Relational Interpreters for Search Problems

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Recognition vs. Search

X — alphabet

$L \subseteq X^*$

if $\omega \in L$, denote the *witness* of this fact p_ω

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$$X \text{ — alphabet}$$

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$$\textit{if}\ \omega \in L, \text{ denote the } \textit{witness} \text{ of this fact } p_\omega$$

$$\text{Recognition:} \quad V(\omega, p_\omega) = \begin{cases} 1, & \omega \in L \\ 0, & \omega \notin L \end{cases}$$

Recognition vs. Search

$$X \text{ — alphabet}$$

$$L \subseteq X^*$$

$$\textit{if}\, \omega \in L, \text{ denote the } \textit{witness} \text{ of this fact } p_\omega$$

$$\text{Recognition:} \quad V(\omega, p_\omega) = \begin{cases} 1, & \omega \in L \\ 0, & \omega \notin L \end{cases}$$

$$\text{Search:} \quad S(\omega) = p_\omega$$

Propositional Formulas: Recognition

```
let rec eval st = function
| Conj (l, r) → eval st l && eval st r
| Disj (l, r) → eval st l || eval st r
| Neg   e     → not (eval st e)
| Var   x     → List.assoc x st
```

Propositional Formulas: Recognition

```
let rec eval st = function
| Conj (l, r) → eval st l && eval st r
| Disj (l, r) → eval st l || eval st r
| Neg   e     → not (eval st e)
| Var   x     → List.assoc x st
```

```
# eval [(`x,true);(`y,false)] (Conj (Var `x) (Neg (Var `y)));;

- : bool = true
```

Propositional Formulas: Search

```
let rec solve st b = function
| Var n → ( match assoc_opt n st with
            | None → [extend st n b]
            | Some b' when b == b' → [st]
            | _     → [])
| Conj (l, r) when b →
    concat @@
    map (λ st → solve st b r) @@
    solve st b l
| Conj (l, r) → solve st b l @ solve st b r
| Neg e → solve st (not b) e
| Disj (l, r) → solve st b (Neg (Conj (Neg l, Neg r)))
```

Search is Hard[1]

Is it possible to generate a search procedure from a recognizer?

[1] compared to recognition

Relational Interpreter

$$V^R(\omega, p_\omega, q)$$

$$V^R(\omega, p_\omega, 1), \quad \mathit{if}\, \omega \in L, p_\omega \text{ — a witness}$$

$$V^R(\omega, p_\omega, 0), \quad \mathit{otherwise}$$

Relational Interpretation for Recognition and Search

$$V^R(\omega, p_\omega, ?) \quad \rightsquigarrow \quad V(\omega, p_\omega)$$

$$V^R(\omega, ?, 1) \quad \rightsquigarrow \quad S(\omega)$$

Only one program to implement!

Propositional Formulas: Relational Interpreter

```
let rec evalᵒ st fm u = ocanren (
 fresh x, y, z, v, w in
  fm == conj x y & evalᵒ st x v & evalᵒ st y w & andᵒ v w u |
  fm == disj x y & evalᵒ st x v & evalᵒ st y w & orᵒ  v w u |
  fm == neg  x   & evalᵒ st x v & notᵒ v u |
  fm == var  z   & assocᵒ z st u
 )
```

Relational Programming is Hard[2]

```
let rec hanoiᵒ a b c moves a' b' c' = ocanren (
 moves == [] & a == a' & b == b' & c == c' |
 fresh f, t, moves', pin_f, pin_t, pin_f_res, pin_t_res, a'', b'', c'' in
   moves == (f, t) :: moves' &
   (f == A & t == B & pin_f == a & pin_f_res == a'' & pin_t == b & pin_t_res == b'' & c'' == c |
    f == A & t == C & pin_f == a & pin_f_res == a'' & pin_t == c & pin_t_res == c'' & b'' == b |
    f == B & t == A & pin_f == b & pin_f_res == b'' & pin_t == a & pin_t_res == a'' & c'' == c |
    f == B & t == C & pin_f == b & pin_f_res == b'' & pin_t == c & pin_t_res == c'' & a'' == a |
    f == C & t == A & pin_f == c & pin_f_res == c'' & pin_t == a & pin_t_res == a'' & b'' == b |
    f == C & t == B & pin_f == c & pin_f_res == c'' & pin_t == b & pin_t_res == b'' & a'' == a) &
   fresh top_f, rest_f in
     pin_f == top_f :: rest_f &
     (pin_t == [] |
      fresh top_t, rest_t in
        pin_t == top_t :: rest_t & ltᵒ top_f top_t true
     ) &
     pin_f_res == rest_f &
     pin_t_res == top_f :: pin_t &
     hanoiᵒ a'' b'' c'' moves' a' b' c'
)
```

