One Algorithm to Evaluate Them All: Unified Linear Algebra Based Approach to Evaluate Both Regular and Context-Free Path Queries

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ABSTRACT

We propose a new algo for CFPQ! Abstract is very abstract.

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CCS CONCEPTS

• **Information systems → Query languages for non-relational engines**; • **Theory of computation → Grammars and context-free languages**; *Parallel computing models*; • **Computing methodologies → Massively parallel algorithms**; • **Computer systems organization →** *Single instruction, multiple data*;

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1 INTRODUCTION

Language-constrained path querying. Regular ath querying (RPQ) is !!!! Context-cfree is more specific, but actively developing last years.

CFPQ as a separated algorithms. Matrix is the fastest.

To make it usable... Integration with graph DB.

Integration with query languages. The problem. We cannot separate regular and context-free queries in general case.

Moreover, grammar transformation for matrix-based (the fastest existing algorithm) is required, !!!

Linear algebra, GraphBLAS, !!!! is a right way.

Contribution

(1) New algorithm. Based on operation over Boolena matrices. Correctness and time complexity.
(2) The way to optimize queries.
(3) Evaluation.

2 PRELIMINARIES

In this section we introduse basic definitions which are used in our work.

Graph

Grammar

CFPQ with different semantics

Finite state machine. Regexp to FSM.

Recursive state machine. Grammar to RSM.

Tensor product definition.

Tensor product for FSM intersection.

Rsm and FSM intersection classical theorem proof?

3 CONTEXT-FREE PATH QUERYING BY KRONECKER PRODUCT

In this section, we introduce the algorithm for CFPQ which is based on Kronecker product of Boolean matrices. The algorithm provides the ability to solve all-pairs CFPQ in all-paths semantics (according to Hellings [?]) and consists of two the following parts.

(1) Index creation. In the first step, the algorithm computes an index which contains information which is necessary to restore paths for specified pairs of vertices. This index can be used to solve the reachability problem without paths extraction. Note that this index is finite even if the set of paths is infinite.

(2) Paths extraction. All paths for the given pair of vertices can be enumerated by using the index computed at the previous step. As far as the set of paths can be infinite, all paths cannot be enumerated explicitly, and advanced techniques such as lazy evaluation are required for implementation. Anyway, a single path can by always extracted by using standard techniques.

We describe both these steps, prove corrctness, and provide time complexity estimations. After thet we provide step-by-step example of query evaluation by using the proposed algorithm.

3.1 Index Creation Algorithm

In this section, we introduce the algorithm for the computation of context-free reachability in a graph $\mathcal{G}$. The algorithm determines the existence of a path, which forms a sentence of the language defined by the input RSM R, between each pair of vertices in the graph $\mathcal{G}$. The algorithm is based on the generalization of the FSM intersection for an RSM, and an input graph. Since a graph can be interpreted as a FSM, in which transitions correspond to the labeled edges between vertices of the graph, and an RSM is composed of a set of FSMs, the intersection of such machines can be computed using the classical algorithm for FSM intersection, presented in [?].

The intersection can be computed as a Kronecker product of the corresponding adjacency matrices for an RSM and a graph. Since we are only determining the reachability of vertices, it is enough to represent intersection result as a Boolean matrix. It simplifies the algorithm implementation and allows one to express it in terms of basic matrix operations.

Listing 1 shows main steps of the algorithm. The algorithm accepts context-free grammar $G = (\Sigma, N, P)$ and graph $\mathcal{G} = (V, E, L)$ as an input. An RSM R is created from the grammar G. Note, that R must have no ε-transitions. M_1 and M_2 are the adjacency matrices for the machine R and the graph $\mathcal{G}$ correspondingly.

Then for each vertex i of the graph $\mathcal{G}$, the algorithm adds loops with non-terminals, which allows deriving ε-word. Here the following rule is implied: each vertex of the graph is reachable by itself through an ε-transition. Since the machine R does not have any ε-transitions, the ε-word could be derived only if a state s in the box B of the R is both initial and final. This data is queried by the $getNonterminals()$ function for each state s.

