Bar-Hillel Theorem Mechanization in Coq*

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Abstract. Formal language theory has a deep connection with such areas as static code analysis, graph database querying, formal verification, and compressed data processing. Many application problems can be formulated in terms of languages intersection. The Bar-Hillel theorem states that context-free languages are closed under intersection with a regular set. This theorem has a constructive proof and thus provides a formal justification of correctness of the algorithms for applications mentioned above. Mechanization of the Bar-Hillel theorem, therefore, is both a fundamental result of formal language theory and a basis for the certified implementation of the algorithms for applications. In this work, we present the mechanized proof of the Bar-Hillel theorem in Coq.

Keywords: Formal languages · Coq · Bar-Hillel theorem · Closure · Intersection · Regular Language · Context-free language.

1 Introduction

Formal language theory has a deep connection with different areas such as static code analysis [37, 40, 41, 36, 25, 31, 42], graph database querying [19, 20, 43, 23], formal verification [12, 9], and others. One of the most frequent uses is to formulate a problem in terms of languages intersection. In verification, one language can serve as a model of a program and another language describe undesirable behaviors. When the intersection of these two languages is not empty, one can conclude that the program is incorrect. Usually, the only concern is the decidability of the languages intersection emptiness problem. But in some cases, a constructive representation of the intersection may prove useful. This is the case, for example, when the intersection of the languages models graph querying: a language produced by intersection is a query result and to be able to process it, one needs the appropriate representation of the intersection result.

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Let us consider several applications starting with the user input validation. The problem is to check if the input provided by the user is correct with respect to some validation template such as a regular expression for e-mail validation. User input can be represented as a one word language. The intersection of such a language with the language specifying the validation template is either empty or contains the only string: the user input. If the intersection is empty, then the input should be rejected.

Checking that a program is syntactically correct is another example. The AST for the program (or lack thereof) is just a constructive representation of the intersection of the one-word language (the program) and the programming language itself.

Graph database regular querying serves as an example of the intersection of two regular languages [1, 23, 2]. Next and one of the most comprehensive cases with decidable emptiness problem is an intersection of a regular language with a context-free language. This case is relevant for program analysis [37, 40, 41], graph analysis [20, 43, 17], context-free compressed data processing [26], and other areas. The constructive intersection representation in these applications is helpful for further analysis.

The intersection of some classes of languages is not generally decidable. For example, the intersection of the linear conjunctive and the regular languages, used in the static code analysis [42], is undecidable while multiple context-free languages (MCFL) is closed under intersection with regular languages and emptiness problem for MCFLs is decidable [39]. Is it possible to express any useful properties in terms of regular and multiple context-free languages intersection? This question is beyond the scope of this paper but provides a good reason for future research in this area. Moreover, the history of pumping lemma for MCFG shows the necessity to mechanize formal language theory. In this paper, we focus on the intersection of regular and context-free languages.

Some applications mentioned above require certifications. For verification this requirement is evident. For databases it is necessary to reason about security aspects and, thus, we should create certified solutions for query executing. Certified parsing may be critical for secure data loading (for example in Web), as well as certified regular expressions for input validation. As a result, there is a significant number of papers focusing on regular expressions mechanization and certification [14], and a number on certified parsers [5, 15, 18]. On the other hand, mechanization (formalization) is important by itself as theoretical results mechanization and verification, and there is a lot of work done on formal languages theory mechanization [16, 34, 4]. Also, it is desirable to have a base to reason about parsing algorithms and other problems of languages intersection.

Context-free languages are closed under intersection with regular languages. It is stated as the Bar-Hillel theorem [3] which provides a constructive proof and construction for the resulting language description. We believe that the mechanization of the Bar-Hillel theorem is a good starting point for certified application development and since it is one of the fundamental theorems, it is

an important part of formal language theory mechanization. And this work aims to provide such mechanization in Coq.

Our current work is the first step: we provide mechanization of theoretical results on context-free and regular languages intersection. We choose the result of Jana Hofmann on context-free languages mechanization [21] as a base for our work. The main contribution of this paper is the constructive proof of the Bar-Hillel theorem in Coq. All code is published on GitHub: `https://github.com/YaccConstructor/YC_in_Coq`.

2 Bar-Hillel Theorem

In this section, we provide the Bar-Hillel theorem and sketch the proof which we use as the base of our work. We also provide some additional lemmas which are used in the proof of the main theorem.

