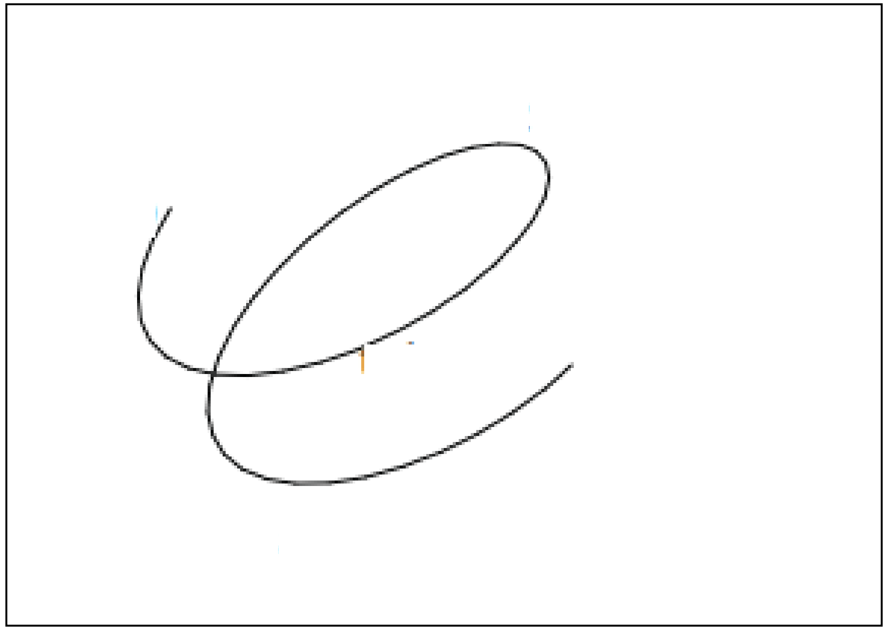
Lecture 7:

Elements of differential geometry and topology (short basics)

1. Curves
2. Surfaces
3. Manifolds: examples
4. Topological spaces
5. Manifolds: definition
6. Tangent spaces
7. Riemannian structure
8. Geodesic lines

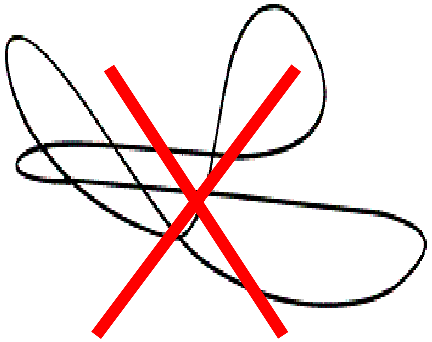
Curves

$$\mathbf{X}(t) = \begin{pmatrix} x_1(t) \\ \dots \\ x_p(t) \end{pmatrix}$$ - **smooth** mapping: $t \in (a, b) \subset R^1 \to R^p$



$\Gamma = \{\mathbf{X}(t),\ a < t < b\}$ - a curve in R^p,

$\mathbf{X}(t)$ - curve parameterization by parameter t (time)



The curve Γ has intrinsic dimension $q = 1$

$t' = \varphi(t)$: **smooth one-to-one** mapping from (a, b) to $(a' = \varphi(a),\ b' = \varphi(b))$ $\quad \varphi'(t) \neq 0$

$$\Gamma = \{\mathbf{X}(t),\ a < t < b\} = \{\mathbf{X}(\varphi^{-1}(t')),\ a' < t' < b'\} = \{\mathbf{X}'(t'),\ a' < t' < b'\}$$

- curve reparameterization

$$\dot{\mathbf{X}}(s) = \begin{pmatrix} \dot{x}_1(s) \\ \dots \\ \dot{x}_p(s) \end{pmatrix} = \frac{\partial \mathbf{X}(t)}{\partial t}(s)$$

$\frac{\partial \mathbf{X}(t)}{\partial t} \neq 0,\ a < t < b,$ - regular curve

Curve regularity does not depend on parameterization

$$t' = \varphi(t) \qquad \rightarrow \qquad t = \varphi^{-1}(t'),\ \ \mathbf{X}(t) = \mathbf{X}(\varphi^{-1}(t')) = \mathbf{X}'(t')$$

$$\frac{\partial \mathbf{X}'(t')}{\partial t'} = \frac{\partial \mathbf{X}(\varphi^{-1}(t'))}{\partial t'} = \frac{\partial \mathbf{X}(t)}{\partial t}(\varphi^{-1}(t')) \times \left(\frac{\partial \varphi(t)}{\partial t}(\varphi^{-1}(t'))\right)^{-1} \neq 0$$

$L(\Gamma) = \int_a^b \left|\dot{\mathbf{X}}(t)\right| dt$ - curve length

Curve length does not depend on parameterization

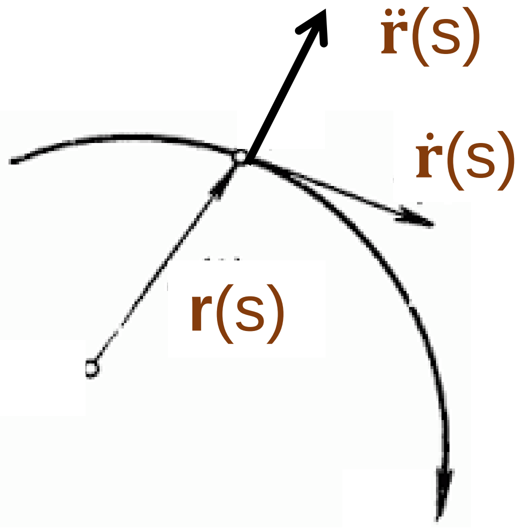
$$\int_{a'}^{b'} \left|\dot{\mathbf{X}}'(t')\right| dt' = \int_{a'}^{b'} \left|\frac{\partial \mathbf{X}(\varphi^{-1}(t'))}{\partial t'}\right| dt' = \int_{a'}^{b'} \left|\frac{\partial \mathbf{X}(t)}{\partial t}(\varphi^{-1}(t')) \times \left(\frac{\partial \varphi(t)}{\partial t}(\varphi^{-1}(t'))\right)^{-1}\right| dt' =$$

