WoLLIC 2019

Bar-Hillel Theorem Mechanization in Coq

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Automated Theorem Proving

- Yet another attemt to automate proof correctness checking
- In some systems — a way to create correct by construction algorithms
  - Coq

Formal Language Theory Mechanization

- Nontrivial proofs checking
- Correctness of algorithms

The Bar-Hillel Theorem

Theorem (Bar-Hillel)

If L_1 is a context-free language and L_2 is a regular language, then $L_1 \cap L_2$ is context-free.

Context-Free Path Quierying (CFPQ)

- $\mathbb{G} = (\Sigma, N, P)$ — context-free grammar
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- $R_A = \{(n, m) \mid \exists n \pi m$, such that $\omega(\pi) \in L(\mathbb{G}, A)\}$
- $P_A = \{\pi \mid \pi$ is a path in G, such that $\omega(\pi) \in L(\mathbb{G}, A)\}$

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- Graph data base querying
    - Yannacacis !!!
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4. Finally, join them together with the operation of the union

Hofmann's Results Generalization

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And now we should carefully rewrite all existing stuff ...

DFA Splitting

If $L \neq \varnothing$ and L is regular then L is the union of regular language $A_1, \ldots, A_n$ where each A_i is accepted by a DFA with precisely one final state

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```
Lemma correct_split:
   forall dfa w,
     dfa_language dfa w <->
     exists sdfa,
        In sdfa (split_dfa dfa) /\ s_dfa_language sdfa w.
```

Chomsky Induction

Lemma

Let G be a grammar in CNF. Consider an arbitrary nonterminal $N \in G$ and phrase which consists only of terminals w. If w is derivable from N and $|w| \geq 2$, then there exists two nonterminals N_1, N_2 and two phrases w_1, w_2 such that: $N \to N_1 N_2 \in G$, $der(G, N_1, w_1)$, $der(G, N_2, w_2)$, $|w_1| \geq 1$, $|w_2| \geq 1$ and $w_1 ++ w_2 = w$.

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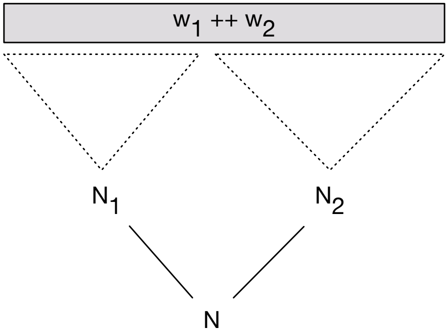

Chomsky Induction in Coq

```
Definition syntactic_analysis_is_possible :=
forall (G : grammar) (A : var) (w : phrase),
   der G A w -> (R A w \in G)
                \/
                (exists rhs, R A rhs \in G /\ derf G rhs w).
```

Languges Union

```
Variable grammars: seq (var * grammar).

Theorem correct_union:
forall word,
  language (grammar_union grammars) (V (start Vt))
           (to_phrase word)
  <->
  exists s_l,
    language (snd s_l) (fst s_l) (to_phrase word)
    /\
    In s_l grammars.
```

The Final Theorem

Theorem

*For any two decidable types **Tt** and **Nt** for types of terminals and nonterminals correspondingly. If there exists a bijection from **Nt** to $\mathbb{N}$ and syntactic analysis is possible (in the sense of our definition), then for any DFA **dfa** and any context-free grammar **G**, there exists the context-free grammar G_{INT}, such that $L(G_{INT}) = L(G) \cap L(dfa)$.*

The Final Theorem in Coq

```
Theorem grammar_of_intersection_exists:
  exists
   (NewNonterminal: Type)
   (IntersectionGrammar: @grammar Terminal NewNonterminal)
   St,
  forall word,
    dfa_language dfa word /\ language G S (to_phrase word)
    <->
    language IntersectionGrammar St (to_phrase word).
```

Conclusion

- We present mechanized in Coq proof of the Bar-Hillel theorem on the closure of context-free languages under intersection with the regular languages
- We generalize the results of Jana Hofmann and Gert Smolka
  - The definition of the terminal and nonterminal alphabets in context-free grammar were made generic
  - All related definitions and theorems were adjusted to work with the updated definition
- All results are published at GitHub and are equipped with automatically generated documentation

Future work

- Ruy J. G. B. de Queiroz vs Jana Hifmann
  - We use results of Jana Hofman
  - Results of Ruy J. G. B. de Queiroz looks more mature
  - Is it possible to create one "true" solution in this area?
    - ★ Wether our grammar-based proof is always better then PDA-based one?

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    - ★ Wether our grammar-based proof is always better then PDA-based one?
- Mechanization of practical algorithms which are just implementation of the Bar-Hillel theorem
  - Context-free path querying algorithm, based on CYK or even on GLL parsing algorithm
  - Certified algorithm for context-free constrained path querying for graph databases

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- Sources: https://github.com/YaccConstructor/YC_in_Coq

Thanks!