One Algorithm to Evaluate Them All: Unified Approach Based on Linear Algebra for Both Regular and Context-Free Path Queries

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ABSTRACT

An algorithm for context-free path querying (CFPQ) was recently proposed by Egor Orachev et. al. We reduce this algorithm to operations over Boolean matrices and extend it with a mechanism to extract all paths of interest. We prove $O(n^3/\log n)$ time complexity of the proposed algorithm, where n is a number of vertices of the input graph. Thus we provide an alternative slightly subcubic algorithm for CFPQ based on linear algebra and on a classical graph-theoretic problem (incremental transitive closure), rather than the algorithm proposed by Swarat Chaudhuri. Our evaluation demonstrates the applicability of the proposed solution to both RPQ and CFPQ over real-world graphs.

CCS CONCEPTS

• **Information systems** → **Graph-based database models**; **Query languages for non-relational engines**; • **Theory of computation** → **Grammars and context-free languages**; **Regular languages**; • **Mathematics of computing** → **Paths and connectivity problems**; *Graph algorithms.*

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1 INTRODUCTION

Language-constrained path querying [5] is a technique for graph navigation querying. This technique allows one to use formal languages as constraints on paths in edge-labeled graphs: path satisfies constraints if labels along it form a word from the specified language.

The utilization of regular languages as constraints, or *Regular Path Querying* (RPQ), is most well-studied and widespread. Different aspects of RPQs are actively studied in graph databases [2, 4, 32], while regular constraints are supported in such popular query languages as PGQL [48] and SPARQL[1] [29] (property paths). Nevertheless, there is certainly room for improvement of RPQ efficiency, and new solutions are being created [37, 51].

At the same time, using more powerful languages, namely context-free languages, as constraints has gained popularity in the last few years. *Context-Free Path Querying* problem (CFPQ) was introduced by Mihalis Yannakakis in 1990 in [52]. Many algorithms were proposed since that time, but recently, Jochem Kuijpers et al. showed in [30] that state-of-the-art CFPQ algorithms are not performant enough for practical use. This motivates us to develop new algorithms for CFPQ.

[1] Specification of regular constraints in SPARQL property paths: https://www.w3.org/TR/sparql11-property-paths/. Access date: 07.07.2020.

One promising way to achieve high-performance solutions for graph analysis problems is to reduce them to linear algebra operations. This way, GraphBLAS [27] API, the description of basic linear algebra primitives, was proposed. Solutions that use libraries that implement this API, such as SuiteSparce [13] and CombBLAS [7], show that reduction to linear algebra is a way to utilize high-performance parallel and distributed computations for graph analysis.

Rustam Azimov shows in [3] how to reduce CFPQ to matrix multiplication. Later, it was shown in [36] and [46] that utilization of appropriate libraries for linear algebra for Azimov's algorithm implementation makes a practical solution for CFPQ. However Azimov's algorithm requires transforming the input grammar to Chomsky Normal Form, which leads to the grammar size increase, and hence worsens performance, especially for regular queries and complex context-free queries.

To solve these problems, an algorithm based on automata intersection was proposed [38]. This algorithm is based on linear algebra and does not require the transformation of the input grammar. We improve the algorithm in this work. We reduce the above mentioned solution to operations over Boolean matrices, thus simplifying its description and implementation. Also, we show that this algorithm is performant enough for regular queries, so it is a good candidate for integration with real-world query languages: one algorithm can be used to evaluate both regular and context-free queries.

Moreover, we show that this algorithm opens the way to tackle a long-standing problem about the existence of truly-subcubic $O(n^{3-\epsilon})$ CFPQ algorithm [10, 52]. Currently, the best result is an $O(n^3/\log n)$ algorithm of Swarat Chaudhuri [10]. Also, there exist truly subcubic solutions which use fast matrix multiplication for some fixed subclasses of context-free languages [6]. Unfortunately, this solutions cannot be generalized to arbitrary CFPQs. In this work, we identify incremental transitive closure as a bottleneck on the way to achieve subcubic time complexity for CFPQ.

To sum up, we make the following contributions.

(1) We rethink and improve the CFPQ algorithm based on tensor-product proposed by Orachev et al. [38]. We reduce this algorithm to operations over Boolean matrices. As a result, all-path query semantics is handled, as opposed to the previous matrix-based solution which handles only the single-path semantics. Also, both regular and context-free grammars can be used as queries. We prove the correctness and time complexity for the proposed algorithm.
(2) We demonstrate the interconnection between CFPQ and incremental transitive closure. We show that incremental transitive closure is a bottleneck on the way to achieve faster CFPQ algorithm for general case of arbitrary graphs as well as for special families of graphs, such as planar graphs.
(3) We implement the described algorithm and evaluate it on real-world data for both RPQ and CFPQ. Results show that the proposed solution is comparable with existing solutions for CFPQ and RPQ, thus it is a promising way to create a unified algorithm for both CFPQ and RPQ evaluation.

2 PRELIMINARIES

In this section we introduce basic notation and definitions from graph theory and formal language theory.

2.1 Language-Constrained Path Querying Problem

We use a directed edge-labeled graph as a data model. To introduce the *Language-Constraint Path Querying Problem* [5] over directed edge-labeled graphs we first give both language and grammar definitions.

Definition 2.1. An *edge-labeled directed graph* $\mathcal{G}$ is a triple $\langle V, E, L\rangle$, where $V = \{0, \ldots, |V|-1\}$ is a finite set of vertices, $E \subseteq V \times L \times V$ is a finite set of edges and L is a finite set of edge labels.

The example of a graph which we use in the further examples is presented in Figure 1.

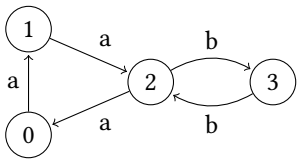


Figure 1: The example of input graph $\mathcal{G}$

Definition 2.2. An *adjacency matrix* for an edge-labeled directed graph $\mathcal{G} = \langle V, E, L\rangle$ is a matrix M, which has size $|V| \times |V|$ and $M[i,j] = \{l \mid e = (i,l,j) \in E\}$.

Adjacency matrix M_2 of the graph $\mathcal{G}$ is

$$M_2 = \begin{pmatrix} . & \{a\} & . & . \\ . & . & \{a\} & . \\ \{a\} & . & . & \{b\} \\ . & . & \{b\} & . \end{pmatrix}.$$

Definition 2.3. The *Boolean matrices decomposition*, or *Boolean adjacency matrix*, for an edge-labeled directed graph $\mathcal{G} = \langle V, E, L\rangle$ with adjacency matrix M is a set of matrices $\mathcal{M} = \{M^l \mid l \in L, M^l[i,j] = 1 \iff l \in M[i,j]\}$.

We use the decomposition of the adjacency matrix into a set of Boolean matrices. As an example, matrix M_2 can be represented as a set of two Boolean matrices M_2^a and M_2^b:

$$M_2^a = \begin{pmatrix} . & 1 & . & . \\ . & . & 1 & . \\ 1 & . & . & . \\ . & . & . & . \end{pmatrix}, M_2^b = \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \end{pmatrix}.$$

This way we reduce operations necessary for our algorithm from operations over custom semiring (over edge labels) to operations over a Boolean semiring with an *addition* $+$ as $\vee$ and a *multiplication* $\cdot$ as $\wedge$ over Boolean values.

We also use notation $\mathcal{M}(\mathcal{G})$ and $\mathcal{G}(\mathcal{M})$ to describe the Boolean decomposition matrices for some graph and the graph formed by its adjacency Boolean matrices.

Definition 2.4. A *path* π in the graph $\mathcal{G} = \langle V, E, L \rangle$ is a sequence $e_0, e_1, \ldots, e_{n-1}$, where $e_i = (v_i, l_i, u_i) \in E$ and for any e_i, e_{i+1}: $u_i = v_{i+1}$. We denote a path from v to u as $v \pi u$.

