Adv. Appl. Prob. **18,** 679–705 (1986)
Printed in N. Ireland

AUTOREGRESSIVE MOVING-AVERAGE PROCESSES WITH NEGATIVE-BINOMIAL AND GEOMETRIC MARGINAL DISTRIBUTIONS

ED MCKENZIE,* *University of Strathclyde*

Abstract

Some simple models are described which may be used for the modelling or generation of sequences of dependent discrete random variates with negative binomial and geometric univariate marginal distributions. The models are developed as analogues of well-known continuous variate models for gamma and negative exponential variates. The analogy arises naturally from a consideration of self-decomposability for discrete random variables. An alternative derivation is also given wherein both the continuous and the discrete variate processes arise simultaneously as measures on a process of overlapping intervals. The former is the process of interval lengths and the latter is a process of counts on these intervals.

DISCRETE-VARIATE ARMA PROCESSES; DISCRETE SELF-DECOMPOSABILITY; INTERVAL PROCESSES; BIVARIATE NEGATIVE BINOMIAL DISTRIBUTIONS; THINNING

1. Introduction

1.1. *Background.* In a series of papers, referenced individually, Gaver, Jacobs, Lawrance and Lewis developed a class of simple models for stationary time-series with negative exponential and gamma marginal distributions. The motivation for such models is discussed in most of these papers and will not be reviewed here in any detail. Essentially, two primary uses are envisaged. These are the modelling of time-series with such marginal distributions, which do occur in practice; and the simulation of sequences or vectors of dependent gammas and negative exponentials. Almost incidentally, new and interesting multivariate distributions are generated. Flexibility and simplicity of form are important features of all the models. They mirror the structure and correlation of the well-known autoregressive–moving average (ARMA) processes used to model time-series with Gaussian marginals. For this reason, acronyms such as EARMA (an ARMA-type process with negative exponential marginals) are used to describe the models, and will be used here.

Received 21 February 1985; revision received 1 August 1985.

* Dept of Mathematics, University of Strathclyde, Livingstone Tower, 26 Richmond St, Glasgow G1 1XH, UK.

The purpose of this work is to develop processes with geometric and negative binomial marginal distributions which are analogues of the negative exponential and gamma processes referred to above. In general, there are few practical models for discrete variate processes. The two main ones are Markov chains and the DARMA processes of Jacobs and Lewis (1978a,b), (1983). The former are overparametrized for practical use in a time-series context, and sometimes too specialized in correlation structure. The latter form a very general family of process, but, as a result, cannot derive any benefit from the structure of particular distributions. The view taken here is that such benefit can be considerable. In particular, much use is made of the relationship between the negative binomial and gamma distributions.

We attempt to construct simple models for time-series with negative binomial marginals using essentially the same approach which succeeded with the gamma processes. Obviously, this must be adapted to allow for the discrete nature of the variates. Consequently, the construction of the models is of some interest in itself. The problem of scalar multiplication for discrete random variables is resolved by the use of thinning, a procedure more familiar in the study of point processes. This use of thinning is appropriate since many discrete-variate processes arise as aggregated point processes, i.e. counts of a point process in consecutive intervals of time. The use of thinning arises naturally from the work of Steutel and Van Harn (1979) on self-decomposability for discrete random variables. Using thinning, it is found that negative binomial analogues exist for all the gamma processes considered.

Finally, an alternative derivation is given of both the gamma processes and their discrete analogues. It has perhaps more intuitive appeal than the earlier, more operational, one, since now the two processes arise naturally and simultaneously as measures on an interval process, i.e. a sequence of overlapping intervals. The gamma processes are the lengths of the intervals and their negative binomial counterparts are counts on the same intervals.

1.2. *The negative binomial distribution.* The negative binomial is one of the most frequently and widely applied discrete distributions. See, for example, Johnson and Kotz (1969) and Patil (1970). The random variable N is said to be negative binomial if

$$P(N=n)=\binom{\beta+n-1}{n}(1-\theta)^{\beta}\theta^{n}, \qquad n=0,1,\cdots,\beta>0, \qquad 0<\theta<1.$$

The corresponding probability generating function (p.g.f.) may be derived as

$$P_N(z)=E(z^N)=\{(1-\theta)/(1-\theta z)\}^{\beta}.$$

For the purposes of this work it is more convenient to use $G_N(z)=P_N(1-z)$.

If we write $\lambda = (1-\theta)/\theta$, it is easily shown that

$$G_N(z) = \left(\frac{\lambda}{\lambda + z}\right)^{\beta}. \tag{1.1}$$

This is immediately recognizable as the Laplace transform of the gamma distribution with shape parameter β and scale parameter λ. Such a gamma random variable will be denoted by $\mathrm{Ga}(\beta, \lambda)$ and the analogous negative binomial random variable corresponding to (1.1) by $\mathrm{NB}(\beta, \lambda)$. In the same way that the negative exponential distribution is $\mathrm{Ga}(1, \lambda)$, so the geometric is $\mathrm{NB}(1, \lambda)$. Note that $G_N(z)$ is not quite the factorial moment generating function of N. We refer to it as the alternate p.g.f. Bondessen (1979) has drawn attention to the use of such a function in a similar context to ours.

The reason for the similarity in the transforms of the gamma and negative binomial distributions is well known. If a homogeneous Poisson process of unit rate is observed for a period of time X which is a $\mathrm{Ga}(\beta, \lambda)$ random variable, then the number of events observed, N, is a $\mathrm{NB}(\beta, \lambda)$ random variable. In terms of the transforms defined above,

$$G_N(z) = E[(1-z)^N] = E[\exp(-Xz)] = L_X(z) \tag{1.2}$$

the Laplace transform of X.

As we shall see, this relationship and the implied underlying structure play a fundamental role in our development of discrete analogues for the exponential and gamma processes.

We shall also exhibit a number of bivariate negative binomial distributions and, for completeness and comparison, we note the two most commonly cited in the literature. The one most often used in theoretical studies is also called the negative multinomial distribution. It is discussed in Johnson and Kotz (1969) and has alternate p.g.f. of the form $\{1 + \alpha u + \beta v\}^{-k}$ where k, α, β are positive. A generalization of this is developed and applied to data by Edwards and Gurland (1961). It was also used to generate a Markov process with a negative binomial marginal by Phatarfod and Mardia (1973). The alternate p.g.f. has the form $\{1 + \alpha u + \alpha v + \alpha\beta uv\}^{-k}$ where $\beta = \alpha - (1+\alpha)\rho$ and ρ is the correlation.

It is worth noting that neither of these distributions appear as the bivariate distributions of any of the processes of this paper. The second, more general one, appears in an asymmetric form as a component of several.

2. Geometric ARMA processes

2.1. *Self-decomposability and thinning.* The model initially considered by Gaver and Lewis (1980) was the first-order autoregressive process, the AR(1).

The classical equation for an AR(1) process $\{X_n\}$ is

$$X_n = \alpha X_{n-1} + E_n, \tag{2.1}$$

where $\{E_n\}$ is a sequence of independent and identically distributed random variables. Thus, if X_n is a positive random variable then $\alpha \in (0, 1)$ instead of the usual $(-1, 1)$. Taking Laplace transforms throughout (2.1) yields

$$L_E(s) = L_X(s)/L_X(\alpha s). \tag{2.2}$$

If X_n is a $\mathrm{Ga}(1, \lambda)$ random variable then Equation (2.2) is easily solved for the distribution of E_n:

$$E_n = \begin{cases} 0 & \text{with probability } \alpha \\ \mathrm{Ga}(1, \lambda) & \text{with probability } 1 - \alpha \ (=\bar{\alpha}). \end{cases} \tag{2.3}$$

Note that, in general, it is possible to construct an AR(1) using (2.1) as described if and only if $L_E(s)$ as given by (2.2) is the Laplace transform of a distribution. As noted by Gaver and Lewis, $L_E(s)$ is the transform of a distribution for $\alpha \in (0, 1)$ if and only if the distribution of X_n is self-decomposable (SD). Thus, only SD-distributions lend themselves to use in an AR(1) using (2.1) with $\alpha \in (0, 1)$. In such a case, we shall refer to the distribution of E_n as the innovation distribution corresponding to that of X_n.

The SD-distributions are of interest as a class of limit laws for certain kinds of normalized sums of random variables. See, for example, Loève (1963). The class is a proper subset of the infinitely divisible distributions and contains all the stable laws. It has received considerable attention in recent years and a great deal is now known about it. Many of the most common distributions are SD, including the gamma distribution. However, all SD-distributions are absolutely continuous and thus represent an approach which cannot succeed if X_n is discrete.