The algorithm terminates when the matrix M_2 stops changing. Kronecker product of matrices M_1 and M_2 is evaluated for each iteration. The result is stored in M_3 as a Boolean matrix. For the given M_3 a C_3 matrix is evaluated by the $transitiveClosure()$ function call. The M_3 could be interpreted as an adjacency matrix for an directed graph with no labels, used to evaluate transitive closure in terms of classical graph definition of this operation. Then the algorithm iterates over cells of the C_3. For the pair of indices (i, j), it computes s and f — the initial and final states in the recursive automata R which relate to the concrete $C_3[i, j]$ of the closure matrix. If the given s and f belong to the same box B of R, $s = q^0_B$, and $f \in F_B$, then $getNonterminals()$ returns the respective non-terminal. If the the condition holds then the algorithm adds the computed non-terminals to the respective cell of the adjacency matrix M_2 of the graph.

The functions $getStates$ and $getCoordinates$ (see listing 2) are used to map indices between Kronecker product arguments and the result matrix. The Implementation appeals to the blocked structure of the matrix C_3, where each block corresponds to some automata and graph edge.

The algorithm returns the updated matrix M_2 which contains the initial graph $\mathcal{G}$ data as well as non-terminals from N. If a cell $M_2[i, j]$ for any valid indices i and j contains symbol $S \in N$, then vertex j is reachable from vertex i in grammar G for non-terminal S.

Lemma 3.1. *Let $\mathcal{G} = (V, E, L)$ be a graph and $G = (\Sigma, N, P)$ be a grammar. Let $\mathcal{G}_k = (V, E_k, L \cup N)$ be graph and M_k its adjacency matrix of the execution some iteration $k \geq 0$ of the*

Listing 1 Kronecker product based CFPQ

function contextFreePathQuerying(G, $\mathcal{G}$)
  $R \leftarrow$ Recursive automata for G
  $M_1 \leftarrow$ Adjacency matrix for R
  $M_2 \leftarrow$ Adjacency matrix for $\mathcal{G}$
  for $s \in 0..dim(M_1) - 1$ **do**
    for $i \in 0..dim(M_2) - 1$ **do**
      $M_2[i, i] \leftarrow M_2[i, i] \cup getNonterminals(R, s, s)$
  while Matrix M_2 is changing **do**
    $M_3 \leftarrow M_1 \otimes M_2$ ▷ Evaluate Kroncker product
    $C_3 \leftarrow transitiveClosure(M_3)$
    $n \leftarrow \dim(M_3)$ ▷ Matrix M_3 size = $n \times n$
    for $(i, j) \in [0..n-1] \times [0..n-1]$ **do**
      if $C_3[i, j]$ **then**
        $s, f \leftarrow getStates(C_3, i, j)$
        if $getNonterminals(R, s, f) \neq \emptyset$ **then**
          $x, y \leftarrow getCoordinates(C_3, i, j)$
          $M_2[x, y] \leftarrow M_2[x, y] \cup getNonterminals(R, s, f)$
  return M_2

Listing 2 Help functions for Kronecker product based CFPQ

function getStates(C, i, j)
  $r \leftarrow dim(M_1)$ ▷ M_1 is adjacency matrix for automata R
  return $\lfloor i/r \rfloor, \lfloor j/r \rfloor$
function getCoordinates(C, i, j)
  $n \leftarrow dim(M_2)$ ▷ M_2 is adjacency matrix for graph $\mathcal{G}$
  return $i \bmod n$, $j \bmod n$

algorithm ??. Then for each edge $e = (m, S, n) \in E_k$, where $S \in N$, the following statement holds: $\exists m\pi n : S \rightarrow_G l(\pi)$.

Proof. (Proof by induction)

Basis: For $k = 0$ and the statement of the lemma holds, since $M_0 = M$, M where is adjacency matrix of the graph G. Non-terminals, which allow to derive ε-word, are also added at algorithm preprocessing step, since each vertex of the graph is reachable by itself through an ε-transition.