Definition 2.5. A *word formed by a path*

$$\pi = (v_0, l_0, v_1), (v_1, l_1, v_2), \ldots, (v_{n-1}, l_{n-1}, v_n)$$

is a concatenation of labels along the path: $\omega(\pi) = l_0 l_1 \ldots l_{n-1}$.

Definition 2.6. A *language* $\mathcal{L}$ over a finite alphabet Σ is a subset of all possible sequences formed by symbols from the alphabet: $\mathcal{L}_\Sigma = \{\omega \mid \omega \in \Sigma^*\}$.

Now we are ready to introduce CFPQ problem for the given graph $\mathcal{G} = \langle V, E, L \rangle$ and the given language $\mathcal{L}$ with reachability and all-path semantics.

Definition 2.7. To evaluate context-free path query with reachability semantics is to construct a set of pairs of vertices (v_i, v_j) such that there exists a path $v_i \pi v_j$ in $\mathcal{G}$ which forms the word from the given language:

$$R = \{(v_i, v_j) \mid \exists \pi : v_i \pi v_j, \omega(\pi) \in \mathcal{L}\}$$

Definition 2.8. To evaluate context-free path query with all-path semantics is to construct a set of paths π in $\mathcal{G}$ which form the word from the given language:

$$\Pi = \{\pi \mid \omega(\pi) \in \mathcal{L}\}$$

Note that Π can be infinite, thus in practice we should provide a way to enumerate such paths with reasonable complexity, instead of explicit construction of the Π.

2.2 Regular Path Queries and Finite State Machine

In *Regular Path Querying* (RPQ) the language $\mathcal{L}$ is regular. This case is widespread and well-studied. The most common way to specify regular languages is by *regular expressions*. We use the following definition of regular expressions.

Definition 2.9. A *regular expression* over the alphabet Σ is a finite combination of patterns, which can be defined as follows: $\varnothing$ (empty language), ε (empty string), $a_i \in \Sigma$ are regular expressions, and if R_1 and R_2 are regular expressions, then $R_1 \mid R_2$ (alternation), $R_1 \cdot R_2$ (concatenation), R_1^* (Kleene star) are also regular expressions.

For example, one can use regular expression $R_1 = ab^*$ to search for paths in the graph $\mathcal{G}$ (Figure 1). The expected query result is a set of paths which start with an a-labeled edge and contain zero or more b-labeled edges after that.

In this work we use the notion of *Finite-State Machine* (FSM) or *Finite-State Automaton* (FSA) for RPQs.

Definition 2.10. A *deterministic finite-state machine without ε-transitions* T is a tuple $\langle \Sigma, Q, Q_s, Q_f, \delta \rangle$, where Σ is an input alphabet, Q is a finite set of states, $Q_s \subseteq Q$ is a set of start (or initial) states, $Q_f \subseteq Q$ is a set of final states and $\delta : Q \times \Sigma \to Q$ is a transition function.

It is well known, that every regular expression can be converted to deterministic FSM without ε-transitions [24]. We use FSM as a representation of RPQ. FSM can be naturally represented by a directed edge-labeled graph: $V = Q$, $L = \Sigma$, $E = \{(q_i, l, q_j) \mid \delta(q_i, l) = q_j\}$, where some vertices have special markers to specify the start and final states. An example of the graph representation of FSM T_1 for the regular expression R_1 is presented in Figure 2.

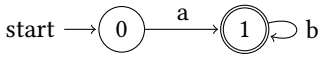


Figure 2: The example of graph representation of FSM for the regular expression ab^*

As a result, FSM also can be represented as a set of Boolean adjacency matrices $\mathcal{M}$ accompanied by the information about the start and final vertices. Such representation of T_1:

$$M^a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, M^b = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note, that an edge-labeled graph can be viewed as an FSM where edges represent transitions, and all vertices are both start and final at the same time. Thus RPQ evaluation is an intersection of two FSMs. The query result can also be represented as FSM, because regular languages are closed under intersection.

2.3 Context-Free Path Querying and Recursive State Machines

An even more general case than RPQ is a *Context-Free Path Querying Problem (CFPQ)*, where one can use context-free languages as constraints. These constraints are more expressive than the regular constraints. For example, a classic

same-generation query can be expressed by a context-free language, but not a regular language.

Definition 2.11. A *context-free grammar* $G = \langle \Sigma, N, S, P \rangle$, where Σ is a finite set of terminals (or terminal alphabet), N is a finite set of nonterminals (or nonterminal alphabet), $S \in N$ is a start nonterminal, P is a finite set of productions (grammar rules) of form $N_i \to \alpha$ where $N_i \in N, \alpha \in (\Sigma \cup N)^*$.

Definition 2.12. The sequence $\omega_2 \in (\Sigma \cup N)^*$ is *derivable* from $\omega_1 \in (\Sigma \cup N)^*$ *in one derivation step*, or $\omega_1 \to \omega_2$, in the grammar $G = \langle \Sigma, N, S, P \rangle$ iff $\omega_1 = \alpha N_i \beta$, $\omega_2 = \alpha \gamma \beta$, and $N_i \to \gamma \in P$.

Definition 2.13. Context-free grammar $G = \langle \Sigma, N, S, P \rangle$ specifies a *context-free language*: $\mathcal{L}(G) = \{\omega \mid S \overset{*}{\to} \omega\}$, where $(\overset{*}{\to})$ denotes zero or more derivation steps $(\to)$.

For instance, a grammar $G_1 = \langle \{a, b\}, \{S\}, S, \{S \to a\, b;\ S \to a\ S\ b\} \rangle$ can be used to search for paths, which form words in the language $\mathcal{L}(G_1) = \{a^n b^n \mid n > 0\}$ in the graph $\mathcal{G}$ (fig. 1).

While a regular expression can be transformed to a FSM, a context-free grammar can be transformed to a *Recursive State Machine* (RSM) in the similar fashion. In our work we use the following definition of RSM based on [1].

Definition 2.14. A *recursive state machine* R over a finite alphabet Σ is defined as a tuple of elements $\langle M, m, \{C_i\}_{i \in M} \rangle$, where M is a finite set of labels of boxes, $m \in M$ is an initial box label, a $C_i = \langle \Sigma \cup M, Q_i, q_i^0, F_i, \delta_i \rangle$ is a *component state machine* or *box*, where:

- $\Sigma \cup M$ is a set of symbols, $\Sigma \cap M = \varnothing$
- Q_i is a finite set of states, where $Q_i \cap Q_j = \varnothing, \forall i \neq j$
- q_i^0 is an initial state for C_i
- $F_i \subseteq Q_i$ is a set of the final states for C_i
- $\delta_i : Q_i \times (\Sigma \cup M) \to Q_i$ is a transition function

RSM behaves as a set of finite state machines (or FSM). Each FSM is called a *box* or a *component state machine*. A box works similarly to the classical FSM, but it also handles additional *recursive calls* and employs an implicit *call stack* to *call* one component from another and then return execution flow back.

The execution of an RSM could be defined as a sequence of the configuration transitions, which are done while reading the input symbols. The pair (q_i, S), where q_i is a current state for box C_i and S is a stack of *return states*, describes an *execution configuration*.

The RSM execution starts from the configuration $(q_m^0, \langle\rangle)$. The following list of rules defines the machine transition from configuration (q_i, S) to (q', S') on some input symbol a:

- $(q_i^k, S) \rightsquigarrow (\delta_i(q_i^k, a), S)$
- $(q_i^k, S) \rightsquigarrow (q_j^0, \delta_i(q_i^k, j) \circ S)$
- $(q_j^k, q_i^t \circ S) \rightsquigarrow (q_i^t, S)$, where $q_j^k \in F_j$

An input word $a_1 \dots a_n$ is accepted, if machine reaches configuration $(q, \langle\rangle)$, where $q \in F_m$. Note, that an RSM makes nondeterministic transitions and does not read the input character when it *calls* some component or *returns*.