Recently, Steutel and Van Harn (1979) proposed analogues for the stable and SD-classes of distributions for discrete random variables on the non-negative integers. Their interests lie in showing that their analogous classes have properties corresponding to those for the continuous case. In particular, Equation (2.2) is reconstituted in terms of probability generating functions (p.g.f.'s). If the discrete random variable N on N_0 has a p.g.f. $P_N(z)$ which is such that $P_\alpha(z)$ defined by

$$P_\alpha(z) = P_N(z)/P_N(\bar{\alpha} + \alpha z) \tag{2.4}$$

is a p.g.f. for all $\alpha \in (0, 1)$ then N is said to be discrete self-decomposable (DSD).

The random variable M with p.g.f. $P_N(\bar{\alpha} + \alpha z)$ can be written as $M = \sum_{i=1}^{N} B_i(\alpha)$, where $\{B_i(\alpha)\}$ is a sequence of i.i.d. binary random variables with

$P(B_i(\alpha) = 1) = \alpha = 1 - P(B_i(\alpha) = 0)$. Thus, conditional on N, M is Bi(N, α). For convenience, we shall write such an M in the form $\alpha^* N$. Clearly, $\alpha^* N = 0$ (or N) if $\alpha = 0$ (or 1).

Now, if N_n is DSD, we can define a stationary AR(1) process by

$$N_n = \alpha^* N_{n-1} + R_n \tag{2.5}$$

where the distribution of R_n may be obtained from its p.g.f. $P_\alpha(z)$ in (2.4). Again, we refer to the distribution of R_n as the innovation distribution corresponding to that of N_n.

In particular, if N_n is NB$(1, \lambda)$ we find that

$$R_n = \begin{cases} 0 & \text{with probability } \alpha, \\ \text{NB}(1, \lambda) & \text{with probability } \bar{\alpha}. \end{cases} \tag{2.6}$$

The similarity of (2.3) and (2.6) is striking, and clearly the procedure for the generation of $\{N_n\}$ via (2.5) is an exact analogue of that for $\{X_n\}$ via (2.1). It is not difficult to find the reason for the similarity of the two AR(1) processes. Rewriting the condition for N to be DSD, i.e. Equation (2.4), in terms of alternate p.g.f.'s yields the following. If the discrete random variable N on N_0 has an alternate p.g.f. $G_N(z)$ such that

$$G_\alpha(z) = G_N(z)/G_N(\alpha z) \tag{2.7}$$

is an alternate p.g.f. for $\alpha \in (0, 1)$, then N is DSD. Hence, from Equation (1.1), the negative binomial distribution is DSD because the gamma is SD. Further, the innovation distribution for the negative binomial has an alternate p.g.f. which is identical to the Laplace transform of the innovation distribution for the gamma. This is the essence of the relationship between (2.3) and (2.6).

The closeness of the two distributions, when viewed in terms of their transforms, and the two forms of self-decomposability is strongly suggestive. The simple act of replacing scalar multiplication (αX) in the negative exponential case by the *-operation in the geometric case yields essentially identical processes. It is natural to explore how far this relationship extends to the more general EARMA models. Before doing so, however, a brief comment on the nature of the *-operation is in order.

We have noted the possibility of viewing the negative binomial random variable N as the count in a Poisson process during an interval whose length is a gamma random variable. In this context, we may view $\alpha^* N$ as the result of considering each of the N events in this interval independently. In each case, we retain it with probability α and delete it with probability $\bar{\alpha}$. Thus $\alpha^* N$ is obtained by thinning the point process within the interval. For a discussion of thinning, see Cox and Isham (1980). Since we may, in general, view our discrete random variable as the result of an aggregated point process, we

should perhaps refer to the *-operation as aggregate thinning. However, for convenience, we shall simply say that $\alpha * N$ is obtained by thinning N.

2.2. *Geometric* AR(1). This process is the one developed in Equations (2.5) and (2.6). For completeness, it is given again below in a single form:

$$N_n = \alpha * N_{n-1} + U_n M_n \tag{2.5'}$$

where $\{U_n\}$ is a sequence of i.i.d. binary random variables with $P(U_n = 0) = \alpha = 1 - P(U_n = 1)$. Further, U_n is independent of M_n and N_{n-1}. It is straightforward to verify that the process $\{N_n\}$ shares many properties with its negative exponential analogue, the EAR(1) process (2.1), (2.3). In particular, $E(N_n \mid N_{n-1})$ is linear in N_{n-1} whereas $E(N_{n-1} \mid N_n)$ is exponential. However, $\mathrm{Var}\,(N_n \mid N_{n-1}) = \alpha(1-\alpha)N_{n-1} + \mathrm{Var}\,(R_n)$, i.e. it is linear in N_{n-1}. This is in marked contrast to the EAR(1) process where $\mathrm{Var}\,(X_n \mid X_{n-1})$ is a constant, $\mathrm{Var}\,(E_n)$, and so independent of X_{n-1}. The difference is due to the fact that $\alpha * N_{n-1}$ is a binomial random variable when N_{n-1} is known whereas αX_{n-1} is constant when X_{n-1} is known. This is the major difference between the processes. Thinning is a random operation and so has an associated variance.

The random nature of thinning also ensures that the geometric AR(1) does not have another problematic aspect of the EAR(1), viz. the zero-deficiency property. From (2.1) and (2.3), it is clear that in any realization we can have sequences where successive values are fixed multiples of the previous value. Such behaviour is somewhat unrealistic for practical applications, and was commented upon by Gaver and Lewis (1980). It led to the development of alternate forms of the negative exponential AR(1) process which will be discussed later. However, the random nature of thinning precludes any such possibility in the discrete analogue. Even when the innovation $U_n M_n$ in (2.7) is 0 the fact that $\alpha * N_{n-1}$ is Bin (N_{n-1}, α) ensures that there is no degeneracy.

The correlation structures of the EAR(1), $\{X_n\}$, and the geometric AR(1), $\{N_n\}$, are identical. In fact, $\{N_n\}$ is a Markov chain with transition matrix $P = [p(y, x)]$ where $p(y, x) = P(N_n = y \mid N_{n-1} = x)$. This condition probability may be derived in the form

$$p(y,x) = \begin{cases} \bar{\alpha}\bar{\theta}\theta^{y-x} \displaystyle\sum_{k=0}^{y} \binom{x}{k} \alpha^k (\bar{\alpha}\theta)^{x-k} + \binom{x}{y} \alpha^{y+1} (\bar{\alpha})^{x-y} & \\ \qquad\qquad\qquad\qquad y = 0, 1, \cdots, x & \\ \bar{\alpha}\bar{\theta}\theta^{y-x}(\alpha + \bar{\alpha}\theta)^x \quad y = x+1, x+2, \cdots. & \end{cases} \tag{2.8}$$

Note that

$$p(x, x) = \bar{\alpha}\bar{\theta}(\alpha + \bar{\alpha}\theta)^x + \alpha^{x+1}, \qquad x = 0, 1, \cdots.$$

This probability is useful in the derivation of the distribution of lengths of runs

of a particular value x. Following Jacobs and Lewis (1978a,b), we can calculate the p.g.f. of T_x, the length of a run of x initiated at time 1, as follows:

$$P_{T_x}(z) = 1 - p_N(x) + p_N(x) \sum_{n=1}^{\infty} [1 - p(x, x)]p(x, x)^{n-1} z^n$$
$$= 1 - p_N(x) \frac{1 - z}{1 - p(x, x)z},$$

where

$$p_N(x) = P(N_n = x) = (1 - \theta)\theta^x, \; x = 0, 1, \cdots.$$

The joint distribution of two consecutive values (N_n, N_{n-1}) may be determined directly from the conditional probabilities (2.8). However, it is more instructive to derive the joint alternate p.g.f. here, i.e.

$$G_{N_n, N_{n-1}}(u, v) = P_{N_n, N_{n-1}}(1 - u, 1 - v)$$
$$= \frac{\lambda(\lambda + \alpha u)}{(\lambda + u)(\lambda + \alpha u + v - \alpha uv)}. \tag{2.9}$$

We may compare this with the joint Laplace transform of (X_n, X_{n-1}) from the EAR(1) process:

$$L_{X_n, X_{n-1}}(u, v) = \frac{\lambda(\lambda + \alpha u)}{(\lambda + u)(\lambda + \alpha u + v)}. \tag{2.10}$$

Apart from the product term, uv, the transforms (2.9) and (2.10) are identical. It is clear that there is some relationship between the two processes leading to this similarity in the second-order transforms. We shall discuss this in more detail later.

