,0.3,0.1,0.1)^\top$ and (B4) $(0.2,1,0.2,0.1,0.1)^\top$;
- ZINB2 models with $(\omega,\alpha_0,\alpha_1,\alpha_2,a)^\top=$ (B5) $(0.4,2,0.3,0.1,0.1)^\top$ and (B6) $(0.2,1,0.2,0.1,0.1)^\top$;

(3) INGARCH $(1,1)$ models:
- ZIP models with $(\omega,\alpha_0,\alpha_1,\beta_1)^\top=$ (C1) $(0.1,1,0.4,0.3)^\top$ and (C2) $(0.15,2,0.3.0.2)^\top$;
- ZINB1 models with $(\omega,\alpha_0,\alpha_1,\beta_1,a)^\top=$ (C3) $(0.1,1,0.4,0.3,0.2)^\top$ and (C4) $(0.1,1,0.4,0.3,0.5)^\top$;
- ZINB2 models with $(\omega,\alpha_0,\alpha_1,\beta_1,a)^\top=$ (C5) $(0.1,1,0.4,0.3,0.2)^\top$ and (C6) $(0.1,1,0.4,0.3,0.5)^\top$;

For the purpose of comparing the two different versions of the ZINB distribution, we choose the parameters of ZINB1 and ZINB2 models are the same. The simulation was conducted in `Matlab`, functions `poissrnd` and `nbinrnd` were used to generate random data, the length of the time series $n=200$ and 500, and the number of replications $m=1000$. For the

Table 1
Mean of estimates, MADEs (within parentheses) for zero-inflated INARCH(1) models.

Model	n	ω	α_0	α_1	a
A1	200	0.4982(0.0314)	2.0160(0.1733)	0.4826(0.0922)	
	500	0.4994(0.0193)	2.0118(0.1069)	0.4937(0.0539)	
A2	200	0.1924(0.0438)	1.0064(0.1106)	0.3883(0.0708)	
	500	0.1995(0.0276)	1.0053(0.0672)	0.3964(0.0453)	
A3	200	0.4960(0.0334)	2.0040(0.1930)	0.4916(0.0982)	0.1820(0.0693)
	500	0.4986(0.0213)	2.0093(0.1236)	0.4946(0.0597)	0.1893(0.0594)
A4	200	0.1884(0.0563)	1.0029(0.1375)	0.3859(0.0824)	0.4908(0.1018)
	500	0.1982(0.0363)	1.0074(0.0872)	0.3955(0.0496)	0.4954(0.0714)
A5	200	0.4969(0.0365)	2.0068(0.2051)	0.4772(0.1198)	0.1891(0.0630)
	500	0.4985(0.0238)	2.0029(0.1307)	0.4900(0.0750)	0.1947(0.0930)
A6	200	0.1892(0.0545)	1.0071(0.1266)	0.3824(0.0959)	0.4884(0.0681)
	500	0.1968(0.0374)	1.0054(0.0809)	0.3891(0.0604)	0.4962(0.0631)

Table 2
Mean of estimates, MADEs (within parentheses) for zero-inflated INARCH(2) models.