According to [1], recursive state machines are equivalent to pushdown systems. Since pushdown systems are capable of accepting context-free languages [24], RSMs are equivalent to context-free languages. Thus RSMs suit to encode query grammars. Any CFG can be easily converted to an RSM with one box per nonterminal. The box which corresponds to a nonterminal A is constructed using the right-hand side of each rule for A.

An example of such RSM R constructed for the grammar G with rules $S \to aSb \mid ab$ is provided in Figure 3.

Since R is a set of FSMs, it can be represented as an adjacency matrix for the graph where vertices are states from $\bigcup_{i \in M} Q_i$ and edges are transitions between q_i^a and q_i^b with the label $l \in \Sigma \cup M$, if $\delta_i(q_i^a, l) = q_i^b$. Thus, similarly to an FSM, an RSM can be represented as a set of Boolean adjacency matrices.

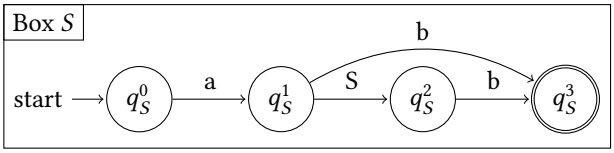


Figure 3: Recursive state machine R for grammar G

As an RPQ, a CFPQ is the intersection of the given context-free language and a FSM specified by the given graph. As far as every context-free language is closed under the intersection with regular languages, such intersection can be represented as an RSM. Also, an RSM can be viewed as an FSM over $\Sigma \cup N$. We use this point of view to propose a unified algorithm to evaluate both regular and context-free path queries with zero overhead for regular queries.

2.4 Graph Kronecker Product and Machines Intersection

In this section we introduce classical Kronecker product definition, describe graph Kronecker product and its relation to Boolean matrices algebra, and RSM and FSM intersection.

Definition 2.15. Given two matrices A and B of sizes $m_1 \times n_1$ and $m_2 \times n_2$ respectively, with element-wise product operation $\cdot$, the *Kronecker product* of these two matrices is a new matrix $C = A \otimes B$ of size $m_1 * m_2 \times n_1 * n_2$ and $C[u * m_1 + v, n_1 * p + q] = A[u, p] \cdot B[v, q]$.

Definition 2.16. Given two edge-labeled directed graphs $\mathcal{G}_1 = \langle V_1, E_1, L_1 \rangle$ and $\mathcal{G}_2 = \langle V_2, E_2, L_2 \rangle$, the *Kronecker product* of these graphs is a edge-labeled directed graph $\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2$,

where $\mathcal{G} = \langle V, E, L\rangle$: $V = V_1 \times V_2$, $E = \{((u,v),l,(p,q)) \mid (u,l,p) \in E_1 \wedge (v,l,q) \in E_2\}$ and $L = L_1 \cap L_2$.

The Kronecker product for graphs produces a new graph with a property that if some path $(u,v)\pi(p,q)$ exists in the result graph then paths $u\pi_1 p$ and $v\pi_2 q$ exist in the input graphs, and $\omega(\pi) = \omega(\pi_1) = \omega(\pi_2)$. These paths π_1 and π_2 could be easily found from π by its definition.

The Kronecker product for directed graphs can be described as the Kronecker product of the corresponding adjacency matrices of graphs, what gives the following definition:

Definition 2.17. Given two adjacency matrices M_1 and M_2 of sizes $m_1 \times n_1$ and $m_2 \times n_2$ respectively for some directed graphs $\mathcal{G}_1$ and $\mathcal{G}_2$, the *Kronecker product* of these two adjacency matrices is the adjacency matrix M of some graph $\mathcal{G}$, where M has size $m_1 * m_2 \times n_1 * n_2$ and $M[u * m_1 + v, n_1 * p + q] = M_1[u,p] \cap M_2[v,q]$.

By the definition, the Kronecker product for adjacency matrices gives an adjacency matrix with the same set of edges as in the resulting graph in the Def. 2.16. Thus, $M(\mathcal{G}) = M(\mathcal{G}_1) \otimes M(\mathcal{G}_2)$, where $\mathcal{G} = \mathcal{G}_1 \otimes \mathcal{G}_2$.

Definition 2.18. Given two FSMs $T_1 = \langle \Sigma, Q^1, Q^1_S, S^1_F, \delta^1\rangle$ and $T_2 = \langle \Sigma, Q^2, Q^2_S, S^2_F, \delta^2\rangle$, the *intersection* of these two machines is a new FSM $T = \langle \Sigma, Q, Q_S, S_F, \delta\rangle$, where $Q = Q^1 \times Q^2$, $Q_S = Q^1_S \times Q^2_S$, $Q_F = Q^1_F \times Q^2_F$, $\delta : Q \times \Sigma \to Q$ and $\delta(\langle q_1, q_2\rangle, s) = \langle q'_1, q'_2\rangle$, if $\delta(q_1, s) = q'_1$ and $\delta(q_2, s) = q'_2$.

According to [24] an FSM intersection defines the machine for which $L(T) = L(T_1) \cap L(T_2)$.

The most substantial part of intersection is the δ function construction for the new machine T. Using adjacency matrices decomposition for FSMs, we can reduce the intersection to the Kronecker product of such matrices over Boolean semiring at some extent, since the transition function δ of the machine T in matrix form is exactly the same as the product result. More precisely:

Definition 2.19. Given two adjacency matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ over Boolean semiring, the *Kronecker product* of these matrices is a new matrix $\mathcal{M} = \mathcal{M}_1 \otimes \mathcal{M}_2$, where $\mathcal{M} = \{M^a_1 \otimes M^a_2 \mid a \in \Sigma\}$ and the element-wise operation is *and* over Boolean values.

Applying the Kronecker product theory for both the FSM and the edge-labeled directed graph, we can intersect these objects as shown in Def. 2.19, since the graph could be interpreted as an FSM with transitions matrix represented as the Boolean adjacency matrix.

In this work we show how to express RSM and FSM intersection in terms of the Kronecker product and transitive closure over Boolean semiring.

3 CONTEXT-FREE PATH QUERYING BY KRONECKER PRODUCT

In this section we introduce the algorithm for CFPQ which is based on Kronecker product of Boolean matrices. The algorithm solves all-pairs CFPQ in all-path semantics (according to Hellings [21]) and works in two steps.

(1) *Index creation.* In this step, the algorithm computes an index which contains information necessary to restore paths for given pairs of vertices. This index can be used to solve the reachability problem without extracting paths. Note that this index is finite even if the set of paths is infinite.
(2) *Paths extraction.* All paths for the given pair of vertices can be enumerated by using the index. Since the set of paths can be infinite, all paths cannot be enumerated explicitly, and advanced techniques such as lazy evaluation are required for the implementation. Nevertheless, a single path can always be extracted with standard techniques.

In the following subsections we describe these steps, prove correctness of the algorithm, and provide time complexity estimations.

3.1 Index Creation Algorithm

The *index creation* algorithm outputs the final adjacency matrix $\mathcal{M}_2$ for the input graph with all pairs of vertices which are reachable through some nonterminal in the input grammar G, as well as the index matrix C_3, which is to be used to extract paths in the *path extraction* algorithm.

The algorithm is based on the generalization of the FSM intersection for an RSM, and the edge-labeled directed input graph. Since the RSM is composed as a set of FSMs, it could be easily presented as an adjacency matrix for some graph over labels set $\Sigma \cup S$. As shown in the Def. 2.19, we can apply Kronecker product from Boolean matrices to *intersect* the RSM and the input graph to some extent. But the RSM contains the nonterminal symbols from N with additional *recursive calls* logic, which requires *transitive closure* step to extract such symbols.

The core idea of the algorithm comes from Kronecker product and transitive closure. The algorithm boils down to the iterative Kronecker product evaluation for the RSM adjacency matrix $\mathcal{M}_1$ and the input graph adjacency matrix $\mathcal{M}_2$, followed by transitive closure, extraction of nonterminals and updating the graph adjacency matrix $\mathcal{M}_2$. Listing 1 shows main steps of the algorithm.

