WoLLIC 2019

Bar-Hillel Theorem Mechanization in Coq

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Automated Theorem Proving

- Automatization of checking of the proofs correctness

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- Automatization of checking of the proofs correctness
- Also a way to create correct-by-construction algorithms
  - Coq proof assistant
    - Based on calculus of inductive constructions
    - Supports extraction of certified programs to executable programming languages

Mechanization of Formal Language Theory

Goals:

- Check nontrivial proofs
- Ensure correctness of algorithms
  - Parsing algorithms
  - Algorithms over regular expressions
  - Algorithms over finite automata

The Bar-Hillel Theorem

Theorem (Bar-Hillel)

If L_1 is a context-free language and L_2 is a regular language, then $L_1 \cap L_2$ is context-free languege.

Context-Free Path Quierying (CFPQ)

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Navigation through an edge-labelled graph

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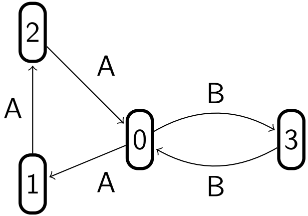

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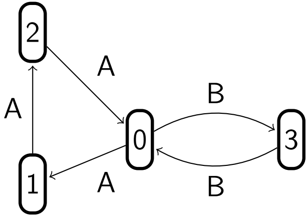
- Are there paths in graph, which form well-balanced sequences over `A` and `B`?



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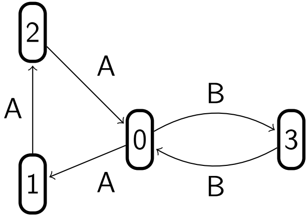

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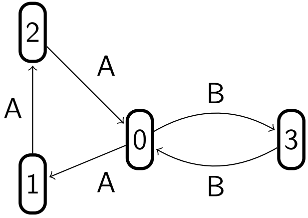

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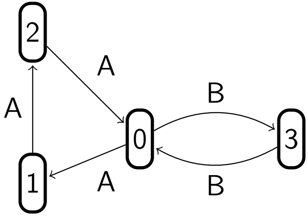

Paths filter (query):

$$s \to A\ s\ B\ s \mid \varepsilon$$

Answer:

- $2 \xrightarrow{A} 0 \xrightarrow{B} 3$
- $1 \xrightarrow{A} 2 \xrightarrow{A} 0 \xrightarrow{B} 3 \xrightarrow{B} 0$
- ...

CFPQ: Formal View

- $\mathbb{G} = (\Sigma, N, P, S)$ — context-free grammar
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- $\omega(\pi) = \omega(v_0 \xrightarrow{t_0} v_1 \xrightarrow{t_1} \cdots \xrightarrow{t_{n-2}} v_{n-1} \xrightarrow{t_{n-1}} v_n) = t_0 t_1 \cdots t_{n-1}$

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- $R = \{(n, m) \mid \exists n \pi m$, such that $\omega(\pi) \in L(\mathbb{G})\}$
- $P = \{\pi \mid \pi$ is a path in G, such that $\omega(\pi) \in L(\mathbb{G})\}$

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- Our problems all are about $L_1 \cap L_2$
- The Bar-Hillel theorem
  - Says that our problems are decidable
  - Shows how to solve our problems

Applications of CFPQ

- Graph database querying
  - Mihalis Yannakakis, "Graph-theoretic methods in database theory" (1990)
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- Static code analysis
  - Thomas Reps. "Program Analysis via Graph Reachability" (1997)
  - Andrei Marian Dan et al, "Finding Fix Locations for CFL-Reachability Analyses via Minimum Cuts" (2017)

Sketch of the Proof[1]

Theorem (Bar-Hillel)

If L_1 is a context-free language and L_2 is a regular language, then $L_1 \cap L_2$ is context-free.

1. Assume that there is a context-free grammar $\mathbb{G}_{CNF}$ in Chomsky Normal Form, such that $L(\mathbb{G}_{CNF}) = L_1$

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    - If $L \neq \varnothing$ and L is regular then L is the union of regular languages $A_1, \ldots, A_n$ where each A_i is accepted by a DFA with a single final state

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4. Finally, join them together with the operation of the union

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Hofmann's Results Generalization

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We had to carefully refactor everything...

