Transparent Boundary Conditions for the Equation of Rod Transverse Vibrations and Compact Approximation Scheme

Vladimir A. Gordin, Aleksandr A. Shemendyuk*

National Research University Higher School of Economics, Moscow, Russia

Hydrometeorological Centre of Russia, Moscow, 123242, Russia

Abstract

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1. Introduction

$$\rho\frac{\partial^2 u}{\partial t^2} - R^2\rho\frac{\partial^4 u}{\partial x^2\partial t^2} + ER^2\frac{\partial^4 u}{\partial x^4} = 0. \tag{1}$$

2. Implicit Compact Scheme

$$d(u^n_{m-2} + u^n_{m+2}) + a(u^{n-1}_{m-1} + u^{n-1}_{m+1} + u^{n+1}_{m-1} + u^{n+1}_{m+1}) + c(u^n_{m-1} + u^n_{m+1}) + bu^n_m + u^{n-1}_m + u^{n+1}_m = 0, \tag{2}$$

where $a = \frac{3}{12\mu+4} - \frac{1}{2}$, $b = \frac{9\nu}{3\mu+1} - 2$, $c = 1 - \frac{12\nu+3}{6\mu+2}$, $d = \frac{3\nu}{6\mu+2}$, and the dimensionless parameters $\nu = \frac{ER^2}{\rho}\cdot\tau^2 h^{-4}$, $\mu = R^2\cdot h^{-2}$.

3. DTBCs for Compact Approximation Scheme

After Z-transform of (2) the homogeneous characteristic equation is

$$dz\lambda^4 + (a(z^2+1) + cz)\lambda^3 + (z^2 + 1 + bz)\lambda^2 + (a(z^2+1) + cz)\lambda + dz = 0. \tag{3}$$

We rearrange terms in (3) and divide both parts by $\lambda^2 \neq 0$ and $z^2 \neq 0$ to get

$$dz^{-1}(\lambda + \lambda^{-1})^2 + (a(1 + z^{-2}) + cz^{-1})(\lambda + \lambda^{-1}) + 1 + z^{-2} + bz^{-1} - 2dz^{-1} = 0. \tag{4}$$

Finally, we introduce change of variables $\eta = \lambda + \lambda^{-1}$ and $\omega = z^{-1}$ to get

$$d\omega\eta^2 + (a(1+\omega^2) + c\omega)\eta + \omega^2 + (b - 2d)\omega + 1 = 0. \tag{5}$$

The roots of (5) are

$$\eta_{1,2}(\omega) = -\frac{a(1+\omega^2) + c\omega}{2d\omega} \pm \frac{1}{2}\sqrt{\left(\frac{a(1+\omega^2)+c\omega}{d\omega}\right)^2 - 4\cdot\frac{\omega^2 + (b-2d)\omega + 1}{d\omega}}. \tag{6}$$

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*Corresponding author
Email addresses: vagordin@mail.ru (Vladimir A. Gordin), aashemendyuk@edu.hse.ru (Aleksandr A. Shemendyuk)

3.1. Eta series

$$\eta_{1,2}(\omega) = \frac{(6\mu-1)\omega^2 - 2(-12\nu+6\mu-1)\omega + 6\mu - 1}{12\nu\omega} \pm \frac{(6\mu-1)(1-\omega)\sqrt{\omega^2 - 2\left(1+\frac{72\nu}{(6\mu-1)^2}\right)\omega + 1}}{12\nu\omega} \tag{7}$$

$$\eta_{1,2}(\omega) = \frac{1}{12\nu\omega}\left[(6\mu-1)\omega^2 - 2(-12\nu+6\mu-1)\omega + 6\mu - 1 \pm (6\mu-1)(1-\omega)\sqrt{\omega^2 - 2\left(1+\frac{72\nu}{(6\mu-1)^2}\right)\omega + 1}\right]$$

Now we use the following formula for the Legendre polynomials generating function:

$$(\omega^2 - 2\varepsilon\omega + 1)^{-1/2} = \sum_{n=0}^{\infty} \mathrm{P}_n(\varepsilon)\omega^n$$

and obtain for $\mu > 1/6$

$$\eta_{1,2}(\omega) = \frac{6\mu-1}{12\nu\omega}\left[\omega^2 - 2\frac{-12\nu+6\mu-1}{6\mu-1}\omega + 1 \pm (1-\omega)\left(\omega^2 - 2\left(1+\frac{72\nu}{(6\mu-1)^2}\right)\omega + 1\right)\sum_{n=0}^{\infty}\mathrm{P}_n\left(1+\frac{72\nu}{(6\mu-1)^2}\right)\omega^n\right] \tag{8}$$

