Relational Interpreters for Search Problems

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Recognition vs. Search

X — alphabet

$L \subseteq X^*$

if $\omega \in L$, denote the *witness* of this fact p_ω

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Recognition: $V(\omega, p_\omega) = \begin{cases} 1, & \omega \in L \\ 0, & \omega \notin L \end{cases}$

Recognition vs. Search

$$X \text{ — alphabet}$$

$$L \subseteq X^*$$

if $\omega \in L$, denote the *witness* of this fact p_ω

Recognition: $V(\omega, p_\omega) = \begin{cases} 1, & \omega \in L \\ 0, & \omega \notin L \end{cases}$

Search: $S(\omega) = p_\omega$

Propositional Formulas: Recognition

```
let rec eval st = function
| Conj (l, r) → eval st l && eval st r
| Disj (l, r) → eval st l || eval st r
| Neg   e     → not (eval st e)
| Var   x     → List.assoc x st
```

Propositional Formulas: Recognition

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let rec eval st = function
| Conj (l, r) → eval st l && eval st r
| Disj (l, r) → eval st l || eval st r
| Neg   e     → not (eval st e)
| Var   x     → List.assoc x st
```

```
# eval [(`x,true);(`y,false)] (Conj (Var `x) (Neg (Var `y)));;

- : bool = true
```

Propositional Formulas: Search

```
let rec solve env b = function
| Var n → ( match assoc_opt n env with
            | None → [extend env n b]
            | Some b' when b == b' → [env]
            | _    → [])
| Conj (l, r) when b →
    concat @@
    map (λ env → solve env b r) @@
    solve env b l
| Conj (l, r) → solve env b l @ solve env b r
| Neg e → solve env (not b) e
| Disj (l, r) → solve env b (Neg (Conj (Neg l, Neg r)))
```

Search is Hard[1]

Is it possible to generate a search procedure from a recognizer?

[1] compared to recognition

Relational Interpreter

$$V^R(\omega, p_\omega, q)$$

$$V^R(\omega, p_\omega, 1), \quad \mathit{if}\, \omega \in L, p_\omega \text{ — a witness}$$

$$V^R(\omega, p_\omega, 0), \quad \mathit{otherwise}$$

Relational Interpretation for Recognition and Search

$$V^R(\omega, p_\omega, ?) \quad \rightsquigarrow \quad V(\omega, p_\omega)$$

$$V^R(\omega, ?, 1) \quad \rightsquigarrow \quad S(\omega)$$

Only one program to implement!

Propositional Formulas: Relational Interpreter

```
let rec evalᵒ st f u =
  fresh (x y z v w) (
    conde [
      ?& [f ≡ conj x y; evalᵒ st x v; evalᵒ st y w; andᵒ v w u];
      ?& [f ≡ disj x y; evalᵒ st x v; evalᵒ st y w; orᵒ v w u];
      ?& [f ≡ neg  x  ; evalᵒ st x v; notᵒ v u];
      ?& [f ≡ var  z  ; assocᵒ z st u];
    ])
```

Relational Programming is Hard[2]

