Tensor Algebra

Outline

- What is Tensor?
- Quick introduction to ***Numpy***
- Basic Vector Algebra
- Basic Matrix Algebra
- Eigen Values - Problems and Applications
- Singular Value Decomposition

Disclaimer

This is not a math class!

- Terms and concepts are introduced as needed
- NOT complete!

What is a Tensor?

What is a Tensor?

In mathematics, a **tensor** is a geometric object that maps in a multi-linear manner geometric vectors, scalars, and other tensors to a resulting tensor. **Vectors and scalars** which are often used in elementary physics and engineering applications, are considered as the simplest tensors...
An elementary example of mapping, describable as a tensor, is the **dot product**, which maps two vectors to a scalar. ..." [Wikipedia]

NumPy - A Python Library for Arrays and Tensors

https://docs.scipy.org/doc/numpy/

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In [1]:
```
#comunity convention to name numpy "np"
import numpy as np
```

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NumPy introduction "on the fly" - detailed intro this afternoon in the Lab session.

Vector Arithmetic

Let's start simple: recall vector notation and some basic vector algebra.

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Notation: we wrtite $\vec{a}$ to denote elements of some vector space, e.g. $\vec{a} \in \mathbb{R}^n$

$\vec{a} := (a_0, a_1, \ldots, a_n), a_i \in \mathbb{R}$

See https://en.wikipedia.org/wiki/Vector_space for formal definition of vector spaces.

In [2]:
```
#in numpy we define vectors as 1D arrays
a=np.array([1,2,3,4])
a
```

Out[2]:
```
array([1, 2, 3, 4])
```

Basic Vector opperations

For some example vector space $\mathbb{R}^n$

- addition: $\vec{c} = \vec{a} + \vec{b} \rightarrow: \forall a, b \in \mathbb{R}^n : c \in \mathbb{R}^n$
- scalar multiplocation: $h\vec{a}, h \in \mathbb{R} := (ha_0, ha_1, \dots ha_n)$
- **dot product**: $< \vec{a}, \vec{b} >:= c$,where $c \in \mathbb{R}$

In [3]:
```python
#in numpy:
a=np.random.random(4)
b=np.random.random(4)
a+b*5
```

Out[3]:
```
array([5.36585606, 4.34191011, 4.58154376, 3.97513302])
```

In [3]:
```
#in numpy:
a=np.random.random(4)
b=np.random.random(4)
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Out[3]:
```
array([5.36585606, 4.34191011, 4.58154376, 3.97513302])
```

In [4]:
```
#WARNING:
a*b #element wise mult
```

Out[4]:
```
array([0.54462089, 0.66476672, 0.62213011, 0.03894683])
```

In [3]:
```
#in numpy:
a=np.random.random(4)
b=np.random.random(4)
a+b*5
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Out[3]:
```
array([5.36585606, 4.34191011, 4.58154376, 3.97513302])
```
In [4]:
```
#WARNING:
a*b #element wise mult
```
Out[4]:
```
array([0.54462089, 0.66476672, 0.62213011, 0.03894683])
```
In [5]:
```
a.dot(b) #this is a dot product !
```
Out[5]:
```
1.8704645486924747
```

Some more important vector opperations:

- vector norm (formal): $||\vec{a}|| := \sqrt{< \vec{a}, \vec{a} >}$
- eucledian norm: $||\vec{a}||_2 := \sqrt{\sum_i a_i^2}$

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- eucledian norm: $||\vec{a}||_2 \; := \sqrt{\sum_i a_i^2}$

In [6]:
```
#in numpy
a=np.array([2,2,2])
np.linalg.norm(a)
```

Out[6]:
```
3.4641016151377544
```

Some more important vector opperations:

- vector norm (formal): $||\vec{a}|| := \sqrt{< \vec{a}, \vec{a} >}$
- eucledian norm: $||\vec{a}||_2 := \sqrt{\sum_i a_i^2}$
- outer product (dyadic product): $\vec{a} \otimes \vec{b} := \vec{a}\vec{b}^T$ (Matrix product)