3.2. Eta branching

Consider the points ω were functions $\eta_{1,2}(\omega)$ have branching. From Eq. (7), this happens when

$$(6\mu-1)(1-\omega)\sqrt{\omega^2 - 2\left(1+\frac{72\nu}{(6\mu-1)^2}\right)\omega + 1} = 0, \tag{9}$$

i.e.

$$\omega_1 = 1, \quad \omega_{2,3} = 1 + \frac{72\nu}{(6\mu-1)^2} \pm \sqrt{\left(1+\frac{72\nu}{(6\mu-1)^2}\right)^2 - 1}.$$

Expanding the argument of the square root as difference of a squares, obtain

$$\omega_{2,3} = 1 + \frac{72\nu}{(6\mu-1)^2} \pm \sqrt{\frac{72\nu}{(6\mu-1)^2}\left(2+\frac{72\nu}{(6\mu-1)^2}\right)}.$$

Therefore, we obtain three branching points:

$$\omega_1 = 1, \quad \omega_{2,3} = 1 + \frac{72\nu}{(6\mu-1)^2} \pm \frac{1}{(6\mu-1)^2}\sqrt{72\nu\,(72\mu^2 - 24\mu + 2 + 72\nu)}, \tag{10}$$

Note that from Eq. (9) according to Vieta's theorem the product $\omega_2 \cdot \omega_3 = 1$, $\omega_2 > 1$ (ω_2 is taken from Eq. (10) with positive sign), and both points are real valued. Therefore, point $0 < \omega_3 < 1$ belongs to the unit circle on the complex plain.

3.3. Lambda series

$$\lambda_{1,3} = \frac{\eta_1(\omega)}{2} \mp \sqrt{\frac{\eta_1^2(\omega)}{4} - 1} \tag{11}$$

$$\lambda_{2,4} = \frac{\eta_2(\omega)}{2} \mp \sqrt{\frac{\eta_2^2(\omega)}{4} - 1} \tag{12}$$

$$\lambda_{1,3} = \frac{\eta_1(\omega)}{2} \mp \sqrt{\frac{\eta_1(\omega)}{2} - 1} \cdot \sqrt{\frac{\eta_1(\omega)}{2} + 1}$$

As mentioned before, we have $\eta_1(\omega) \to -\frac{1}{a} + r(\omega)$, where $r(\omega) \to 0$ as $\omega \to 0$. Therefore,

$$\lambda_{1,3} = \frac{\eta_1(\omega)}{2} \mp \sqrt{-\frac{1}{2a} + \frac{r(\omega)}{2} - 1} \cdot \sqrt{-\frac{1}{2a} + \frac{r(\omega)}{2} + 1}$$

$$\lambda_{1,3} = \frac{\eta_1(\omega)}{2} \mp \sqrt{-\frac{1}{2a} - 1} \cdot \sqrt{1 + \frac{r(\omega)}{-\frac{1}{a} - 2}} \cdot \sqrt{-\frac{1}{2a} + 1} \cdot \sqrt{1 + \frac{r(\omega)}{-\frac{1}{a} + 2}}$$

$$\lambda_{1,3} = \frac{\eta_1(\omega)}{2} \mp \sqrt{\frac{1}{4a^2} - 1} \cdot \sqrt{1 - \frac{r(\omega)}{\frac{1}{a} + 2}} \cdot \sqrt{1 + \frac{r(\omega)}{-\frac{1}{a} + 2}}$$

Then we use the Maclaurin series expansion of as square root

$$\sqrt{1 + x} = \sum_{n=0}^{\infty} \frac{(-1)^n \, (2n)!}{(1 - 2n) \, n! \, 4^n} x^n, \quad |x| < 1$$

to obtain

$$\lambda_{1,3} = \frac{\eta_1(\omega)}{2} \mp \sqrt{\frac{1}{4a^2} - 1} \cdot \sum_{n=0}^{\infty} \frac{(2n)!}{(1 - 2n) \, n! \, 4^n \left(\frac{1}{a} + 2\right)^n} r^n(\omega) \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \, (2n)!}{(1 - 2n) \, n! \, 4^n \left(2 - \frac{1}{a}\right)^n} r^n(\omega).$$

4. Initial Data Construction

5. Results of Numerical Experiments

6. Conclusion

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References