J. Appl. Prob. **27**, 314–324 (1990)
Printed in Israel

AN INTEGER-VALUED pth-ORDER AUTOREGRESSIVE STRUCTURE (INAR(p)) PROCESS

A. A. ALZAID AND
M. AL-OSH,* *King Saud University*

Abstract

An extension of the INAR(1) process which is useful for modelling discrete-time dependent counting processes is considered. The model investigated here has a form similar to that of the Gaussian AR(p) process, and is called the integer-valued pth-order autoregressive structure (INAR(p)) process. Despite the similarity in form, the two processes differ in many aspects such as the behaviour of the correlation, Markovian property and regression. Among other aspects of the INAR(p) process investigated here are the limiting as well as the joint distributions of the process. Also, some detailed discussion is given for the case in which the marginal distribution of the process is Poisson.

INTEGER-VALUED RANDOM VARIABLE; AUTOCORRELATION; REGRESSION; LIMITING AND JOINT DISTRIBUTIONS; STATE SPACE REPRESENTATION; POISSON INAR(p) PROCESS

1. Introduction

Recently there has been growing interest in modelling discrete-time stationary processes with discrete marginal distributions. In developing such models the integer-valued first-order autoregressive (INAR(1)) process, introduced independently by McKenzie (1985) and Al-Osh and Alzaid (1987), has received considerable attention. The INAR(1) model is of the form

$$X_n = \alpha \circ X_{n-1} + \varepsilon_n,$$

where $\alpha \circ X_{n-1}$ denotes a sum of X_{n-1} independent 0–1 random variables $Y_i^{(n-1)}$, independent of X_{n-1} with $P_r(Y_i^{(n-1)} = 1) = 1 - P_r(Y_i^{(n-1)} = 0) = \alpha$ and $\{\varepsilon_n\}$ is a sequence of i.i.d. non-negative integer-valued random variables with mean μ_ε and finite variance σ_ε^2. Some properties of the Poisson and geometric INAR(1) processes are discussed in Alzaid and Al-Osh (1988). McKenzie (1986) also developed processes with geometric and negative binomial marginal distributions. These authors noted that the INAR(1) process is not only similar in form to the standard AR(1), but also has the same autocorrelation and regression behaviour.

Received 23 December 1987; revision received 19 May 1989.
* Postal address for both authors: Department of Statistics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia.

The purpose of the present paper is to introduce an extension of the INAR(1) process with the aim of having more flexible models for counting processes. The form of the model investigated here is that of the standard AR(p) process with the replacement of the scalar multiplication by $\circ$-operations as in the INAR(1) model. The new process will be called integer-valued pth-order autoregressive structure (INAR(p)) process. It will be seen that, unlike the first-order process, the similarity between the standard AR(p) and the INAR(p) processes does not extend beyond the form. In fact, because of the existence of a certain type of dependence among the components of the INAR(p) model, the autocorrelation of this process behaves like that of the standard ARMA(p, $p-1$) process. Also the two processes differ in terms of the regression. In addition, whereas the INAR(p) process itself does not have the Markovian property, a state vector of this process is Markovian. Consequently, our designation of the process as autoregressive should be taken loosely and the word 'structure' is intended to indicate similarity in form with the AR(p) process.

The organization of the paper is as follows. In the following section we present the INAR(p) process and discuss some assumptions and peculiarities as well as the limiting distribution of the process. A detailed derivation of the autocorrelation function is given in Section 3. In Section 4, we consider a state space representation for the process and discuss the joint distribution. In Section 5, we discuss the Poisson INAR(p) process and compare it with the multiple Poisson process introduced by McKenzie (1988).

2. Integer-valued pth-order autoregressive structure (INAR(p)) process

The INAR(1) model defined in (1.1) is appropriate for modelling data of the branching process type. However, realizations of some counting process $\{X_n\}$ might be attributed not only to its immediate past X_{n-1} but also to previous realizations of the process $\{X_{n-j}\}_{j=2}^{p}$ for some constant p. Consequently for modelling such processes the need arises to extend the INAR(1) process to take into account these previous realizations. A direct way of extending the INAR(1) process is to consider the form of the standard AR(p) process with replacement of the scalar multiplication, $\alpha_i X_{n-i}$, by the operation '$\alpha_i \circ X_{n-i}$' for $i = 1, 2, \cdots, p$. However, with such replacement, some assumptions on the model are needed in order to govern the dependence structure of the process and for the model to be well defined. The process $\{X_n\}$ is said to be INAR(p) if it admits the representation