Model	n	ω	α_0	α_1	α_2	a
B1	200	0.3969(0.0303)	2.0152(0.1874)	0.2958(0.0813)	0.0891(0.0759)	
	500	0.4003(0.0205)	2.0054(0.1208)	0.3019(0.0471)	0.0963(0.0484)	
B2	200	0.5976(0.0273)	3.0192(0.2393)	0.1925(0.0896)	0.2925(0.0910)	
	500	0.5997(0.0180)	3.0081(0.1470)	0.1954(0.0567)	0.2965(0.0585)	
B3	200	0.3966(0.0328)	2.0106(0.1694)	0.2920(0.0738)	0.0926(0.0725)	0.0911(0.0877)
	500	0.3993(0.0216)	2.0066(0.1177)	0.2954(0.0511)	0.0980(0.0494)	0.0848(0.0699)
B4	200	0.1918(0.0557)	1.0110(0.1266)	0.1910(0.0754)	0.0865(0.0702)	0.0906(0.0798)
	500	0.1937(0.0376)	1.0089(0.0816)	0.1918(0.0489)	0.0920(0.0466)	0.0930(0.0739)
B5	200	0.3989(0.0337)	2.0369(0.2120)	0.2851(0.0891)	0.0824(0.0861)	0.0906(0.0457)
	500	0.3991(0.0229)	2.0062(0.1315)	0.2963(0.0530)	0.0981(0.0505)	0.0939(0.0735)
B6	200	0.1863(0.0560)	1.0089(0.1344)	0.1940(0.0795)	0.0784(0.0738)	0.0915(0.0880)
	500	0.1946(0.0389)	1.0063(0.0846)	0.1943(0.0476)	0.0923(0.0460)	0.0959(0.0843)

Table 3
Mean of estimates, MADEs (within parentheses) for zero-inflated INGARCH(1,1) models.

Model	n	ω	α_0	α_1	β_1	a
C1	200	0.0805(0.0322)	0.8333(0.1931)	0.4504(0.0724)	0.2020(0.1156)	
	500	0.0821(0.0226)	0.8453(0.1598)	0.4624(0.0665)	0.1858(0.1178)	
C2	200	0.1507(0.0228)	1.8193(0.1807)	0.3697(0.0799)	0.1630(0.0714)	
	500	0.1510(0.0146)	1.8676(0.1324)	0.3759(0.0779)	0.1438(0.0716)	
C3	200	0.1233(0.0339)	0.9521(0.1964)	0.4102(0.0570)	0.2909(0.0856)	0.1778(0.0773)
	500	0.1248(0.0279)	0.9300(0.1512)	0.4150(0.0386)	0.2940(0.0626)	0.1833(0.0588)
C4	200	0.1274(0.0377)	0.9624(0.2037)	0.3938(0.0597)	0.3055(0.0876)	0.4561(0.1155)
	500	0.1297(0.0323)	0.9227(0.1593)	0.4012(0.0376)	0.3131(0.0666)	0.4818(0.1152)
C5	200	0.1207(0.0357)	0.9684(0.2026)	0.4160(0.0616)	0.2773(0.0929)	0.1884(0.0856)
	500	0.1236(0.0301)	0.9547(0.1488)	0.4215(0.0443)	0.2782(0.0682)	0.1911(0.0722)
C6	200	0.1187(0.0403)	0.9947(0.2092)	0.4097(0.0756)	0.2694(0.0986)	0.4754(0.1024)
	500	0.1214(0.0308)	0.9619(0.1561)	0.4176(0.0493)	0.2781(0.0740)	0.4847(0.0477)

maximization of the profile likelihood function $l(\hat{\Theta}(a),a)$, the constrained nonlinear optimization function `fmincon` in `Matlab` was used and the constrained conditions are $a>0$. Moreover, the initial values for ω is 0.5, and the initial values for θ and a are chosen randomly from a uniform distribution.

A summary of the simulation results is given in Tables 1–3, which represent INARCH(1) models, INARCH(2) models and INGARCH(1, 1) models, respectively. These studies indicate that the estimation approach has reasonable estimates and small absolute deviation errors. As the sample size increases, the estimates seem to converge to the true parameter values. Neither ZINB1 nor ZINB2 can turn out to be clearly superior when their parameters are the same.