Lemma 1. *If L is a context-free language and $\varepsilon \notin L$ then there is a grammar in Chomsky Normal Form that generates L.*

Lemma 2. *If $L \neq \varnothing$ and L is regular then L is the union of regular language $A_1, \ldots, A_n$ where each A_i is accepted by a DFA with precisely one final state.*

Theorem 1 (Bar-Hillel). *If L_1 is a context-free language and L_2 is a regular language, then $L_1 \cap L_2$ is context-free.*

Sketch of the proof.

1. By Lemma 1 we can assume that there is a context-free grammar G_{CNF} in Chomsky normal form, such that $L(G_{CNF}) = L_1$
2. By Lemma 2 we can assume that there is a set of regular languages $\{A_1 \ldots A_n\}$ where each A_i is recognized by a DFA with precisely one final state and $L_2 = A_1 \cup \ldots \cup A_n$
3. For each A_i we can explicitly define a grammar of the $L(G_{\text{CNF}}) \cap A_i$
4. Finally, we join them together with the union operation

As far as Bar-Hillel theorem operates with arbitrary context-free languages and the selected proof requires grammar in CNF, it is necessary to implement a certified algorithm for the conversion of an arbitrary CF grammar to CNF. We wanted to reuse existing mechanized proof for the conversion. We chose the one provided in Smolka's work and discussed it in the context of our work in section 3.1.

3 Bar-Hillel Theorem Mechanization in Coq

In this section, we describe in detail all the fundamental parts of the proof. We also briefly describe the motivation to use the chosen definitions. In addition, we discuss the advantages and disadvantages of using third-party proofs.

The overall goal of this section is to provide a step-by-step algorithm which constructs the context-free grammar of the intersection of two languages. The final formulation of the theorem can be found in the last subsection.

3.1 Hofmann's Results Generalization

A substantial part of this proof relies on the work of Jana Hofmann [21][5] from which many definitions and theorems were taken. Namely, the definition of a grammar, the definitions of a derivation in grammar, some auxiliary lemmas about the decidability of properties of grammar and derivation. We also use the theorem that states that there always exists the transformation from a context-free grammar to a grammar in Chomsky Normal Form.

However, the proof of the existence of the transformation to CNF had one major flaw that we needed to fix: the representation of terminals and nonterminals. In the definition of the grammar, a terminal is an element of the set of terminals—the alphabet of terminals. It is sufficient to represent each terminal by a unique natural number—conceptually, the index of the terminal in the alphabet.

The same observation is correct for nonterminals. Sometimes it is useful when the alphabet of nonterminals bears some structure. For the purposes of our proof, nonterminals are better represented as triples. We decided to make terminals and nonterminals to be polymorphic over the alphabet. We are only concerned that the representation of symbols is a type with decidable relation of equality. Namely, let Tt and Vt be such types, then we can define the types of terminals and nonterminals over Tt and Vt respectively.

Fortunately, the proof of Hofmann has a clear structure, and there was only one aspect of the proof where the use of natural numbers was essential. The grammar transformation which eliminates long rules creates new nonterminals. In the original proof, it was done by taking the maximum of the nonterminals included in the grammar. It is not possible to use the same mechanism for an arbitrary type.

To tackle this problem, we introduced an additional assumption on the alphabet types for terminals and nonterminals. We require the existence of the bijection between natural numbers and the alphabet of terminals as well as nonterminals.

Another difficulty is that the original work defines grammar as a list of rules and does not specify the start nonterminal. Thus, in order to define the language described by a grammar, one needs to specify the start terminal explicitly. It leads to the fact that the theorem about the equivalence of a CF grammar and the corresponding CNF grammar is not formulated in the most general way, namely, it guarantees equivalence only for non-empty words.

The predicate "is grammar in CNF" as defined in Hofmann [21] does not treat the case when the empty word is in the language. That is, with respect to the definition in [21], a grammar cannot have epsilon rules at all.

The question of whether the empty word is derivable is decidable for both the CF grammar and the DFA. Therefore, there is no need to adjust the definition

[5] Jana Hofmann, Verified Algorithms for Context-Free Grammars in Coq. Related sources in Coq: `https://www.ps.uni-saarland.de/~hofmann/bachelor/coq_src.zip`. Documentation: `https://www.ps.uni-saarland.de/~hofmann/bachelor/coq/toc.html`. Access date: 10.10.2018.

of the grammar (and subsequently all proofs). It is possible just to consider two cases (1) when the empty word is derivable in the grammar (and acceptable by DFA) and (2) when the empty word is not derivable. We use this feature of CNF definition to prove some of the lemmas presented in this paper.