```
let eval_hanoi a b c moves a' b' c' =
  conde [
    ?& [moves ≡ nil (); a ≡ a'; b ≡ b'; c ≡ c';];
    fresh (f t moves' pin_f pin_t pin_f_res pin_t_res a'' b'' c'') (
      ?& [ moves ≡ (pair f t) % moves';
           conde [
             ?& [f ≡ !!A; t ≡ !!B; pin_f ≡ a; pin_f_res ≡ a''; pin_t ≡ b; pin_t_res ≡ b''; c'' ≡ c];
             ?& [f ≡ !!A; t ≡ !!C; pin_f ≡ a; pin_f_res ≡ a''; pin_t ≡ c; pin_t_res ≡ c''; b'' ≡ b];
             ?& [f ≡ !!B; t ≡ !!A; pin_f ≡ b; pin_f_res ≡ b''; pin_t ≡ a; pin_t_res ≡ a''; c'' ≡ c];
             ?& [f ≡ !!B; t ≡ !!C; pin_f ≡ b; pin_f_res ≡ b''; pin_t ≡ c; pin_t_res ≡ c''; a'' ≡ a];
             ?& [f ≡ !!C; t ≡ !!A; pin_f ≡ c; pin_f_res ≡ c''; pin_t ≡ a; pin_t_res ≡ a''; b'' ≡ b];
             ?& [f ≡ !!C; t ≡ !!B; pin_f ≡ c; pin_f_res ≡ c''; pin_t ≡ b; pin_t_res ≡ b''; a'' ≡ a];
           ];
           fresh (top_f rest_f) (
             ?& [
                  pin_f ≡ top_f % rest_f;
                  conde [ pin_t ≡ nil ();
                          fresh (top_t rest_t) (
                            ?& [pin_t ≡ top_t % rest_t;
                                lt° top_f top_t tru°;])];
                  pin_f_res ≡ rest_f;
                  pin_t_res ≡ top_f % pin_t;
                  eval_hanoi a'' b'' c'' moves' a' b' c';])])])
```

This took 3 people 6 hours to implement it

[2] compared to functional programming

Ways to Create Relational Interpreters

- Manual implementation
- Interpretation of functional programs with a relational interpreter
- Relational conversion

Ways to Create Relational Interpreters

- Manual implementation
- **Interpretation of functional programs with a relational interpreter**
- Relational conversion

Relational Interpretation of Functional Programs

- Implement good relational interpreter of a Turing-complete language
- Implement functional recognizer
- Run functional recognizer with a relational interpreter

Interpretation Overhead

Running relational interpreter comes with a price

Are there ways to get rid of it?

Specialization

Interpreter:

```
eval prog input == output
```

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Consider that a part of the input is known: `input == (static, dynamic)`

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Consider that a part of the input is known: `input == (static, dynamic)`

Specializer:

$$\texttt{spec prog static} \Rightarrow \texttt{prog}_{spec}$$
$$\texttt{eval prog (static, dynamic)} == \texttt{eval prog}_{spec}\ \texttt{dynamic}$$

Jones-Optimality

- Specializers can fail to remove all interpretation overhead
- Jones-optimal specializer: the specialized interpreter is as efficient as the program it was specialized for
- There exists a Jones-optimal specializer for a logical language [Leuschel, 2004]
- Not for miniKanren
- Jones-optimality is hard to achieve

Ways to Create Relational Interpreters

- Manual implementation
- Interpretation of functional programs with a relational interpreter
- **Relational conversion**

Relational Conversion for Relational Interpreter

- Implement a functional recognizer (verifier): $V(\omega, p_\omega)$
- Transform it into a relation: $V^R(\omega, p_\omega, q)$
- Specialize
  - Redundancy introduced with the relational conversion
  - Direction ($q == 1$)
  - Known data (ω)
- The result is a search routine

Relational Conversion (Unnesting) [Byrd 2009]

- Introduce a new variable for each subexpression
- For every n-ary function create an (n+1)-ary relation, where the last argument is unified with the result
- Transform **if**-expressions and pattern matchings into disjunctions with unifications for patterns
- Introduce into scope free variables (with **fresh**)
- Pop unifications to the top

Relational Conversion: Step 1

Introduce a new variable for each subexpression

```
let rec append a b =
  match a with
  | []        →  b
  | x :: xs  →
    x :: append xs b
```

```
let rec append a b =
  match a with
  | []        →  b
  | x :: xs  →
    let q = append xs b in
    x :: q
```

Relational Conversion: Step 2

Introduce a new variable for each subexpression

```
let rec append a b = ...
```

```
let rec append^o a b c = ...
```

Relational Conversion: Step 3

Transform **if**-expressions and pattern matchings into disjunctions with unifications for patterns

```
let rec append a b =
  match a with
  | []      → b
  | x :: xs →
    let q = append xs b in
    x :: q
```