$$\int_a^b \left|\frac{\partial \mathbf{X}(t)}{\partial t}\right| \times \left|\left(\frac{\partial \varphi(t)}{\partial t}\right)^{-1}\right| \times d\varphi(t) = \int_a^b \left|\frac{\partial \mathbf{X}(t)}{\partial t}\right| \times \left|\frac{\partial \varphi(t)}{\partial t}\right|^{-1} \times \left|\frac{\partial \varphi(t)}{\partial t}\right| dt = \int_a^b \left|\dot{\mathbf{X}}(t)\right| dt$$

$s = \varphi(t)$ - natural parameter, $a' = \varphi(a)$, $b' = \varphi(b)$

$a' < c < d < b'$: $\int_c^d |\dot{\mathbf{r}}(s)| ds = d - s$

$\mathbf{r}(s) = \mathbf{X}(\varphi^{-1}(s))$ - natural parameterization, $a' < s < b'$



$\Gamma = \{\mathbf{r}(s),\ 0 < s < L(\Gamma)\}$ $\qquad$ $|\dot{\mathbf{r}}(s)| = 1$

$\dot{\mathbf{r}}(s)$ - tangent vector to the curve Γ at the point $\mathbf{r}(s)$

$$|\dot{\mathbf{r}}(s)|^2 = (\dot{\mathbf{r}}(s), \dot{\mathbf{r}}(s)) = 1$$

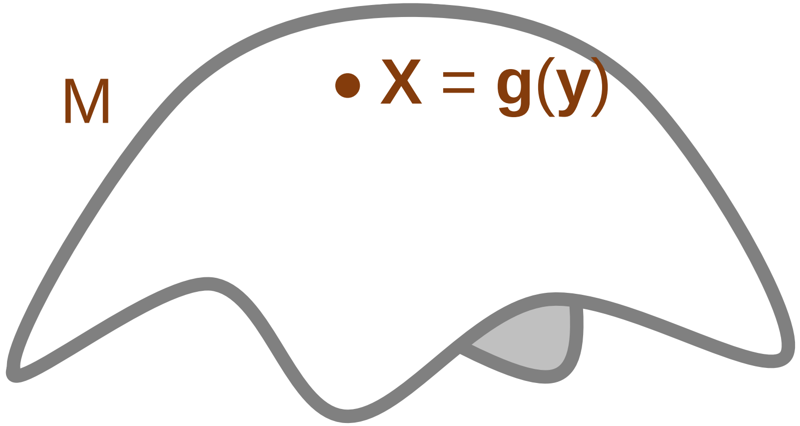
$$(\ddot{\mathbf{r}}(s), \dot{\mathbf{r}}(s)) + (\dot{\mathbf{r}}(s), \ddot{\mathbf{r}}(s)) = 0 \qquad (\dot{\mathbf{r}}(s), \ddot{\mathbf{r}}(s)) = 0$$

$k(s) = |\ddot{\mathbf{r}}(s)|$ - curvature of the curve Γ at the point $\mathbf{r}(s)$

$\mathbf{n}(s) = \ddot{\mathbf{r}}(s) / |\ddot{\mathbf{r}}(s)|$ - normal vector to the curve Γ at the point $\mathbf{r}(s)$

Surfaces

$$\mathbf{X} = \begin{pmatrix} x_1 \\ \cdots \\ x_p \end{pmatrix} = \mathbf{g}(\mathbf{y}) = \begin{pmatrix} g_1(\mathbf{y}) \\ \cdots \\ g_p(\mathbf{y}) \end{pmatrix}, \mathbf{y} = \begin{pmatrix} y_1 \\ \cdots \\ y_q \end{pmatrix} \in \mathbf{Y} \subset R^q, q < p, \mathbf{g}: R^q \to R^p$$ - **smooth** mapping



$M = \{\mathbf{g}(\mathbf{y}), \mathbf{y} \in \mathbf{Y}\}$ - q-dimensional surface in R^p

parameterized by q-dimensional parameter $\mathbf{y}$

The surface M has intrinsic dimension q

$M = \{\mathbf{g}(\mathbf{y}), \mathbf{y} \in \mathbf{Y}\}$ - q-dimensional surface in R^p parameterized by q-dimensional parameter $\mathbf{y}$

$\mathbf{X}' = \mathbf{g}(\mathbf{y}') = \mathbf{g}(\mathbf{y}) + J(\mathbf{y}) \times (\mathbf{y}' - \mathbf{y}) + \mathbf{o}(\mathbf{y}' - \mathbf{y})$ - Taylor expansion

$J(\mathbf{y})$ - $p \times q$ Jacobian matrix of the mapping $\mathbf{g}$: $R^q \to R^p$