3.2 Basic Definitions

In this section, we introduce the basic definitions used in the paper, such as alphabets, context-free grammar, and derivation.

We define a symbol as either a terminal or a nonterminal. Next, we define a word and a phrase as lists of terminals and symbols respectively. One can think that word is an element of the language defined by the grammar, and a phrase is an intermediate result of derivation. Also, a right-hand side of any derivation rule is a phrase.

The notion of nonterminal does not make sense for DFA, but in order to construct the derivation in grammar, we need to use nonterminals in intermediate states. For phrases, we introduce a predicate that defines whenever a phrase consists of only terminals. If it is the case, the phrase can be safely converted to the word.

We inherit the definition of CFG from [21]. The rule is defined as a pair of a nonterminal and a phrase, and a grammar is a list of rules. Note, that this definition of a grammar does not include the start nonterminal, and thus does not specify the language by itself.

An important step towards the definition of a language specified by a grammar is the definition of derivability. Proposition $der(G, A, p)$ means that the phrase p is derivable in the grammar G starting from the nonterminal A.

Also, we use the proof of the fact that every grammar is convertible into CNF from [21] because this fact is important for our proof.

We define the language as follows. We say that a phrase (not a word) w belongs to the language generated by a grammar G from a nonterminal A, if w is derivable from nonterminal A in grammar G and w consists only of terminals.

3.3 General Scheme of the Proof

A general scheme of our proof is based on the constructive proof presented in [8]. This proof does not use push-down automata explicitly and operates with grammars, so it is pretty simple to mechanize it. Overall, we will adhere to the following plan.

1. We consider the trivial case when DFA has no states.
2. We state that every CF language can be converted to CNF.
3. We show that every DFA can be presented as a union of DFAs with the single final state.
4. We construct an intersection of grammar in CNF with DFA with one final state.
5. We prove that the union of CF languages is CF language.

6. We putting everything mentioned above together. Additionally, we handle the fact that the initial CF language may contain the ε word. By the definition which we reuse from [21], the grammar in CNF has no epsilon rules, but we still need to consider the case when the empty word is derivable in the grammar. We postpone this consideration to the last step. Only one of the following statements is true: $\varepsilon \in L(G)$ *and* $\varepsilon \in L(dfa)$ or $\neg\varepsilon \in L(G)$ *or* $\neg\varepsilon \in L(dfa)$. So, we should just check emptiness of languages as a separated case.

3.4 Trivial Cases

First, we consider the case when the number of the DFA states is zero. In this case, we immediately derive a contradiction. By definition, any DFA has an initial state. It means that there is at least one state, which contradicts the assumption that the number of states is zero.

It is worth to mention, that in the proof [8] cases when the empty word is derivable in the grammar or a DFA specifies the empty language are discarded as trivial. It is assumed that one can carry out themselves the proof for these cases. In our proof, we include the trivial cases in the corresponding theorems.

3.5 Regular Languages and Automata

In this section, we describe definitions of DFA and DFA with exactly one final state, we also present the function that converts any DFA to a set of DFAs with one final state and lemma that states this split in some sense preserves the language specified.

We assume that a regular language is described by a DFA. We do not impose any restrictions on the type of input symbols and the number of states in DFA. Thus, the DFA is a 5-tuple: (1) a type of states, (2) a type of input symbols, (3) a start state, (4) a transition function, and (5) a list of final states.

Next, we define a function that evaluates the finish state of the automaton if it starts from the state s and receives a word w.

We say that the automaton accepts a word w being in state s if the function (*final_state s w*) returns a final state. Finally, we say that an automaton accepts a word w, if the DFA starts from the initial state and stops in a final state.

The definition of the DFA with exactly one final state differs from the definition of an ordinary DFA in that the list of final states is replaced by one final state. Related definitions such as *accepts* and *dfa_language* are slightly modified.

We define functions *s_accepts* and *s_dfa_language* for DFA with one final state in the same fashion. In the function *s_accepts*, it is enough to check for equality the state in which the automaton stopped with the finite state. Function *s_dfa_language* is the same as *dfa_language* except for that the function for a DFA with one final state should use *s_accepts* instead of *accepts*.