3.1.1 Application of Dynamic Transitive Closure. It is easy to see that the most time-consuming steps of the algorithm are the Kronecker product and transitive closure computations. Note that the adjacency matrix $\mathcal{M}_2$ is always changed

Listing 1 Kronecker product based CFPQ using dynamic transitive closure

```
1:  function contextFreePathQuerying(G, 𝒢)
2:      R ← Recursive automata for G
3:      ℳ1 ← Boolean adjacency matrix for R
4:      ℳ2, 𝒜2 ← Boolean adjacency matrix for 𝒢
5:      C3 ← The empty matrix
6:      for s ∈ 0..dim(ℳ1) − 1 do
7:          for S ∈ getNonterminals(R, s, s) do
8:              for i ∈ 0..dim(ℳ2) − 1 do
9:                  M2^S[i, i] ← 1
10:     while ℳ2 is changing do
11:         M3' ← ⋁_{M^S ∈ ℳ1⊗𝒜2} M^S
12:         𝒜2 ← The empty matrix
13:         C3' ← The empty matrix
14:         for (i, j) | M3'[i, j] ≠ 0 do
15:             C3' ← add(C3, C3', i, j)   ▷ Updating the transitive closure
16:             C3 ← C3 + C3'
17:         for (i, j) | C3'[i, j] ≠ 0 do
18:             s, f ← getStates(C3', i, j)
19:             x, y ← getCoordinates(C3', i, j)
20:             for S ∈ getNonterminals(R, s, f) do
21:                 M2^S[x, y] ← 1
22:                 A2^S[x, y] ← 1
23:     return ℳ2, C3
24: function getStates(C, i, j)
25:     r ← dim(ℳ1)                    ▷ ℳ1 is adjacency matrix for R
26:     return ⌊i/r⌋, ⌊j/r⌋
27: function getCoordinates(C, i, j)
28:     n ← dim(ℳ2)                    ▷ ℳ2 is adjacency matrix for 𝒢
29:     return i mod n, j mod n
```

incrementally i. e. elements (edges) are added to $\mathcal{M}_2$ (and are never deleted from it) at each iteration of the algorithm. So it is not necessary to recompute the whole product or transitive closure if an appropriate data structure is maintained.

To compute the Kronecker product, we employ the fact that it is left-distributive. Let $\mathcal{A}_2$ be a matrix with newly added elements and $\mathcal{B}_2$ be a matrix with the all previously found elements, such that $\mathcal{M}_2 = \mathcal{A}_2 + \mathcal{B}_2$. Then by the left-distributivity of the Kronecker product we have $\mathcal{M}_1 \otimes \mathcal{M}_2 = \mathcal{M}_1 \otimes (\mathcal{A}_2 + \mathcal{B}_2) = \mathcal{M}_1 \otimes \mathcal{A}_2 + \mathcal{M}_1 \otimes \mathcal{B}_2$. Note that $\mathcal{M}_1 \otimes \mathcal{B}_2$ is known and is already in the matrix $\mathcal{M}_3$ and its transitive closure also is already in the matrix C_3, because it has been calculated at the previous iterations, so it is left to update some elements of $\mathcal{M}_3$ by computing $\mathcal{M}_1 \otimes \mathcal{A}_2$.

The fast computation of transitive closure can be obtained by using incremental dynamic transitive closure technique. Now we describe the function *add* from Listing 1. Let C_3 be a transitive closure matrix of the graph G with n vertices. We use an approach by Ibaraki and Katoh [25] to maintain dynamic transitive closure. The key idea of their algorithm is to recalculate reachability information only for those vertices, which become reachable after insertion of the certain edge. We have modified it to achieve a logarithmic speed-up.

For each newly inserted edge (i, j) and every node $u \neq j$ of G such that $C_3[u, i] = 1$ and $C_3[u, j] = 0$, one needs to perform operation $C_3[u, v] = C_3[u, v] \wedge C_3[j, v]$ for every node v, where $1 \wedge 1 = 0 \wedge 0 = 1 \wedge 0 = 0$ and $0 \wedge 1 = 1$. Notice that these operations are equivalent to the element-wise (Hadamard) product of two vectors of size n, where multiplication operation is denoted as $\wedge$. To check whether $C_3[u, i] = 1$ and $C_3[u, j] = 0$ one needs to multiply two vectors: the first vector represents reachability of the given vertex i from other vertices $\{u_1, u_2, ..., u_n\}$ of the graph and the second vector represents the same for the given vertex j. The operation $C_3[u, v] \wedge C_3[j, v]$ also can be reduced to the computation of the Hadamard product of two vectors of size n for the given u_k. The first vector contains the information whether vertices $\{v_1, v_2, ..., v_n\}$ of the graph are reachable from the given vertex u_k and the second vector represents the same for the given vertex j. The element-wise product of two vectors can be calculated naively in time $O(n)$. Thus, the time complexity of the transitive closure can be reduced by speeding up element-wise product of two vectors of size n.

To achieve logarithmic speed-up, we use the Four Russians' trick. First we split each vector into $n/\log n$ parts of size $\log n$. Then we create a table S such that $S(a, b) = a \wedge b$ where $a, b \in \{0, 1\}^{\log n}$. This takes time $O(n^2 \log n)$, since there are $2^{\log n} = n$ variants of Boolean vectors of size $\log n$ and hence n^2 possible pairs of vectors (a, b) in total, and each component takes $O(\log n)$ time. With table S, we can calculate product of two parts of size $\log n$ in constant time. There are $n/\log n$ such parts, so the element-wise product of two vectors of size n can be calculated in time $O(n/\log n)$ with $O(n^2 \log n)$ preprocessing.

Theorem 3.1. *Let $\mathcal{G} = (V, E, L)$ be a graph and $G = \langle \Sigma, N, S, P \rangle$ be a grammar. Let $\mathcal{M}_2$ be a resulting adjacency matrix after the execution of the algorithm in Listing 1. Then for any valid indices i, j and for each nonterminal $A \in N$ the following statement holds: the cell $M^A_{2,(k)}[i, j]$ contains $\{1\}$, iff there is a path $i \pi j$ in the graph $\mathcal{G}$ such that $A \xrightarrow{*} l(\pi)$.*

Proof. The main idea of the proof is to use induction on the height of the derivation tree obtained on each iteration. □

Theorem 3.2. *Let $\mathcal{G} = (V, E, L)$ be a graph and $G = \langle \Sigma, N, S, P \rangle$ be a grammar. The algorithm from Listing 1 calculates resulting matrices $\mathcal{M}_2$ and C_3 in $O(n^3/\log n)$ time where $n = |V|$. Moreover, the maintaining of the dynamic transitive closure dominates the cost of the algorithm.*

Proof. Let $|\mathcal{A}|$ be a number of non-zero elements in a matrix $\mathcal{A}$. Consider the total time which is needed for computing the Kronecker products. The elements of the matrices

$\mathcal{A}_2^{(i)}$ are pairwise distinct on every i-th iteration of the algorithm therefore the total number of operations is

$$\sum_i T(\mathcal{M}_1 \otimes \mathcal{A}_2^{(i)}) = |\mathcal{M}_1| \otimes \sum_i |\mathcal{A}_2^{(i)}| = |\mathcal{M}_1| O(n^2).$$

Now we derive the time complexity of maintaining the dynamic transitive closure. Since C_3 has size of $O(n^2)$, no more than $O(n^2)$ edges will be added during all iterations of the algorithm. Checking condition whether $C_3[u, i] = 1$ and $C_3[u, j] = 0$ for every node $u \in V$ for each newly inserted edge (i, j) requires one multiplication of vectors per insertion, thus total time is $O(n^3/\log n)$. Note that after checking the condition, at least one element $C[u', j]$ changes value from 0 to 1 and then never becomes 0 for some u' and j. Therefore the operation $C_3[u', v] = C_3[u', v] \wedge C_3[j, v]$ for all $v \in V$ is executed at most once for every pair of vertices (u', j) during the entire computation implying that the total time is equal to $O(n^2 n/\log n) = O(n^3/\log n)$ (using multiplication of vectors).