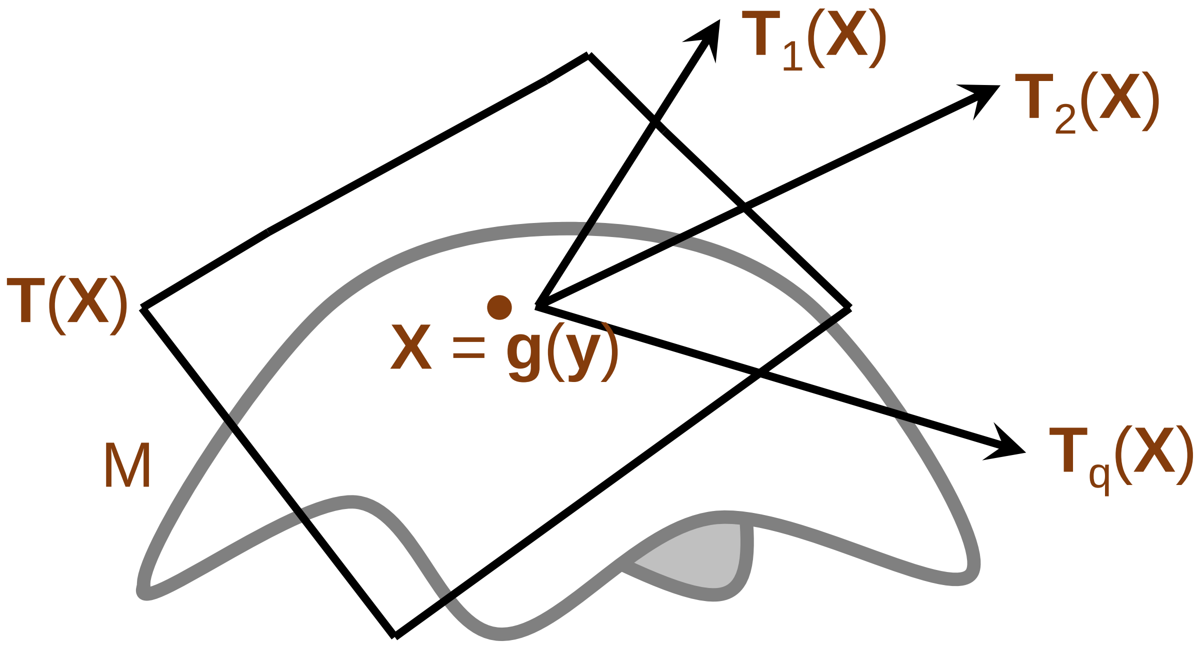
$$J(\mathbf{y}) = (\mathbf{t}_1(\mathbf{y}), \mathbf{t}_2(\mathbf{y}), \dots , \mathbf{t}_q(\mathbf{y}))$$

$$\mathbf{t}_k(\mathbf{y}) = \begin{pmatrix} t_{k1}(\mathbf{y}) \\ \dots \\ t_{kq}(\mathbf{y}) \end{pmatrix}, \qquad \mathbf{t}_{kj}(\mathbf{y}) = \frac{\partial g_k(\mathbf{y})}{\partial y_j}$$

$$\mathbf{T}(\mathbf{X}) = \{\mathbf{X} + J(\mathbf{y}) \times (\mathbf{e} - \mathbf{y}) : \mathbf{e} = \begin{pmatrix} e_1 \\ \cdots \\ e_q \end{pmatrix} \in R^q\}$$

- q-**dimensional** affine subspace in R^p passing through point $\mathbf{X} = \mathbf{g}(\mathbf{y}) \in M$

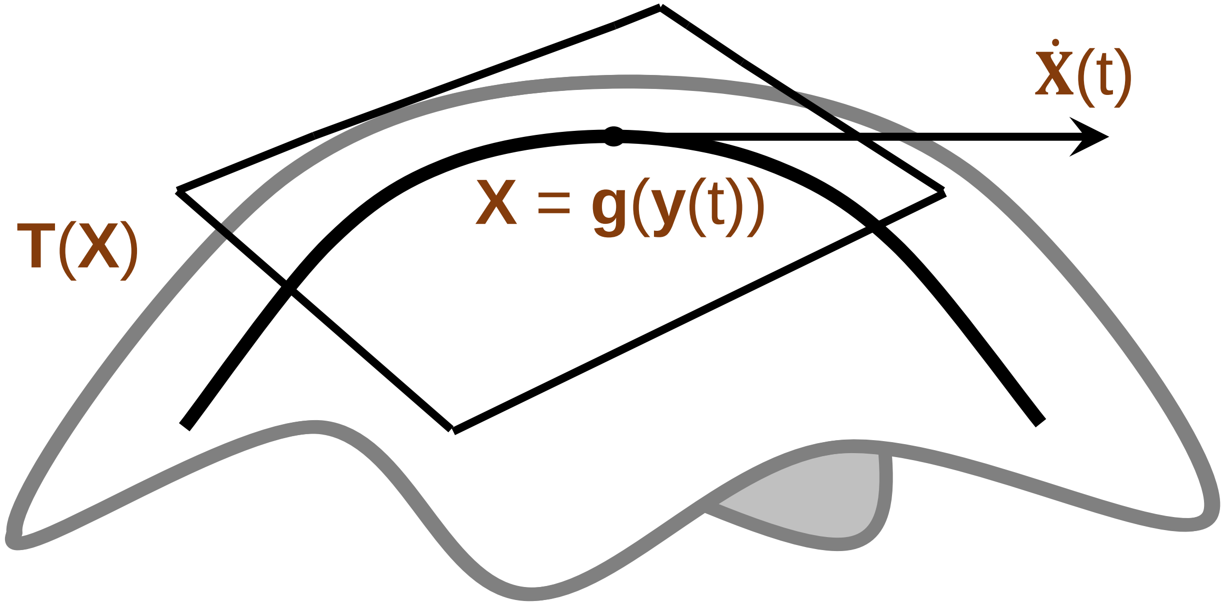
$\mathbf{T}(\mathbf{X}) = \{\mathbf{X} + \sum_{k=1}^{q} \mathbf{T}_k(\mathbf{X}) \times (e_k - y_k)\}$ - tangent space to the surface M at the point $\mathbf{X} = \mathbf{g}(y) \in M$