```
let rec appendᵒ a b c =
  (a ≡ [] ∧ b ≡ c) ∨
  (  (a ≡ x :: xs) ∧
     (appendᵒ xs b q) ∧
     (c ≡ x :: q))
```

Relational Conversion: Step 4

Introduce free variables into scope (with **fresh**)

```
let rec appendᵒ a b c =
  (a ≡ [] ∧ b ≡ c) ∨
  (  (a ≡ x :: xs) ∧
     (appendᵒ xs b q) ∧
     (c ≡ x :: q))
```

```
let rec appendᵒ a b c =
  (a ≡ [] ∧ b ≡ c) ∨
  (fresh (x xs q) (
     (a ≡ x :: xs) ∧
     (appendᵒ xs b q) ∧
     (c ≡ x :: q)))
```

Relational Conversion: Step 5

Pop unifications to the top

```
let rec appendᵒ a b c =
  (a ≡ [] ∧ b ≡ c) ∨
  (fresh (x xs q) (
     (a ≡ x :: xs) ∧
     (appendᵒ xs b q) ∧
     (c ≡ x :: q)))
```

```
let rec appendᵒ a b c =
  (a ≡ [] ∧ b ≡ c) ∨
  (fresh (x xs q) (
     (a ≡ x :: xs) ∧
     (c ≡ x :: q) ∧
     (appendᵒ xs b q))
```

Forward Execution is Efficient, Backward Execution is not

Forward execution is efficient, since it mimics the execution of a function

Relational conversion for $\mathtt{f}_1\ \mathtt{x}_1$ && $\mathtt{f}_2\ \mathtt{x}_2$:

$$
\begin{array}{l}
\lambda\ \mathtt{res}\ \rightarrow \\
\quad \textbf{fresh}\ (\mathtt{p})\ (\\
\qquad (\mathtt{f}_1\ \mathtt{x}_1\ \mathtt{p})\ \wedge \\
\qquad (\textbf{conde}\ [\\
\qquad\quad (\mathtt{p} \equiv \uparrow\textbf{false}\ \wedge\ \mathtt{res} \equiv \uparrow\textbf{false}); \\
\qquad\quad (\mathtt{p} \equiv \uparrow\textbf{true}\ \ \wedge\ \mathtt{f}_2\ \mathtt{x}_2\ \mathtt{res})]))
\end{array}
$$

Computes $\mathtt{f}_2\ \mathtt{x}_2\ \mathtt{res}$ only if $\mathtt{f}_1\ \mathtt{x}_1\ \mathtt{p}$ fails

It is not the best strategy, if `res` is known

This coversion of $\texttt{f}_1\ \texttt{x}_1$ && $\texttt{f}_2\ \texttt{x}_2$ is better for the backward execution, but not for forward

$$
\begin{array}{l}
\lambda\ \texttt{res}\ \rightarrow \\
\quad \textbf{conde}\ [\\
\qquad (\texttt{res} \equiv \uparrow\textbf{false}\ \wedge\ \texttt{f}_1\ \texttt{x}_1\ \uparrow\textbf{false}); \\
\qquad (\texttt{f}_1\ \texttt{x}_1\ \uparrow\textbf{true}\ \ \wedge\ \texttt{f}_2\ \texttt{x}_2\ \texttt{res})]
\end{array}
$$

There is no single strategy suitable for all cases

There is no Single Good Strategy

Is there a way to automatically generate relations efficient in the specified directions?

Specialization: Not Only for Direction

When solving a search problem, we know its search space

$$V^R(\omega, ?, 1) \rightsquigarrow S(\omega)$$

Partial Deduction: Specialization for Logic Language

- Given:
  - Logic program
  - Goal
- Result: specialized program
- How:
  - Construct a *partial* SLD-tree
  - Generate a program from the tree
- Hopefully, all excessive computations are done statically and do not come to the specialized program

Partial Deduction: Example

```
last([x|xs], r) ← l(xs, x, r).
l([], x, x).
l([z|zs], x, r) ← l(zs, z, r).