Now we can define a function that converts an ordinary DFA into a set of DFAs with exactly one final state. Let d be a DFA. Then the list of its final states is known. For each such state, one can construct a copy of the original DFA, but with one selected final state.

As a result prove the theorem that the function of splitting preserves the language.

Theorem 2. *Let dfa be an arbitrary DFA and w be a word. Then the fact that dfa accepts w implies that there exists a single-state DFA s_dfa, such that s_dfa ∈ split_dfa(dfa). And vice versa, for any s_dfa ∈ split_dfa(dfa) the fact that s_dfa accepts a word w implies that dfa also accepts w.*

3.6 Chomsky Induction

Many statements about properties of words in a language can be proved by induction over derivation structure. Although a one can get a phrase as an intermediate step of derivation, DFA only works on words, so we can not simply apply induction over the derivation structure. To tackle this problem, we created a custom induction principle for grammars in CNF.

The current definition of derivability does not imply the ability to "reverse" the derivation back. That is, nothing about the rules of the grammar or properties of derivation follows from the fact that a phrase w is derived from a nonterminal A in a grammar G. Because of this, we introduce an additional assumption on derivations that is similar to the syntactic analysis of words. Namely, we assume that if the phrase w is derived from the nonterminal A in grammar G, then either there is a rule $A \to w \in G$ or there is a rule $A \to rhs \in G$ and w is derivable from rhs.

Any word derivable from a nonterminal A in the grammar in CNF is either a solitary terminal or can be split into two parts, each of which is derived from nonterminals B and C, when the derivation starts with the rule $A \to BC$. Note that if we naively take a step back, we can get a nonterminal which derives some substring in the middle of the word. Such a situation does not make any sense for DFA.

By using induction, we always deal with subtrees that describe a substring of the word. To put it more formally:

Lemma 3. *Let G be a grammar in CNF. Consider an arbitrary nonterminal $N \in G$ and phrase which consists only of terminals w. If w is derivable from N and $|w| \geq 2$, then there exists two nonterminals N_1, N_2 and two phrases w_1, w_2 such that: $N \to N_1 N_2 \in G$, $der(G, N_1, w_1)$, $der(G, N_2, w_2)$, $|w_1| \geq 1$, $|w_2| \geq 1$ and $w_1 +\!+ w_2 = w$.*

Lemma 4. *Let G be a grammar in CNF. And P be a predicate on nonterminals and phrases (i.e. $P : var \to phrase \to Prop$). Let's also assume that the following two hypotheses are satisfied: (1) for every terminal production (i.e. in the form $N \to a$) of grammar G, $P(r, [Ts\ r])$ holds and (2) for every $N, N_1, N_2 \in G$ and two phrases that consist only of terminals w_1, w_2, if $P(N_1, w_1)$, $P(N_2, w_2)$, $der(G, N_1, w_1)$ and $der(G, N_2, w_2)$ then $P(N, w_1 +\!+ w_2)$. Then for any nonterminal N and any phrase consisting only of terminals w, the fact that w is derivable from N implies $P(N, w)$.*

3.7 Intersection of CFG and Automaton

Since we already have lemmas about the transformation of a grammar to CNF and the transformation of a DFA to a DFA with exactly one state, further we assume that we only deal with (1) DFA with exactly one final state—*dfa* and (2) grammar in CNF—G. In this section, we describe the proof of the lemma that states that for any grammar in CNF and any automaton with exactly one state there is a grammar for an intersection of the languages.

Construction of Intersection We present the adaptation of the algorithm given in [8].

Let G_{INT} be the grammar of intersection. In G_{INT}, nonterminals are presented as triples ($from \times var \times to$) where *from* and *to* are states of *dfa*, and *var* is a nonterminal of G.

Since G is a grammar in CNF, it has only two types of productions: (1) $N \to a$ and (2) $N \to N_1 N_2$, where N, N_1, N_2 are nonterminals and a is a terminal.

For every production $N \to N_1 N_2$ in G we generate a set of productions of the form $(from, N, to) \to (from, N_1, m)(m, N_2, to)$ where: *from*, m, *to* enumerate all *dfa* states.

For every production of the form $N \to a$ we add a set of productions of the form $(from, N, (dfa\_step(from, a))) \to a$ where *from* enumerates all *dfa* states and *dfa_step (from, a)* is the state in which the *dfa* appears after receiving terminal a in the state *from*.