$$X_n = \sum_{i=1}^{p} \alpha_i \circ X_{n-i} + \varepsilon_n \quad \text{for } n = 0, \pm 1, \cdots, \tag{2.1}$$

where $\{\varepsilon_n\}$ is a sequence of i.i.d. non-negative integer-valued random variables with mean μ_ε and finite variance σ_ε^2; and α_i ($i = 1, 2, \cdots, p$) are non-negative constants such that $\Sigma_{i=1}^{p} \alpha_i < 1$. The conditional distribution of the vector $(\alpha_1 \circ X_n, \alpha_2 \circ X_n, \cdots, \alpha_p \circ X_n)$ given $X_n = x_n$ is multinomial with parameters $(\alpha_1, \alpha_2, \cdots, \alpha_p, x_n)$ and is independent of the past history of the process. That is to say, given $X_n = x_n$ the random variable $\alpha_i \circ X_n$ is independent of X_{n-k} and its survivals $\alpha_j \circ X_{n-k}$ for $i, j = 1, 2, \cdots, p$ and $k > 0$.

The INAR(p) process defined in (2.1) can be viewed as an extension (or special type) of a branching process with immigration. For this, consider a human or biological population in which a female can reproduce at most one female offspring during her reproductive span which is split into p non-overlapping periods. The probability that such a female will have her offspring during her ith period of reproduction is α_i. Now it is clear that the size of the nth generation $\{X_n\}$ is just the total offspring of the last p generations in addition to the immigrant process $\{\varepsilon_n\}$ which have entered the system during the time interval $(n-1, n]$. It should be observed that for $\Sigma_{i=1}^{p} \alpha_i \geqq 1$ the generation size $\{X_n\}$ will increase monotonically and we have an explosive process. The previous interpretation is merely an extension of the branching (INAR(1)) process in which only the last generation can reproduce.

We may point out that even though the form of the INAR(p) model appears to be a pth-order autoregressive process, the dependence across time of the $\circ$-operation in (2.1) makes this process differ from the standard AR(p) process. To explain this point let us consider the case $p = 2$, for simplicity, that is:

$$X_n = \alpha_1 \circ X_{n-1} + \alpha_2 \circ X_{n-2} + \varepsilon_n.$$

In the standard AR(2) process X_n is obtained by a direct multiplication of the constants α_1 and α_2 to X_{n-1} and X_{n-2}, respectively, at time n and independent of all previous stochastic structure. This is not the case for the INAR(2) process. For this process the random variables $\alpha_1 \circ X_{n-2}$ and $\alpha_2 \circ X_{n-2}$, which are elements of X_{n-1} and X_n respectively, are intimately connected in a much more powerful way than is the case in the standard AR(2) where only the presence of X_{n-2} would connect them. Here the operations '$\alpha_1 \circ$' and '$\alpha_2 \circ$' on X_{n-2} are dependent but appear to be performed at different times. The structure is perhaps clearest when simulating the process. At time n, X_n is observed and we have available X_{n-1} and $V_{n-1} = \alpha_2 \circ X_{n-1}$, having used $\alpha_1 \circ X_{n-1}$ to derive X_n. The first step is to form $U_n = \alpha_1 \circ X_n$ and $V_n = \alpha_2 \circ X_n$ and then $X_{n+1} = U_n + V_{n-1} + \varepsilon_{n+1}$, and V_n is available for use in the derivation of X_{n+2}.

The mutual dependence structure between the components of X_n i.e. $\alpha_i \circ X_n$, $i = 1, 2, \cdots, p$ appearing at different times induces a moving-average structure into the process. In fact it will be seen in the following section that the behaviour of the autocorrelation function of the INAR(p) process mimics that of the standard ARMA(p, $p-1$) process.

In the remainder of this section we discuss the limiting distribution of the process $\{X_n\}$ as defined in (2.1). For this let us define $\{w_j\}_{j=0}^{\infty}$ as a sequence of weights such that:

$$w_0 = 1 \tag{2.2}$$
$$w_j = \sum_{i=1}^{\min(j,p)} \alpha_i w_{j-i}.$$

Obviously $\{w_j\}_{j=0}^{\infty}$ are the coefficients of the usual moving average representation for the INAR(p) process given in (2.1). Thus

$$\left(\sum_{k=0}^{\infty} w_k z^k\right)\left(1-\sum_{k=1}^{p} \alpha_k z^k\right)=1.$$