Inductive step: Assume that the statement of the lemma holds for any $k \leq (p-1)$ and show that it also holds for $k = p$, where $p \geq 1$.

For the algorithm iteration p the Kronecker product K_p and transitive closure C_p are evaluated as described in the algorithm. By the properties of this operations, some edge $e = ((s, m), (f, n))$ exists in the directed graph, represented by adjacency matrix C_p, if and only if $\exists s\pi^{'} f$ in the RSM graph, represented by matrix M_r, and $\exists m\pi n$ in graph, represented by M_{p-1}. Concatenated symbols along the path $\pi^{'}$ form some derivation string v, composed from terminals and non-terminals, where $v \rightarrow_G l(\pi)$ by the inductive assumption.

The new edge $e = (m, S, n)$ will be added to the E_p only if s and f are initial and final states of some box B of the RSM corresponding to the non-terminal S_B. In this case, the grammar G has the derivation rule $S_B \rightarrow_G v$, by the inductive assumption $v \rightarrow_G l(\pi)$. Therefore, $S_B \rightarrow_G l(\pi)$ and this completes the proof of the lemma.

□

Lemma 3.2. *Let $\mathcal{G} = (V, E, L)$ be a graph and $G = (\Sigma, N, P)$ be a grammar. Let $\mathcal{G}_k = (V, E_k, L \cup N)$ be graph and M_k its adjacency matrix of the execution some iteration $k \geq 1$ of the algorithm ??. For any path $m\pi n$ in graph $\mathcal{G}$ with word $l = l(\pi)$ if exists the derivation tree of l for the grammar G and starting non-terminal S with the height $h \leq k$, then $\exists e = (m, S, n) : e \in E_k$.*

Proof. (Proof by induction)

Basis: Show that statement of the lemma holds for the $k = 1$. Matrix M and edges of the graph $\mathcal{G}$ contains only labels from L. Since the derivation tree of height $h = 1$ contains only one non-terminal S as a root and only symbols from $\Sigma \cup \varepsilon$ as leafs, for all paths, which form a word with derivation tree of the height $h = 1$, the corresponding nonterminals will be added to the M_1 via preprocessing step and first iteration of the algorithm. Thus, the lemma statement holds for the $k = 1$.

Inductive step: Assume that the statement of the lemma hold for any $k \leq (p-1)$ and show that it also holds for $k = p$, where $p \geq 2$.

For the algorithm iteration p the Kronecker product K_p and transitive closure C_p are evaluated as described in the algorithm. By the properties of this operations, some edge $e = ((s, m), (f, n))$ exists in the directed graph, represented by adjacency matrix C_p, if and only if $\exists s\pi_1 f$ in the RSM graph, represented by matrix M_{RSM}, and $\exists m\pi n$ in graph, represented by M_{p-1}.

For any path $m\pi n$, such that exist derivation tree of height $h < k$ for the word $l(\pi)$ with root non-terminal S, there exists edge $e = (m, S, n) : e \in E_k$ by inductive assumption.

Suppose, that exists derivation tree T of height $h = p$ with the root non-terminal S for the path $m\pi n$. The tree T is formed as $S \rightarrow a_1..a_d, d \geq 1$ where $\forall i \in [1..d]$ a_i is sub-tree of height $h_i \leq p - 1$ for the sub-path $m_i\pi_i n_i$. By inductive hypothesis, there exists path π_i for each derivation sub-tree, such that $m = m_1\pi_1 m_2..m_d\pi_d m_{d+1} = n$ and concatenation of these paths forms $m\pi n$, and the root non-terminals of this sub-trees are included in the matrix M_{p-1}.

Therefore, vertices m_i $\forall i \in [1..d]$ form path in the graph, represented by matrix M_{p-1}, with complete set of labels. Thus, new edge between vertices m and n with the respective non-terminal S will be added to the matrix M_p and this completes the proof of the lemma.