Next, we join the functions above to get a generic function that works for both types of productions. Note that since the grammar is in CNF, the third alternative can never be the case.

Note that at this point we do not conduct any manipulations with the start nonterminal. Nevertheless, the hypothesis of the uniqueness of the final state of the DFA helps to define the start nonterminal of the grammar of intersection unambiguously. The start nonterminal for the intersection grammar is the following nonterminal: *(start, S, final)* where: *start*—the start state of DFA, S—the start nonterminal of the initial grammar, and *final*—the final state of DFA. Without the assumption that the DFA has only one final state it is not clear how to unequivocally define the start nonterminal over the alphabet of triples.

Correctness of Intersection In this subsection, we present a high-level description of the proof of correctness of the intersection function.

In the interest of clarity of exposition, we skip some auxiliary lemmas and facts like that we can get the initial grammar from the grammar of intersection by projecting the triples back to the corresponding terminals/nonterminals. Also note that grammar remains in CNF after the conversion, since the transformation of rules does not change the structure of them, but only replaces their terminals and nonterminals with attributed ones.

Next, we prove the following lemmas. First, the fact that a word can be derived in the initial grammar and is accepted by *s_dfa* implies it can be derived

in the grammar of the intersection. And the other way around, the fact that a word can be derived in the grammar of the intersection implies that it is derived in the initial grammar and is accepted by *s_dfa*.

Let G be a grammar in CNF. In order to use Chomsky Induction, we also assume that syntactic analysis is possible.

Theorem 3. *Let s_dfa be an arbitrary DFA, let r be a nonterminal of grammar G, let from and to be two states of the DFA. We also pick an arbitrary word—w. If it is possible to derive w from r and the s_dfa starting from the state from finishes in the state to after consuming the word w, then the word w is also derivable in grammar (convert_rules G next) from the nonterminal (V (from, r, to)).*

On the other side, now we need to prove the theorems of the form "if it is derivable in the grammar of triples, then it is accepted by the automaton and is derivable in the initial grammar".

We start with the DFA.

Theorem 4. *Let from and to be states of the automaton, var be an arbitrary nonterminal of G. We prove that if a word w is derived from the nonterminal (from, var, to) in the grammar (convert_rules G), then the automaton starting from the state from accepts the word w and stops in the state to.*

Next, we prove the similar theorem for the grammar.

Theorem 5. *Let from and to be the states of the automaton, let var be an arbitrary nonterminal of grammar G. We prove that if a word w is derivable from the nonterminal (from, var, to) in the grammar (convert_rules G), then w is also derivable in the grammar G from the nonterminal var.*

In the end, one needs to combine both theorems to get a full equivalence. By this, the correctness of the intersection is proved.

3.8 Union of Languages

During the previous step, we constructed a list of context-free grammars. In this section, we provide a function which constructs a grammar for the union of the languages.

First, we need to make sure the sets of nonterminals for each of the grammars under consideration have empty intersections. To achieve this, we label nonterminals. Each grammar of the union receives a unique ID number and all nonterminals within one grammar will have the same ID as the grammar. In addition, it is necessary to introduce a new start nonterminal of the union.

The function that constructs the union grammar takes a list of grammars, then, it (1) splits the list into head [h] and tail [tl], (2) labels [$length\ \ tl$] to h, (3) adds a new rule from the start nonterminal of the union to the start nonterminal of the grammar [h], finally (4) the function is recursively called on the tail [tl] of the list.

Proof of Languages Equivalence We prove that the function *grammar_union* constructs a correct grammar of the union language. Namely, we prove the following theorem.

Theorem 6. *Let grammars be a sequence of pairs of starting nonterminals and grammars. Then for any word w, the fact that w belongs to the language of the union is equivalent to the fact that there exists a grammar $(st, gr) \in$ grammars such that w belongs to the language generated by (st, gr).*

3.9 Putting All Parts Together

Now we can put all previously described lemmas together to prove the main statement of this paper.

```
Theorem grammar_of_intersection_exists:
 exists
  (NewNonterminal: Type)
  (IntersectionGrammar: @grammar Terminal NewNonterminal) St,
 forall word,
   dfa_language dfa word /\ language G S (to_phrase word) <->
   language IntersectionGrammar St (to_phrase word).
```