Now let $\phi_n(s)$ and $\psi_n(s)$ denote the probability generating functions (p.g.f.s) of X_n and ε_n respectively, then (2.1) implies that

$$\begin{aligned}\phi_n(s) &= E(s^{\sum_{i=1}^{p}\alpha_i\circ X_{n-i}+\varepsilon_n})\\ &= E[(1-w_1+w_1 s)^{X_{n-1}} s^{\sum_{i=2}^{p}\alpha_i\circ X_{n-i}}]\psi_n(1-w_0+w_0 s).\end{aligned}$$

Using an iteration argument one gets

(2.3) $$\phi_n(s)=\prod_{j=0}^{k-1}\psi_{n-j}(1-w_j+w_j s)E\left[\prod_{l=1}^{p}(1-w_{k-l}+w_{k-l}s)^{\sum_{i=l}^{p}\alpha_i\circ X_{n-k+l-i}}\right] \text{ for } k\geqq p.$$

Now, to find the limiting distribution of $\{X_n\}$, we need the following lemma.

Lemma 2.1 (Fuller (1976), p. 57). Suppose that the weights $\{w_j\}$ are as defined in (2.2) and let the roots of the polynomial

(2.4) $$y^p-\alpha_1 y^{p-1}-\cdots-\alpha_p=0 \quad \text{with } \alpha_p\neq 0$$

be less than 1 in absolute value. Then there exists $0<\lambda<1$ such that

$$0\leqq w_j\leqq c\lambda^j \qquad j=0,1,2,\cdots$$

for some constant c.

The proof of this lemma is contained in the proof of Theorem 2.6.1 of Fuller (1976). The following theorem establishes the limiting distribution of the process $\{X_n\}$.

Theorem 2.1. Let $\{X_n: n=0,1,2,\cdots\}$ be an INAR(p) process with parameters $\alpha_1,\alpha_2,\cdots,\alpha_p$ such that $\alpha_p>0$. Suppose that the roots of Equation (2.4) are less than 1 in absolute value and $\sum_{j=0}^{\infty}(j+1)^{-1}r_j<\infty$ with $r_j=\sum_{k=j+1}^{\infty}p(\varepsilon_1=k)$. Then the process $\{X_n\}$ has a limiting distribution with p.g.f. given by

(2.5) $$\phi(s)=\prod_{j=0}^{\infty}\psi(1-w_j+w_j s).$$

Proof. From (2.3) we have

$$\phi_{n+p}(s)=\prod_{j=0}^{n-1}\psi(1-w_j+w_j s)E\left[\prod_{l=1}^{p}(1-w_{n-l}+w_{n-l}s)^{\sum_{i=l}^{p}\alpha_i\circ X_{p+l-i}}\right].$$

Now by Lemma 2.1, it is clear, in view of the bounded convergence theorem, that

$$\lim_{n\to\infty} E\left[\prod_{l=1}^{p}(1-w_{n-l}+w_{n-l}s)^{\sum_{i=l}^{p}\alpha_i\circ X_{p+l-i}}\right]=1.$$

Therefore it is sufficient to show that $\prod_{j=0}^{n-1}\psi(1-w_j+w_j s)$ is convergent. But the last product is convergent if and only if $\sum_{j=1}^{n}[1-\psi(1-w_j+w_j s)]$ is convergent. Since ψ is non-decreasing, we have, by Lemma 2.1,

$$1 - \psi(1 - w_j + w_j s) \leqq 1 - \psi(1 - c\lambda^j(1 - s))$$
$$\leqq 1 - \psi(1 - \lambda^j) \quad \text{for } 1 - 1/c \leqq s < 1.$$

Now, using the argument of Heathcote (1966), we find that $\Pi_{j=0}^{n-1} \psi(1 - w_j + w_j s)$ is convergent if and only if $\Sigma_{j=0}^{\infty} (j + 1)^{-1} r_j$ is convergent. This completes the proof.

The following theorem states that the limiting distribution of the process X_n, as given in (2.5), is still valid if the process is stationary and has started from the infinite past.

Theorem 2.2. Let $\{X_n : n = 0, \pm 1, \pm 2, \cdots\}$ be a stationary INAR(p) process with parameters $\alpha_1, \alpha_2, \cdots, \alpha_p$. Then under the assumption of Theorem 2.1, the p.g.f. of X_n is given by (2.5).

Proof. The proof is essentially similar to that of Theorem 2.1 by taking the limit as k tends to infinity in Equation (2.3).