□

Theorem 3.3. *Let $\mathcal{G} = (V, E, L)$ be a graph and $G = (\Sigma, N, P)$ be a grammar. Let $\mathcal{G}_R = (V, E_R, L)$ be a result graph for the execution of the algorithm ??. The following statement holds: $e = (m, S, n) \in E_R$, where $S \in N$, if and only if $\exists m\pi n : S \rightarrow_G l(\pi)$.*

Proof. This theorem is a consequence of the Lemma 3.1 and Lemma 3.2. □

Theorem 3.4. *Let $\mathcal{G} = (V, E, L)$ be a graph and $G = (\Sigma, N, P)$ be a grammar. The algorithm* **??** *terminates in finite number of steps.*

Proof. The main algorithm *while-loop* is executed while graph adjacency matrix M is changing. Since the algorithm only adds the edges with non-terminals from N, the maximum required number of iterations is $|N| \times |V| \times |V|$, where each component has finite size. This completes the proof of the theorem. □

3.2 Paths Extraction Algoritm

After index created one can enumerate all paths betwen cpecified vertices.

Listing 3 Paths extraction algorithm

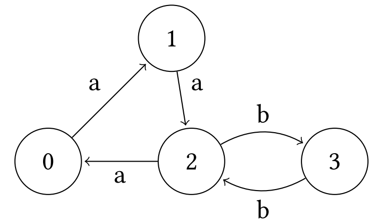
function getPaths(v_s, v_f, N, C_3, M_1, M_2)
  $s \leftarrow$ Start states of automata for N
  $f \leftarrow$ Final states of automata for N
  $res \leftarrow$ getPInner(i, j, C_3, M_1, M_2)
  return res
function getSubPaths(i, j, k, C_3, M_1, M_2)
  l $\leftarrow$ $\{(i.g, t, k.g) \mid M_1[t][i.r, k.r] = 1 \ \& \ M_2[t][i.g, k.g]\} \cup \bigcup_{N \mid M_1[N][i.r,k.r]}$ getPaths($i.g$, $k.g$, N, C_3, M_1, M_2) $\cup$ getPInner(i, k, C_3, M_1, M_2)
  r $\leftarrow$ $\{(k.g, t, j.g) \mid M_1[t][k.r, j.r] = 1 \ \& \ M_2[t][k.g, j.g]\} \cup \bigcup_{N \mid M_1[N][k.r,j.r]}$ getPaths($k.g$, $j.g$, N, C_3, M_1, M_2) $\cup$ getPInner(k, j, C_3, M_1, M_2)
  return $l \cdot r$
function getPInner(i, j, C_3, M_1, M_2)
  $parts \leftarrow \{k \mid C_3[i, k] = 1 \ \& \ C_3[k, j] = 1\}$
  return $\bigcup_{k \in parts}$ getSubPaths(i, j, k, C_3, M_1, M_2)

Ideas and description.
Correcness.
Time complexity.

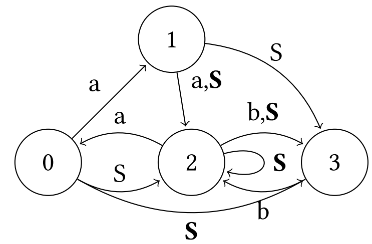
3.3 Example

Adjacency matrices M_1 and M_2 for automata R and graph $\mathcal{G}$ respectively are initialized as follows:

$$M_1 = \begin{pmatrix} . & . & \{a\} & . \\ . & . & \{S\} & \{b\} \\ . & . & . & \{b\} \\ . & . & . & . \end{pmatrix}, \ M_2^0 = \begin{pmatrix} . & \{a\} & . & . \\ . & . & \{a\} & . \\ \{a\} & . & . & \{b\} \\ . & . & \{b\} & . \end{pmatrix}.$$



(a) The input graph $\mathcal{G}$



(b) The result graph $\mathcal{G}$

Figure 1: The input and result graphs for example

Matrix M_1 can be represented as a set of Boolean matrices as follows:

$$M_1^S = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} . & . & . & . \\ . & . & 1 & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \end{array}, \ M_1^a = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} . & . & 1 & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \end{array},$$

$$M_1^b = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} . & . & . & . \\ . & . & . & 1 \\ . & . & . & 1 \\ . & . & . & . \end{pmatrix} \end{array}$$

Matrix M_2 can be represented as a set of Boolean matrices as follows:

$$M_2^{S,0} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & . \\ . & . & . & . \end{pmatrix} \end{array}, \ M_2^{a,0} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} . & 1 & . & . \\ . & . & 1 & . \\ 1 & . & . & . \\ . & . & . & . \end{pmatrix} \end{array},$$

$$M_2^{b,0} = \begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} . & . & . & . \\ . & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \end{pmatrix} \end{array}$$

First iteration.