2.3. *Geometric moving-average processes.* The negative exponential moving-average process of order 1, the EMA(1), is described and discussed in detail by Lawrance and Lewis (1977). It is given by

$$X_n = \beta E_n + V_n E_{n-1} \tag{2.11}$$

where $\{E_n\}$ is a sequence of i.i.d. $\mathrm{Ga}(1, \lambda)$ random variables and $\{V_n\}$ is a sequence of i.i.d. binary random variables with $P(V_n = 0) = \beta$. Since (2.11) was derived directly from the EAR(1) it is natural to consider the direct analogue derived from (2.5), i.e.

$$N_n = \beta^* M_n + V_n M_{n-1}. \tag{2.12}$$

If $\{M_n\}$ is a sequence of i.i.d. $\mathrm{NB}(1, \lambda)$ random variables and $\{V_n\}$ as before then $\{N_n\}$ defined by (2.12) is a dependent sequence of $\mathrm{NB}(1, \lambda)$ random

variables. Further, $\{N_n\}$ and $\{X_n\}$ have a common correlation structure, viz.

$$\rho_X(1) = \rho_N(1) = \beta(1-\beta); \qquad \rho_X(k) = \rho_N(k) = 0, \qquad k > 1.$$

Clearly, (2.12) is a first-order moving-average process and we shall refer to it as the geometric MA(1). This is not a Markov process and distributions of run lengths are consequently much more complex and not dealt with here. As with the EMA(1), however, the distributions of sums of $\{N_n\}$ are easily obtained using alternate p.g.f.'s. The basic component is the alternate p.g.f. of (N_n, N_{n-1}) which may be obtained in the form

$$G_{N_n,N_{n-1}}(u, v) = \frac{\lambda^2(\lambda + \beta u + \beta v - \beta^2 uv)}{(\lambda + \beta u)(\lambda + v)(\lambda + u + \beta v - \beta uv)}. \tag{2.13}$$

As with the AR(1) processes, we find that $G_{N_n,N_{n-1}}(u, v)$ is identical to $L_{X_n,X_{n-1}}(u, v)$ as given by Lawrance and Lewis (1977) if the product terms in uv are removed from the former, i.e. in (2.13).

Higher-order geometric moving-average processes are derived from the MA(1) in exactly the same way that they were developed for the negative exponential distribution in Lawrance and Lewis (1977), (1980). Thus, for example, a geometric MA(2) may be constructed as follows:

$$N_n = \begin{cases} \beta_1^* M_n & \text{w.p. } \beta_1 \\ \beta_1^* M_n + \beta_2^* M_{n-1} & \text{w.p. } \beta_2\bar{\beta}_1 \\ \beta_1^* M_n + \beta_2^* M_{n-1} + M_{n-2} & \text{w.p. } \bar{\beta}_2\bar{\beta}_1 . \end{cases} \tag{2.14}$$

2.4. *The geometric* ARMA(1, 1). As with the previous processes, the geometric ARMA(1, 1) is derived by taking the EARMA(1) process and replacing all Ga(1, λ) random variables by NB(1, λ) random variables and scalar multiplication by thinning. The resulting process $\{N_n\}$ is given by the two equations

$$N_n = \beta^* M_n + V_n W_{n-1} \tag{2.15a}$$

$$W_n = \alpha^* W_{n-1} + U_n M_n \tag{2.15b}$$

where $\{M_n\}$, $\{U_n\}$ and $\{V_n\}$ are independent sequences of i.i.d. random variables as described in Sections 2.2 and 2.3. Comparing (2.15b) with (2.7) shows that $\{W_n\}$ is a geometric AR(1) process and W_n is NB(1, λ). Thus, (2.15a) has the same form as (2.12) and so N_n is also NB(1, λ). Thus, if we begin with W_0 an NB(1, λ) random variable the equations (2.15a, b) generate a stationary process $\{N_n\}$. We refer to $\{N_n\}$ thus defined as the geometric ARMA(1, 1) process. It is the discrete analogue of the EARMA(1, 1) process discussed in detail by Jacobs and Lewis (1977), and shares with it a common correlation

structure:

$$\rho_N(k) = \bar{\beta}(\beta\bar{\alpha} + \alpha\bar{\beta})\alpha^{k-1}, \qquad k = 1, 2, \cdots.$$

As with the EARMA(1, 1) process the geometric version reduces to the AR(1) if $\beta = 0$, or the MA(1) if $\alpha = 0$.

Again the alternate p.g.f. of (N_n, N_{n-1}) is identical apart from product terms to the joint Laplace transform of two consecutive values in the EARMA(1, 1) process. It is given by

$$G_{N_n,N_{n-1}}(u, v) = \frac{\lambda^2}{(\lambda + \beta u)(\lambda + \beta v)}\left\{\frac{\beta(\lambda + \beta v)}{(\lambda + u)} + \frac{\lambda(1-\beta)(\lambda + \alpha u + \beta v - \alpha\beta uv)^2}{(\lambda + \alpha u)(\lambda + u + \beta v - \beta uv)(\lambda + \alpha u + v - \alpha uv)}\right\}$$

and may be compared with $L_{X_n,X_{n-1}}(u, v)$ given in Jacobs and Lewis (1977).

2.5. *The geometric* ARMA(p, q) *process.* The extension of the EARMA(p, q) process to the geometric analogue is achieved in the usual way. Negative exponential random variables are replaced by geometric random variables and thinning replaces scalar multiplication. For completeness, the equations are given below. They are exact analogues of those given for the EARMA(p, q) in Lawrance and Lewis (1980).

$$N_n = \begin{cases} \sum_{i=0}^{k} \beta^*_{q-i} M_{n-i} & \text{w.p. } b_{k+1} \\ & k = 1, 2, \ldots, q-1 \\ \sum_{i=0}^{q-1} \beta^*_{q-i} M_{n-i} + W_{n-q} & \text{w.p. } b_1 \end{cases}$$

and

$$W_n = \alpha_i^* W_{n-i} + \eta_n \qquad \text{w.p. } a_i, \quad i = 1, 2, \ldots, p.$$

Here, $\{M_n\}$ is a sequence of i.i.d. NB$(1, \lambda)$ random variables, and $\{\eta_n\}$ is also a sequence of i.i.d. random variables but each is a mixture of thinned versions of M_n, i.e. $\eta_n = r_i^* M_n$ with probability p_i, $i = 1, 2, \cdots, p$. The coefficients $\{r_i\}$ and the probabilities $\{p_i\}$ depend upon the values $\{\alpha_i\}$ only. The relationships between $\{b_i\}$ and $\{\beta_i\}$ and between $\{a_i\}$ and $\{\alpha_i\}$ are exactly as described in Lawrance and Lewis (1980). The example $p = q = 2$ is given explicitly.

The moving-average component is given by Equations (2.14) with M_{n-2}

replaced by W_{n-2}. The autoregressive component can be written in the form

$$W_n = Z_n(\alpha_1^* W_{n-1}) + (1 - Z_n)(\alpha_2^* W_{n-2}) + \gamma^* M_n$$

where $\{Z_n\}$ is a sequence of i.i.d. binary random variables with $P(Z_n = 1) = 1 - \alpha_2$, and

$$\gamma = \begin{cases} 0 & \text{w.p.} \quad \alpha_1\alpha_2/\delta \\ 1 & \text{w.p.} \quad \bar{\alpha}_1\bar{\alpha}_2/\bar{\delta} \\ \delta & \text{w.p.} \quad \alpha_2\bar{\alpha}_2(\alpha_1 - \alpha_2)^2/\delta\bar{\delta} \end{cases}$$

where $\delta = \alpha_2(1 + \alpha_1 - \alpha_2)$.

3. The NEAR(1) and gamma AR processes

3.1. *The gamma autoregressive process,* GAR(1). Gaver and Lewis (1980) also derived the GAR(1) process. This is a first-order autoregressive process as given by (2.1) with X_n a Ga(β, λ) random variable. As noted the gamma distribution is SD, and so (2.2) defines an innovation distribution for the random variable E. It has Laplace transform

$$L_E(s) = (\lambda + \alpha s)^{\beta}(\lambda + s)^{-\beta}. \tag{3.1}$$

Although this demonstrates the existence of a suitable innovation process $\{E_n\}$, it did not provide Gaver and Lewis with any means of generating it. Lawrance (1982), however, provided an elegant solution to the problem derived from the shot-noise process. In essence, he showed that the random variable E with Laplace transform (3.1) could be expressed in the form

$$E = \sum_{i=1}^{N} \alpha^{R_i} Y_i \tag{3.2}$$

where $\{Y_i\}$ is a sequence of i.i.d. Ga(1, λ) random variables, $\{R_i\}$ are i.i.d. random variables uniformly distributed on (0, 1), and N is a Poisson random variable with mean $(-\beta \ln \alpha)$. If $N = 0$, the sum is 0.