Remark. Theorem 2.1 is equivalent to saying that the marginal distribution of X_n is identical to that obtained from an integer-valued moving average (INMA) of infinite order with parameters w_j, $j = 0, 1, 2, \cdots$ and innovation sequence $\{\varepsilon_n\}$. That is,

$$X_n \overset{\mathrm{d}}{=} \sum_{j=0}^{\infty} w_j \circ \varepsilon_{n-j}, \qquad n = 0, \pm 1, \pm 2, \cdots.$$

It is worth mentioning that, unlike the standard moving-average process, given ε_n the random variables $w_j \circ \varepsilon_n$ and $w_i \circ \varepsilon_n$ are dependent (see Al-Osh and Alzaid (1988) for the definition of the INMA(q) process and some of its properties).

3. Correlation properties

In studying the correlation properties of the INAR(p) process we make use of the assumption that given $X_n = x_n$ the random variable $\alpha_i \circ x_n$ is independent of the past history of the process $\{X_{n-k}\}$ and of $\alpha_j \circ X_{n-k}$ for $i, j = 1, 2, \cdots, p$ and $k \geqq 1$. Now it can be seen from (2.1) that

$$E(X_n) = \sum_{i=1}^{p} \alpha_i E(X_{n-i}) + \mu_\varepsilon \tag{3.1}$$

where $\mu_\varepsilon = E(\varepsilon_n)$. Under the weak stationarity assumption we have

$$\mu_X \equiv E(X_n) = \mu_\varepsilon \bigg/ \left(1 - \sum_{i=1}^{p} \alpha_i\right).$$

For derivation of the autocovariance at lag k, $\gamma(k)$, we have

$$\gamma(k) = \mathrm{Cov}(X_{n-k}, X_n) = \sum_{i=1}^{p} \mathrm{Cov}(X_{n-k}, \alpha_i \circ X_{n-i}) + \delta_k(0)\sigma_\varepsilon^2 \tag{3.2}$$
$$= \sum_{i=1}^{p} \gamma(k-i, \alpha_i) + \delta_k(0)\sigma_\varepsilon^2,$$

where $\gamma(l, \alpha_i) \equiv \mathrm{Cov}(X_{n-l}, \alpha_i \circ X_n)$ and $\delta_k(0) = 1$ if $k = 0$ and zero otherwise.

It is obvious from the definition of $\gamma(l, \alpha_i)$ that

$$\gamma(l, \alpha) = \alpha\gamma(l) \quad \text{for } l \geqq 0. \tag{3.3}$$

Now, by using a conditional argument and the assumption that given $X = x$ the vector $(\alpha_1 \circ x, \alpha_2 \circ x, \cdots, \alpha_p \circ x)$ has a multinomial distribution with parameters $(\alpha_1, \alpha_2, \cdots, \alpha_p, x)$ we have

$$\mathrm{Cov}(\alpha_j \circ X_{n-k}, \alpha_i \circ X_{n-l}) = \begin{cases} \alpha_j\gamma(k-l, \alpha_i) & \text{for } k < l \\ \alpha_j\alpha_i\gamma(0) + \alpha_j(\delta_j(i) - \alpha_i)\mu_X & \text{for } k = l \\ \alpha_i\gamma(l-k, \alpha_j) & \text{for } k > l. \end{cases} \tag{3.4}$$

Now $\gamma(-l, \alpha)$ can be determined recursively by using (3.4). Starting with $l = 1$, we have

$$\gamma(-1, \alpha_i) = \alpha_1\alpha_i\gamma(0) + \alpha_1(\delta_i(1) - \alpha_i)\mu_X + \alpha_i \sum_{j=2}^{p} \gamma(\alpha_j, j-1) \tag{3.5}$$
$$= \alpha_i\gamma(-1) + \alpha_1(\delta_i(1) - \alpha_i)\mu_X.$$

Now let $\mu(l, \alpha_i) \equiv \gamma(l, \alpha_i) - \alpha_i\gamma(l)$; then Equation (3.3) gives $\mu(l, \alpha_i) = 0$ for $l \geqq 0$, and by Equation (3.5), we have $\mu(-1, \alpha_i) = \alpha_1(\delta_i(1) - \alpha_i)\mu_X$. Similarly

$$\mu(-2, \alpha_i) = \sum_{j=1}^{p} \mathrm{Cov}(\alpha_j \circ X_{n-j}, \alpha_i \circ X_{n-2}) - \alpha_i\gamma(-2)$$
$$= \alpha_1\gamma(-1, \alpha_i) + \alpha_2\alpha_i\gamma(0) + \alpha_2(\delta_i(2) - \alpha_i)\mu_X$$
$$+ \alpha_i \sum_{j=3}^{p} \mathrm{Cov}(\alpha_j \circ X_{n-j}, X_{n-2}) - \alpha_i\gamma(-2)$$
$$= \alpha_1\mu(-1, \alpha_i) + \alpha_2(\delta_i(2) - \alpha_i)\mu_X.$$