$M_3^1 = M_1^a \otimes M_2^{a,0} + M_1^b \otimes M_2^{b,0} + M_1^S \otimes M_2^{S,0} =$

$$\begin{array}{r|cccc|cccc|cccc|cccc}
 & (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) & (3,0) & (3,1) & (3,2) & (3,3) \\
(0,0) & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . \\
(0,1) & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\
(0,2) & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
(0,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
(1,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(1,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(1,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(1,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

$C_3^1 = tc(M_3^1) =$

$$\begin{array}{r|cccc|cccc|cccc|cccc}
 & (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) & (3,0) & (3,1) & (3,2) & (3,3) \\
(0,0) & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . \\
(0,1) & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & \mathbf{1} \\
(0,2) & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
(0,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
(1,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(1,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(1,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(1,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

Second iteration.

$M_3^2 = M_1^a \otimes M_2^{a,0} + M_1^b \otimes M_2^{b,0} + M_1^S \otimes M_2^{S,1} =$

$$\begin{array}{r|cccc|cccc|cccc|cccc}
 & (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) & (3,0) & (3,1) & (3,2) & (3,3) \\
(0,0) & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . \\
(0,1) & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . \\
(0,2) & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
(0,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
(1,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(1,1) & . & . & . & . & . & . & . & . & . & . & . & \mathbf{1} & . & . & . & . \\
(1,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(1,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

$C_3^2 = tc(M_3^2) =$

$$\begin{array}{r|cccc|cccc|cccc|cccc}
 & (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) & (3,0) & (3,1) & (3,2) & (3,3) \\
(0,0) & . & . & . & . & . & 1 & . & . & . & . & . & \mathbf{1} & . & . & \mathbf{1} & . \\
(0,1) & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 \\
(0,2) & . & . & . & . & 1 & . & . & . & . & . & . & . & . & . & . & . \\
(0,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
(1,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(1,1) & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & \mathbf{1} & . \\
(1,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(1,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

$C_3^3 =$

$$\begin{array}{r|cccc|cccc|cccc|cccc}
 & (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) & (3,0) & (3,1) & (3,2) & (3,3) \\
(0,0) & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & 1 & . \\
(0,1) & . & . & . & . & . & . & 1 & . & . & . & . & . & . & . & . & 1 \\
(0,2) & . & . & . & . & 1 & . & . & . & . & . & \mathbf{1} & . & . & . & . & \mathbf{1} \\
(0,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
(1,0) & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & \mathbf{1} \\
(1,1) & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . \\
(1,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(1,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

$C_3^4 =$

$$\begin{array}{r|cccc|cccc|cccc|cccc}
 & (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) & (3,0) & (3,1) & (3,2) & (3,3) \\
(0,0) & . & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & 1 & . \\
(0,1) & . & . & . & . & . & . & 1 & . & . & . & . & \mathbf{1} & . & . & \mathbf{1} & 1 \\
(0,2) & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & 1 \\
(0,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
(1,0) & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & 1 \\
(1,1) & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & . \\
(1,2) & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & \mathbf{1} & 1 \\
(1,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

$C_3^5 =$

$$\begin{array}{r|cccc|cccc|cccc|cccc}
 & (0,0) & (0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) & (3,0) & (3,1) & (3,2) & (3,3) \\
(0,0) & . & . & . & . & . & 1 & . & . & . & . & \mathbf{1} & 1 & . & . & 1 & \mathbf{1} \\
(0,1) & . & . & . & . & . & . & 1 & . & . & . & . & 1 & . & . & 1 & 1 \\
(0,2) & . & . & . & . & 1 & . & . & . & . & . & 1 & . & . & . & . & 1 \\
(0,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
(1,0) & . & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . & 1 \\
(1,1) & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & 1 & \mathbf{1} \\
(1,2) & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & 1 \\
(1,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