As noted in Section 2.1, for the NB(β, λ) case, Equation (3.1) gives the alternate p.g.f. of the innovation random variable M. Thus, the corresponding AR(1) process for the negative binomial is

$$N_n = \alpha^* N_{n-1} + M_n$$

where M_n has alternate p.g.f. given by (3.1). It is straightforward to verify that we can derive M from the usual reconstruction of E, given by (3.2), i.e. replace Ga(β, λ) random variables by NB(β, λ) random variables and scalar

multiplication by thinning. Thus, Equation (3.2) becomes

$$M = \sum_{i=1}^{N} (\alpha^{R_i})^* W_i \tag{3.3}$$

where $\{R_i\}$ and N are exactly as in (3.2), and $\{W_i\}$ are i.i.d. NB$(1, \lambda)$ random variables.

The GAR(1) and its negative binomial analogue share an interesting property. The Laplace transform of a Ga(β, λ) random variable is simply that of a negative exponential random variable raised to the power β. If the marginal distribution of the GAR(1) process is Ga(β, λ), then the joint Laplace transforms of consecutive values of the process is again the βth power of the transform of the same values for the EAR(1) process. This results from the fact that these joint transforms can be written as the quotient of products of transforms of the marginal distribution with different arguments. It is clear that the same property will hold for the alternate p.g.f.'s in the negative binomial case.

This gamma process was not developed or extended by Gaver and Lewis because of the complexity of its innovation process. A more flexible and simpler approach for gamma processes in general was introduced by Lewis (1982) and we consider these next.

3.2. *The gamma beta autoregressive process,* GBAR(1). Lewis (1982) presented a linear, random coefficient autoregression of order 1 for the Ga(β, λ) distribution. This is the basis of an important class of models for the gamma distribution which will be discussed elsewhere. The construction involves the use of the beta distribution. A random variable with a beta distribution with parameters (p, q) has density

$$f(x) = x^{p-1}(1-x)^{q-1}/B(p, q), \qquad 0 < x < 1.$$

We refer to such a random variable as Be(p, q). The GBAR(1) process uses two well-known distributional results. The first is that the sum of independent Ga(α, λ) and Ga(β, λ) random variables is a Ga$(\alpha + \beta, \lambda)$ random variable, and the second is that the product of independent Be$(\alpha, \beta - \alpha)$ and Ga(β, λ) random variables is a Ga(α, λ) random variable. Hence, if X_{n-1} is Ga(β, λ), A_n is Be$(\alpha, \beta - \alpha)$ and B_n is Ga$(\beta - \alpha, \lambda)$, and these three are mutually independent, then the equation

$$X_n = A_n X_{n-1} + B_n \tag{3.4}$$

defines a stationary stochastic process $\{X_n\}$ with a marginal Ga(β, λ) distribution. Further, the autocorrelation function of the process is $\rho_X(k) = E(A)^k = (\alpha/\beta)^k$.

A negative binomial analogue for this process (3.4) will exist if the two

noted results have discrete analogues. The first is well known. The sum of independent NB(α, λ) and NB(β, λ) random variables is an NB($\alpha + \beta, \lambda$) random variable. The second necessitates thinning replacing a product again. Consider $Y = \alpha * N$. Conditional on N, Y is Bin(N, α). Now let α be a Be(p, q) random variable. We find that

$$p_{Y|N}(y \mid N) = \binom{N}{y} B(p + y, q + N - y)/B(p, q) \qquad y = 0, 1, \cdots, N. \tag{3.5}$$

This distribution is known variously as the negative hypergeometric, the beta–binomial and the Pólya–Eggenberger distribution. It is of some importance as a contagious distribution and is discussed in detail in a number of guises in Johnson and Kotz (1969). For our purposes, it is enough to note that if the distribution of Y conditional on N is given by (3.5), and N is NB($p + q, \lambda$), then Y is a NB(p, λ) random variable. This may be verified directly by forming the joint distribution of (Y, N) and integrating.

Hence, if N_{n-1} is NB(β, λ), A_n is Be($\alpha, \beta - \alpha$) and M_n is NB($\beta - \alpha, \lambda$) and all are mutually independent, then the equation

$$N_n = A_n^* N_{n-1} + M_n \tag{3.6}$$

defines a stationary stochastic process $\{N_n\}$ with a marginal NB(β, λ) distribution. This process, (3.6), is evidently an exact analogue of the GBAR(1) process, (3.4). It has the same autocorrelation function and shares many of its properties. The joint alternate p.g.f. is given by

$$G_{N_n, N_{n-1}}(u, v) = (1 + u)^{\alpha - \beta}(1 + v)^{\alpha - \beta}(1 + u + v - uv)^{-\alpha}. \tag{3.7}$$

Further, as with previous models, we find that the Laplace transform of (X_n, X_{n-1}) is the same as (3.7) if the product uv does not appear in it.

3.3. *The* NEAR(1) *process.* Motivated by the lack of realism incurred by the zero-deficiency aspect of the EAR(1) process, Lawrance and Lewis (1981) developed a new exponential AR(1) model, the NEAR(1). This is a much more flexible model than the EAR(1) in that it allows for a much broader range of sample path behaviour. It takes the form

$$X_n = \beta U_n X_{n-1} + [1 - V_n + \bar{\alpha}\beta V_n]E_n \tag{3.8}$$

where $\{U_n\}$ and $\{V_n\}$ are independent sequences of i.i.d. binary random variables such that $P(U_n = 1) = \alpha$, $P(V_n = 1) = \alpha\beta/(1 - \bar{\alpha}\beta)$, and $\{E_n\}$ is a sequence of i.i.d. Ga($1, \lambda$) random variables. This process is also a random coefficient autoregression.

The discrete analogue is derived directly in the usual way as

$$N_n = (\beta U_n) * N_{n-1} + [1 - V_n + \bar{\alpha}\beta V_n] * M_n \tag{3.9}$$

where $\{U_n\}$, $\{V_n\}$ are as before and $\{M_n\}$ is a sequence of i.i.d. NB$(1, \lambda)$ random variables.

The NEAR(1) process can be extended to higher-order processes, and Lawrance and Lewis (1981) detail this and a procedure for generating negative correlation. These are not discussed here. We merely note that the simple procedure of replacing the products by thinning and Ga$(1, \lambda)$ by NB$(1, \lambda)$ random variables yields analogues of most of their results.

Again, we note that the alternate p.g.f. of (N_n, N_{n-1}) differs from the Laplace transform of (X_n, X_{n-1}) only in that the former contains terms in (uv) which do not appear in the latter. In particular,

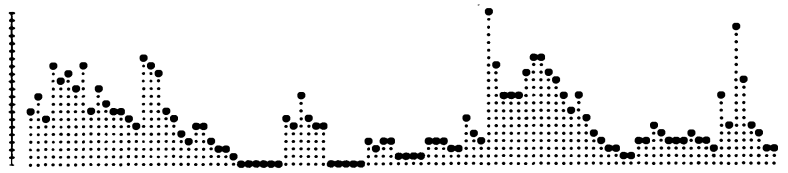
$$G_{N_n, N_{n-1}}(u, v) = \frac{\lambda^2(\lambda + \beta u)(\lambda + \bar{\alpha}\beta u + v - \bar{\alpha}\beta uv)}{(\lambda + u)(\lambda + v)(\lambda + \bar{\alpha}\beta u)(\lambda + \beta u + v - \beta uv)}.$$

3.4. *Some sample realizations.* To illustrate the sample path behaviour of some of these processes, simulations of length 100 were run of the geometric analogue of the NEAR(1) process, as given by (3.9), and of the negative binomial analogue of the GBAR(1) process, as given by (3.6). These are displayed in Figures 1, 2, 3 and 4. To emphasize the discrete nature of the variables, not only is the observed value plotted but all the non-negative integers below it are also shown. Thus, in Figure 1(a), the observations begin $4, 4, 2, 2, \cdots$, in 1(b), the sequence starts $7, 9, 6, \cdots$, and in 1(c) it is $1, 1, 1, 1, 8, \cdots$.

