Statistics Part II

Outline

- Basic Probability Theory
- Bayes' Theorem
- Hypotheses and Statistical Tests

Basic Probability Theory

Basic Probability Theory

Probability is the measure of the likelihood that an event will occur. ***Probability*** quantifies as a number between 0 and 1, where, loosely speaking, 0 indicates impossibility and 1 indicates certainty. The higher the probability of an event, the more likely it is that the event will occur. [wikipedia]

Notation and Probability Axioms

we write $P(E) \in [0, 1]$ to denote the ***probability*** of event E. $P(E)$ follows the following axioms:

- $P(E) \in \mathbb{R}$, and $P(E) \geq 0$ for all possible E
- $P(\Omega) = 1$: it is certain that at least one event will occur
- $P\left(\cup_i E_i\right) = \sum_i P(E_i)$

Dependence of Events

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→ **multiplication** of probabilities if events are independent!

Example 1: Coin flip

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Example 2: Sex of children

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Conditional Probability

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where $P(E_1|E_2)$ is the ***conditional*** probability of event E_1 , aka the probability that E_1 occurs **after** $P(E_2) = 1$

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This results in:

- $P(E_1|E_2) = P(E_1)$ if events are independent
- $P(E_1|E_2) = P(E_1, E_2) \div P(E_2)$

Probability can be quite counter intuitive:

Examples based on child sex (assuming 50% chance for a boy or girl an independence for several children):

- **Example 1**: What is the probability of both children to be girls $P(B) := 0.25$, given the conditional event that the first child is a Girl $P(G) := 0.5$

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$$P(B|G) = P(B, G) \div P(G) = P(B) \div P(G) = 0.25 \div 0.5 = 0.5$$

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- **Example 2**: What is the probability of both children to be girls $P(B) := 0.25$, given the conditional event that at least one child is a girl $P(L) := 0.75$

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- **Example 2**: What is the probability of both children to be girls $P(B) := 0.25$, given the conditional event that at least one child is a girl $P(L) := 0.75$

$$P(B|L) = P(B, L) \div P(L) = P(B) \div P(L) = 0.25 \div 0.75 = 0.333$$

Bayes' Theorem

Bayes' Theorem

In probability theory and statistics, ***Bayes' theorem*** describes the ***probability*** of an event, based on ***prior knowledge*** of conditions that might be related to the event.

Bayes' Theorem

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} \text{ for } P(B) > 0$$

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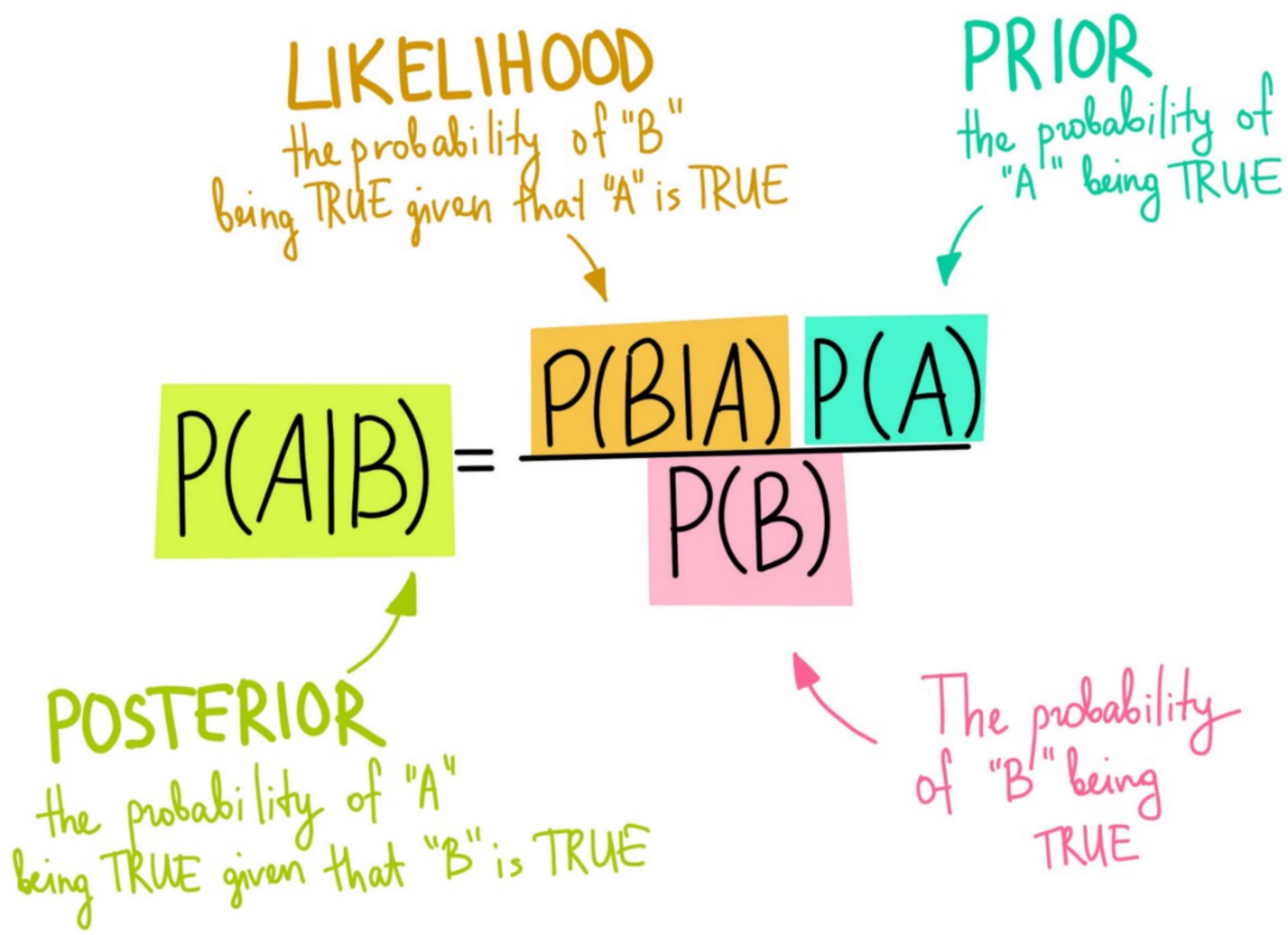
Interpretation:

- reversing the deduction of a conditional probability
- a feature often needed in inference (machine learning) settings (later more)


LIKELIHOOD
the probability of "B" being TRUE given that "A" is TRUE
PRIOR
the probability of "A" being TRUE
P(A|B) = P(B|A) P(A) / P(B)
POSTERIOR
the probability of "A" being TRUE given that "B" is TRUE
The probability of "B" being TRUE

Example:

Suppose that a test for using a particular drug is 99% sensitive and 99% specific. That is, the test will produce 99% true positive results for drug users and 99% true negative results for non-drug users. Suppose that 0.5% of people are users of the drug. What is the probability that a randomly selected individual with a positive test is a drug user?

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$$\begin{aligned}
P(\text{User} \mid +) &= \frac{P(+ \mid \text{User}) P(\text{User})}{P(+)} \\
&= \frac{P(+ \mid \text{User}) P(\text{User})}{P(+ \mid \text{User}) P(\text{User}) + P(+ \mid \text{Non-user}) P(\text{Non-user})} \\
&= \frac{0.99 \times 0.005}{0.99 \times 0.005 + 0.01 \times 0.995} \\
&\approx 33.2\%
\end{aligned}$$

Experiments: Hypotheses and Statistical Tests

Ultimately, we use **statistics** and analyze **probabilities** in order to **draw conclusions**. A common approach in statistics is to start an experiment with a ***hypotheses***, which is then validated by a statistical ***test***.

Statistical Inference Pipeline

- Formulate Hypotheses
- Design Experiment
- Collect Data
- Test / draw conclusions

Null-Hypotheses

- The statement being tested in a test of statistical significance is called the ***null hypothesis***. The test of significance is designed to assess the strength of the evidence against the null hypothesis. Usually, the null hypothesis is a statement of ***'no effect'*** or ***'no difference'***.
  It is often symbolized as H_0.
- The statement that is being tested against the null hypothesis is the alternative hypothesis H_1.
- ***Statistical significance test***: Very roughly, the procedure for deciding goes like this: Take a random sample from the population. If the sample data are consistent with the null hypothesis, then do not reject the null hypothesis; if the sample data are inconsistent with the null hypothesis, then reject the null hypothesis and conclude that the alternative hypothesis is true.

Significant Tests

- p-Value (or significance): is, for a given statistical model, the probability that, when the null hypothesis is true, the statistical **test summary** would be greater than or equal to the actual observed results.
- at experiment design, a significance threshold α is chosen
- typically, $\alpha = 0.05$ for scientific experiments

t-Test

The ***t-Test*** (also called Student's t-Test) compares two averages (means) and tells you if they are different from each other. The t-Test also tells you how significant the differences are; In other words it lets you know if those differences could have happened by chance.

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$t = \sqrt{n}\frac{\bar{X}-\mu_0}{\sigma}$, where

- σ standard deviation (from n samples)
- $\bar{X}$ sampled mean from n samples
- μ_0 mean hypotheses

In [11]:
```
#Sample Size
N = 1000000
#Gaussian distributed data with mean = 0 and var = 0.1
a = np.random.normal(0,0.3,N)
#Gaussian distributed data with with mean = 0 and var = 0.2
b = np.random.normal(-0.0001,0.1,N)
plt.hist(a,bins=100)
plt.hist(b,bins=100)
## Cross Checking with the internal scipy function
t2, p2 = stats.ttest_ind(a,b)
print("means: ", a.mean(), b.mean())
print("t = " + str(t2))
print("p = " + str(p2))
```

```
means:  0.00026424482988052976 -3.7931362338061 98e-07
t = 0.836169199886757
p = 0.40305982287283526
```