$C_3^6 =$

$$\begin{array}{r|cccc|cccc|cccc|cccc}
 & {}^{0}(0,0) & {}^{1}(0,1) & (0,2) & (0,3) & (1,0) & (1,1) & (1,2) & (1,3) & (2,0) & (2,1) & (2,2) & (2,3) & (3,0) & (3,1) & (3,2) & (3,3) \\
0{:}(0,0) & . & . & . & . & . & 1 & . & . & . & . & 1 & 1 & . & . & 1 & 1 \\
1{:}(0,1) & . & . & . & . & . & . & 1 & . & . & . & . & 1 & . & . & 1 & 1 \\
2{:}(0,2) & . & . & . & . & 1 & . & . & . & . & . & 1 & \mathbf{1} & . & . & \mathbf{1} & 1 \\
3{:}(0,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
\hline
4{:}(1,0) & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & \mathbf{1} & 1 \\
5{:}(1,1) & . & . & . & . & . & . & . & . & . & . & 1 & 1 & . & . & 1 & 1 \\
6{:}(1,2) & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . & 1 & 1 \\
7{:}(1,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
8{:}(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
9{:}(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
10{:}(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
11{:}(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
\hline
12{:}(2,0) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
13{:}(2,1) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
14{:}(2,2) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . \\
15{:}(2,3) & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & .
\end{array}$$

Reachability is done. Now we can to restore paths. Let we try to restore path from 2 to 2.

getPaths(2, 2, S, C_3^6, M_1, M_2^0)
–––getPInner(2, 14, C_3^6, M_1, M_2^0)
––––parts= {4}
––––getSubpaths(2, 14, 4, C_3^6, M_1, M_2^0)
–––––l={$2 \xrightarrow{a} 0$}
–––––getPInner(4, 14, C_3^6, M_1, M_2^0)
––––––parts={11}
––––––getSubpaths(4, 14, 11, C_3^6, M_1, M_2^0)
–––––––getPaths(0, 3, S, C_3^6, M_1, M_2^0)
––––––––getPInner(0, 15, C_3^6, M_1, M_2^0))
–––––––––parts={5, 10}
–––––––––getSubpaths(0, 15, 5, C_3^6, M_1, M_2^0)
––––––––––l={$0 \xrightarrow{a} 1$}
––––––––––getPInner(5, 15, C_3^6, M_1, M_2^0)
–––––––––––parts={10}
–––––––––––getSubpaths(5, 15, 10, C_3^6, M_1, M_2^0)
––––––––––––getPaths(1, 2, S, C_3^6, M_1, M_2^0)
–––––––––––––getPInner(1, 14, C_3^6, M_1, M_2^0)
––––––––––––––parts={6, 11}
––––––––––––––getSubpaths(1, 14, 6, C_3^6, M_1, M_2^0)
–––––––––––––––l={$1 \xrightarrow{a} 2$}
–––––––––––––––getPInner(6, 14, C_3^6, M_1, M_2^0)
––––––––––––––––parts={11}
––––––––––––––––getSubpaths(6, 14, 11, C_3^6, M_1, M_2^0)
–––––––––––––––––getPaths(2, 3, S, C_3^6, M_1, M_2^0)
––––––––––––––––––getPInner(2, 15, C_3^6, M_1, M_2^0)
–––––––––––––––––––parts={4, 10}

—————————————————————getSubpaths(2, 15, 4, C_3^6, M_1, M_2^0)

——————————————————————l=$\{2 \xrightarrow{a} 0\}$

——————————————————————getPInner(4, 15, C_3^6, M_1, M_2^0)

———————————————————————parts=$\{10\}$

———————————————————————getSubpaths(4, 15, 10, C_3^6, M_1, M_2^0)

————————————————————————getPaths(0, 2, S, C_3^6, M_1, M_2^0)