$\mathbf{t}_k(\mathbf{y}) = \mathbf{T}_k(\mathbf{X})$

$\mathbf{y}(t) = \begin{pmatrix} y_1(t) \\ \dots \\ y_q(t) \end{pmatrix} \in \mathbf{Y},\ t \in R^1$ - **smooth** curve in R^q

$\mathbf{X}(t) = \mathbf{g}(\mathbf{y}(t)) = \begin{pmatrix} g_1(\mathbf{y}(t)) \\ \dots \\ g_p(\mathbf{y}(t)) \end{pmatrix}$ - **smooth** curve in M



$\dot{\mathbf{X}}(t)$ - tangent vector to the curve $\mathbf{X}(t)$ at the point $\mathbf{X} = \mathbf{X}(t)$

$$\dot{\mathbf{X}}(t) = \frac{\partial}{\partial t}\mathbf{g}(\mathbf{y}(t)) = \frac{\partial}{\partial \mathbf{y}}\mathbf{g}(\mathbf{y}(t)) \times \dot{\mathbf{y}}(t) = \mathrm{J}(\mathbf{y}(t)) \times \dot{\mathbf{y}}(t) = \sum_{k=1}^{q} \mathbf{T}_k(X) \times \dot{y}_k(t) \in \mathbf{T}(\mathbf{X})$$

Curve length: $L(\Gamma) = \int_a^b |\dot{\mathbf{X}}(t)| dt = \int_a^b \left(\dot{\mathbf{X}}^T(t) \times \dot{\mathbf{X}}(t)\right)^{1/2} dt$

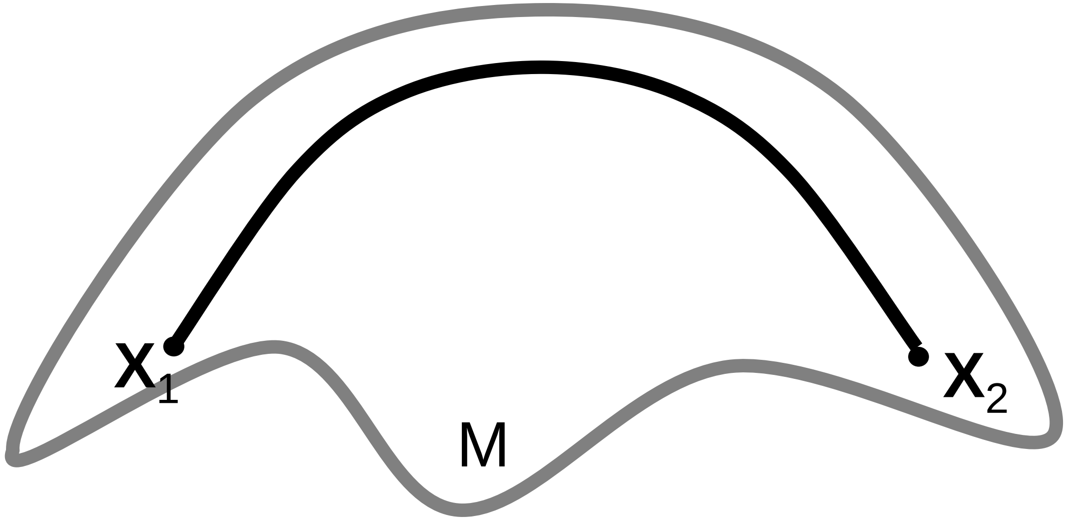
$$= \int_a^b \left(\dot{\mathbf{y}}^T(t) \times [J^T(\mathbf{y}(t)) \times J(\mathbf{y}(t))] \times \dot{\mathbf{y}}(t)\right)^{1/2} dt$$

$J^T(\mathbf{y})\times J(\mathbf{y})$ - $q\times q$ matrix $\rightarrow$ $G(\mathbf{X}) = J^T(\mathbf{y})\times J(\mathbf{y})$ - Riemannian (metric) tensor at point $\mathbf{X} = \mathbf{g}(\mathbf{y})$

$$G(\mathbf{X}) = \left\| \left(\mathbf{T}_i(\mathbf{X}), \mathbf{T}_j(\mathbf{X}) \right) \right\|$$

Curve length: $L(\Gamma) = \int_a^b \left(\dot{\mathbf{y}}^T(t) \times G(\mathbf{X}(t)) \times \dot{\mathbf{y}}(t)\right)^{1/2} dt$

$\mathbf{X}_1 , \mathbf{X}_2 \in M$



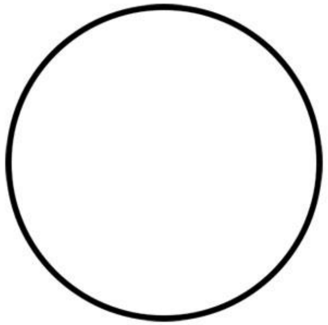
$\boldsymbol{\Gamma} = \boldsymbol{\Gamma}(\mathbf{X}_1 , \mathbf{X}_2) = \{\Gamma = \{\mathbf{X}(t), 0 \leq t \leq 1\} \in M: \mathbf{X}(0) = \mathbf{X}_1, \mathbf{X}(1) = \mathbf{X}_1\}$