Using this iterative procedure, we get

$$\mu(-l, \alpha_i) = \sum_{j=1}^{l-1} \alpha_j\mu(j-l, \alpha_i) + \mu_{i,l} \tag{3.6}$$

where $\mu_{i,l} = \alpha_i(\delta_i(l) - \alpha_l)\mu_X$. This means that $\mu(-l, \alpha_i)$ is a linear function of the mean of the process μ_X. By utilizing the definition of $\mu(l, \alpha_i)$, it can be seen that $\gamma(k-i, \alpha_i)$ in (3.2) satisfies

$$\gamma(k-i, \alpha_i) = \alpha_i\gamma(k-i) + \mu(k-i, \alpha_i), \tag{3.7}$$

where $\mu(k-i, \alpha_i)$ is determined by (3.6) for $k < i$ and $\mu(k-i, \alpha_i) = 0$ for $k \geqq i$ as has been shown in the discussion following Equation (3.5). Now, by using (3.7), we have

$$\gamma(k) = \sum_{i=1}^{p} \alpha_i \gamma(k-i) + \sum_{i=k+1}^{p} \mu(k-i, \alpha_i) + \delta_k(0)\sigma_\varepsilon^2. \tag{3.8}$$

It can be seen from (3.8) that the autocovariance of the INAR(p) process is of the same form as that of the Gaussian ARMA($p, p-1$) process. This behaviour of $\gamma(k)$ is due to the mutual dependence structure between components of X_n, i.e. $\alpha_i \circ X_{n-i}$ for $i = 1, 2, \cdots, p$, appearing in different times as discussed in the previous section.

4. State space representation of the INAR(p) process

By analogy with the Gaussian AR(p) process which admits state space representation, we consider in this section such a representation for the INAR(p) process. This state space representation will be utilized to find the joint distribution of the process and for comparing our process with the multiple Poisson AR(1) process considered by McKenzie (1988). We discuss the case of the INAR(2) process with some details and indicate the results for higher-order autoregression structure.

The state vector which we consider for the process is $\boldsymbol{X}_n^* = (X_n, \alpha_2 \circ X_{n-1})'$. This vector can be manipulated more easily than the corresponding state vector $\boldsymbol{X}_n = (X_n, X_{n-1})'$. Obviously the joint distribution (and hence the marginal distributions) of the process $\{X_n\}$ can be determined from the joint distribution of $\{\boldsymbol{X}_n^*\}$. The process $\{\boldsymbol{X}_n^*\}$ is Markovian since

$$E(s_1^{X_n} s_2^{\alpha_2 \circ X_{n-1}} \mid \boldsymbol{X}_{n-1}^*, \boldsymbol{X}_{n-2}^*, \cdots)$$
$$= \psi_n(s_1)(1 - \alpha_1 - \alpha_2 + \alpha_1 s_1 + \alpha_2 s_2)^{X_{n-1}} s_1^{\alpha_2 \circ X_{n-2}}.$$

Hence it is sufficient to consider the joint distribution of $\boldsymbol{X}_n^*$ and $\boldsymbol{X}_{n-1}^*$ to determine the joint distribution of the process. Now using the assumption that given X_{n-1}, $(\alpha_1 \circ X_{n-1}, \alpha_2 \circ X_{n-1})$ has multinomial distribution independent of X_{n-2} and its survivals, the joint p.g.f. $\phi_{n,n-1}^*$ of $\boldsymbol{X}_n^*$ and $\boldsymbol{X}_{n-1}^*$ is given by

$$\begin{aligned} \phi_{n,n-1}^*(\boldsymbol{s}, \boldsymbol{t}) &= E(s_1^{X_n} s_2^{\alpha_2 \circ X_{n-1}} t_1^{X_{n-1}} t_2^{\alpha_2 \circ X_{n-2}}) \\ &= \psi^*(\boldsymbol{s}) \phi_{n-1}^*(t_1(1 - \alpha_1 - \alpha_2 + \alpha_1 s_1 + \alpha_2 s_2), s_1 t_2) \end{aligned} \tag{4.1}$$