—————————————————————————getPInner(0, 14, C_3^6, M_1, M_2^0)

——————————————————————————parts=$\{5, 11\}$

——————————————————————————getSubpaths(0, 14, 5, C_3^6, M_1, M_2^0)

———————————————————————————l=$\{0 \xrightarrow{a} 1\}$

———————————————————————————getPInner(5, 14, C_3^6, M_1, M_2^0)

————————————————————————————parts=$\{11\}$

————————————————————————————getSubpaths(5, 14, 11, C_3^6, M_1, M_2^0)

—————————————————————————————getPaths(1, 3, S, C_3^6, M_1, M_2^0)

——————————————————————————————getPInner(1, 15, C_3^6, M_1, M_2^0)

———————————————————————————————parts=$\{6\}$

———————————————————————————————getSubpaths(1, 15, 6, C_3^6, M_1, M_2^0)

————————————————————————————————l = $\{1 \xrightarrow{a} 2\}$

————————————————————————————————r = $\{2 \xrightarrow{b} 3\}$

————————————————————————————————$l \cdot r = \{1 \xrightarrow{a} 2 \xrightarrow{b} 3\}$

———————————————————————————————$\{1 \xrightarrow{a} 2 \xrightarrow{b} 3\}$

——————————————————————————————$\{1 \xrightarrow{a} 2 \xrightarrow{b} 3\}$

—————————————————————————————l=$\{1 \xrightarrow{a} 2 \xrightarrow{b} 3\}$

—————————————————————————————r=$\{3 \xrightarrow{b} 2\}$

—————————————————————————————$l \cdot r = \{1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}$

————————————————————————————$\{1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}$

———————————————————————————r = $\{1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}$

———————————————————————————$l \cdot r = \{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}$

——————————————————————————getSubpaths(0, 14, 11, C_3^6, M_1, M_2^0)

———————————————————————————getPInner(0, 11, C_3^6, M_1, M_2^0)

————————————————————————————parts=$\{5\}$

————————————————————————————getSubpaths(0, 11, 5, C_3^6, M_1, M_2^0)

—————————————————————————————l=$\{0 \xrightarrow{a} 1\}$

—————————————————————————————getPaths(1, 3, S, C_3^6, M_1, M_2^0) (*recursive call!!! Infinite set of paths*)

—————————————————————————————r = $r_\infty^{1\rightsquigarrow 3}$

—————————————————————————————$l \cdot r = \{0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3}$

————————————————————————————$\{0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3}$

———————————————————————————l = $\{0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3}$

———————————————————————————r = $\{3 \xrightarrow{b} 2\}$

———————————————————————————$l \cdot r = \{0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2\}$

——————————————————————————$\{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (\{0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2\})$

—————————————————————————$\{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (\{0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2\})$

————————————————————————l=$\{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (\{0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2\})$

————————————————————————r=$\{2 \xrightarrow{b} 3\}$

————————————————————————$l \cdot r = \{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (\{0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3\})$

———————————————————————$\{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (\{0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3\})$

——————————————————————r=$\{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (\{0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3\})$

——————————————————————$l \cdot r = \{\{2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (\{2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3\})$

—————————————————————getSubpaths(2, 15, 10, C_3^6, M_1, M_2^0)

——————————————————————getPInner(2, 10, C_3^6, M_1, M_2^0)

———————————————————————parts=$\{4\}$

———————————————————————getSubpaths(2, 10, 4, C_3^6, M_1, M_2^0)

————————————————————————l=$\{2 \xrightarrow{a} 0\}$

————————————————————————getPaths(0, 2, S, C_3^6, M_1, M_2^0)(*recursive call!!! Infinite set of paths*)

————————————————————————r = $r_\infty^{0\rightsquigarrow 2}$

————————————————————————$l \cdot r = \{2 \xrightarrow{a} 0\} \cdot r_\infty^{0\rightsquigarrow 2}$

———————————————————————$\{2 \xrightarrow{a} 0\} \cdot r_\infty^{0\rightsquigarrow 2}$