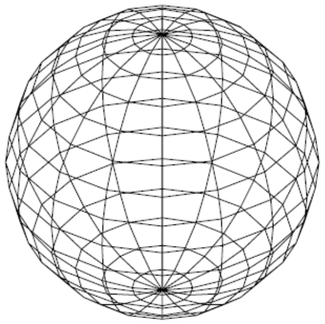
$\Gamma^* \in \boldsymbol{\Gamma}(\mathbf{X}_1 , \mathbf{X}_2) = \mathrm{argmin}_{\Gamma \in \boldsymbol{\Gamma}} L(\Gamma)$ - geodesic curve

Manifolds

Not always sets that are subsets of P-dimensional space R^P and having an intrinsic dimension $Q < P$

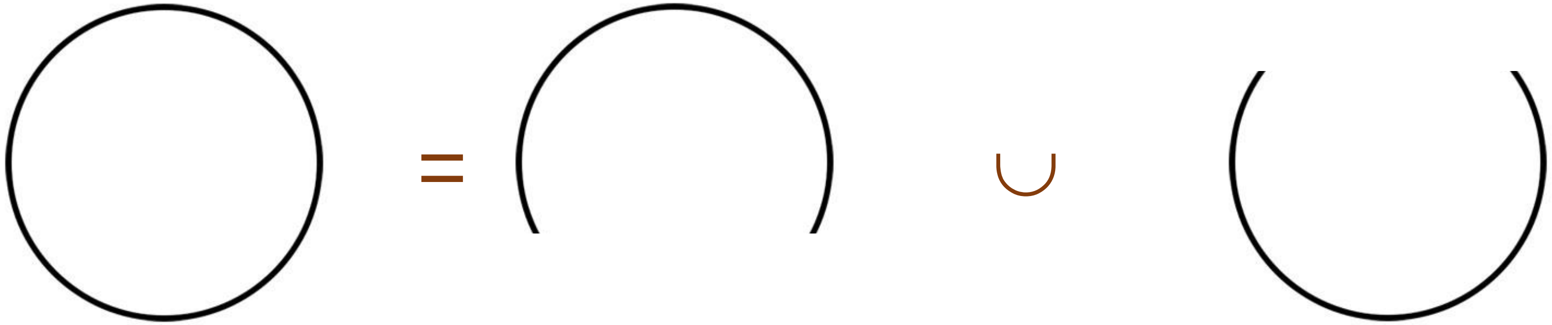
can be described as Q-dimensional surfaces in R^P





can not be described
using single mapping **g**

1-D sphere in 2-D   2-D sphere in 3-D



each ‘subsphere’ can be described
using single certain mapping

$$S(p, q) = \{p\times q \text{ orthogonal matrices } M:\ M^T \times M = I_q\}$$

$$M = \|X_{ki}\| = (\mathbf{X}_1 \quad \cdots \quad \mathbf{X}_q),\ \mathbf{X}_k = \begin{pmatrix} X_{k1} \\ \ldots \\ X_{kp} \end{pmatrix} \in R^p,\ k = 1, 2, \ldots, q$$

$$M = \mathbf{X} = \begin{pmatrix} \mathbf{X}_1 \\ \ldots \\ \mathbf{X}_q \end{pmatrix} \in R^{pq} \qquad S(p, q) \subset R^{pq}$$

$\varphi(\mathbf{X})$ - vector-function with components $\varphi_{kj}(\mathbf{X}) = (\mathbf{X}_k, \mathbf{X}_j) - \delta_{kj},\ 1 \le k \le j \le q$

$$\mathrm{Dim}(\varphi(\mathbf{X})) = q(q+1)/2 \qquad S(p, q) = \{\mathbf{X} \in R^{pq}: \varphi(\mathbf{X}) = \mathbf{0} \in R^{q(q+1)/2}\}$$

$$\mathrm{Dim}(S(p, q)) = pq - q(q+1)/2$$

$$S(p, q) \subset R^{pq} \qquad \mathrm{Dim}(S(p, q)) = pq - q(q+1)/2$$

- M is not $(pq - q(q+1)/2)$-dimensional surface in R^{pq}: it can not be described by using single mapping from $R^{pq-q(q+1)/2}$ to R^{pq}

- Inverse function theorem: for any point $\mathbf{X}_0 \in S(p, q)$, there exists an open subset $V(\mathbf{X}_0) \subset S(p, q)$, $\mathbf{X}_0 \in V$ ($V = B(X_0, \varepsilon) \cap S(p, q)$, $\varepsilon > 0$) and **smooth** mapping $\mathbf{f}: V \to \mathbf{Y} = \mathbf{f}(M) \subset R^{pq-q(q+1)/2}$, such that there exists smooth inverse mapping $\mathbf{g} = \mathbf{f}^{-1}: \mathbf{Y} \subset R^{pq-q(q+1)/2} \to V(\mathbf{X}_0) = g(\mathbf{Y}) \subset S(p, q)$

$$V(\mathbf{X}_0) = \{g(y), y \in \mathbf{Y} \subset R^{pq-q(q+1)/2}\}$$

Topological space

Let $\mathbf{X}$ be a set whose the elements X called points can be any mathematical objects

$\Sigma = \{\Omega\}$ consists of subsets $\Omega \subset \mathbf{X}$ called **open sets** satisfying the following properties:

- $\mathbf{X}, \varnothing \in \Sigma$
- $\Omega, \Omega' \in \Sigma \quad \rightarrow \qquad \Omega \cap \Omega' \in \Sigma$
- for any index set $A = \{\alpha\}$: such that $\Omega_\alpha \in \Sigma$: $\cup_{\alpha \in A}\Omega_\alpha \in \Sigma$