← last([A,B|xs], r).
```

Partial Deduction: Example

```
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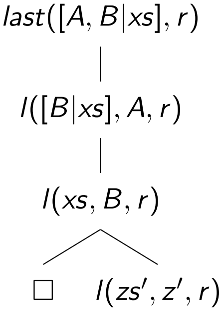
```



Partial Deduction: Example

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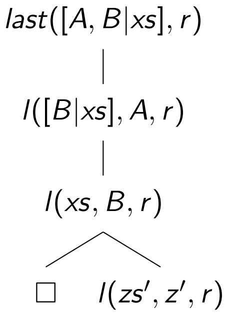


```
last([A,B], B).
last([A,B,z'|zs'], r) ← l(zs', z', r).
l([], x, x).
l([z|zs], x, r) ← l(zs, z, r).
```

Partial Deduction: Conjunctions

$$f(x, y) \vee f(y, z)$$

$$f(x, y) \qquad f(y, z)$$

$\checkmark$

Partial Deduction: Conjunctions

$$f(x, y) \vee f(y, z)$$

$f(x, y) \quad f(y, z)$

✓

$$f(x, y) \wedge f(y, z)$$

$f(x, y) \quad f(y, z)$

X

Conjunctive Partial Deduction

- Fully automatic program transformation
- For pure logic language
- Features:
  - Specialization
  - Deforestation
  - Tupling

Deforestation

Deforestation — program transformation which eliminates intermediate data structures

```
let doubleAppendᵒ x y z xyz =
  (fresh (t) (
     (appendᵒ x y t) ∧
     (appendᵒ t z xyz)))

let rec appendᵒ x y xy = conde [
  (x ≡ nil () ∧ xy ≡ y);
  (fresh (h t ty) (
     (x  ≡ h % t)   ∧
     (xy ≡ h % t')  ∧
     (appendᵒ t y t')))]
```

```
let rec doubleAppendᵒ x y z xyz = conde [
   (x ≡ nil () ∧ appendᵒ y z xyz);
   (fresh (h t t') (
      (x ≡ h % t)   ∧
      (xyz ≡ h % t')  ∧
      (doubleAppendᵒ t y z t')))]
```

Tupling

Tupling — program transformation which eliminates multiple traversals of the same data structure

```
let maxLength° xs m l = max° xs m ∧ length° xs l

let rec length° xs l = conde [
  (xs ≡ nil () ∧ l ≡ zero ());
  (fresh (h t m) (
    xs ≡ h % t ∧ l ≡ succ m ∧ length° t m))]

let max° xs m = max°₁ xs (zero ()) m

let rec max°₁ xs n m = conde [
  (xs ≡ nil () ∧ m ≡ n);
  (fresh (h t) (
    (xs ≡ h % t) ∧
    (conde [
      (le° h n ↑true ∧ max°₁ t n m);
      (gt° h n ↑true ∧ max°₁ t h m)]))))]
```

Tupling

Tupling — program transformation which eliminates multiple traversals of the same data structure