Matrix Algebra

Definition

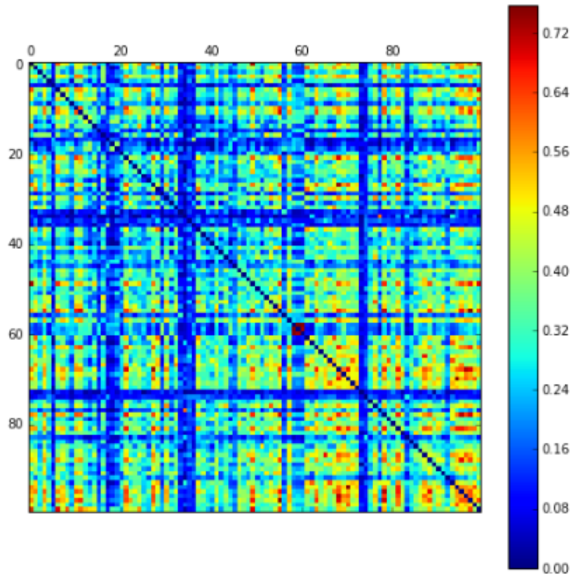
A **matrix** A is defined as a $m \times n$ 2d tensor (rank 2): $A := \begin{pmatrix} a_{00} & a_{01} & \dots & a_{0n} \\ \vdots & & & \vdots \\ a_{m0} & a_{m1} & \dots & a_{mn} \end{pmatrix}$

- e.g. with m **row vectors** $\in \mathbb{R}^n$ and n **column vectors** $\in \mathbb{R}^m$
- $\forall a_{ij} \in \mathbb{R}$

```
In [7]: #in numpy
        A=np.array([[1,2,3,4],[1,2,3,4],[5,6,7,8]])
        A

Out[7]: array([[1, 2, 3, 4],
               [1, 2, 3, 4],
               [5, 6, 7, 8]])
```

Motivation I: Matrix as data structure

- e.g. to store and process distances between objects



- we will see many examples where matricies hold
  - distances
  - correlations
  - afinity in graphs
  - ...

Motivation II: linear mappings / equation systems

Write system of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m, \end{aligned}$$

as $Ax = b$ with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

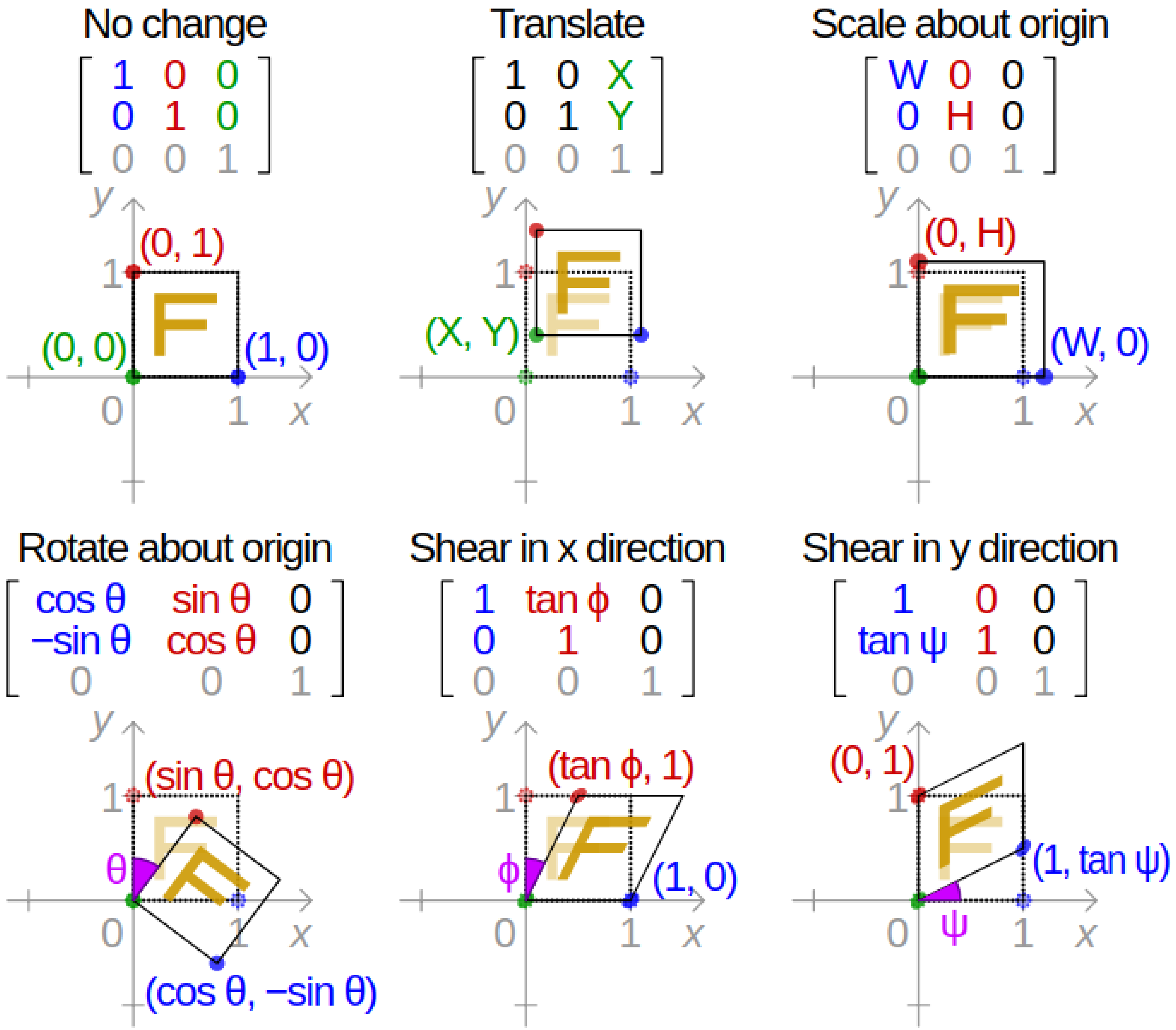
```
In [9]: # NumPy: Solve the system of equations
        #
        # 3 * x0 + x1 = 9 and
        # x0 + 2 * x1 = 8:
        a = np.array([[3,1], [1,2]])
        b = np.array([9,8])
        np.linalg.solve(a, b)

Out[9]: array([2., 3.])
```