The set $\mathbf{X}$ with subsets' system Σ is called **Topological space**

If set $\mathbf{X}$ is metric space with metric $\rho(X, X')$, consider a system $\Sigma_0 = \{B(X, \varepsilon)\}$ consisting of the balls

$$B(X, \varepsilon) = \{X' \in \mathbf{X}: \rho(X, X') < \varepsilon\}$$

for all points $X \in \mathbf{X}$ and numbers $\varepsilon > 0$

Define system $\Sigma = \{\Omega\}$ in which subsets $\Omega \subset \mathbf{X}$ are unions of the balls

A subset $\Omega \subset \mathbf{X}$ is open one if for any point $X \in \Omega$ there exists a number $\varepsilon > 0$ such that $B(X, \varepsilon) \subset \Omega$

Example: $G(p, q)$ consists of all q-dimensional linear subspaces L in R^p

$L \in G(p, q)$: orthonormal vectors $e_1, e_2, \ldots , e_q \in R^d$ that form basis in $L = \mathrm{Span}(e_1, e_2, \ldots , e_q)$

Let E - $p \times q$ basis-matrix with columns $e_1, e_2, \ldots , e_q$: $L = \mathrm{Span}(E)$

$L = \mathrm{Span}(E)$, $L' = \mathrm{Span}(E') \in G(p, q)$: Binet-Cauchy metric $\rho_{BC}(L, L') = (1 - \mathrm{Det}^2(E^T \times E'))^{1/2}$

Binet-Cauchy metric $\rho_{BC}(L, L')$ does not depend on choice of basis-matrices E and E': other arbitrary basis-matrices E^* and $E^{*\prime}$ are: $E^* = E \times O$ and $E^{*\prime} = E' \times O'$, O and O' - orthogonal $q \times q$ matrices

$$\mathrm{Det}(E^{*T} \times E^{*\prime}) = \mathrm{Det}[(E \times O)^T \times (E' \times O')] = \mathrm{Det}[O^T \times (E^T \times E') \times O'] =$$

$$\mathrm{Det}(O^T) \times \mathrm{Det}(E^T \times E') \times \mathrm{Det}(O') = \mathrm{Det}(E^T \times E')$$

$B(L, \varepsilon) = \{L' \in G(p, q): \rho_{BC}(L, L') < \varepsilon\}$ - open ball

For point $x \in \mathbf{X}$ such that $x \in \Omega \in \Sigma$, subset Ω is called a neighborhood of point x

Topological space $(\mathbf{X}, \Sigma)$ is called **Hausdorff space** if for any two distinct points x and x' there exist their neighborhoods Ω and Ω', respectively such that $\Omega \cap \Omega' = \varnothing$

Let $(\mathbf{X}, \Sigma)$ and $(\mathbf{X}', \Sigma')$ be topological spaces. A mapping $f: (\mathbf{X}, \Sigma) \rightarrow (\mathbf{X}', \Sigma')$ is called **continuous** one if for any point $x \in \mathbf{X}$ and arbitrary neighborhood Ω' of the point $f(x) \in \mathbf{X}'$ there exists neighborhood Ω of the point x such that $f(\Omega) \subset \Omega'$

A continuous mapping $f: (\mathbf{X}, \Sigma) \rightarrow (\mathbf{X}', \Sigma')$ that has inverse mapping $f^{-1}: (\mathbf{X}', \Sigma') \rightarrow (\mathbf{X}, \Sigma')$, which is also continuous, is called **homeomorphism** - the mapping ‘continuous in both directions’

A subset $\mathbf{M} \subset \mathbf{X}$ of Hausdorff topological space $(\mathbf{X}, \Sigma)$ every point of which point $X \in \mathbf{M}$ has neighborhood U that is **homeomorphic** to an open subspace $\Omega \subset R^q$ of q-dimensional Euclidean space is called q-dimensional **manifold**

A homeomorphic mapping $h: U \subset \mathbf{M} \to h(U) = \Omega \subset R^q$

An inverse continuous mapping $g = h^{-1}: \Omega \subset R^q \to U \subset \mathbf{M}: U = \{X = g(y): y \in \Omega \subset R^q\}$

$U = \{X = g(y) \in \mathbf{M}: y \in \Omega \subset R^q\}$ - q-dimensional surface

Homeomorphism h - a **coordinate chart**; a pair (U, h) - a **map**

Let $A = \{\alpha\}$ - index set, (U_α, h_α) - a system of maps such that $\cup_{\alpha \in A}\Omega_\alpha = \mathbf{M}$ **-** a cover

A system $\{(U_\alpha, h_\alpha), \alpha \in A\}$ is called **atlas**

$X \in \mathbf{M}: X \in U_\alpha, h_\alpha(X) \in R^q$ - local coordinates

$X \in \mathbf{M}$: $X \in U = U_1 \cap U_2$, (U_1, h_1) and (U_2, h_2) - two different maps