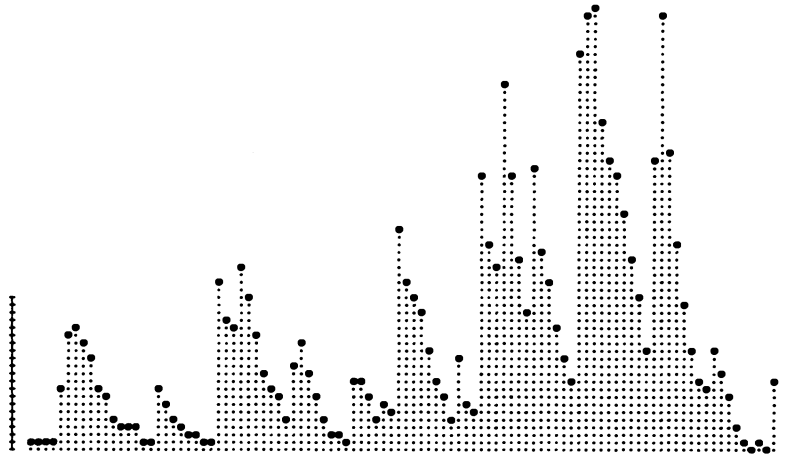
Figures 1 and 2 may be compared with Figures 1a and 1b of Lawrance and Lewis (1981). Their figures are of realizations of the NEAR(1) processes for $\alpha = 0{\cdot}99$, $\beta = 0{\cdot}758$ in 1a, and $\alpha = 0{\cdot}758$, $\beta = 0{\cdot}99$ in 1b. Figure 1 of this paper has $\alpha = 0{\cdot}99$, $\beta = 0{\cdot}758$ also, but the parameter of the geometric marginal is different in each case, with the mean value, μ, increasing from the top to the bottom of the figure. It is clear that as the mean increases the behaviour of the geometric process approaches that of the negative exponential version. Indeed, Figure 1(c) exhibits behaviour very similar to that shown in Figure 1a of Lawrance and Lewis. However, the smoothness of the continuous-variate sample path is not present in the discrete analogues. Nevertheless, the general nature of the paths are the same in both cases: sudden rises are followed by much slower recessions. Similar comments may be applied to Figure 2 of this paper and Figure 1b of Lawrance and Lewis. The only difference is that now runs tend to be upwards and are suddenly terminated.

Figure 3 also displays the geometric analogue of the NEAR(1) process. In fact, 3(a), (b) and (c) exhibit processes with the same univariate geometric marginal distribution, with mean unity, and $\rho = 0{\cdot}0$ (i.e. i.i.d.), $0{\cdot}5$ and $0{\cdot}75$ respectively. The effects of increasing positive correlation are clearly seen in these three exhibits. Figure 3(d) also has $\rho = 0{\cdot}75$, but the marginal geometric has

(a): $\theta = 0{\cdot}5$, $\mu = 1{\cdot}0$; $\beta = 0{\cdot}758$, $\rho = 0{\cdot}75$



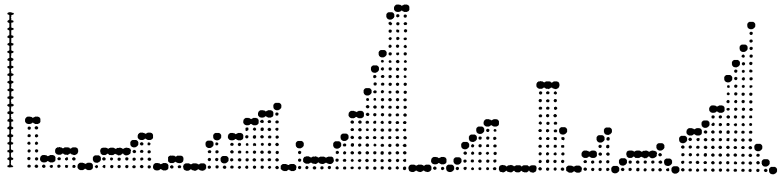
(b): $\theta = 0{\cdot}8$, $\mu = 4{\cdot}0$; $\beta = 0{\cdot}758$, $\rho = 0{\cdot}75$



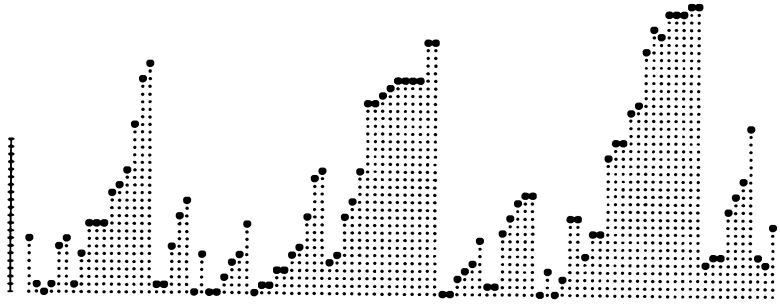
(c): $\theta = 0{\cdot}9$, $\mu = 9{\cdot}0$; $\beta = 0{\cdot}758$, $\rho = 0{\cdot}75$

Figure 1. Geometric analogue of NEAR(1) process with $\alpha = 0{\cdot}99$, i.e. approximately the EAR(1) process

(a): $\theta = 0{\cdot}5$, $\mu = 1{\cdot}0$; $\alpha = 0{\cdot}758$, $\rho = 0{\cdot}75$



(b): $\theta = 0{\cdot}8$, $\mu = 4{\cdot}0$; $\alpha = 0{\cdot}758$, $\rho = 0{\cdot}75$



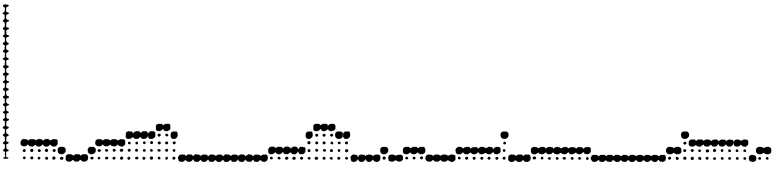
(c): $\theta = 0{\cdot}9$, $\mu = 9{\cdot}0$; $\alpha = 0{\cdot}758$, $\rho = 0{\cdot}75$

Figure 2. Geometric analogue of NEAR(1) process with $\beta = 0{\cdot}99$

mean 4. Comparisons of 3(d) with 1(b) and 2(b), which also have $\rho = 0{\cdot}75$, $\mu = 4$; and 3(c) with 1(a) and 2(a), which have $\rho = 0{\cdot}5$, $\mu = 1$, reveal the wide variety of sample-path behaviour available from this model by choice of parameter values. This aspect of the NEAR(1) model is also discussed by Lawrance and Lewis (1981).

(a): $\theta = 0{\cdot}5$, $\mu = 1{\cdot}0$; $\rho = 0{\cdot}0$, i.e. independent random variables

(b): $\theta = 0{\cdot}5$, $\mu = 1{\cdot}0$; $\alpha = 0{\cdot}707$, $\beta = 0{\cdot}707$, $\rho = 0{\cdot}5$



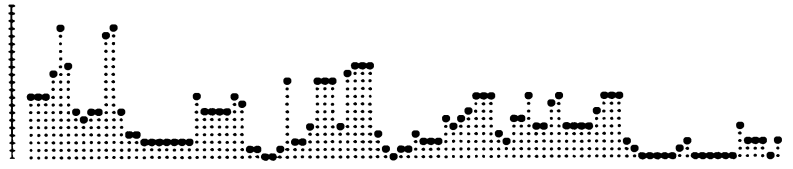
(c): $\theta = 0{\cdot}5$, $\mu = 1{\cdot}0$; $\alpha = 0{\cdot}866$, $\beta = 0{\cdot}866$, $\rho = 0{\cdot}75$

(d): $\theta = 0{\cdot}8$, $\mu = 4{\cdot}0$; $\alpha = 0{\cdot}866$, $\beta = 0{\cdot}866$, $\rho = 0{\cdot}75$

Figure 3. Geometric analogue of NEAR(1) process

(a): $\theta = 0{\cdot}2,\ \mu = 0{\cdot}25;\ \alpha = \rho = 0{\cdot}75$

(b): $\theta = 0{\cdot}5,\ \mu = 1{\cdot}0;\ \alpha = \rho = 0{\cdot}75$



(c): $\theta = 0{\cdot}8,\ \mu = 4{\cdot}0;\ \alpha = \rho = 0{\cdot}75$

(d): $\theta = 0{\cdot}5,\ \mu = 1{\cdot}0;\ \alpha = \rho = 0{\cdot}5$

Figure 4. Geometric analogue of the GBAR(1) process

4. Interval processes

4.1. *Background.* The discrete analogues of the last two sections were derived initially from a consideration of self decomposability for continuous and discrete random variables. This led naturally to the possibility of replacing multiplication in processes of continuous random variables by thinning for discrete ones. This simple operational change proved to be most successful. Indeed, it would be remarkable were it not for the close relationship between the gamma and negative binomial distributions. It is clear that this relationship plays a vital role in all the constructions considered. As we have noted, the proximity of these two distributions may be interpreted as being due to the fact that they arise naturally in a single context. The number of events in a homogeneous Poisson process observed during a period whose length is a gamma random variable is a negative binomial random variable.

A natural question to ask is whether there is some more general structure from which both the continuous and discrete processes arise not only naturally but simultaneously. The purpose of this section is to describe such a structure. It consists of a sequence of overlapping intervals and will be referred to as an interval process.

We begin with some notation and a basic result. $I(\alpha)$ is defined to be an α-subinterval of the interval I, $(0 \leqq \alpha \leqq 1)$, if $I(\alpha) \subset I$ and $|I(\alpha)| = \alpha\,|I|$, where $|I|$ is the length of I. We shall also be interested in sequences of nested subintervals and, with a slight abuse of notation, we write $I(\alpha\delta)$ for a δ-subinterval of $I(\alpha)$ and $I(\alpha\delta\gamma)$ for a γ-subinterval of $I(\alpha\delta)$, etc.