DFA Splitting

If $L \neq \varnothing$ and L is regular, then L is the union of regular languages $A_1, \ldots, A_n$ where each A_i is accepted by a DFA with precisely one final state

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```
Lemma correct_split:
   forall dfa w,
     dfa_language dfa w <->
     exists sdfa,
        In sdfa (split_dfa dfa) /\ s_dfa_language sdfa w.
```

Chomsky Induction

Lemma

Let $\mathbb{G}$ *be a grammar in CNF. Consider an arbitrary nonterminal* $N \in \mathbb{G}$ *and phrase which consists only of terminals* w*. If* w *is derivable from* N *(*$der(\mathbb{G}, N, w)$*) and* $|w| \geq 2$*, then there exists two nonterminals* N_1, N_2 *and two phrases* w_1, w_2 *such that:* $N \to N_1 N_2 \in \mathbb{G}$*,* $der(\mathbb{G}, N_1, w_1)$*,* $der(\mathbb{G}, N_2, w_2)$*,* $|w_1| \geq 1$*,* $|w_2| \geq 1$ *and* $w_1 ++ w_2 = w$*.*

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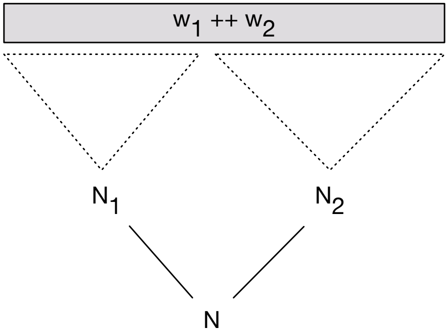
Let $\mathbb{G}$ be a grammar in CNF. Consider an arbitrary nonterminal $N \in \mathbb{G}$ and phrase which consists only of terminals w. If w is derivable from N ($der(\mathbb{G}, N, w)$) and $|w| \geq 2$, then there exists two nonterminals N_1, N_2 and two phrases w_1, w_2 such that: $N \to N_1 N_2 \in \mathbb{G}$, $der(\mathbb{G}, N_1, w_1)$, $der(\mathbb{G}, N_2, w_2)$, $|w_1| \geq 1$, $|w_2| \geq 1$ and $w_1 ++ w_2 = w$.



Chomsky Induction in Coq

```
Definition syntactic_analysis_is_possible :=
forall (G : grammar) (A : var) (w : phrase),
   der G A w -> (R A w \in G)
                \/
                (exists rhs, R A rhs \in G /\ derf G rhs w).
```

Languges Union

```
Variable grammars: seq (var * grammar).

Theorem correct_union:
forall word,
  language (grammar_union grammars) (V (start Vt))
           (to_phrase word)
  <->
  exists s_l,
    language (snd s_l) (fst s_l) (to_phrase word)
    /\
    In s_l grammars.
```

The Final Theorem

Theorem

For any two decidable types ***Tt*** *and* ***Nt*** *for types of terminals and nonterminals correspondingly. If there exists a bijection from* ***Nt*** *to* $\mathbb{N}$ *and syntactic analysis is possible (in the sense of our definition), then for any DFA* ***dfa*** *and any context-free grammar* $\mathbb{G}$*, there exists the context-free grammar* $\mathbb{G}_{INT}$*, such that* $L(\mathbb{G}_{INT}) = L(\mathbb{G}) \cap L(dfa)$*.*

The Final Theorem in Coq

```
Theorem grammar_of_intersection_exists:
  exists
   (NewNonterminal: Type)
   (IntersectionGrammar: @grammar Terminal NewNonterminal)
   St,
  forall word,
    dfa_language dfa word /\ language G S (to_phrase word)
    <->
    language IntersectionGrammar St (to_phrase word).
```

Conclusion

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- We present mechanized in Coq proof of the Bar-Hillel theorem on the closure of context-free languages under intersection with the regular languages
- We generalize the results of Jana Hofmann and Gert Smolka
  - The definition of the terminal and nonterminal alphabets in context-free grammar were made generic
  - All related definitions and theorems were adjusted to work with the updated definition
- All results are published at GitHub and are equipped with automatically generated documentation

Future work

- Marcus Ramos vs Jana Hifmann
  - We use results of Jana Hofman
  - Results of Marcus Ramos seem more mature
  - Is it possible to create one "true" solution in this area?
    - Is our grammar-based proof better then PDA-based one in all contexts?

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    - Is our grammar-based proof better then PDA-based one in all contexts?
- Mechanization of practical algorithms which are just implementation of the Bar-Hillel theorem
  - Context-free path querying algorithm, based on CYK or even on GLL parsing algorithm
  - Certified algorithm for context-free constrained path querying for graph databases

Contact Information

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  - sbozhko@mpi-sws.com
- Leyla Khatbullina:
  - St.Petersburg Electrotechnical University "LETI", St.Petersburg, Russia
  - leila.xr@gmail.com
- Sources: https://github.com/YaccConstructor/YC_in_Coq

Thanks!