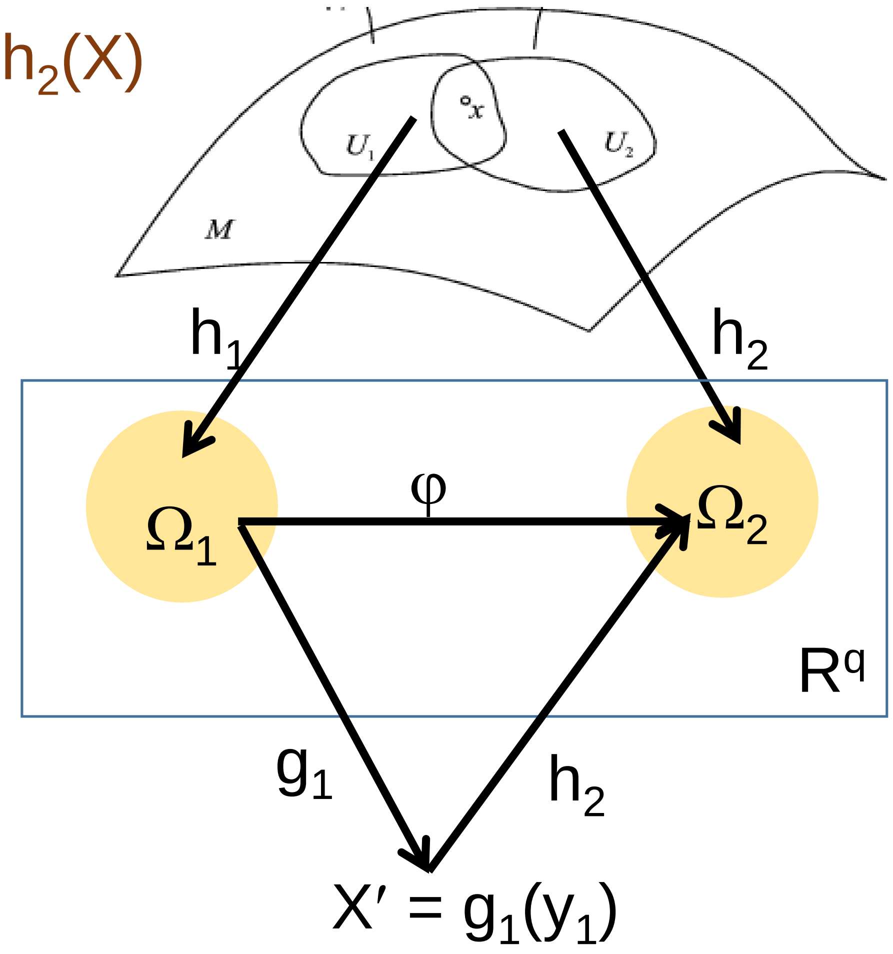
$X \in U$ has two different q-dimensional local coordinates: $h_1(X)$ and $h_2(X)$

$\Omega_1 = h_1(U) \subset R^q$ and $\Omega_2 = h_2(U) \subset R^q$

$g_1(y_1)$, $y_1 \in \Omega_1 \subset R^q$ - inverse mapping to h_1

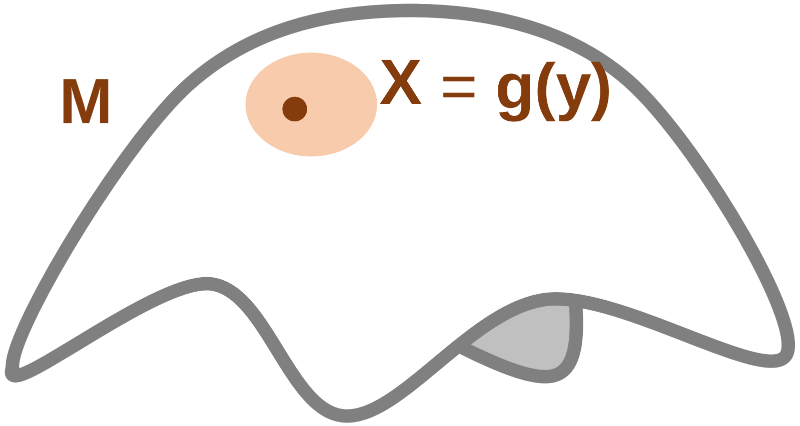
$\varphi: \Omega_1 \rightarrow \Omega_2$: $y_1 \in \Omega_1 \rightarrow y_2 = \varphi(y_1) = h_2(g_1(y_1)) \in \Omega_2$

$\varphi^{-1}(y_2) = h_1(g_2(y_2))$ - inverse mapping to φ



A q-dimensional **manifold M** is called **differentiable manifold** if the mappings φ (φ^{-1}): $R^q \rightarrow R^q$ are Infinitely differentiable functions (have derivatives of all orders)

Tangent space



Assume for simplicity that manifold **M** is a subset of Euclidean space

$\mathbf{X} \in U \subset \mathbf{M}$, (U, h) - map

$U = \{X' = g(y) \in \mathbf{M}: y \in \Omega \subset R^q\}$ - q-dimensional surface

Define the **Tangent space** $\mathbf{T(X)}$ to the manifold **M** at point **X**