[2] compared to functional programming

Relational Programming is Hard[2]

```
let rec hanoi^o a b c moves a' b' c' = ocanren (
 moves == [] & a == a' & b == b' & c == c' |
 fresh f, t, moves', pin_f, pin_t, pin_f_res, pin_t_res, a'', b'', c'' in
   moves == (f, t) :: moves' &
   (f == A & t == B & pin_f == a & pin_f_res == a'' & pin_t == b & pin_t_res == b'' & c'' == c |
    f == A & t == C & pin_f == a & pin_f_res == a'' & pin_t == c & pin_t_res == c'' & b'' == b |
    f == B & t == A & pin_f == b & pin_f_res == b'' & pin_t == a & pin_t_res == a'' & c'' == c |
    f == B & t == C & pin_f == b & pin_f_res == b'' & pin_t == c & pin_t_res == c'' & a'' == a |
    f == C & t == A & pin_f == c & pin_f_res == c'' & pin_t == a & pin_t_res == a'' & b'' == b |
    f == C & t == B & pin_f == c & pin_f_res == c'' & pin_t == b & pin_t_res == b'' & a'' == a) &
   fresh top_f, rest_f in
     pin_f == top_f :: rest_f &
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     ) &
     pin_f_res == rest_f &
     pin_t_res == top_f :: pin_t &
     hanoi^o a'' b'' c'' moves' a' b' c'
)
```

It took 3 people 6 hours to implement

[2] compared to functional programming

Ways to Create Relational Interpreters

- Manual implementation
- Interpretation of functional programs with a relational interpreter
- Relational conversion

Ways to Create Relational Interpreters

- Manual implementation
- **Interpretation of functional programs with a relational interpreter**
- Relational conversion

Relational Interpretation of Functional Programs

- Implement good relational interpreter of a Turing-complete language
- Implement functional recognizer
- Run functional recognizer with a relational interpreter

Interpretation Overhead

Running relational interpreter comes with a price

Are there ways to get rid of it?

Specialization

Interpreter:

```
eval prog input == output
```

Specialization

Interpreter:

`eval prog input == output`

Consider that a part of the input is known: `input == (static, dynamic)`

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`eval prog input == output`

Consider that a part of the input is known: `input == (static, dynamic)`

Specializer:

$$\texttt{spec prog static} \Rightarrow \texttt{prog}_{spec}$$
$$\texttt{eval prog (static, dynamic)} == \texttt{eval prog}_{spec}\ \texttt{dynamic}$$

Jones-Optimality

- Specializers can fail to remove all interpretation overhead
- Jones-optimal specializer: the specialized interpreter is as efficient as the program it was specialized for
- There exists a Jones-optimal specializer for a logical language [Leuschel, 2004]
- Not for miniKanren
- Jones-optimality is hard to achieve

Ways to Create Relational Interpreters

- Manual implementation
- Interpretation of functional programs with a relational interpreter
- **Relational conversion**

Relational Conversion for Relational Interpreter

- Implement a functional recognizer (verifier): $V(\omega, p_\omega)$
- Transform it into a relation: $V^R(\omega, p_\omega, q)$
- Specialize
  - Redundancy introduced with the relational conversion
  - Direction ($q == 1$)
  - Known data (ω)
- The result is a search routine

Relational Conversion (Unnesting) [Byrd 2009]

- Introduce a new variable for each subexpression
- For every n-ary function create an (n+1)-ary relation, where the last argument is unified with the result
- Transform **if**-expressions and pattern matchings into disjunctions with unifications for patterns
- Introduce into scope free variables (with **fresh**)
- Pop unifications to the top

Relational Conversion: Step 1

Introduce a new variable for each subexpression

```
let rec append a b =
  match a with
  | []       →  b
  | x :: xs  →
    x :: append xs b
```

```
let rec append a b =
  match a with
  | []       →  b
  | x :: xs  →
    let q = append xs b in
    x :: q
```