Motivation III: Matrices as operators (tensors)

Matrices are widely use as operators (tensors) to apply transformations (mappings) on data vectors. E.g., to represent affine transformations with matrices, we can use **homogeneous coordinates**. This means representing a 2-vector (x, y) as a 3-vector $(x, y, 1)$.

$$\vec{a}' = A\vec{a}$$

Example: transformations of the unit square in $\mathbb{R}^2$ [wikipedia].



Special types of Matrices

- **Unit Matrix** (or Identity Matrix): $I_n := \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}$ with $I_m A = A I_n = A$

- **Symmetric Matrix**: square Matrix $(m = n)$ where $A^T = A$

- **Diagonal Matrix**: square Matrix where $a_{mn} := 0, \quad \forall m \neq m$

In [10]:
```
#in numpy
np.diag([1,2,3,4])
```

Out[10]:
```
array([[1, 0, 0, 0],
       [0, 2, 0, 0],
       [0, 0, 3, 0],
       [0, 0, 0, 4]])
```

In [10]:
```
#in numpy
np.diag([1,2,3,4])
```

Out[10]:
```
array([[1, 0, 0, 0],
       [0, 2, 0, 0],
       [0, 0, 3, 0],
       [0, 0, 0, 4]])
```

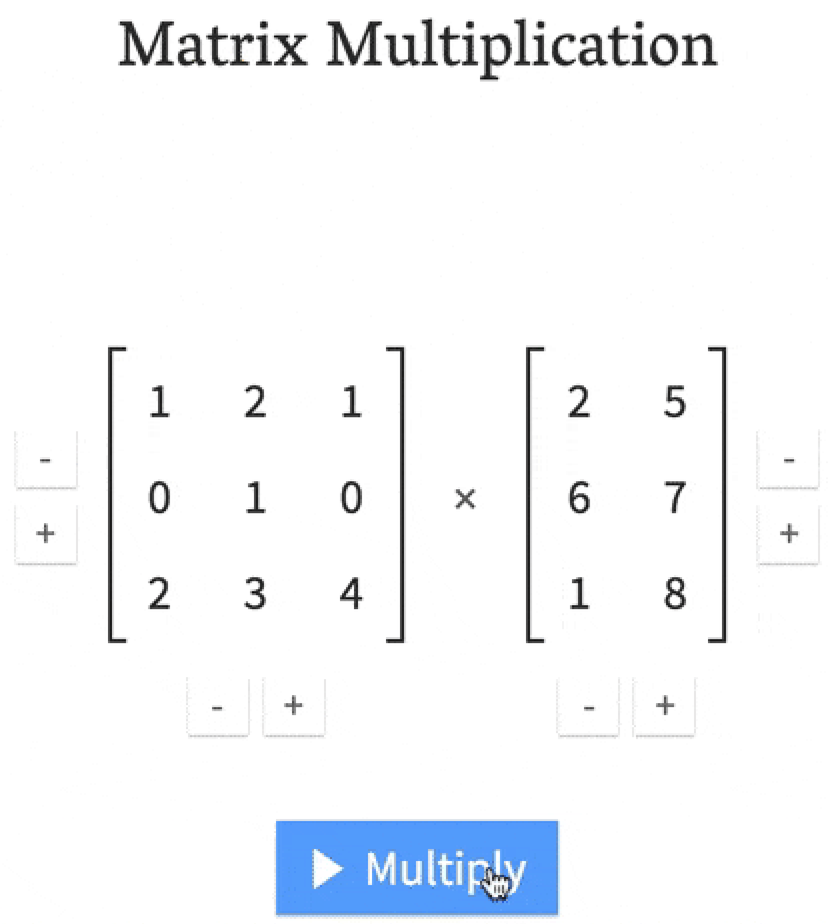
In [11]:
```
np.identity(4)
```

Out[11]:
```
array([[1., 0., 0., 0.],
       [0., 1., 0., 0.],
       [0., 0., 1., 0.],
       [0., 0., 0., 1.]])
```

Matrix Arithmetic

- **adding matrices** (element wise): $C := A + B$, where $c_{ij} := a_{ij} + b_{ij} \forall i \in m, j \in n$
- **skalar multiplication**: $C := hA$, where $c_{ij} := ha_{ij} \forall i \in m, j \in n$

- **inner product** (matrix multiplication):



https://www.reddit.com/r/educationalgifs/comments/5il2xm/matrix_multiplication/

Matrix Inverse

The inverse A^{-1} of a quadratic Matrix A is defined as: $A^{-1}A = AA^{-1} = I$

Complexity: $O(n^3)$ whens solving the above equation via Gauss-Jorden.
https://en.wikipedia.org/wiki/Computational_complexity_of_mathematical_operations

Changing the Basis of a Vector Space

Changing the Basis of a Vector Space

Definition:

a **set** B of elements (**vectors**) in a **vector space** V is called a **basis**, if every element of V may be written in a unique way as a (finite) **linear combination** of elements of B. The coefficients of this linear combination are referred to as components or **coordinates** on B of the vector. The elements of a basis are called basis vectors [wikipedia].

Simple Example:

In $\mathbb{R}^2$, the ***eucledian*** basis is the set of the two vectors $\overrightarrow{e_0} := (1,0)^T$, $\overrightarrow{e_1} := (0,1)^T$.

Every point $\vec{p} := (p_0, p_1) \in \mathbb{R}^2$ can be expressed by it's **coordinates** x, y in the form of $p_0 := x\overrightarrow{e_0}$ and $p_1 := y\overrightarrow{e_1}$