Consider now an interval I whose length X is a Ga(β, λ) random variable. As noted, if we observe a homogeneous Poisson process of unit rate, HPP(1), throughout the interval I, the count $N(I)$ is an NB(β, λ) random variable. Suppose that $I(\alpha)$ is an α-subinterval of I chosen at random but independently of the observed HPP(1) on I. Our interest lies in $N(I(\alpha))$. Given that $N(I)$ events occur in I we may assume, by virtue of the properties of the Poisson process, that these are uniformly and independently distributed throughout I. Thus, each has independently the same probability of falling in $I(\alpha)$, i.e. α. Hence, given $N(I) = n$, $N(I(\alpha))$ is Bin(n, α), i.e. $N(I(\alpha)) = \alpha^* N(I)$. We note in passing that the unconditional distribution of $N(I(\alpha))$ is NB$(\beta, \lambda/\alpha)$, i.e. the original θ has become $\alpha\theta/(1 - \bar{\alpha}\theta)$.

4.2. *An autoregressive interval process.* Consider an ordinary renewal process with intervals $I_0 = J_0, J_1, J_2, \cdots$, where $|J_k| = E_k$ are i.i.d. random variables. Let us observe an HPP(1) throughout the duration of this renewal process, and denote the counts in successive intervals by $N_0, M_1, M_2, \cdots$,

respectively. A sequence of intervals $\{W_n : n = 1, 2, \cdots\}$ is now defined by

$$W_n = \begin{cases} \varphi \text{ (empty set)} & \text{w.p.} \quad \alpha \\ J_n & \text{w.p.} \quad \bar{\alpha}. \end{cases}$$

A sequence of overlapping intervals $\{I_n : n = 0, 1, 2, \cdots\}$ is constructed from $\{J_n\}$ and $\{W_n\}$ as follows.

An α-subinterval $I_0(\alpha)$ is chosen from I_0 at random and independently of the HPP(1) realization on I_0. A new interval I_1 is defined by

$$I_1 = I_0(\alpha) \cup W_1:$$

a repetition of this procedure with one modification yields I_2. The α-subinterval $I_1(\alpha)$ of I_1 is the union of an α-subinterval of $I_0(\alpha)$ and an α-subinterval of W_1, i.e.

$$I_2 = I_0(\alpha^2) \cup W_1(\alpha) \cup W_2.$$

To construct I_3, $I_2(\alpha)$ is the union of α-subintervals of $I_0(\alpha^2)$, $W_1(\alpha)$ and W_2. Thus,

$$I_3 = I_0(\alpha^3) \cup W_1(\alpha^2) \cup W_2(\alpha) \cup W_3.$$

Proceeding in this way, we generate a sequence of overlapping intervals $\{I_n : n = 0, 1, \cdots\}$ defined by

$$I_n = I_0(\alpha^n) \bigcup_{k=1}^{n} W_k(\alpha^{n-k}). \tag{4.1}$$

Note that the sequences of α-subintervals $\{I_0(\alpha^k)\}$ and $\{W_k(\alpha^i)\}$ are nested as described in Section 4.1. Explicitly, we have

$$I_0 \supset I_0(\alpha) \supset I_0(\alpha^2) \supset \cdots$$

$$W_k \supset W_k(\alpha) \supset W_k(\alpha^2) \supset \cdots.$$

Also, the intervals I_n and I_{n-1} overlap. We can write $I_n \cap I_{n-1} = I_{n-1}(\alpha)$ provided we interpret this interval as the appropriate union of α-subintervals. With such an interpretation, (4.1) may be rewritten in a more suggestive form as

$$I_n = I_{n-1}(\alpha) \cup W_n. \tag{4.2}$$

The similarity to an autoregressive process of order 1 is evident, and we refer to $\{I_n\}$ thus defined as an autoregressive interval process.

To illustrate the construction sequence and to highlight the fact that Equation (4.2) requires the detailed interpretation given in (4.1), an example

is given in Figure 5 in which I_n is derived for $n = 0, 1, 2$ and 3. The first and second lines denote the observed HPP and the renewal process (RP) specifying the intervals of observation, $I_0, J_1, J_2, \cdots$ Here, the counts in these intervals are $N_0 = 3$, $M_1 = 1$, $M_2 = 2$, $M_3 = 1$ and $M_4 = 2$. In this example $\alpha = 0{\cdot}8$ and the middle 80% of the interval I is always chosen as $I(0{\cdot}8)$ for simplicity. Thus, the third line shows the derivation of $I_0(\alpha)$ and W_1, which we suppose in this case is J_1. Hence, I_1 is derived as their union, and $N_1 = 4$.

The fifth line shows the derivation of $I_1(\alpha)$ and W_2, and again we assume that $W = J$. Note that $I_1(\alpha)$ is derived as the union of α-subsets of $I_0(\alpha)$ and W_1, and also that the count on $I_1(\alpha)$ is only 3, one event being lost in the 'α-thinning' of I_1. The union of $I_1(\alpha)$ and W_2 is I_2, which appears on the sixth line with $N_2 = 5$. The next line gives the derivation of $I_2(\alpha)$ and W_3. Now, the latter is assumed to be the empty set, and the entire interval J_3 is lost. Note that $I_2(\alpha)$ is the union of three α-subintervals, and that $I_3 = I_2(\alpha)$, since $W_3 = \varphi$. Thus, $N_3 = 5$.

Consider now the lengths of the intervals $\{I_n\}$. Since $|J_n| = E_n$ so $|W_n| = U_n E_n$, where $\{U_n\}$ is a sequence of i.i.d. binary random variables with $P(U_n = 1) = 1 - \alpha$. If $|I_n| = X_n$, then, since $|I_{n-1}(\alpha)| = \alpha X_{n-1}$, we have from

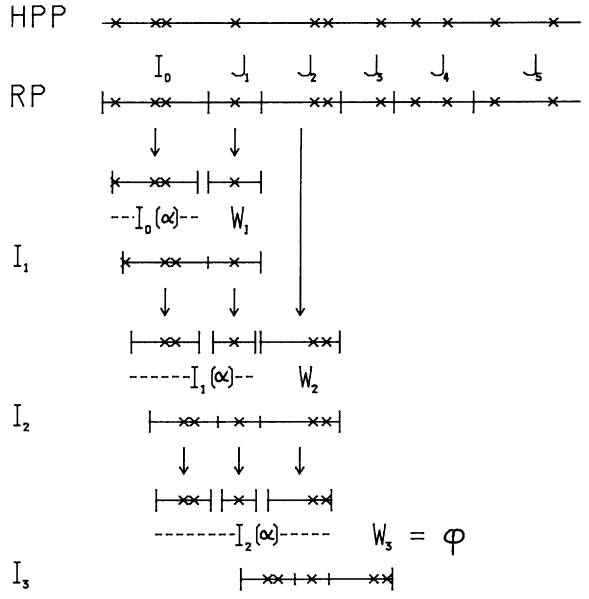


Figure 5. Construction of an autoregressive interval process

(4.2)
$$X_n = \alpha X_{n-1} + U_n E_n, \qquad n = 1, 2, \cdots$$

and

$$X_0 = E_0.$$

Hence, if E_n is negative exponential for every n, then $\{X_n\}$ is the EAR(1) process.

If the count of the observed HPP(1) on I_n is N_n then, as shown at the end of Section 4.1, the count on $I_{n-1}(\alpha)$ is $\alpha^* N_{n-1}$. Further, the count on W_n is $U_n M_n$, and so, from (4.2), we have

$$N_n = \alpha^* N_{n-1} + U_n M_n, \qquad n = 1, 2, \cdots$$

and

$$N_0 = M_0.$$

Hence, E_n is negative exponential, N_n is geometric, and $\{N_n\}$ is the geometric AR(1) process discussed in Section 2.2, the discrete analogue of the EAR(1) process.

Furthermore, note that both of these AR-processes, $\{E_n\}$ and $\{N_n\}$, are aspects of the *same* interval process $\{I_n\}$. The construction of this interval process has been given in some detail so that its components and the relationships between them may be clearly identified. We have constructed a sequence of overlapping intervals of an HPP with the property that the lengths of consecutive intervals describe the EAR(1) process, and the counts in these intervals describe the analogous geometric AR(1) process.

The other interval processes will be discussed in rather less detail.