```
let maxLengtho xs m l = maxLength1o xs m (zero ()) l

let rec maxLength1o xs m n l = conde [
  (xs ≡ nil () ∧ m ≡ n ∧ l ≡ zero ());
  (fresh (h t l1)
     (xs ≡ h % t) ∧
     (l ≡ succ l1) ∧
     (conde [
       (leo h n ∧ maxLength1o t m n l);
       (gto h n ∧ maxLength1o t m h l)]))]
```

CPD: Intuition

- Local control: compute a partial SLDNF-tree per a relation of interest
  - Having a conjunction of atoms, which atom should be selected?
  - When to stop building a tree?
- Global control: determine which relations are of interest
  - Do not process the same conjunction twice
  - If a conjunction *embeds* something processed before, *generalize* it
  - How to define *embedding*?
  - How to *generalize*?

CPD: Implementation

- Local control
  - Deterministic unfold (only one nondeterministic unfold per tree)
  - Selectable conjunct: leftmost atom which does not have any predecessor embedded into it
  - Variant check
  - Stop when there are no selectable atoms
- Global control
  - Variant check
  - Generalization: split conjunction in maximally connected subconjunctions + most specific generalization
  - Homeomorphic embedding extended for conjunctions
- Residualization
  - A definition per a partial SLDNF-tree
  - Redundant Argument Filtering

Evaluation

Compare

- Unnesting
- Unnesting strategy aimed at backward execution
- Unnesting + CPD
- Interpretation of functional verifier with relational interpreter

Tasks

- Path search
- Search for a unifier of two terms

Path Search

Directed graph is a tuple $(N, E, start, end)$, where:

- N — set of nodes
- E — set of edges
- Functions $start, end : E \to N$ return a start (end) node of an edge

Path Search

Directed graph is a tuple $(N, E, start, end)$, where:

- N — set of nodes
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- Functions $start, end : E \to N$ return a start (end) node of an edge

Path is a sequence $\langle n_0, e_0, n_1, e_1, \ldots, n_k, e_k, n_{k+1} \rangle$, such that

$$\forall i \in \{0 \ldots k\} \; : \; n_i = start\,(e_i) \text{ and } n_{i+1} = end\,(e_i)$$

Path Search

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- N — set of nodes
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$$\forall i \in \{0 \ldots k\} \ : \ n_i = start\,(e_i) \text{ and } n_{i+1} = end\,(e_i)$$

Path search problem is to find the set of paths in a given graph

Path Search: Relational Conversion

```
let rec isPath ns g =
  match ns with
  | x₁ :: x₂ :: xs → elem (x₁, x₂) g && isPath (x₂ :: xs) g
  | [_]            → true
```

Path Search: Relational Conversion

```
let rec isPath ns g =
  match ns with
  | x1 :: x2 :: xs → elem (x1, x2) g && isPath (x2 :: xs) g
  | [_]            → true
```