Listing 1: Final theorem

Theorem 7. *For any two decidable types Tt and Nt for types of terminals and nonterminals correspondingly. If there exists a bijection from Nt to $\mathbb{N}$ and syntactic analysis in the sense of definition is possible, then for any DFA dfa that define language over Tt and any context-free grammar G, there exists the context-free grammar G_{INT} , such that $L(G_{INT}) = L(G) \cap L(dfa)$.*

4 Related Works

There is a large number of contributions in the mechanization of different parts of formal languages theory and certified implementations of parsing algorithms and algorithms for graph database querying. These works use various tools, such as Coq, Agda, Isabelle/HOL, and are aimed at different problems such as the theory mechanization or executable algorithm certification. We discuss only a small part which is close enough to the scope of this work.

4.1 Formal Language Theory in Coq

The massive amount of work was done by Ruy de Queiroz who formalized different parts of formal language theory, such as pumping lemma [33], context-free

grammar simplification [35] and closure properties [32] in Coq. The work on closure properties contains mechanization of such properties as closure under union, Kleene star, but it does not contain mechanization of the intersection with a regular language. All these results are summarized in [34].

Gert Smolka et al. also provide a large number of contributions on regular and context-free languages formalization in Coq [11, 10, 22, 21]. The paper [21] describes the certified transformation of an arbitrary context-free grammar to the Chomsky normal form which is required for our proof of the Bar-Hillel theorem. Initially, we hoped to reuse these both parts because the Bar-Hillel theorem is about both context-free and regular languages, and it was the reason to choose results of Gert Smolka as the base for our work. But the works on regular and on context-free languages are independent, and we are faced with the problems of reusing and integration, so in the current proof, we use only results on context-free languages.

4.2 Formal Language Theory in Other Languages

In the parallel with works in Coq there exist works on formal languages mechanization in other languages and tools such as Agda or Isabelle/HOL.

Firstly, there are works of Denis Firsov who implements some parts of the formal language theory and parsing algorithms in Agda. In particular, Firsov implements CYK parsing algorithm [15, 13] and Chomsky Normal Form [16], and some other results on regular languages [14].

There are also works on the formal language theory mechanization in HOL-4 [4, 6, 7] by Aditi Bartwall and Michael Norrish. This work contains basic definitions and a big number of theoretical results, such as Chomsky normal form and Greibach normal form for context-free grammars. As an application of the mechanized theory authors, provide certified implementation of the SLR parsing algorithm [5].

5 Conclusion

We present mechanized in Coq proof of the Bar-Hillel theorem, the fundamental theorem on the closure of context-free languages under intersection with the regular set. By this, we increase mechanized part of formal language theory and provide a base for reasoning about many applicative algorithms which are based on language intersection. We generalize the results of Gert Smolka and Jana Hofmann: the definition of the terminal and nonterminal alphabets in context-free grammar were made generic, and all related definitions and theorems were adjusted to work with the updated definition. It makes previously existing results more flexible and eases reusing. All results are published at GitHub and are equipped with automatically generated documentation.

The first open question is the integration of our results with other results on formal languages theory mechanization in Coq. There are two independent sets of results in this area: works of Ruy de Queiroz and works of Gert Smolka. We

use part of Smolka's results in our work, but even here we do not use existing results on regular languages. We believe that theory mechanization should be unified and results should be generalized. We think that these and other related questions should be discussed in the community.

One direction for future research is mechanization of practical algorithms which are just implementation of the Bar-Hillel theorem. For example, context-free path querying algorithm, based on CYK [20, 43] or even on GLL [38] parsing algorithm [17]. Final target here is the certified algorithm for context-free constrained path querying for graph databases.

Another direction is mechanization of other problems on language intersection which can be useful for applications. For example, the intersection of two context-free grammars one of which describes finite language [27, 29]. It may be useful for compressed data processing [24] or speech recognition [28, 30]. And we believe all these works should share the common base of mechanized theoretical results.