4.3. *A moving-average interval process.* The moving-average interval process is much simpler in structure. Again, consider the interval sequences $\{J_n\}$ and $\{W_n\}$ defined in Section 4.2. Now, define a sequence of overlapping intervals $\{I_n : n = 0, 1, 2, \cdots\}$ by

(4.3)
$$I_n = J_n(\alpha) \cup W_{n-1}.$$

There is no nesting of subintervals here, and the sequence is described completely by (4.3). The lengths $\{X_n\}$ of the intervals $\{I_n\}$ are easily derived as before. They satisfy

$$X_n = \alpha E_n + U_n E_{n-1}$$

which is evidently the EMA(1) process of Lawrance and Lewis (1977) discussed

in Section 2.3. Similarly, the count process $\{N_n\}$ may be derived as

$$N_n = \alpha^* M_n + U_n M_{n-1},$$

the geometric MA(1) of Section 2.3.

4.4. *The* ARMA *interval processes.* To construct this process it is necessary to combine the previous two. We begin by deriving the AR sequence of intervals $\{I_n\}$ exactly as described in Section 4.2. Another sequence of intervals is derived from this via

$$V_n = \begin{cases} \varphi & \text{w.p. } \beta \\ I_n & \text{w.p. } \bar{\beta}. \end{cases}$$

The required sequence of overlapping intervals $\{K_n\}$ is now given by

$$K_n = J_n(\beta) \cup V_{n-1}. \tag{4.4}$$

It is now straightforward to verify that $\{|K_n|\}$ is the EARMA(1, 1) process of Jacobs and Lewis (1977) and $\{N(K_n)\}$ is the geometric ARMA(1, 1) which is its discrete analogue. Both are described in Section 2.4.

Higher-order processes can be derived from the construction of interval processes of higher order, but still based on the renewal process intervals $\{J_n\}$. For example, the MA(2) processes derive directly from an interval process $\{I_n\}$ defined by

$$I_n = J_n(\beta_1) \cup V_{n-1}$$

where

$$V_n = \begin{cases} \varphi & \text{w.p. } \beta_1 \\ J_n(\beta_2) \cup W_{n-1} & \text{w.p. } \bar{\beta}_1 \end{cases}$$

and

$$W_n = \begin{cases} \varphi & \text{w.p. } \beta_2 \\ J_n & \text{w.p. } \bar{\beta}_2. \end{cases}$$

However, it is clear that the increasing order will be accompanied by an increasing complexity of construction and we do not pursue this further here.

4.5. *More general autoregressive interval processes.* Interval processes leading to the other AR(1) processes of Section 3 may be similarly constructed. Consider a modified renewal process where the interevent intervals are again $I_0 = J_0, J_1, J_2, \cdots$ and let $|I_0| = X_0$ and $|J_k| = B_k$, $k \geqq 1$, where $\{B_k\}$ is a sequence of i.i.d. random variables. As before, an HPP(1) is observed throughout the renewal process and the counts on the intervals are $N_0, M_1, M_2, \cdots$ respectively. Suppose $\{A_n : n = 1, 2, \cdots\}$ is a sequence of i.i.d.

random variables independent of $\{B_n\}$ and such that $0 \leqq A_n \leqq 1$, for all n. Now, an A_1-subinterval of I_0 is selected independently of the observations on I_0, and I_1 is derived as $I_0(A_1) \cup J_1$. The second interval is similarly constructed, i.e. $I_2 = I_1(A_2) \cup J_2$, but $I_1(A_2)$ is the union of A_2-subintervals of $I_0(A_2)$ and J_1. Proceeding in this way yields explicitly

$$I_n = I_0(A_1 A_2 \cdots A_n) \cdot \bigcup_{k=1}^{n-1} J_k(A_{k+1} A_{k+2} \cdots A_n) \cup J_n$$

which we write as

$$I_n = I_{n-1}(A_n) \cup J_n \tag{4.5}$$

interpreting in $I_{n-1}(A_n)$ as the union of A_n-subintervals of $I_0(A_1 A_2 \cdots A_{n-1})$ and $\{J_k(A_{k+1} \cdots A_{n-1}) : k = 1, 2, \cdots, n-2\}$. As in earlier cases, the nesting of subintervals may be made explicit:

$$I_0 \supset I_0(A_1) \supset I_0(A_1 A_2) \supset \cdots$$

$$J_k \supset J_k(A_{k+1}) \supset J_k(A_{k+1} A_{k+2}) \supset \cdots .$$

Now, if $|I_n| = X_n$, $n = 0, 1, 2, \cdots$, then, since $|I_{n-1}(A_n)| = A_n X_{n-1}$, and $|J_n| = B_n$, $n = 1, 2, \cdots$, by definition, we have, from (4.5)

$$\begin{cases} X_n = A_n X_{n-1} + B_n, & n = 1, 2, \cdots \\ X_0 \quad \text{specified.} \end{cases} \tag{4.6}$$

Similarly, if $N(I_n) = N_n$, then $N(I_{n-1}(A_n)) = A_n^* N_{n-1}$ and, since $N(J_n) = M_n$, so (4.5) yields

$$\begin{cases} N_n = A_n^* N_{n-1} + M_n & n = 1, 2, \cdots \\ N_0 \quad \text{specified.} \end{cases} \tag{4.7}$$

The distributions of $\{X_n\}$ and $\{N_n\}$ and the behaviour and properties of the processes (4.6) and (4.7) are determined by our choices of X_0, $\{B_n\}$ and $\{A_n\}$. The cases of direct interest here are detailed below:

(i) X_0 is a Ga(β, λ) random variable, $A_n = \alpha$ w.p. 1, and B_n has Laplace transform $(\lambda + \alpha t)^{\beta}(\lambda + t)^{-\beta}$ as in Equation (3.1). Then $\{X_n\}$ is the GAR(1) process described in Section 3.1 and $\{N_n\}$ is its negative binomial analogue discussed in the same section.

(ii) A particular case of (i), obtained by taking $\beta = 1$, is the EAR(1) process for $\{X_n\}$ and the geometric AR(1) for $\{N_n\}$, as discussed in Section 4.2.

(iii) X_0 is a Ga(β, λ) random variable, A_n is Be$(\alpha, \beta - \alpha)$ and B_n is Ga$(\beta - \alpha, \lambda)$. In this case, $\{X_n\}$ given by (4.6) is the GBAR(1) process of Lewis (1982), and $\{N_n\}$ given by (4.7) is its negative binomial analogue discussed in Section 3.2.

(iv) X_0 is a Ga$(1, \lambda)$ random variable, $A_n = \beta U_n$ and $B_n = (1 - V_n + \bar{\alpha}\beta V_n)E_n$ where $\{U_n\}$, $\{V_n\}$ and $\{E_n\}$ are as defined in Section 3.3. In this case (4.6) yields the NEAR(1) process and (4.7) its geometric analogue discussed in Section 3.3.

4.6. *Joint distributions.* We have noted the close relationship between the Laplace transform of (X_n, X_{n-1}) and the alternate p.g.f. of (N_n, N_{n-1}) in all the processes discussed. Here we examine these transforms as they evolve from the distributions associated with the interval processes. Consider first the more general AR(1) interval process of Section 4.5. The alternate p.g.f. of two consecutive counts is

$$G_{N_n, N_{n-1}}(u, v) = E[\bar{u}^{N(I_n)}\bar{v}^{N(I_{n-1})}].$$

From (4.5), $N(I_n) = N(I_{n-1}(A_n)) + N(J_n)$. Also I_{n-1} can be decomposed into $I_{n-1}(A_n)$ and its relative complement in I_{n-1}, i.e. $I_{n-1}\backslash I_{n-1}(A_n)$. For convenience, this last is written as $I'_{n-1}(A_n)$. Thus, $N(I_{n-1}) = N(I_{n-1}(A_n)) + N(I'_{n-1}(A_n))$ and so

$$\begin{aligned} G_{N_n, N_{n-1}}(u, v) &= E\{\bar{u}^{N(J_n)}(\overline{uv})^{N(I'_{n-1}(A_n))}\bar{v}^{N(I'_{n-1}(A_n))}\} \\ &= E\{\exp[-uB_n - (u + v - uv)A_nX_{n-1} - v(1 - A_n)X_{n-1}]\} \\ &= L_B(u) \,.\, E\{L_X[v + Au(1 - v)]\}, \end{aligned} \tag{4.8}$$

where L_B and L_X are the Laplace transforms of B and X. Also

$$L_{X_n, X_{n-1}}(u, v) = L_B(u) \,.\, E\{L_X(v + Au)\}, \tag{4.9}$$

where the expectation is taken with respect to the random variable A.