where $\boldsymbol{s} = (s_1, s_2)'$, $\boldsymbol{t} = (t_1, t_2)'$, ψ^* is the p.g.f. of $\boldsymbol{\varepsilon}_n = (\varepsilon_n, 0)'$. This implies that the joint distribution of the process $\{\boldsymbol{X}_n^*\}$ is determined by the marginal distributions of the process itself together with that of the sequence $\{\boldsymbol{\varepsilon}_n\}$. In the following, we discuss the determination of the marginal and the limiting distributions of $\{\boldsymbol{X}_n^*\}$. We have

$$\begin{aligned}\phi_n^*(\mathbf{s}) &= E(s_1^{X_n} s_2^{\alpha_2 \circ X_{n-1}}) \\ &= E(s_1^{\alpha_1 \circ X_{n-1} + \alpha_2 \circ X_{n-2} + \varepsilon_n} \cdot s_2^{\alpha_2 \circ X_{n-1}}) \\ &= \psi^*(\mathbf{s}) \phi_{n-1}^*(1 - \alpha_1 - \alpha_2 + \alpha_1 s_1 + \alpha_2 s_2, s_1) \\ &= \psi^*(\mathbf{s}) \phi_{n-1}^*(\mathbf{1} - A'(\mathbf{1} - \mathbf{s}))\end{aligned} \tag{4.2}$$

where $\mathbf{1} = (1, 1)'$, and

$$A = \begin{bmatrix} \alpha_1 & 1 \\ \alpha_2 & 0 \end{bmatrix}.$$

The form (4.2) of $\phi_n^*(\mathbf{s})$ suggests that the process $\{X_n^*\}$ can be viewed as a special multitype branching process with immigration. (A probability generating function of the form (4.2) with general matrix A has been considered by Steutel and van Harn (1986) to define multivariate discrete self-decomposability and by McKenzie (1988) to define a discrete vector autoregressive process of order 1.)

The above approach can be extended to a higher order INAR process. For the general INAR(p) process the corresponding state vector is

$$\mathbf{X}_n^* = (X_n, \Sigma_{i=2}^p \alpha_i \circ X_{n+1-i}, \Sigma_{i=3}^p \alpha_i \circ X_{n+2-i}, \cdots, \alpha_p \circ X_{n-1})'$$

which is of the same form as the state vector for the Gaussian AR(p) process (see e.g. Harvey and Phillips (1979)).

Now following an iteration argument on (4.2) we get

$$\phi_n^*(\mathbf{s}) = \phi_1^*(\mathbf{1} - \mathbf{A}'^n(\mathbf{1} - \mathbf{s})) \prod_{i=1}^{n-1} \psi(\mathbf{1} - \mathbf{A}'^i(\mathbf{1} - \mathbf{s}))\psi^*(\mathbf{s}). \tag{4.3}$$

Therefore the marginal distributions (and hence the joint distribution) of $\{X_n^*\}$ will be determined by the initial distribution of X_1^* and the distribution of the sequence $\{\varepsilon_n\}$. However, if the process is stationary then the distribution of X_n^* will be the same as the limiting distribution. In the following we discuss the limiting distribution of $\{X_n^*\}$.

By direct multiplication of the matrix $\mathbf{A}'$, it can be seen that $\mathbf{A}'^n$, for $n \geqq 1$, has the form

$$\mathbf{A}'^n = \begin{bmatrix} w_n & \alpha_2 w_{n-1} \\ w_{n-1} & \alpha_2 w_{n-2} \end{bmatrix},$$

where the weights w_j's are defined in (2.2) with $w_j = 0$ for $j < 0$. Using Lemma 2.1, we have $\lim_{n\to\infty} w_n = 0$. This leads to the fact that $\lim_{n\to\infty} \mathbf{A}'^n = 0$. Consequently, following an argument similar to that used to prove Theorem 2.1, the limiting distribution of X_n^* is

$$\phi^*(\mathbf{s}) = \prod_{i=0}^{\infty} \psi(1 - w_i(1 - s_1) - \alpha_2 w_{i-1}(1 - s_2)). \tag{4.4}$$

This limiting distribution agrees with the conjecture of Steutel and van Harn (1986) (see Theorem 2 of their paper) concerning the limiting distribution for general transition matrix $\mathbf{A}$.

Thus, if the distribution of the innovation sequence $\{\varepsilon_n\}$ is given, one can determine $\phi^*(\mathbf{s})$. In the following section we present the solution to (4.2) when $\{\varepsilon_n\}$ is a sequence of independent Poisson variables.