The matrix C_3' contains only new elements, therefore C_3 can be updated directly using only $|C_3'|$ operations and hence $O(n^2)$ operations in total. The same holds for cycle in line 17 of the algorithm from Listing 1, because operations are performed only for non-zero elements of the matrix $|C_3'|$. Finally, the time complexity of the algorithm is $O(n^2) + O(n^2 \log n) + O(n^3/\log n) + O(n^2) + O(n^2) = O(n^3/\log n)$. □

The complexity analysis of the Algorithm 1 shows that the maintaining of the incremental transitive closure dominates the cost of the algorithm. Thus, CFPQ can be solved in truly subcubic $O(n^{3-\varepsilon})$ time if there is an incremental dynamic algorithm for the transitive closure for a graph with n vertices with preprocessing time $O(n^{3-\varepsilon})$ and total update time $O(n^{3-\varepsilon})$. Unfortunately, such an algorithm is unlikely to exist: it was proven that there is no incremental dynamic transitive closure algorithm for a graph with n vertices and at most m edges with preprocessing time $poly(m)$, total update time $mn^{1-\varepsilon}$, and query time $m^{\delta-\varepsilon}$ for any $\delta \in (0, 1/2]$ per query that has an error probability of at most 1/3 assuming the widely believed Online Boolean Matrix-Vector Multiplication (OMv) Conjecture [23]. OMv Conjecture introduced by Henzinger et al. [23] states that for any constant $\varepsilon > 0$, there is no $O(n^{3-\varepsilon})$-time algorithm that solves OMv with an error probability of at most 1/3.

3.1.2 Index creation for RPQ. In case of the RPQ, the main **while** loop takes only one iteration to actually append data. Since the input query is provided in the form of the regular expression, one can construct the corresponding RSM, which consists of the single *component state machine*. This CSM is built from the regular expression and is labeled as S, for example, which has no *recursive calls*. The adjacency matrix of the machine is build over Σ only. Therefore, calculating the Kronecker product, all relevant information is taken into account at the first iteration of the loop.

3.2 Paths Extraction Algorithm

After the index has been created, one can enumerate all paths between specified vertices. The index stores information about all reachable pairs for all nonterminals. Thus, the most natural way to use this index is to query paths between the specified vertices derivable from the specified nonterminal.

To do so, we provide a function getPaths(v_s, v_f, N), where v_s is a start vertex of the graph, v_f — the final vertex, and N is a nonterminal. Implementation of this function is presented in Listing 2.

Listing 2 Paths extraction algorithm

```
1:  C_3 ← result of index creation algorithm: final transitive closure
2:  M_1 ← the set of adjacency matrices of the input RSM
3:  M_2 ← the set of adjacency matrices of the final graph
4:  function getPaths(v_s, v_f, N)
5:      q_N^0 ← Start state of automata for N
6:      F_N ← Final states of automata for N
7:      res ← ⋃_{f∈F_N} getPathsInner((q_N, v_s), (f, v_f))
8:      return res
9:  function getSubpaths((s_i, v_i), (s_j, v_j), (s_k, v_k))
10:     l ← {(v_i, t, v_k) | M_2^t[s_i, s_k] ∧ M_1^t[v_i, v_k]}
              ∪ ⋃_{N|M_2^N[s_i,s_k]} getPaths(v_i, v_k, N)
              ∪ getPathsInner((s_i, v_i), (s_k, v_k))
11:     r ← {(v_k, t, v_j) | M_2^t[s_k, s_j] ∧ M_1^t[v_k, v_j]}
              ∪ ⋃_{N|M_2^N[s_k,s_j]} getPaths(v_k, v_j, N)
              ∪ getPathsInner((s_k, v_k), (s_j, v_j))
12:     return l · r
13: function getPathsInner((s_i, v_i), (s_j, v_j))
14:     parts ← {(s_k, v_k) | C_3[(s_i, v_i), (s_k, v_k)] = 1 ∧
        C_3[(s_k, v_k), (s_j, v_j)] = 1}
15:     return ⋃_{(s_k,v_k)∈parts} getSubpaths((s_i, v_i), (s_j, v_j), (s_k, v_k))
```

Paths extraction is implemented as three mutually recursive functions. The entry point is getPaths(v_s, v_f, N). This function returns a set of the paths between v_s and v_f such that the word formed by a path is derivable from the nonterminal N.

To compute such paths, it is necessary to compute paths from vertices of the form (q_N^s, v_s) to vertices of the form (q_N^f, v_f) in the result of transitive closure, where q_N^s is an initial state of RSM for N and q_N^f is a final state. The function getPathsInner$((s_i, v_i), (s_j, v_j))$ is used to do it. This function finds all possible vertices (s_k, v_k) which split a path from (s_i, v_i) to (s_j, v_j) into two subpaths. After that, function getSubpaths$((s_i, v_i), (s_j, v_j), (s_k, v_k))$ computes the corresponding subpaths. Each subpath may be at least a single

edge. If single-edge subpath is labeled by terminal then corresponding edge should be added to the result else (label is nonterminal) getPaths should be used to restore paths. If subpath is longer then one edge, getPaths should be used to restore paths.

It is assumed that the sets are computed lazily, so as to ensure termination in the case of an infinite number of paths. We also do not check paths for duplication manually, since they are assumed to be represented as sets.

4 IMPLEMENTATION DETAILS

Currenly, our goal is to evaluate the applicability of the proposed algorithm, thus we implemented its naive version. We compute the transitive closure from scratch on each iteration and do not use any incremental techniques. In our implementation we use PyGraphBLAS[2] — a Python wrapper for SuiteSparse library [13][3]. SuiteSparse is a C implementation of GraphBLAS [27] standard which introduces linear algebra building blocks for implementation of graph analysis algorithms. Thus we provide a highly-optimized parallel CPU implementation of the naive version of the algorithm[4].

At present, we do not integrate with a graph database and a graph query language. We suppose that the input graph is stored in a file, while the query is expressed in terms of a context-free grammar and is also stored in file. As it was shown in [46], it is possible to integrate SuiteSparse based implementation in the RedisGraph database. Providing integration with a query language requires a lot of technical effort to extend the language. There are existing proposals, for example to extend the Cypher language[5].

Paths extraction is implemented in Python by using PyGraphBLAS. Since lazy evaluation is not natural for Python, we cap the maximal number of paths to extract in the implementation.

5 EVALUATION

The goal of this evaluation was to investigate the applicability of the proposed algorithm to both regular and context-free path querying. We measured the execution time of the index creation, which solves the reachability problem, for both kinds of queries. The execution time for CFPQ was compared with the Azimov's algorithm for CFPQ reachability. We also investigated the practical applicability of paths extraction algorithm for both regular and context-free path queries.

For evaluation, we used a PC with Ubuntu 18.04 installed. It has Intel core i7-6700 CPU, 3.4GHz, and DDR4 64Gb RAM. We only measure the execution time of the algorithms themselves, thus we assume an input graph is loaded into RAM in the form of its adjacency matrix in the sparse format. Note, that the time needed to load an input graph into the RAM is excluded from the time measurements.

Table 1: Graphs for RPQ evaluation

Graph	#V	#E
LUBM1k	120 926	484 646
LUBM3.5k	358 434	144 9711
LUBM5.9k	596 760	2 416 513
LUBM1M	1 188 340	4 820 728
LUBM1.7M	1 780 956	7 228 358
LUBM2.3M	2 308 385	9 369 511
Uniprotkb	6 442 630	24 465 430
Proteomes	4 834 262	12 366 973
Taxonomy	5 728 398	14 922 125
Geospecies	450 609	2 201 532
Mappingbased_properties	8 332 233	25 346 359

5.1 RPQ Evaluation

To investigate the applicability of the proposed algorithm for regular path querying we gathered a dataset which consists of both real-world and synthetically generated graphs. We generated the queries from the most popular RPQ templates.