——————————————————————l=$\{2 \xrightarrow{a} 0\} \cdot r_\infty^{0\rightsquigarrow 2}$

——————————————————————r=$\{2 \xrightarrow{b} 3\}$

——————————————————————$l \cdot r = \{2 \xrightarrow{a} 0\} \cdot r_\infty^{0\rightsquigarrow 2} \cdot \{2 \xrightarrow{b} 3\}$

————————————————————$\{2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (\{2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3\}) \cup (\{2 \xrightarrow{a} 0\} \cdot r_\infty^{0\rightsquigarrow 2} \cdot \{2 \xrightarrow{b} 3\})$ (*last is the same as middle so omit it in our example*)

———————————————————l=$\{2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (\{2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3\}))$

———————————————————r=$\{3 \xrightarrow{b} 2\}$

———————————————————$l \cdot r = \{2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (\{2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

——————————————————$\{2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (\{2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

—————————————————r=$\{2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (\{2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

—————————————————$l \cdot r = \{1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

————————————————getSubpaths(1, 14, 11, C_3^6, M_1, M_2^0) (*Leads to recursion. Omited*)

————————————————$\{1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

———————————————$\{1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

——————————————l=$\{1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

——————————————r=$\{2 \xrightarrow{b} 3\}$

——————————————$l \cdot r = \{1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\}))$

—————————————$\{1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\}))$

———————————r=$\{1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\}))$

———————————$l \cdot r = \{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\}))$

——————————getSubpaths(0, 15, 10, C_3^6, M_1, M_2^0) (*Leads to recursion. Omited*)

——————————$\{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\}))$

—————————$\{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\}))$

————————l=$\{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\}))$

————————r=$\{3 \xrightarrow{b} 2\}$

————————$l \cdot r = \{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

———————$\{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

——————r=$\{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

——————$l \cdot r = \{2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

—————$\{2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_\infty^{1\rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

————$\{2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\} \cup (2 \xrightarrow{a} 0 \xrightarrow{a}$

$1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_{\infty}^{1 \rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2\}))$

`getPaths`$(2, 2, S, C_3^6, M_1, M_2^0)$
└ `getPInner`$(2, 14, C_3^6, M_1, M_2^0)$
  └ `parts=` $\{4\}$
  └ `getSubpaths`$(2, 14, 4, C_3^6, M_1, M_2^0)$
    └ `l=`$\{2 \xrightarrow{a} 0\}$
      └ $l \cdot r = \{0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\} \cup (0 \xrightarrow{a} 1 \xrightarrow{a} 2 \xrightarrow{a} 0 \xrightarrow{a} 1\} \cdot r_{\infty}^{1 \rightsquigarrow 3} \cdot \{3 \xrightarrow{b} 2 \xrightarrow{b} 3 \xrightarrow{b} 2 \xrightarrow{b} 3\}))$

4 IMPLEMENATION DETAILS

Linear algebra, GraphBLAS.

Specific details. Sparsity parameters. How to express some steps efficiently.

Integration with RedisGraph.

Grammar is a file.

Om paths extraction algorithm. I think that we shuold implement single path extraction, and paths without recursive calls. Lazy evalustion is not good idea for C implementation.

5 EVALUATION

Questions.

(1) Compare classical RPQ algorithms and our agorithm
(2) Compare other CFPQ algorithms and our algorithms
(3) Ivestigate effect of grammar optimization

5.1 RPQ

Dataset description, tools selection.

5.2 CFPQ

Comparison with matrix-based.

5.3 Grammar transformation

On query optimization.

Memory aliases.

Synthetic???

6 RELATED WORK

CFPQ algorithms: Hellings [?], Bradford [?], Azimov [?], Verbitskaya [?], Ciro [?], form static code analysis [?],

RPQ algorithms: !!!! [?]

Query optimization.

7 CONCLUSION

On RSM optimization and query optimization.

HiCOO format.

GPGPU-based implementation. Multi-GPU version.

Full integration with Graph DB.

Other semantics: shortest path, simple path and so on.

Streaming graph querying.

Specialization on query.

!!!

REFERENCES