6. Real data example

In this section, we discuss possible application of the introduced ZIP-INGARCH and ZINB-INGARCH models. We consider a time series from The Forecasting Principles site (http://www.forecastingprinciples.com), in the section about Crime data. The data represent counts of arson in the 13th police car beat plus in Pittsburgh, during one month. The data consist of 144 observations, starting from January 1990 and ending in December 2001. Empirical mean and variance of the data are 1.0417 and 1.3829, respectively. A histogram of the series is shown in Fig. 1, from which we know there are 61 zeros, which is 42.36% of the series. Puig and Valero (2006) defined the following zero-inflation index to measure the departure from the Poisson model

$$zi = 1+\log(p_0)/\mu,$$

where p_0 is the proportion of 0's and μ is the mean. Here the zero-inflation index is 0.1815, which indicates that there is zero inflation.

The original series, the sample autocorrelation and partial autocorrelation function of the series are plotted in Fig. 2. We shall identify and fit some models to the data, such as Poisson INARCH(2) (Ferland et al., 2006), NB1-INARCH(2), NB2-INARCH(2), ZIP-INARCH(2), ZINB1-INARCH(2) and ZINB2-INARCH(2). The initial values for ω is 0.5, and the initial values for θ is the CLS estimators. We first use 0.1 as the initial value for a, then iterate the estimation procedure by using the

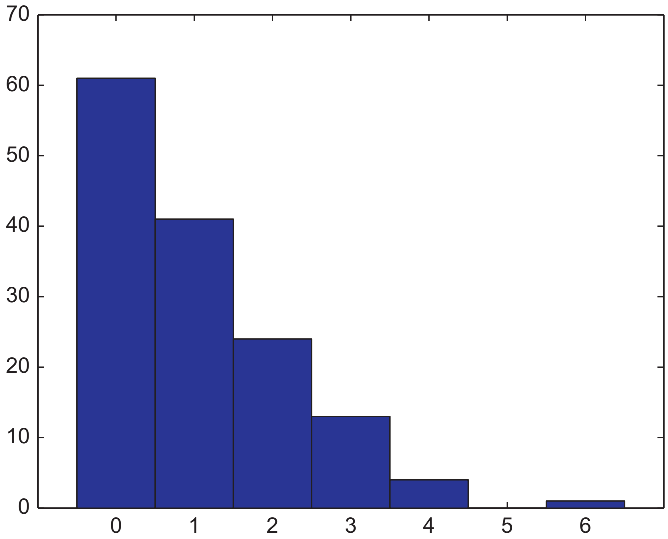

Fig. 1. Histogram of the arson counts series.