5.1.1 Dataset. We gathered several graphs which represent real-world data from different areas and are frequently used for evaluation of the graph querying algorithms. Namely, the dataset consists of three parts. The first part is the set of LUBM graphs[6] [17] which have different numbers of vertices. The second one is the set of graphs from Uniprot database[7]: *proteomes*, *taxonomy* and *uniprotkb*. The last part consists of the RDF files *mappingbased_properties* from DBpedia[8] and *geospecies*[9]. A brief description of the graphs in the dataset is presented in Table 1.

Queries for evaluation were generated from the templates for the most popular RPQs, specifically the queries presented

[2]GitHub repository of PyGraphBLAS, a Python wrapper for GraphBLAS API: https://github.com/michelp/pygraphblas. Access date: 07.07.2020.
[3]Web page of SuiteSparse:GraphBLAS library: http://faculty.cse.tamu.edu/davis/GraphBLAS.html. Access date: 07.07.2020.
[4]Implementation of the described algorithm is published here: https://github.com/JetBrains-Research/CFPQ_PyAlgo. Access date: 07.07.2020.
[5]Cypher language extension proposal which introduces a syntax to express context-free queries: https://github.com/thobe/openCypher/blob/rpq/cip/1.accepted/CIP2017-02-06-Path-Patterns.adoc. Access date: 07.07.2020.
[6]Lehigh University Benchmark (LUBM) web page: http://swat.cse.lehigh.edu/projects/lubm/. Access date: 07.07.2020.
[7]Universal Protein Resource (UniProt) web page: https://www.uniprot.org/. All files used can be downloaded via the link: ftp://ftp.uniprot.org/pub/databases/uniprot/current_release/rdf/. Access date: 07.07.2020.
[8]DBpedia project web site: https://wiki.dbpedia.org/. Access date: 07.07.2020.
[9]The Geospecies RDF: https://old.datahub.io/dataset/geospecies. Access date: 07.07.2020.

Table 2: Queries templates for RPQ evaluation

Name	Query	Name	Query
Q_1	a^*	Q_9^5	$(a \mid b \mid c \mid d \mid e)^+$
Q_2	$a \cdot b^*$	Q_{10}^2	$(a \mid b) \cdot c^*$
Q_3	$a \cdot b^* \cdot c^*$	Q_{10}^3	$(a \mid b \mid c) \cdot d^*$
Q_4^2	$(a \mid b)^*$	Q_{10}^4	$(a \mid b \mid c \mid d) \cdot e^*$
Q_4^3	$(a \mid b \mid c)^*$	Q_{10}^5	$(a \mid b \mid c \mid d \mid e) \cdot f^*$
Q_4^4	$(a \mid b \mid c \mid d)^*$	Q_{10}^2	$a \cdot b$
Q_4^5	$(a \mid b \mid c \mid d \mid e)^*$	Q_{11}^3	$a \cdot b \cdot c$
Q_5	$a \cdot b^* \cdot c$	Q_{11}^4	$a \cdot b \cdot c \cdot d$
Q_6	$a^* \cdot b^*$	Q_{11}^5	$a \cdot b \cdot c \cdot d \cdot f$
Q_7	$a \cdot b \cdot c^*$	Q_{12}	$(a \cdot b)^+ \mid (c \cdot d)^+$
Q_8	$a? \cdot b^*$	Q_{13}	$(a \cdot (b \cdot c)^*)^+ \mid (d \cdot f)^+$
Q_9^2	$(a \mid b)^+$	Q_{14}	$(a \cdot b \cdot (c \cdot d)^*)^+ \cdot (e \mid f)^*$
Q_9^3	$(a \mid b \mid c)^+$	Q_{15}	$(a \mid b)^+ \cdot (c \mid d)^+$
Q_9^4	$(a \mid b \mid c \mid d)^+$	Q_{16}	$a \cdot b \cdot (c \mid d \mid e)$

in Table 2 in [39] and in Table 5 in [51]. These query templates are presented in Table 2. We generate 10 queries for each template and each graph. The most frequent relations from the given graph were used as symbols in the query template[10]. We used the same set of queries for all LUBM graphs to investigate scalability of the proposed algorithm.

5.1.2 Results. We averaged the execution time of index creation over 5 runs for each query. Index creation time for LUBM graphs set is presented in Figure 4. We can see that evaluation time depends on the query: there are queries which evaluate in less than 1 second even for the largest graphs (Q_2, Q_5, Q_{11}^2, Q_{11}^3), while the worst time is 6.26 seconds (Q_{14}). The execution time of our algorithm is comparable with the recent results for the same graphs and queries implemented on a distributed system over 10 nodes [51], while we use only one node. We conclude that our algorithm demonstrates reasonable performance to be applied for real-world data analysis.

Index creation time for each query on the real-world graphs is presented in Figure 5. We can see that in some cases querying small graphs requires more time than querying bigger graphs. For example, conseder Q_{10}^4: querying the *geospecies* graph (450k vertices) in some cases requires more time than querying of *mappingbased_properties* (8.3M vertices) and *taxonomy* (5.7M vertices). We conclude that the evaluation time depends on the inner structure of a graph. On the other hand, *taxonomy* querying in many cases requires significantly more time, than for other graphs, while

[10]Used generator is available as part of CFPQ_data project: https://github.com/JetBrains-Research/CFPQ_Data/blob/master/tools/gen_RPQ/gen.py. Access data: 07.07.2020.

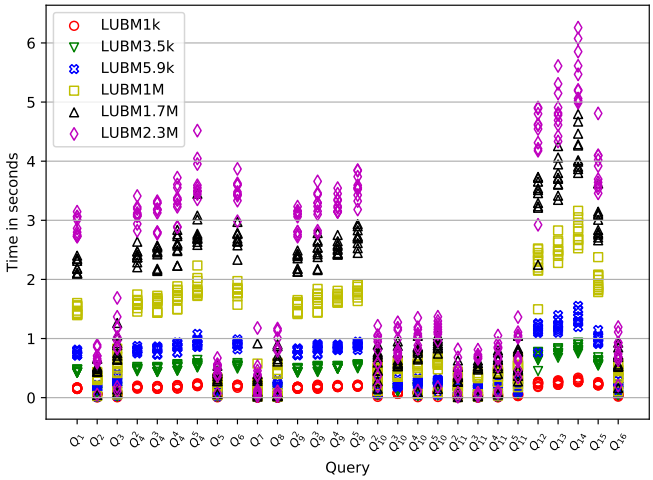


Figure 4: Index creation time for LUBM graphs

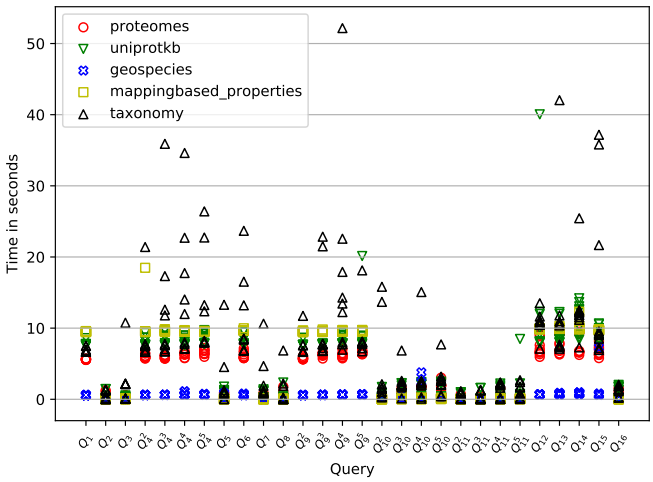
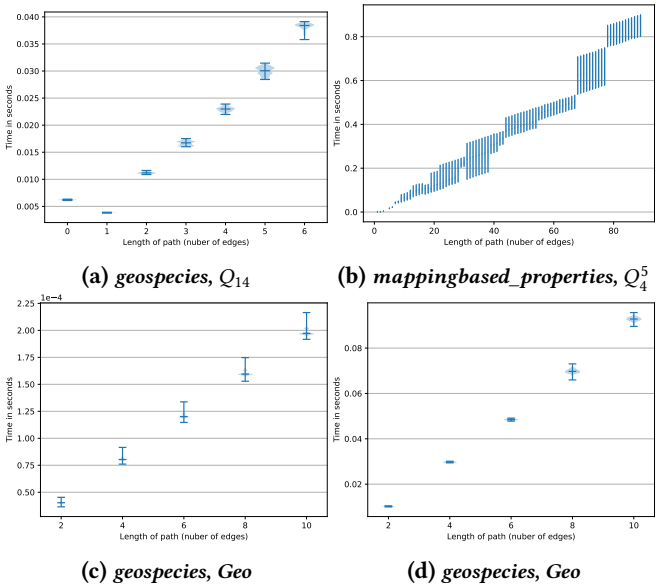