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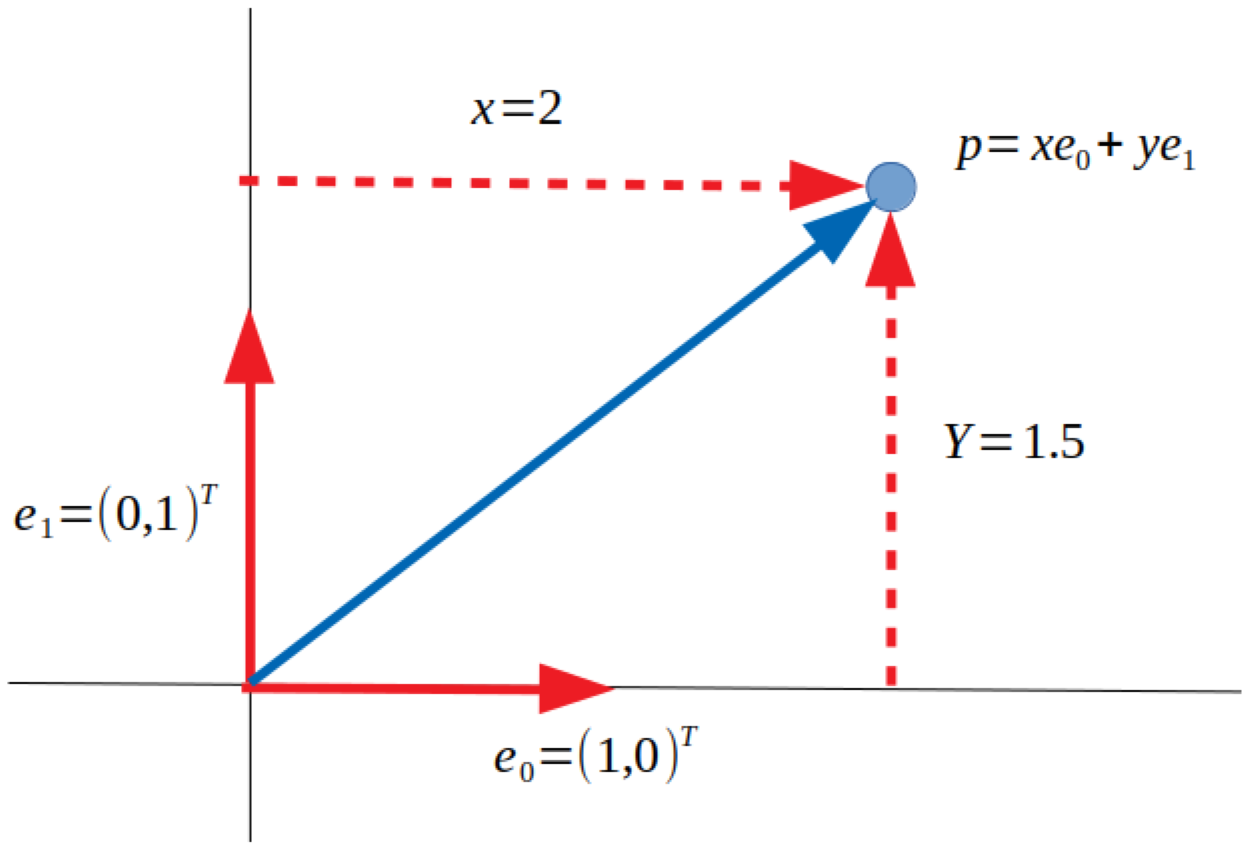
Every point $\vec{p} := (p_0, p_1) \in \mathbb{R}^2$ can be expressed by it's **coordinates** x, y in the form of $p_0 := x\vec{e_0}$ and $p_1 := y\vec{e_1}$