Comparison of (4.8) and (4.9) yields

$$G_{N_n, N_{n-1}}(u, v) = L_{X_n, X_{n-1}}(u - uv, v) \,.\, L_B(u)/L_B(u - uv). \tag{4.10}$$

Note that, in the case when $A_n = \alpha$ w.p. 1, (4.8) may be rewritten to obtain the simpler result

$$G_{N_n, N_{n-1}}(u, v) = L_{X_n, X_{n-1}}(u, v - \alpha uv). \tag{4.11}$$

In this way, the second-order distributions for the discrete analogues can be obtained directly from those for the continuous random variable processes. Since all the joint distributions of an AR(1) process may be obtained from the bivariate ones this may be considered adequate. However, it is interesting to note that the nesting of the subintervals in the interval processes allows the derivation of similar results for higher dimensional distributions. For example, taking the case $A_n = \alpha$ w.p. 1, the joint alternate p.g.f. of (N_n, N_{n-1}, N_{n-2}) can be obtained using the decompositions of I_n, I_{n-1} and I_{n-2} into unions of non-overlapping common subintervals. The result corresponding to (4.11) is

$$G_{N_n, N_{n-1}, N_{n-2}}(u, v, w) = L_{X_n, X_{n-1}, X_{n-2}}(u, v - \alpha uv, w - \alpha^2 uw - \alpha vw + \alpha^2 uvw).$$

The moving-average process of Section 4.3 can also be treated in this way. Using Equation (4.3), $N(I_n)$ can be expressed as $N(J_n(\alpha)) + U_n N(J_{n-1}(\alpha)) + U_n N(J'_{n-1}(\alpha))$, where $J'_{n-1}(\alpha)$ is $J_{n-1} \backslash J_{n-1}(\alpha)$. Since $N(I_{n-1}) = N(J_{n-1}(\alpha)) + U_{n-1} N(J_{n-2})$, the joint alternate p.g.f. can be derived in the form

$$G_{N_n, N_{n-1}}(u, v) = L_E(\alpha u) L_{EU}(v) E\{L_E[\alpha v + Uu(1 - \alpha v)]\}.$$

Further,

$$L_{X_n, X_{n-1}}(u, v) = L_E(\alpha u) L_{EU}(v) E\{L_E(\alpha v + Uu)\},$$

and so

$$G_{N_n, N_{n-1}}(u, v) = L_{X_n, X_{n-1}}(u - \alpha uv, v) \,.\, L_E(\alpha u)/L_E(\alpha u - \alpha^2 uv).$$

The relationship between the joint distributions of $\{N_n\}$ and $\{X_n\}$ for the ARMA interval process does not lend itself easily to the approach used so far. Nevertheless, the basic pattern is easily demonstrated. For example, the ARMA interval process given by (4.4) and (4.2) yields

$$G_{N_n, N_{n-1}}(u, v) = L(\beta u) \,.\, E\{L[\beta v + u(1 - \beta v) V_n U_{n-1}]\} \,.\, E\{L[\alpha u V_n + v V_{n-1} - \alpha uv V_n V_{n-1}]\}$$

and

$$L_{X_n, X_{n-1}}(u, v) = L(\beta u) \,.\, E\{L(\beta v + u V_n U_{n-1})\} \,.\, E\{L(\alpha u V_n + v V_{n-1})\}$$

where L is the Laplace transform of the $\mathrm{Ga}(1, \lambda)$ distribution. It is not easy to relate these two transforms simply. However, it is clear that they differ, as in all previous cases, only in that the alternate p.g.f. contains terms in the product uv which are always absent in the Laplace transform.

5. Extensions, generalizations and further work

We have developed models for discrete-valued stationary stochastic processes whose univariate marginal distributions are negative binomial and geometric. They are analogues of continuous variate processes with gamma and negative exponential marginals. The major part of our presentation here is the development of the models since they share many of their properties with their continuous analogues, and it is usually straightforward to verify such. In the same way, some of the extensions being considered mirror those under consideration in the continuous case. These include the development of multivariate models and the extension to negative correlation structures. The same procedures attempted in the continuous case, e.g. the coupling in Gaver and Lewis (1980) and Lawrance and Lewis (1981), are also available for the discrete models. Some work is also under way in developing inferential

techniques for the discrete models. Apparently, similar but not identical problems arise.

An area where the usefulness of the analogue is less clear is the handling of deterministic components. Many of the naturally occurring time-series in discrete random variables exhibit a deterministic trend and/or seasonal pattern. Work is in progress using thinning to remove such components, but it is by no means completely satisfactory so far.

The models of this paper were initially developed using thinning, as described in Sections 2 and 3. It is natural, in the light of the success of this approach for the gamma distribution, to consider its extension to other discrete distributions. In fact, such models have been developed for the binomial distribution. This is particularly interesting since we may easily show that the binomial is not DSD. The roles played by self-decomposability and the interval processes of Section 4 are central to the development of the models of this paper, and would appear to be so for extensions to other distributions. Equation (1.2) relating the transforms of the discrete and continuous random variables does not depend on their being negative binomial and gamma. Coupling this equation with the SD and DSD definitions (2.2) and (2.7) allows us to extend many of the ideas of this paper to other pairs of distributions. The corresponding processes may be constructed using the interval processes. For example, the degenerate distribution i.e. mass concentrated entirely at a single value, is trivially SD. The corresponding discrete random variable is the count in an HPP(1) in a fixed length interval, i.e. a Poisson random variable. Thus, from the interval processes of Section 4, simple ARMA processes with a Poisson marginal distribution may be constructed.

Details of the Poisson and binomial processes noted here will be reported elsewhere.

Acknowledgements

I should like to express a deep sense of gratitude to Professor P. A. W. Lewis and Dr A. J. Lawrance for their help and encouragement afforded me in many discussions, and also for their valuable comments on an earlier draft of this paper. I am also pleased to acknowledge the support of a National Research Council Associateship at the Naval Postgraduate School in Monterey, California while this work was in progress.

References

BONDESSON, L. (1979) On generalized gamma and generalized negative binomial convolutions. Part I. *Scand. Act. J.*, 125–146.

COX, D. R. AND ISHAM, V. (1980) *Point Processes*. Chapman and Hall, London.

EDWARDS, C. B. AND GURLAND, J. (1961) A class of distributions applicable to accidents. *J. Amer. Statist. Assoc.* **56,** 5503–5517.

GAVER, D. P. AND LEWIS, P. A. W. (1980) First order autoregressive gamma sequences and point processes. *Adv. Appl. Prob.* **12,** 727–745.

JACOBS, P. A. AND LEWIS, P. A. W. (1977) A mixed autoregressive-moving average exponential sequence and point process (EARMA 1, 1). *Adv. Appl. Prob.* **9,** 87–104.

JACOBS, P. A. AND LEWIS, P. A. W. (1978a) Discrete time series generated by mixtures I: Correlational and runs properties. *J. R. Statist. Soc.* B **40,** 94–105.

JACOBS, P. A. AND LEWIS, P. A. W. (1978b) Discrete time series generated by mixtures II: Asymptotic properties. *J. R. Statist. Soc.* B **40,** 222–228.

JACOBS, P. A. AND LEWIS, P. A. W. (1983) Stationary discrete autoregressive-moving average time series generated by mixtures. *J. Time Series Anal.* **4,** 19–36.

JOHNSON, N. L. AND KOTZ, S. (1969) *Discrete Distributions*. Houghton-Mifflin, Boston.

LAWRANCE, A. J. (1982) The innovation distribution of a gamma distributed autoregressive process. *Scand. J. Statist.* **9,** 234–236.

LAWRANCE, A. J. AND LEWIS, P. A. W. (1977) An exponential moving-average sequence and point process (EMA 1). *J. Appl. Prob.* **14,** 98–113.

LAWRANCE, A. J. AND LEWIS, P. A. W. (1980) The exponential autoregressive-moving average EARMA (p, q) process. *J. R. Statist. Soc.* B **42,** 150–161.

LAWRANCE, A. J. AND LEWIS, P. A. W. (1981) A new autoregressive time-series model in exponential variables (NEAR(1)). *Adv. Appl. Prob.* **13,** 826–845.

LEWIS, P. A. W. (1982) Simple multivariate time-series for simulations of complex systems. *Proc. 1981 Winter Simulation Conf.*, ed. T. I. Oren, C. M. Delfosse and C. M. Shab, 389–390.

LOÈVE, M. (1963) *Probability Theory,* 3rd edn. Van Nostrand, New York.

PATIL, G. P. (ED.) (1970) *Random Counts in Scientific Work,* Vol. I. The Pennsylvania State University Press, London.

PHATARFOD, R. M. AND MARDIA, K. V. (1973) Some results for dams with Markovian inputs. *J. Appl. Prob.* **10,** 166–180.

STEUTEL, F. W. AND VAN HARN, K. (1979) Discrete analogues of self-decomposability and stability. *Ann. Prob.* **7,** 893–899.