Figure 5: Index creation time for real-world RDFs

taxonomy is not the biggest graph. Finally, in most cases query execution lasts less than 10 seconds, even for bigger graphs, and no query requires more than 52.17 seconds.

We evaluate path extraction for queries which result in possibly long paths. Long paths usually require many iterations of transitive closure evaluation, thus we used the number of the iterations as a criterion to select the inputs for the evaluation. For each selected graph and query we measure paths extraction time for each reachable pair. Since the index can be reused from the previous step, we omit the time necessary to create the index. We limit by 10 the number of paths to extract.

In Figures 6a and 6b we show the time needed to extract a path of a specific length when only one path was extracted. The main observation is that time is linear on the path length, even if a generic path extraction procedure is used.



(a) *geospecies*, Q_{14} (b) *mappingbased_properties*, Q_4^5

(c) *geospecies*, *Geo* (d) *geospecies*, *Geo*

Figure 6: Single path extraction for specific graph and query for our solution (a, b, d), and Azimov's (c)

5.2 CFPQ Evaluation

We evaluate the applicability of the proposed algorithm for CFPQ processing over real-world graphs on a number of classical cases and compare them with the Azimov's algorithm. Currently only a single path version of Azimov's algorithm exists, and we use its implementation using PyGraphBLAS.

5.2.1 Dataset. We use CFPQ_Data[11] dataset for evaluation. Namely, we use relatively big RDF files and respective same-generation queries G_1 (Eq. 1) and G_2 (Eq. 2) which are used in other works for CFPQ evaluation. We also use the *Geo* (Eq. 3) query provided by J. Kuijpers et. al [30] for *geospecies* RDF. Note that we use $\overline{x}$ notation in queries to denote the inverse of x relation and the respective edge.

$$\begin{aligned} S \rightarrow & \overline{subClassOf}\ S\ subClasOf \mid \overline{type}\ \ S\ type \\ & \mid \overline{subClassOf}\ \ subClasOf \mid \overline{type}\ type \end{aligned} \tag{1}$$

$$S \rightarrow \overline{subClassOf}\ S\ subClasOf \mid subClassOf \tag{2}$$

$$\begin{aligned} S \rightarrow & broaderTransitive\ \ S\ \overline{broaderTransitive} \\ & \mid broaderTransitive\ \ \overline{broaderTransitive} \end{aligned} \tag{3}$$

$$\begin{aligned} & S \rightarrow \overline{d}\ V\ d \\ & V \rightarrow ((S?)\overline{a})^*(S?)(a(S?))^* \end{aligned} \tag{4}$$

[11] CFPQ_Data is a dataset for CFPQ evaluation which contains both synthetic and real-world data and queries https://github.com/JetBrains-Research/CFPQ_Data. Access date: 07.07.2020.

Table 3: Graphs for CFPQ evaluation: *bt* is broader-Transitive, *sco* is subCalssOf

Graph	#V	#E	#sco	#type	#bt	#a	#d
eclass_514en	239 111	523 727	90 512	72 517	—	—	—
enzyme	48 815	109 695	8 163	14 989	—	—	—
geospecies	450 609	2 201 532	0	89 062	20 867	—	—
go	272 770	534 311	90 512	58 483	—	—	—
go-hierarchy	45 007	980 218	490 109	0	—	—	—
taxonomy	5 728 398	14 922 125	2 112 637	2 508 635	—	—	—
arch	3 448 422	5 940 484	—	—	—	671 295	2 298 947
crypto	3 464 970	5 976 774	—	—	—	678 408	2 309 979
drivers	4 273 803	7 415 538	—	—	—	858 568	2 849 201
fs	4 177 416	7 218 746	—	—	—	824 430	2 784 943

Table 4: CFPQ evaluation results, time is measured in seconds

Name	G_1		G_2		*Geo*		*MA*	
	Tns	Mtx	Tns	Mtx	Tns	Mtx	Tns	Mtx
eclass_514en	0.25	0.27	0.23	0.26	—	—	—	—
enzyme	0.04	0.04	0.04	0.01	—	—	—	—
geospecies	0.09	0.06	0.01	0.01	34.12	16.58	—	—
go-hierarchy	0.19	1.43	0.29	0.86	—	—	—	—
go	1.68	1.74	1.37	1.14	—	—	—	—
pathways	0.02	0.01	0.01	0.01	—	—	—	—
taxonomy	5.37	2.71	3.28	1.56	—	—	—	—
arch	—	—	—	—	—	—	390.05	195.51
crypto	—	—	—	—	—	—	395.98	195.54
drivers	—	—	—	—	—	—	2114.16	1050.78
fs	—	—	—	—	—	—	745.97	370.73

Additionally we evaluate our algorithm on memory aliases analysis problem: a well-known problem which can be reduced to CFPQ [54]. To do it, we use some graphs built for different parts of Linux OS kernel (*arch*, *crypto*, *drivers*, *fs*) and the query *MA* (Eq. 4) [50]. The detailed data about all the graphs used is presented in Table 3.

5.2.2 Results. We averaged the index creation time over 5 runs for both Azimov's algorithm (**Mtx**) and the proposed algorithm (**Tns**) (see Table 4).

We can see that while in some cases our solution is comparable or just slightly better than Azimov's algorithm (*enzyme*, *eclass_514en*, *go*), there are cases when our solution is significantly faster (*go-hierarchy*, up to 9 times faster), and when Azimov's algorithm about 2 times faster (all memory aliases and *geospecies* with *Geo* query). Thus we can conclude that our solution is competitive with Azimov's algorithm, and a detailed analysis of different cases is required.

Comparison of paths extraction is presented in Figures 6c and 6d. While both methods demonstrate linear time on the length of the extracted path, our generic solution is more than 1000 times slower than Azimov's single path extraction procedure. We conclude that current generic all-path extraction procedure is not optimal for single path extraction.

5.3 Conclusion

We conclude that the proposed algorithm is applicable for real-world data processing: the algorithm allows one to solve both the reachability problem and to extract paths of interest in a reasonable time even using naive implementation. While index creation time (reachability query evaluation) is comparable with other existing solutions, paths extraction procedure should be improved in the future. Finally, a detailed comparison of the proposed solution with other algorithms for CFPQ and RPQ is required.

To summarize the overall evaluation, the proposed algorithm is applicable for both RPQ and CFPQ over real-world graphs. Thus, the proposed solution is a promising unified algorithm for both RPQ and CFPQ evaluation.

6 RELATED WORK

Language constrained path querying is widely used in graph databases, static code analysis, and other areas. Both, RPQ and CFPQ (known as CFL reachability problem in static code analysis) are actively studied in the recent years.