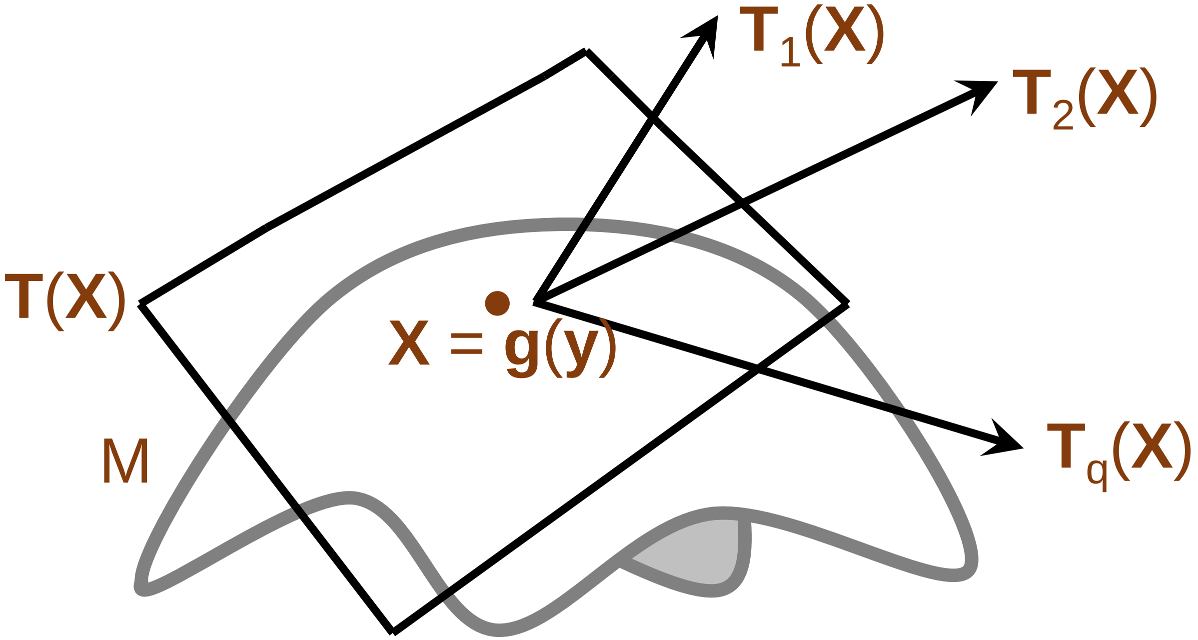
as tangent space to the surface $U = U(\mathbf{X})$

The Tangent space $\mathbf{T(X)}$ does not depend on

chosen neighborhood U of the point X**!!!**

$\{\mathbf{T}_k(\mathbf{X})\}$ - tangent vectors

$G(\mathbf{X}) = \left\|\left(\mathbf{T}_i(\mathbf{X}), \mathbf{T}_j(\mathbf{X})\right)\right\|$ - Gram matrix



Smooth Riemannian manifold

$\mathbf{T}(\mathbf{X})$ - tangent space, $\{\mathbf{T}_k(\mathbf{X}), k = 1, 2, \ldots, q\}$ - basis in $\mathbf{T}(\mathbf{X})$,

Gram matrix $G(\mathbf{X}) = \left\| \left(G_{ij}(\mathbf{X}) \right) \right\| = \left\| \left(\mathbf{T}_i(\mathbf{X}), \mathbf{T}_j(\mathbf{X}) \right) \right\|$

$t, t' \in \mathbf{T}(\mathbf{X})$ - tangent vectors:

$t = \sum_{k=1}^{q} a_k \times \mathbf{T}_k(X)$

$t' = \sum_{k=1}^{q} b_k \times \mathbf{T}_k(X)$

$(t, t') = \sum_{k,j=1}^{q} G_{kj}(\mathbf{X}) a_k b_j$ - inner product in tangent space

determined by matrix $G(\mathbf{X})$

Let manifold $\mathbf{M}$ is equipped with an inner product on the tangent space $\mathbf{T}(X)$ at each point $X \in \mathbf{M}$ that varies smoothly from point to point: if tangent vectors $t'(X), t'(X) \in T(X)$ are smooth vector fields on $\mathbf{M}$ ($t'(X)$ and $t'(X)$ are smooth functions) that mapping $X \rightarrow (t(X), t'(X))$ is smooth function.

Math: Every ***paracompact*** differentiable manifold admits a Riemannian metric

A family $\{G(X)\}$ is called Riemannian metric (tensor),

$(\mathbf{M}, G)$ is called Riemannian manifold

A Riemannian metric (tensor) makes it possible to define various geometric notions on a Riemannian manifold, such as angles, lengths of curves, curvature, gradients of functions, etc.

Geodesic curves

$(\mathbf{M}, G)$ - Riemannian manifold

$\mathbf{X}(t) \in \mathbf{M}$ for each $t \in (a, b)$ - $\Gamma = \{\mathbf{X}(t): t \in (a, b)\}$ - **smooth** curve in $\mathbf{M}$

$\dot{\mathbf{X}}(t) \in \mathbf{T}(\mathbf{X}(t))$ - tangent vector to the curve $\mathbf{X}(t)$ at the point $\mathbf{X}(t) \in \mathbf{M}$

Curve length: $L(\Gamma) = \int_a^b \left(\dot{\mathbf{X}}^T(t) \times G(\mathbf{X}(t)) \times \dot{\mathbf{X}}(t)\right)^{1/2} dt$

$\mathbf{X}_1, \mathbf{X}_2 \in \mathbf{M}$ $\quad \Gamma = \Gamma(\mathbf{X}_1, \mathbf{X}_2) = \{\Gamma = \{\mathbf{X}(t), 0 \le t \le 1\} \in \mathbf{M}: \mathbf{X}(0) = \mathbf{X}_1, \mathbf{X}(1) = \mathbf{X}_1\}$

$\Gamma^* \in \Gamma(\mathbf{X}_1, \mathbf{X}_2) = \mathrm{argmin}_{\Gamma \in \Gamma} L(\Gamma)$ - geodesic curve

$d(\mathbf{X}_1, \mathbf{X}_2) = \min_{\Gamma \in \Gamma} L(\Gamma) = L(\Gamma^*)$ - Riemannian metric on manifold

Math: For any point $X \in \mathbf{M}$ and for any vector $p \in \mathbf{T}(X)$ there exists a unique geodesic

$$\Gamma = \{\mathbf{X}(t) : t \in (a, b)\},\ 0 \in (a, b)$$

such that:

$$X(0) = X$$

$$\dot{\mathbf{X}}(0) = p$$