5. Poisson INAR(p)

The Poisson distribution is perhaps the most commonly used distribution in modelling counting processes. In the INAR(1) process the Poisson distribution plays a role similar to that of the Gaussian distribution in the known AR(1) process. Specifically, it has the property that if the innovation sequence $\{\varepsilon_j\}$ and the initial distribution are Poisson, the marginal distribution of X_n is also Poisson. In this section we consider an extension of the Poisson INAR(1) process to a higher-order autoregression structure and compare this process with the multiple Poisson process introduced recently by McKenzie (1988).

5.1. *The embedded multiple Poisson process.* Let us assume that the innovation sequence $\{\varepsilon_n\}$ in (2.1) has Poisson distribution with mean $\lambda > 0$ and the INAR(p) process is stationary. Then it follows from (4.4), for the INAR(2) process, that

$$\phi^*(\mathbf{s}) = \exp\{-\lambda W[(1-s_1) + \alpha_2(1-s_2)]\}, \tag{5.1}$$

where

$$W = \sum_{n=0}^{\infty} w_n = \frac{1}{1-\alpha_1-\alpha_2} < \infty.$$

That is, X_n and $\alpha_2 \circ X_{n-1}$ are independent Poisson variables with means λW and $\lambda W \alpha_2$, respectively. Consequently, $\{\mathbf{X}_n^*\}$ is a multiple Poisson AR(1) process as defined by McKenzie (1988) (a p-dimension integer-valued process $\{\mathbf{y}_n\}$ is said to be multiple Poisson if the components of $\mathbf{y}_n$ are independent and each have Poisson distribution). Using (4.1), it can be easily seen that the joint p.g.f. of $\mathbf{X}_n^*$ and $\mathbf{X}_{n-1}^*$ is given by

$$\phi^*(\mathbf{s}, \mathbf{u}) = \exp[-\boldsymbol{\theta}'(\mathbf{1}-\mathbf{s}) - \boldsymbol{\theta}'(\mathbf{1}-\mathbf{u}) + (\mathbf{1}-\mathbf{s})'\Gamma^*(1)'(\mathbf{1}-\mathbf{u})], \tag{5.2}$$

where $\boldsymbol{\theta}' = (\lambda W, \lambda \alpha_2 W)$ and

$$\Gamma^*(1) = \begin{bmatrix} \lambda W \alpha_1 & \lambda W \alpha_2 \\ \lambda W \alpha_2 & 0 \end{bmatrix}$$

is the lag -1 covariance matrix of the process $\{\mathbf{X}_n^*\}$. Since the process $\{\mathbf{X}_n^*\}$ is Markovian and the joint p.g.f. of $\mathbf{X}_n^*$ and $\mathbf{X}_{n-1}^*$ is symmetric in $\mathbf{s}$ and $\mathbf{u}$, the process $\{\mathbf{X}_n^*\}$ is time reversible (see McKenzie (1988)).

5.2. *The Poisson INAR*(2) *process.* As has been stated in Section 4, the joint p.g.f. of the INAR(p) process can be obtained from that of the corresponding embedded simple multitype branching process $\{\mathbf{X}_n^*\}$. Specifically, the p.g.f. of $\mathbf{X}_n = (X_n, X_{n-1})'$, $\phi_n(\mathbf{s})$, in INAR(2) process can be obtained from (5.2) and it is of the form:

$$\phi_n(s_1, s_2) = \exp\{-\lambda W(1-\alpha_1)[(1-s_1) + (1-s_2)] - \lambda W \alpha_1(1-s_1 s_2)\}. \tag{5.3}$$

More generally the joint distribution of $X_1, X_2, \cdots, X_n$ is a marginal distribution of the joint distribution of the corresponding embedded process $\mathbf{X}_1^*, \mathbf{X}_2^*, \cdots, \mathbf{X}_n^*$. Now from the time reversibility of the embedded process $\{\mathbf{X}_n^*\}$ one concludes that the process $\{X_n\}$ is time reversible. This can be seen by observing that

$$\begin{aligned}\phi_{X_1,\cdots,X_n}(s_1,\cdots,s_n) &= \phi_{\mathbf{X}_1^*,\cdots,\mathbf{X}_n^*}(\mathbf{s}_1^*,\cdots,\mathbf{s}_n^*)\\ &= \phi_{\mathbf{X}_n^*,\cdots,\mathbf{X}_1^*}(\mathbf{s}_n^*,\cdots,\mathbf{s}_1^*)\\ &= \phi_{X_1,\cdots,X_n}(s_n,\cdots,s_1),\end{aligned}$$