There is a huge number of theoretical research on RPQ and its specific cases. RPQ with single-path semantics was investigated from the theoretical point of view by Barrett et al. [5]. In order to research practical limits and restrictions of RPQ, a number of high-performance RPQ algorithms were provided. For example, derivative-based solution provided by Maurizio Nolé and Carlo Sartiani which is implemented on the top of Pregel-based system [37], or solution of André Koschmieder et al. [28]. But only a limited number of practical solutions provide the ability to restore paths of interest. A recent work of Xin Wang et al. [51] provides a Pregel-based provenance-aware RPQ algorithm which utilizes a Glushkov's construction [15]. There is a lack of research of the applicability of linear algebra-based RPQ algorithms with paths-providing semantics.

On the other hand, many CFPQ algorithms with various properties were proposed recently. They employ the ideas of different parsing algorithms, such as CYK in works of Jelle Hellings [20] and Phillip Bradford [6], (G)LR and (G)LL in works of Ekaterina Verbitskaia et al. [49], Semyon Grigorev et al. [16], Fred Santos et al. [42], Ciro Medeiros et al. [33]. Unfortunately, none of them has better than cubic time complexity in terms of the input graph size. The algorithm of Azimov [3] is, best to our knowledge, the first algorithm for CFPQ based on linear algebra. It was shown by Arseniy Terekhov et al. [46] that this algorithm can be applied for real-world graph analysis problems, while Jochem Kuijpers et al. shows in [30] that other state-of-the-art CFPQ algorithms are not performant enough to handle real-world graphs.

It is improtant in both RPQ and CFPQ to be able to restore paths of interest. Some of the mentioned algorithms can solve only the reachability problem, while it may be important to provide at least one path which satisfies the query. While Arseniy Terekhov et al. [46] provide the first CFPQ algorithm with single path semantics based on linear algebra, Jelle Hellings in [22] provides the first theoretical investigation of this problem. He also provides an overview of the related works and shows that the problem is related to the string generation problem and respective results from the formal language theory. He concludes that both theoretical and empirical investigation of CFPQ with single-path and all-path semantics are at early stage. We agree with this point of view, and we only demonstrate applicability of our solution for paths extraction and do not investigate its properties in details.

While CFPQ on n-node graph has a relatively straightforward $O(n^3)$ time algorithm, it is a long-standing open problem whether there is a truly subcubic $O(n^{3-\varepsilon})$ algorithm for this problem. The question on the existence of a subcubic CFPQ algorithm was stated by Mihalis Yannakakis in 1990 in [52]. A bit later Thomas Reps proposed the CFL reachability as a framework for interprocedural static code analysis [41]. Melski and Reps gave a dynamic programming formulation of the problem which runs in $O(n^3)$ time [34]. The problem of the cubic bottleneck of context-free language reachability is also discussed by Heintze and McAllester [19], and Melski and Reps [34]. The slightly subcubic algorithm with $O(n^3/\log n)$ time complexity was provided by Swarat Chaudhuri in [11]. This result is inspired by recursive state machine reachability. The first truly subcubic algorithm with $O(n^\omega polylog(n))$ time complexity (ω is the best exponent for matrix multiplication) for an arbitrary graph and 1-Dyck language was provided by Phillip Bradford in [6], and Andreas Pavlogiannis and Anders Alnor Mathiasen in [40]. Other partial cases were investigated by Krishnendu Chatterjee et al. in [9], Qirun Zhang in [53].

The utilization of linear algebra is a promising way to high-performance graph analysis. There are many works on specific graph algorithm formulation in terms of linear algebra, for example, classical algorithms for transitive closure and all-pairs shortest paths. Recently this direction was summarized in GrpahBLAS API [27] which provides building blocks to develop a graph analysis algorithm in terms of linear algebra. There is a number of implementations of this API, such as SuiteSparse:GraphBLAS [13] or CombBLAS [7]. Approaches to evaluate different classes of queries in different systems based on linear algebra is being actively researched. This approach demonstrates significant performance improvement when applied for SPARQL queries evaluation [26, 35] and for Datalog queries evaluation [43]. Finally, RedisGraph [8], a linear-algebra powered graph database, was created and it was shown that in some scenarios it outperforms many other graph databases.

7 CONCLUSION AND FUTURE WORK

In this work we present an improved version of the tensor-based algorithm for CFPQ: we reduce the algorithm to operations over Boolean matrices, and we provide the ability to extract all paths which satisfy the query. Moreover, the provided algorithm can handle grammars in EBNF, thus it does not require CNF transformation of the grammar and avoids grammar explosion. As a result, the algorithm demonstrates practical performance not only on CFPQ queries but also on RPQ ones, which is shown by our evaluation. Thus, we provide a universal linear algebra based algorithm for RPQ and CFPQ evaluation with all-paths semantics. Moreover our algorithm opens a way to tackle the long-standing problem of subcubic CFPQ by reducing it to incremental transitive closure: incremental transitive closure with $O(n^{3-\varepsilon})$ total update time for n^2 updates, such that each update returns all of the new reachable pairs, implies $O(n^{3-\varepsilon})$ CFPQ algorithm. We prove $O(n^3/\log n)$ time complexity by providing $O(n^3/\log n)$ incremental transitive closure algorithm.

Recent hardness results for dynamic graph problems demonstrates that any further improvement for incremental transitive closure (and, hence, CFPQ) will imply a major breakthrough for other long-standing dynamic graph problems. An algorithm for incremental dynamic transitive closure with total update time $O(mn^{1-\varepsilon})$ (n denotes the number of graph vertices, m is the number of graph edges) even with polynomial $poly(n)$ time preproccessing of the input graph and $m^{\delta-\varepsilon}$ query time per query for any $\delta \in (0, 1/2]$ will refute the Online Boolean Matrix-Vector Multiplication (OMv) Conjecture, which is used to prove conditional lower bounds for many dynamic problems [23, 47].

Thus, the first task for the future is to improve the logarithmic factor in the obtained bound. It is also interesting to improve bounds in partial cases for which dynamic transitive closure can be supported faster than in general case, for example, planar graphs [45], undirected graph and others. In the case of planarity, it is interesting to investigate properties of the input graph and grammar which allow us to preserve planarity during query evaluation.

An important task for future research is a detailed investigation of the paths extraction algorithm. Jelle Hellings in [22] provides a theoretical investigation of single-path extraction and shows that the problem is related to the formal language theory. Extraction of all paths is more complicated and should be investigated carefully in order to provide an optimal algorithm.

From the practical perspective, it is necessary to analyze the usability of advanced algorithms for dynamic transitive closure. In the current work, we evaluate naive implementation in which transitive closure is recalculated from scratch on each iteration. It is shown in [18] that some advanced algorithms for dynamic transitive closure can be efficiently implemented. Can one of these algorithms be efficiently parallelized and utilized in the proposed algorithm?

Also, it is necessary to evaluate GPGPU-based implementation. Evaluation of Azimov's algorithm shows that it is possible to improve performance by using GPGPU because operations of linear algebra can be efficiently implemented on GPGPU [36, 46]. Moreover, for practical reason, it is interesting to provide a multi-GPU version of the algorithm and to utilize unified memory, which is suitable for linear algebra based processing of out-of-GPGPU-memory data and traversing on large graphs [12, 14].

In order to simplify the distributed processing of huge graphs, it may be necessary to investigate different formats for sparse matrices, such as HiCOO format [31]. Another interesting question in this direction is about utilization of virtualization techniques: should we implement distributed version of algorithm manually or it can be better to use CPU and RAM virtualization to get a virtual machine with huge amount of RAM and big number of computational cores. The experience of the Trinity project team shows that it can make sense [44].

Finally, it is necessary to provide a multiple-source version of the algorithm and integrate it with a graph database. RedisGraph[12] [8] is a suitable candidate for this purpose. This database uses SuiteSparse—an implementation of GraphBLAS standard—as a base for graph processing. This fact allowed to Arseny Terkhov et al. to integrate Azimov's algorithm to RedisGraph with minimal effort [46].

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