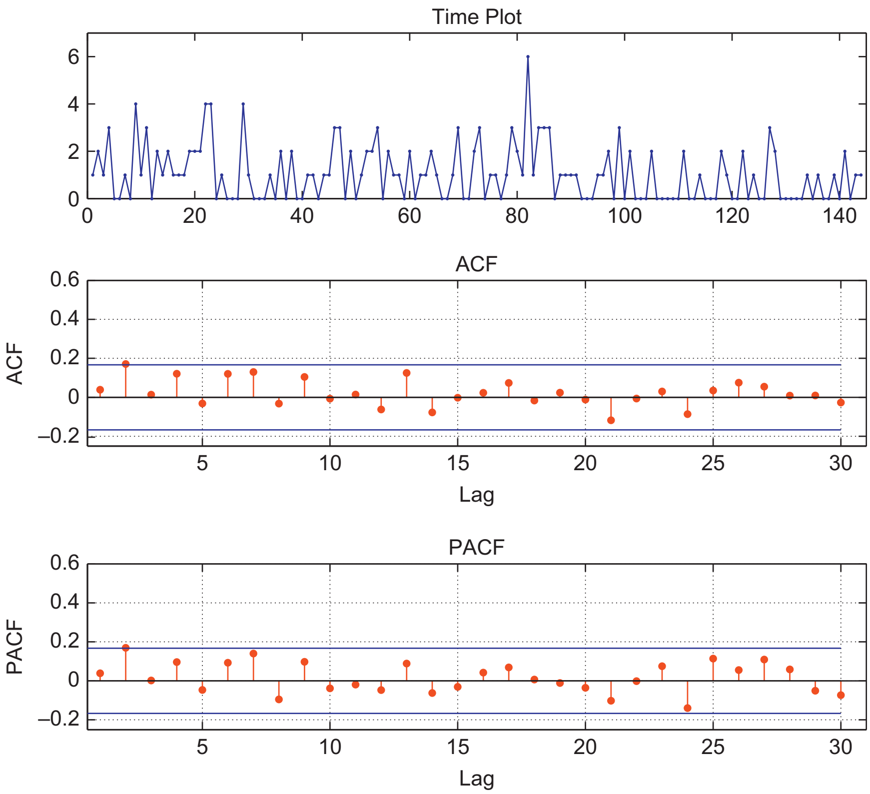

Fig. 2. Arson counts series: the time plot, the sample autocorrelation and partial autocorrelation function.

previous estimated value of a as the next initial value, we stop the procedure when the values of Akaike information criterion (AIC) and Bayesian information criterion (BIC) do not change. The results are summarized in Table 4. Note that in the framework of ZINB2, the zero-inflated coefficient is estimated as zero. Based on AIC and BIC, we find that the data are best fitted by the ZIP-INARCH(2) model.

To assess the adequacy of the ZIP-INARCH(2) model over the Poisson INARCH(2) model, i.e., test $H_0 : \omega = 0$, we use the following likelihood ratio test statistic:

$$\mathrm{LRT} = -2[\log L(\theta^{*}) - \log L(\theta^{\star})],$$

Table 4
Estimated parameters, AIC and BIC values for the arson counts.

Model	$\hat{\omega}$	$\hat{\alpha}_0$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{a}$	AIC	BIC
Poisson		0.8253	0.0269	0.1744		401.3054	410.1729
NB1		0.8395	0.0216	0.1662	0.3291	398.2326	410.0559
NB2		0.8243	0.0327	0.1696	0.3233	397.9348	409.7581
ZIP	0.2149	1.0220	0.0560	0.2321		395.7527	407.5760
ZINB1	0.1979	1.0025	0.0528	0.2278	0.0230	397.7948	412.5739
ZINB2	0	0.8258	0.0249	0.1763	0.3233	399.9228	414.7020

where $\log L(\theta^*)$ is the log-likelihood function under the null hypothesis and $\log L(\theta^\star)$ is the log-likelihood function under the alternative hypothesis. Under the null hypothesis, LRT does not have an asymptotic $\chi^2(1)$ distribution as expected, because $\omega = 0$ belongs to the boundary of the domain of parameters. Following Self and Liang (1987), it can be established that in this situation, the asymptotic distribution of LRT is a 50:50 mixture of the constant zero and the $\chi^2(1)$ distribution. The α upper-tail percentage points for this mixture are the same as the 2α upper-tail percentage points for a $\chi^2(1)$ distribution. We compute LRT to be 7.5527. On comparing $\chi^2_{0.98}(1) = 5.4119$, we notice that LRT is significant, which confirms that there is zero-inflation phenomenon in this data. Within the fitted model the estimated mean and variance are 1.0369 and 1.3952, which are very close to the corresponding empirical ones, respectively. In conclusion, the ZIP-INARCH(2) model is appropriate for this time series.

7. Discussions

This paper proposes zero-inflated Poisson and negative binomial INGARCH models for modeling integer-valued time series with zero inflation and overdispersion. A real data example leads to superior performance of the proposed models compared with other competitive models in the literature.

More research is needed for some aspects of the proposed methodology.

The first issue is the existences of the zero-inflated processes. Many authors have discussed the existences of the integer-valued time series processes, see Du and Li (1991), Latour (1997, 1998), Doukhan et al. (2006), Ferland et al. (2006), and Franke (2010). To my best knowledge, there are two methods for proving the existence of the INGARCH process. The first one is the construction technique used by Ferland et al. (2006), while the second one is the coupling technique discussed in Franke (2010). Unfortunately, the above two methods cannot be applied to the zero-inflated processes. New techniques should be developed to prove the existences of such processes.