Relational Conversion: Step 2

Introduce a new argument for result

```
let rec append a b = ...
```

$\texttt{let rec append}^o\ \texttt{a b c = ...}$

Relational Conversion: Step 3

Transform **if**-expressions and pattern matchings into disjunctions with unifications for patterns

```
let rec append a b =
  match a with
  | []       →  b
  | x :: xs  →
    let q = append xs b in
    x :: q
```

```
let rec appendᵒ a b c =
  ocanren (
    (a ≡ [] & b ≡ c) |
    (  a ≡ x :: xs &
       appendᵒ xs b q &
       c ≡ x :: q))
```

Relational Conversion: Step 4

Introduce free variables into scope (with **fresh**)

```
let rec appendᵒ a b c =
  ocanren (
    (a == [] & b == c) |
    (  (a == x :: xs) &
       (appendᵒ xs b q) &
       (c == x :: q)))
```

```
let rec appendᵒ a b c =
  ocanren (
    (a == [] & b == c) |
    (fresh x, xs, q in
       a == x :: xs &
       appendᵒ xs b q &
       c == x :: q))
```

Pop unifications to the top

```
let rec appendᵒ a b c =
  ocanren (
    (a == [] & b == c) |
    (fresh x, xs, q in
       a == x :: xs &
       appendᵒ xs b q &
       c == x :: q))
```

```
let rec appendᵒ a b c =
  ocanren (
    (a == [] & b == c) |
    (fresh x, xs, q in
       a == x :: xs &
       c == x :: q &
       appendᵒ xs b q))
```

Forward Execution is Efficient, Backward Execution is not

Forward execution is efficient, since it mimics the execution of a function

Relational conversion for $\texttt{f}_1\ \texttt{x}_1$ && $\texttt{f}_2\ \texttt{x}_2$:

```
λ res → ocanren (
  fresh p in
     f₁ x₁ p &
     ( p == false & res == false |
       p == true  & f₂ x₂ res))
```

Computes $\texttt{f}_2\ \texttt{x}_2$ res only if $\texttt{f}_1\ \texttt{x}_1$ p fails

It is not the best strategy, if res is known

Relational Conversion Aimed at Backward Execution

This coversion of $\texttt{f}_1\ \texttt{x}_1$ && $\texttt{f}_2\ \texttt{x}_2$ is better for the backward execution, but not for forward

```
λ res → ocanren (
  res == false & f₁ x₁ false |
  f₁ x₁ true  & f₂ x₂ res)
```

There is no single strategy suitable for all cases

There is no Single Good Strategy

Is there a way to automatically generate relations efficient in the specified directions?

Specialization: Not Only for Direction

When solving a search problem, we know its search space

$$V^R(\omega, ?, 1) \rightsquigarrow S(\omega)$$

Partial Deduction: Specialization for Logic Language

- Given:
  - Logic program
  - Goal
- Result: specialized program
- How:
  - Construct a *partial* SLD-tree
  - Generate a program from the tree
- Hopefully, all excessive computations are done statically and do not come to the specialized program

Partial Deduction: Example

```
last([x|xs], r) ← l(xs, x, r).
l([], x, x).
l([z|zs], x, r) ← l(zs, z, r).

← last([A,B|xs], r).
```

Partial Deduction: Example

```
last([x|xs], r) ← l(xs, x, r).
l([], x, x).
l([z|zs], x, r) ← l(zs, z, r).

← last([A,B|xs], r).
```

$$last([A, B|xs], r)$$

$$l([B|xs], A, r)$$

$$l(xs, B, r)$$

$$\square \qquad l(zs', z', r)$$

Partial Deduction: Example

```
last([x|xs], r) ← l(xs, x, r).
l([], x, x).
l([z|zs], x, r) ← l(zs, z, r).

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```

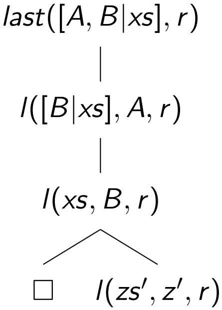


```
last([A,B], B).
last([A,B,z'|zs'], r) ← l(zs', z', r).
l([], x, x).
l([z|zs], x, r) ← l(zs, z, r).
```

Partial Deduction: Conjunctions

$$f(x, y) \vee f(y, z)$$

$$f(x, y) \qquad f(y, z)$$

$\checkmark$

Partial Deduction: Conjunctions

$$\begin{array}{ccc} & f(x,y) \lor f(y,z) & \\ & \swarrow \quad \searrow & \\ f(x,y) & & f(y,z) \end{array}$$

$$\begin{array}{ccc} & f(x,y) \land f(y,z) & \\ & \swarrow \quad \searrow & \\ f(x,y) & & f(y,z) \end{array}$$

✓ X

Conjunctive Partial Deduction

- Fully automatic program transformation
- For pure logic language
- Features:
  - Specialization
  - Deforestation
  - Tupling

Deforestation

Deforestation — program transformation which eliminates intermediate data structures