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A Coq Listing

This listing contains main theorems and definitions from our work.

```
Inductive ter : Type := | T : Tt -> ter.
Inductive var : Type := | V : Vt -> var.

Lemma language_normal_form (G:grammar) (A: var) (u: word):
  u <> [] -> (language G A u <-> language (normalize G) A u).

Inductive symbol : Type :=
```

```
  | Ts : ter -> symbol
  | Vs : var -> symbol.

Definition word := list ter.
Definition phrase := list symbol.
Inductive rule : Type := | R : var -> phrase -> rule.
Definition grammar := list rule.

Inductive der (G : grammar) (A : var) : phrase -> Prop :=
  | vDer : der G A [Vs A]
  | rDer l : (R A l) el G -> der G A l
  | replN B u w v :
    der G A (u ++ [Vs B] ++ w) ->
    der G B v -> der G A (u ++ v ++ w).

Definition language (G : grammar) (A : var) (w : phrase) :=
  der G A w /\ terminal w.

Context {State T: Type}.

Record dfa: Type :=
  mkDfa {
    start: State;
    final: list State;
    next: State -> ter T -> State;
  }.

Fixpoint final_state (next_d: dfa_rule) (s: State) (w: word): State :=
  match w with
  | nil => s
  | h :: t => final_state next_d (next_d s h) t
  end.

Record s_dfa : Type :=
  s_mkDfa {
    s_start: State;
    s_final: State;
    s_next: State -> (@ter T) -> State;
  }.

Fixpoint split_dfa_list (st_d : State) (next_d : dfa_rule)
                        (f_list : list State): list (s_dfa) :=
  match f_list with
  | nil => nil
  | h :: t => (s_mkDfa st_d h next_d) :: split_dfa_list st_d next_d t
```

```
  end.

Definition split_dfa (d: dfa) :=
  split_dfa_list (start d) (next d) (final d).

Lemma correct_split:
  forall dfa w,
    dfa_language dfa w <->
    exists sdfa, In sdfa (split_dfa dfa) /\ s_dfa_language sdfa w.

Definition syntactic_analysis_is_possible :=
  forall (G : grammar) (A : var) (w : phrase),
  der G A w -> (R A w \in G) \/ (exists rhs, R A rhs \in G /\ derf G rhs w).

Definition convert_nonterm_rule_2 (r r1 r2: _) (state1 state2 : _) :=
  map (fun s3 => R (V (s1, r, s3))
                   [Vs (V (s1, r1, s2)); Vs (V (s2, r2, s3))])
      list_of_states.

Definition convert_nonterm_rule_1 (r r1 r2: _) (s1 : _) :=
  flat_map (convert_nonterm_rule_2 r r1 r2 s1) list_of_states.

Definition convert_nonterm_rule (r r1 r2: _) :=
  flat_map (convert_nonterm_rule_1 r r1 r2) list_of_states.

Definition convert_terminal_rule
  (next: _) (r: _) (t: _): list TripleRule :=
  map (fun s1 => R (V (s1, r, next s1 t)) [Ts t]) list_of_states.

Definition convert_rule (next: _) (r: _ ) :=
   match r with
   | R r [Vs r1; Vs r2] =>
       convert_nonterm_rule r r1 r2
   | R r [Ts t] =>
       convert_terminal_rule next r t
   | _  => []    (* Never called *)
   end.

Definition convert_rules
  (rules: list rule) (next: _): list rule :=
  flat_map (convert_rule next) rules.

Definition convert_grammar grammar s_dfa :=
  convert_rules grammar (s_next s_dfa).
```

```
Inductive labeled_Vt : Type :=
  | start : labeled_Vt
  | lV : nat -> Vt -> labeled_Vt.

Definition label_var (label: nat) (v: @var Vt): @var labeled_Vt :=
  V (lV label v).

Definition label_grammar_and_add_start_rule label grammar :=
  let '(st, gr) := grammar in
  (R (V start) [Vs (V (lV label st))]) :: label_grammar label gr.

Fixpoint grammar_union (grammars : seq (@var Vt * (@grammar Tt Vt)))
       : @grammar Tt labeled_Vt :=
  match grammars with
  | [] => []
  |  (g::t) => label_grammar_and_add_start_rule (length t)
                                                g ++ (grammar_union t)
end.

Variable grammars: seq (var * grammar).

Theorem correct_union:
  forall word,
    language (grammar_union grammars)
    (V (start Vt)) (to_phrase word) <->
  exists s_l,
    language (snd s_l) (fst s_l) (to_phrase word) /\ In s_l grammars.

Theorem grammar_of_intersection_exists:
  exists
    (NewNonterminal: Type)
    (IntersectionGrammar: @grammar Terminal NewNonterminal) St,
  forall word,
     dfa_language dfa word /\ language G S (to_phrase word) <->
     language IntersectionGrammar St (to_phrase word).
```