where $\mathbf{s}_i^* = (s_i, 1)'$, $i = 1, \cdots, n$. As a result of this property the forward regression in the Poisson INAR(2) process is equal to the backward regression, that is, $E(X_n \mid X_{n-1} = x_1, X_{n-2} = x_2) = E(X_{n-2} \mid X_{n-1} = x_1, X_n = x_2)$. These conditional expectations can be found by utilizing the joint p.g.f. $\phi_{X_{n-2},X_{n-1},X_n}(\mathbf{s})$ which results in:

$$\begin{aligned}&E(X_n \mid X_{n-1} = x, X_{n-2} = z)\\ &= \lambda\left[1 + (1-\alpha_1)W\frac{p(x-1,z)}{p(x,z)} + \alpha_2 W\frac{p(x,z-1)}{p(x,z)} + \alpha_1^2 W\frac{p(x-1,z-1)}{p(x,z)}\right]\end{aligned} \tag{5.4}$$

where $p(x, z)$ is the joint probability mass function for the random variables X_{n-1} and X_{n-2}. Equation (5.4) suggests that the regression in the INAR(2) process may not be linear in general.

6. Concluding remarks and extensions

The theme of this paper is an extension of the INAR(1) process in the aim of having a more flexible model for counting processes. The form of the model investigated here is similar to that of the standard AR(p) process. Despite the similarity in form, the two processes differ in many aspects. Due to the role which the Poisson distribution plays in counting processes, the case in which the marginal distribution of the process is Poisson has been discussed with some details. It should be observed that the Poisson INAR(p) processes, $p \geqq 2$, $\{X_n\}$ is constructed in two steps. The first step dealt with the Poisson process $\mathbf{X}_n^*$, which has independent components as those of McKenzie (1988). The second step presents a method for obtaining the process X_n, which has dependent components, by using the embedded process $\{\mathbf{X}_n^*\}$.

In comparison with the standard time series models, there are many aspects that need to be investigated. Among these are: (i) Extension of the correlation to negative values. (ii) Estimation of the parameters of the model. The approach used in Al-Osh and Alzaid (1987) can be generalized to the INAR(p) process, however, the MLE for the general model seems intractable. (iii) Combination of the INAR(p) and INMA(q) in a manner similar to that of the standard ARMA(p, q) would provide a richer class of models for counting processes. (iv) Since many observed counting processes exhibit trend and/or seasonality pattern, it is of interest to extend the model to include such factors. Results for some of the problems discussed here will be reported elsewhere in due course.

Acknowledgements

We are thankful to the referee for various comments which led to substantial improvement in the presentation and to the removal of some inaccuracies in the original version. Also, we wish to thank Dr Ed McKenzie of Strathclyde University for allowing us to read his paper before publication.

References

AL-OSH, M. A. AND ALZAID, A. A. (1987) First order integer-valued autoregressive (INAR(1)) process. *J. Time Series Anal.* **8**, 261–275.

AL-OSH, M. A. AND ALZAID, A. A. (1988) Integer-valued moving average (INMA) process. *Statistische Hefte* **29**, 281–300.

ALZAID, A. A. AND AL-OSH, M. A. (1988) First order integer-valued autoregressive process: distributional and regression properties. *Statist. Nederlandica* **42**, 53–61.

FULLER, W. A. (1976) *Introduction to Statistical Time Series*. Wiley, New York.

HARVEY, A. C. AND PHILLIPS, G. D. A. (1979) Maximum likelihood estimation of regression models with autoregressive-moving average disturbances. *Biometrika* **66**, 49–58.

HEATHCOTE, C. R. (1966) A branching process allowing immigration. *J. R. Statist. Soc. B.* **28**, 213–217.

MCKENZIE, E. (1985) Some simple models for discrete variate time series. *Water Res. Bull.* **21**, 645–650.

MCKENZIE, E. (1986) Autoregressive moving-average processes with negative binomial and geometric marginal distributions. *Adv. Appl. Prob.* **18**, 679–705.

MCKENZIE, E. (1988) Some ARMA models for dependent sequences of Poisson counts. *Adv. Appl. Prob.* **20**, 822–835.

STEUTEL, F. W. AND VAN HARN, K. (1986) Discrete operator self-decomposability and queueing networks. *Commun. Statist. Stoch. Models* **2**, 161–169.

STEUTEL, F. W. AND VAN HARN, K. (1987) Queueing networks and operator-self decomposability. Preprint.