The second issue is the unconditional distributions of X_t. For this, we need the Poisson–Charlier expansion discussed in Weiß (2010a). Because of the complex expressions for factorial cumulants, the derivation of the unconditional distributions for the zero-inflated models is lengthy, but it deserves further investigation.

The third issue is the ergodicities of the proposed processes. Fokianos et al. (2009) proved the ergodicity of the Poisson INGARCH(1,1) process by perturbing the model and proving that its perturbed version is ergodic under simple restrictions on the parameter space via Markov chain theory. Recently, Neumann (forthcoming) proved the ergodicity of a very general Poisson INGARCH(1,1) process under a contractive condition without using Markov chain technology. He first proved the count process is absolutely regular (β-mixing) and then concluded from the mixing property of the count process that the bivariate process (the count and intensity processes) is ergodic. These approaches are needed to be investigated for the proposed processes in this paper.

The last issue is to investigate the estimator property under the medium sample size.

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References

Bakouch, H.S., Ristić, M.M., 2010. Zero truncated Poisson integer-valued AR(1) model. Metrika 72, 265–280.
Benjamin, M.A., Rigby, R.A., Stasinopoulos, D.M., 2003. Generalized autoregressive moving average models. Journal of the American Statistical Association 98, 214–223.
Böhning, D., 1998. Zero-inflated Poisson models and C.A.MAN: a tutorial collection of evidence. Biometrical Journal 40, 833–843.
Doukhan, P., Latour, A., Oraichi, D., 2006. A simple integer-valued bilinear time series model. Advances in Applied Probability 38, 559–578.
Du, J.-G., Li, Y., 1991. The integer-valued autoregressive (INAR(p)) model. Journal of Time Series Analysis 12, 129–142.