```
let double_appendᵒ x y z xyz =
  ocanren (
    fresh t in
      appendᵒ x y t &
      appendᵒ t z xyz )

let rec appendᵒ x y xy =
  ocanren (
    x == [] & xy == y |
    fresh h, t, ty in
      x  == h :: t  &
      xy == h :: t' &
      appendᵒ t y t')
```

```
let rec double_appendᵒ x y z xyz =
  ocanren (
    x == [] & appendᵒ y z xyz |
    (fresh h, t, t' in
       x == h :: t  &
       xyz == h :: t' &
       double_appendᵒ t y z t')
```

Tupling

Tupling — program transformation which eliminates multiple traversals of the same data structure

```
let max_lengthᵒ xs m l = ocanren (maxᵒ xs m & lengthᵒ xs l)

let rec lengthᵒ xs l = ocanren (
  xs == [] & l == 0 |
  (fresh h, t, m in
    xs == h :: t & l == succ m & lengthᵒ t m))

let maxᵒ xs m = max₁ᵒ xs 0 m

let rec max₁ᵒ xs n m = ocanren (
  xs == [] & m == n |
  (fresh h, t in
    xs == h :: t &
    ( leᵒ h n true & max₁ᵒ t n m |
      gtᵒ h n true & max₁ᵒ t h m)))
```

Tupling

Tupling — program transformation which eliminates multiple traversals of the same data structure

```
let max_lengthᵒ xs m l = ocanren (max_length₁ᵒ xs m 0 l)

let rec max_length₁ᵒ xs m n l = ocanren (
  xs == [] & m == n & l == 0 |
  (fresh h, t, l₁ in
     xs == h :: t &
     l == succ l₁ &
     ( leᵒ h n & max_length₁ᵒ t m n l |
       gtᵒ h n & max_length₁ᵒ t m h l)))
```

CPD: Intuition

- Local control: compute a partial SLD-tree per a relation of interest
  - Having a conjunction of atoms, which atom should be selected?
  - When to stop building a tree?
- Global control: determine which relations are of interest
  - Do not process the same conjunction twice
  - If a conjunction *embeds* something processed before, *generalize* it
  - How to define *embedding*?
  - How to *generalize*?

CPD: Implementation

- Local control
  - Deterministic unfold (only one nondeterministic unfold per tree)
  - Selectable conjunct: leftmost call which does not have any predecessor embedded into it
  - Variant check
  - Stop when there are no selectable conjuncts
- Global control
  - Variant check
  - Generalization: split conjunction in maximally connected subconjunctions + most specific generalization
  - Homeomorphic embedding extended for conjunctions
- Residualization
  - A definition per a local tree
  - Redundant Argument Filtering

Evaluation

Compare

- Unnesting
- Unnesting strategy aimed at backward execution
- Unnesting + CPD
- Interpretation of functional verifier with relational interpreter

Tasks

- Path search
- Search for a unifier of two terms

Path Search

Directed graph is a tuple $(N, E, start, end)$, where:

- N — set of nodes
- E — set of edges
- Functions $start, end : E \to N$ return a start (end) node of an edge

Path Search

Directed graph is a tuple $(N, E, start, end)$, where:

- N — set of nodes
- E — set of edges
- Functions $start, end : E \to N$ return a start (end) node of an edge

Path is a sequence $\langle n_0, e_0, n_1, e_1, \dots, n_k, e_k, n_{k+1} \rangle$, such that

$$\forall i \in \{0 \dots k\} \ : \ n_i = start\,(e_i) \text{ and } n_{i+1} = end\,(e_i)$$

Path Search

Directed graph is a tuple $(N, E, start, end)$, where:

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$$\forall i \in \{0 \ldots k\} \ : \ n_i = start\,(e_i) \text{ and } n_{i+1} = end\,(e_i)$$

Path search problem is to find the set of paths in a given graph

Path Search: Relational Conversion