Alternative Basis

The basis of vector space is not unique: it is very easy to find new sets of basis vectors.

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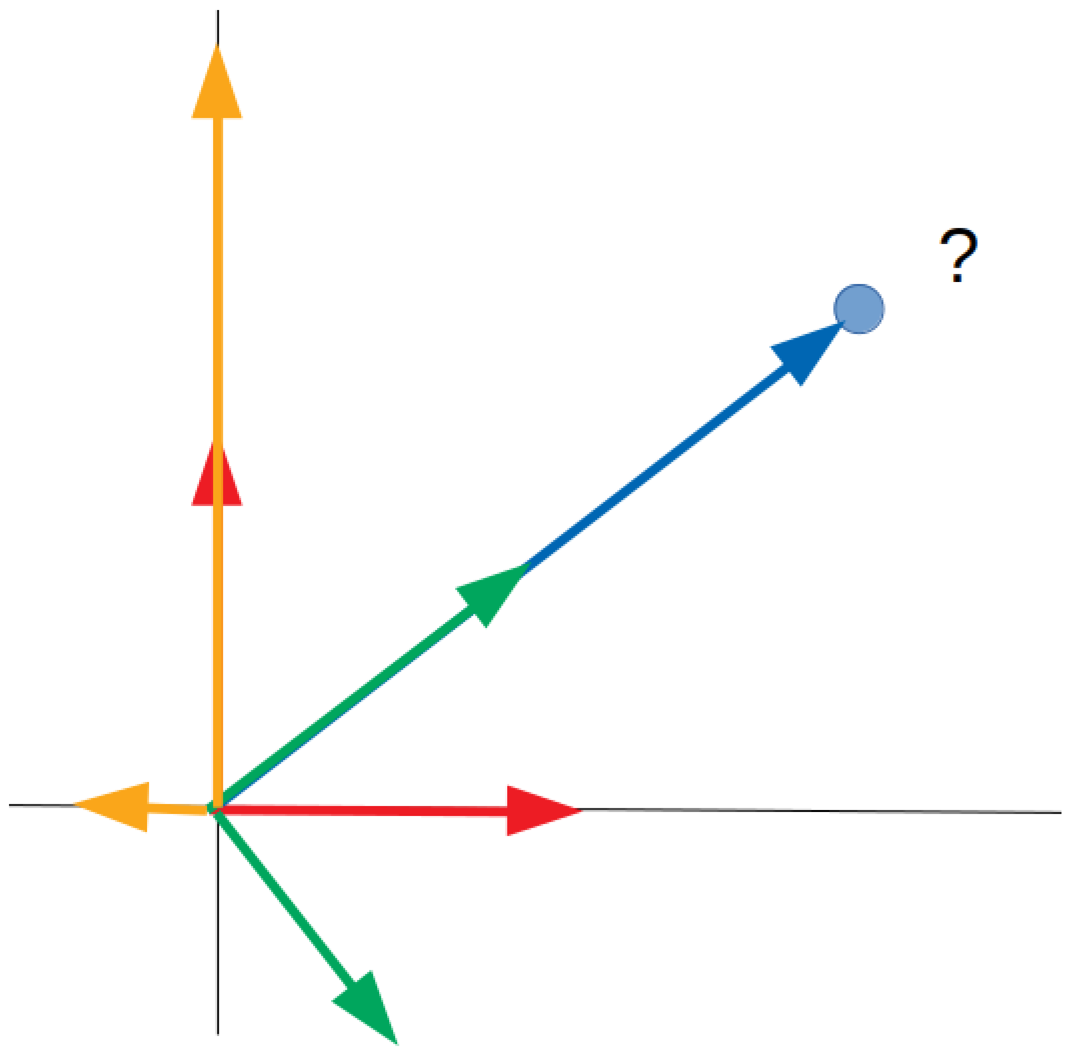