```
let rec isPath^o ns g res = conde [
  (fresh (el) ((ns ≡ el % nil ()) ∧ (res ≡ ↑true));
  (fresh (x1 x2 xs resElem resIsPath) (
    (ns ≡ x1 % (x2 % xs)) ∧
    (elem^o (pair x1 x2) g resElem) ∧
    (isPath^o (x2 % xs) g resIsPath) ∧
    (conde [
      (resElem ≡ ↑false ∧ res ≡ ↑false);
      (resElem ≡ ↑true  ∧ res ≡ resIsPath)]))))]
```

This relation is inefficient for “isPath^o q <graph> **true**”

Path Search: Specialized Relation

$$
\begin{array}{l}
\textbf{let rec}\ \texttt{isPath}^o\ \texttt{ns g res} = \textbf{conde}\ [\\
\quad (\textbf{fresh}\ (\texttt{el})\ ((\texttt{ns} \equiv \texttt{el} \,\%\, \texttt{nil}\ ())\ \wedge\ (\texttt{res} \equiv \uparrow\textbf{true})));\\
\quad (\textbf{fresh}\ (\texttt{x}_1\ \texttt{x}_2\ \texttt{xs resElem resIsPath})\ (\\
\qquad (\texttt{resElem} \equiv \uparrow\textbf{true})\ \wedge\\
\qquad (\texttt{resIsPath} \equiv \uparrow\textbf{true})\ \wedge\\
\qquad (\texttt{ns} \equiv \texttt{x}_1 \,\%\, (\texttt{x}_2 \,\%\, \texttt{xs}))\ \wedge\\
\qquad (\texttt{elem}^o\ (\texttt{pair}\ \texttt{x}_1\ \texttt{x}_2)\ \texttt{g resElem})\ \wedge\\
\qquad (\texttt{isPath}^o\ (\texttt{x}_2 \,\%\, \texttt{xs})\ \texttt{g resIsPath})))]
\end{array}
$$

Better performance for "$\texttt{isPath}^o\ \texttt{q}\ \texttt{<graph>}\ \textbf{true}$"

Path Search: Specialized Relation

$$
\begin{array}{l}
\texttt{let}\ \textbf{rec}\ \texttt{isPath}^o\ \texttt{ns g res} = \textbf{conde}\ [\\
\quad (\textbf{fresh}\ (\texttt{el})\ ((\texttt{ns} \equiv \texttt{el} \,\%\, \texttt{nil}\ ())\ \wedge\ (\texttt{res} \equiv \uparrow\textbf{true})));\\
\quad (\textbf{fresh}\ (\texttt{x}_1\ \texttt{x}_2\ \texttt{xs resElem resIsPath})\ (\\
\qquad (\texttt{resElem} \equiv \uparrow\textbf{true})\ \wedge\\
\qquad (\texttt{resIsPath} \equiv \uparrow\textbf{true})\ \wedge\\
\qquad (\texttt{ns} \equiv \texttt{x}_1 \,\%\, (\texttt{x}_2 \,\%\, \texttt{xs}))\ \wedge\\
\qquad (\texttt{elem}^o\ (\texttt{pair}\ \texttt{x}_1\ \texttt{x}_2)\ \texttt{g resElem})\ \wedge\\
\qquad (\texttt{isPath}^o\ (\texttt{x}_2 \,\%\, \texttt{xs})\ \texttt{g resIsPath})))]
\end{array}
$$

Better performance for "$\texttt{isPath}^o\ \texttt{q} <\texttt{graph}>$ **true**"

This can be achieved automatically with CPD

Evaluation: Path Search

Path length	5	7	9	11	13	15
Only conversion	0.01	1.39	82.13	>300	—	—
Backward oriented conversion	0.01	0.37	2.68	2.91	4.88	10.63
Conversion and CPD	0.01	0.06	0.34	2.66	3.65	6.22
Scheme interpreter	0.80	8.22	88.14	191.44	>300	—

Table: Searching for paths in the graph (seconds)

Unification

Term:

- Variable $(X, Y, \dots)$
- Some constructor applied to terms $(nil, cons(H, T), \dots)$

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Substitution can be *applied* to a term by simultaneously substituting variables for their images

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Unifier is a substitution σ which equalizes terms: $t\sigma = s\sigma$

Unification

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Substitution maps variables to terms

Substitution can be *applied* to a term by simultaneously substituting variables for their images

Unifier is a substitution σ which equalizes terms: $t\sigma = s\sigma$

Problem: given two terms with free variables, find their unifier

Unification: Functional Verifier

```
let rec check_uni subst t1 t2 =
  match t1, t2 with
   | Constr (n1, a1), Constr (n2, a2)  →
       eq_nat n1 n2 && forall2 subst a1 a2
   | Var_ v          , Constr (n, a)    →
     begin match get_term v subst with
     | None    →  false
     | Some t  →  check_uni subst t t2
     end
   | Constr (n, a)   , Var_ v           →
     begin match get_term v subst with
     | None    →  false
     | Some t  →  check_uni subst t1 t
     end
   | Var_ v1         , Var_ v2          →
     match get_term v1 subst with
     | Some t1'  →  check_uni subst t1' t2
     | None      →  match get_term v2 subst with
                    | Some _  →  false
                    | None    →  eq_nat v1 v2
```

Unification: Relational Conversion

Does not fit the slide.

Evaluation: Unification

Terms	f(X, a) f(a, X)	f(a % b % nil, c % d % nil, L) f(X % XS, YS, X % ZS)	f(X, X, g(Z, t)) f(g(p, L), Y, Y)
Only conversion	0.01	>300	>300
Backward oriented conversion	0.01	0.11	2.26
Conversion and CPD	0.01	0.07	0.90
Scheme interpreter	0.04	5.15	>300

Table: Searching for a unifier of two terms (seconds)

Conclusion & Future Work

Funcional verifier + unnesting + specialization = solver

Future work

- Generate functional program from relational to reduce interpretation overhead
- Some other specialization technique, less ad-hoc than CPD