Ferland, R., Latour, A., Oraichi, D., 2006. Integer-valued GARCH process. Journal of Time Series Analysis 27, 923–942.
Fokianos, K., 2011. Some recent progress in count time series. Statistics 45, 49–58.
Fokianos, K., Fried, R., 2010. Interventions in INGARCH processes. Journal of Time Series Analysis 31, 210–225.
Fokianos, K., Rahbek, A., Tjøstheim, D., 2009. Poisson autoregression. Journal of the American Statistical Association 104, 1430–1439.
Fokianos, K., Tjøstheim, D., 2011. Log-linear Poisson autoregression. Journal of Multivariate Analysis 102, 563–578.
Fong, P.W., Li, W.K., Yau, C.W., Wong, C.S., 2007. On a mixture vector autoregressive model. Canadian Journal of Statistics 35, 135–150.
Franke, J., 2010. Weak Dependence of Functional INGARCH Processes. Technical Report, University of Kaiserslautern.
Garay, A.M., Hashimoto, E.M., Ortega, E.M.M., Lachos, V.H., 2011. On estimation and influence diagnostics for zero-inflated negative binomial regression models. Computational Statistics and Data Analysis 55, 1304–1318.
Goldberg, S., 1958. Introduction to Difference Equations. Wiley, New York.
Gradshteyn, I.S., Ryzhik, I.M., 2007. Table of Integrals, Series, and Products, 7th ed. Academic Press, San Diego.
Hall, D.B., Shen, J., 2010. Robust estimation for zero-inflated Poisson regression. Scandinavian Journal of Statistics 37, 237–252.
Heilbron, D., 1994. Zero-altered and other regression models for count data with added zeros. Biometrical Journal 36, 531–547.
Johnson, N.L., Kemp, A.W., Kotz, S., 2005. Univariate Discrete Distributions, 3rd ed. Wiley, New Jersey.
Lambert, D., 1992. Zero-inflated Poisson regression, with an application to defects in manufacturing. Technometrics 34, 1–14.
Latour, A., 1997. The multivariate GINAR(p) process. Advances in Applied Probability 29, 228–248.
Latour, A., 1998. Existence and stochastic structure of a non-negative integer-valued autoregressive process. Journal of Time Series Analysis 19, 439–455.
Lawless, J.F., 1987. Negative binomial and mixed Poisson regression. Canadian Journal of Statistics 15, 209–225.
Matteson, D.S., McLean, M.W., Woodard, D.B., Henderson, S.G., 2011. Forecasting emergency medical service call arrival rates. Annals of Applied Statistics 5, 1379–1406.
Min, A., Czado, C., 2010. Testing for zero-modification in count regression models. Statistica Sinica 20, 323–341.
Mullahy, J., 1986. Specification and testing of some modified count data models. Journal of Econometrics 33, 341–365.
Neumann, M.H. Poisson count processes: ergodicity and goodness-of-fit. Bernoulli, forthcoming.
Puig, P., Valero, J., 2006. Count data distributions: some characterizations with applications. Journal of the American Statistical Association 101, 332–340.
Ridout, M., Demétrio, C.G.B., Hinde, J., 1998. Models for count data with many zeros. In: Proceedings of the 19th International Biometrics Conference, Cape Town, South Africa, pp. 179–190.
Ridout, M., Hinde, J., Demétrio, C.G.B., 2001. A score test for testing a zero-inflated Poisson regression model against zero-inflated negative binomial alternatives. Biometrics 57, 219–223.
Self, S.G., Liang, K.-Y., 1987. Asymptotic properties of maximum likelihood estimators and likelihood ratio tests under nonstandard conditions. Journal of the American Statistical Association 82, 605–610.
Weiß, C.H., 2009. Modelling time series of counts with overdispersion. Statistical Methods and Applications 18, 507–519.
Weiß, C.H., 2010a. The INARCH(1) model for overdispersed time series of counts. Communications in Statistics—Simulation and Computation 39, 1269–1291.
Weiß, C.H., 2010b. INARCH(1) processes: higher-order moments and jumps. Statistics and Probability Letters 80, 1771–1780.
Xie, F.C., Wei, B.C., Lin, J.G., 2009. Score tests for zero-inflated generalized Poisson mixed regression models. Computational Statistics and Data Analysis 53, 3478–3489.
Yang, Z., Hardin, J.W., Addy, C.L., 2009. Testing overdispersion in the zero-inflated Poisson model. Journal of Statistical Planning and Inference 139, 3340–3353.
Zeileis, A., Kleiber, C., Jackman, S., 2008. Regression models for count data in R. Journal of Statistical Software 27 (8), 1–25. URL ⟨http://www.jstatsoft.org/v27/i08/⟩.
Zhu, F., 2011. A negative binomial integer-valued GARCH model. Journal of Time Series Analysis 32, 54–67.
Zhu, F., Li, Q., 2009. Moment and Bayesian estimation of parameters in the INGARCH(1,1) model. Journal of Jilin University (Science Edition) 47, 899–902 (in Chinese).
Zhu, F., Li, Q., Wang, D., 2010. A mixture integer-valued ARCH model. Journal of Statistical Planning and Inference 140, 2025–2036.
Zhu, F., Wang, D., 2010. Diagnostic checking integer-valued ARCH(p) models using conditional residual autocorrelations. Computational Statistics and Data Analysis 54, 496–508.
Zhu, F., Wang, D., 2011. Estimation and testing for a Poisson autoregressive model. Metrika 73, 211–230.
Zhu, F., Wang, D., Li, F., Li, H., 2008. Empirical likelihood inference for an integer-valued ARCH(p) model. Journal of Jilin University (Science Edition) 46, 1042–1048 (in Chinese).
Zuur, A.F., Ieno, E.N., Walker, N.J., Saveliev, A.A., Smith, G.M., 2009. Mixed Effects Models and Extensions in Ecology with R. Springer, New York.