```
let rec is_path ns g =
  match ns with
  | x₁ :: x₂ :: xs → elem (x₁, x₂) g && is_path (x₂ :: xs) g
  | [_]            → true
```

Path Search: Relational Conversion

```
let rec is_path ns g =
  match ns with
  | x1 :: x2 :: xs → elem (x1, x2) g && is_path (x2 :: xs) g
  | [_]            → true

let rec is_path^o ns g res = ocanren (
  fresh el in (ns == [el]  & res == true) |
  (fresh x1, x2, xs, res_elem, res_is_path in
    ns == x1 :: (x2 :: xs) &
    elem^o (x1, x2) g res_elem &
    is_path^o (x2 :: xs) g res_is_path &
    ( res_elem == false & res == false |
      res_elem == true  & res == res_is_path )))
```

This relation is inefficient for “`is_path^o q <graph> true`”

Path Search: Specialized Relation

```
let rec is_pathᵒ ns g res = ocanren (
  fresh el in (ns == [el] & res == true) |
  (fresh x₁, x₂, xs, res_elem, res_is_path in
    res_elem == true &
    res_is_path == true &
    ns == x₁ :: (x₂ :: xs) &
    elemᵒ (x₁, x₂) g res_elem &
    is_pathᵒ (x₂ :: xs) g res_is_path))
```

Better performance for "`is_path`^{o} `q` `<graph>` **true**"

```
let rec is_path^o ns g res = ocanren (
  fresh el in (ns == [el] & res == true) |
  (fresh x1, x2, xs, res_elem, res_is_path in
    res_elem == true &
    res_is_path == true &
    ns == x1 :: (x2 :: xs) &
    elem^o (x1, x2) g res_elem &
    is_path^o (x2 :: xs) g res_is_path)))
```

Better performance for "is_path^o q <graph> **true**"

This can be achieved automatically with CPD

Evaluation: Path Search

Path length	5	7	9	11	13	15
Only conversion	0.01	1.39	82.13	>300	—	—
Backward oriented conversion	0.01	0.37	2.68	2.91	4.88	10.63
Conversion and CPD	0.01	0.06	0.34	2.66	3.65	6.22
Scheme interpreter	0.80	8.22	88.14	191.44	>300	—

Table: Searching for paths in the graph (seconds)

Unification

Term:

- Variable $(X, Y, \ldots)$
- Some constructor applied to terms $(nil, cons(H, T), \ldots)$

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Substitution can be *applied* to a term by simultaneously substituting variables for their images

Unification

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Unifier is a substitution σ which equalizes terms: $t\sigma = s\sigma$

Unification

Term:

- Variable $(X, Y, \dots)$
- Some constructor applied to terms $(nil, cons(H, T), \dots)$

Substitution maps variables to terms

Substitution can be *applied* to a term by simultaneously substituting variables for their images

Unifier is a substitution σ which equalizes terms: $t\sigma = s\sigma$

Problem: given two terms with free variables, find their unifier

Unification: Functional Verifier

```
let rec check_uni subst t1 t2 =
  match t1, t2 with
   | Constr (n1, a1), Constr (n2, a2) →
       eq_nat n1 n2 && forall2 subst a1 a2
   | Var_ v          , Constr (n, a)    →
     begin match get_term v subst with
     | None    →  false
     | Some t  →  check_uni subst t t2
     end
   | Constr (n, a)   , Var_ v          →
     begin match get_term v subst with
     | None    →  false
     | Some t  →  check_uni subst t1 t
     end
   | Var_ v1         , Var_ v2         →
     match get_term v1 subst with
     | Some t1'  →  check_uni subst t1' t2
     | None      →  match get_term v2 subst with
                    | Some _  →  false
                    | None    →  eq_nat v1 v2
```

Unification: Relational Conversion

Does not fit the slide.

Evaluation: Unification

Terms	f(X, a) f(a, X)	f(a % b % nil, c % d % nil, L) f(X % XS, YS, X % ZS)	f(X, X, g(Z, t)) f(g(p, L), Y, Y)
Only conversion	0.01	>300	>300
Backward oriented conversion	0.01	0.11	2.26
Conversion and CPD	0.01	0.07	0.90
Scheme interpreter	0.04	5.15	>300

Table: Searching for a unifier of two terms (seconds)

Conclusion & Future Work

Functional recognizer + unnesting + specialization = search

Future work

- Generate functional program from relational to reduce interpretation overhead
- Some other specialization technique, less ad-hoc than CPD