Optional Properties of Basis Sets

- othortogonal
- orthonormal

Eigen Decomposition

Eigen Decomposition

One of the re-accuring questions is: how to find the ***"best"*** basis for a given problem/data. Decomposition into ***Eigen Values*** and ***Eigen Vectors*** provide a comon solution:

Definition

An eigenvector of a linear transformation T is a non-zero vector that changes by only a scalar factor when that linear transformation is applied to it. This condition can be written as:

$$T(\mathbf{v}) = \lambda \mathbf{v}$$

where λ is a scalar, known as the ***eigenvalue*** associated with the ***eigenvector*** v.

If the vector space V is finite-dimensional, then the linear transformation T can be represented as a **square matrix** A, and the vector v by a **column vector**, rendering the above mapping as a matrix multiplication on the left-hand side and a scaling of the column vector on the right-hand side in the equation [wikipedia]

$$A\mathbf{v} = \lambda \mathbf{v}$$

Eigen Decomposition

for a squared, diagonizable matrix M of size $n \times n$:

- we can find n **eigen vectors** q_i with **eigen values** λ_i
- we can decompose M into $M = Q\Lambda Q^{-1}$
  - where Q is a matrix of the eigenvectors
  - Λ a diagonal matrix with the λ_i on the diagonal

Intuition

- Number of non zero eigenvalues gives the "intrinsic dimension/rank" of the data
- Eigenvectors form **new basis**

In [12]:
```
#a numpy example
A=np.diag((1, 1, 0))
A
```

Out[12]:
```
array([[1, 0, 0],
       [0, 1, 0],
       [0, 0, 0]])
```

In [12]:
```
#a numpy example
A=np.diag((1, 1, 0))
A
```

Out[12]:
```
array([[1, 0, 0],
       [0, 1, 0],
       [0, 0, 0]])
```

In [13]:
```
v,V=np.linalg.eig(A)
print(v)
```

```
[1. 1. 0.]
```

In [ ]:
```
print(V)
```

In [14]:
```
#a bit more complex example
A=np.array([[1,1,0,0],[3,3,0,0],[2,2,0,0],[4,4,0,0]])
v,V=np.linalg.eig(A)
```

In [14]:
```
#a bit more complex example
A=np.array([[1,1,0,0],[3,3,0,0],[2,2,0,0],[4,4,0,0]])
v,V=np.linalg.eig(A)
```

In [15]:
```
v[3]*V[3,:]
```

Out[15]:
```
array([ 0.        ,  4.        ,  0.        , -2.92118697])
```

Problems with Eigen Decompositions

- only for diagonizable, quared marticies
- but, matricies hoding data are usually not square (more data samples than data dimensions)

Sigular Value Decomposition

Definition [wikipedia]:

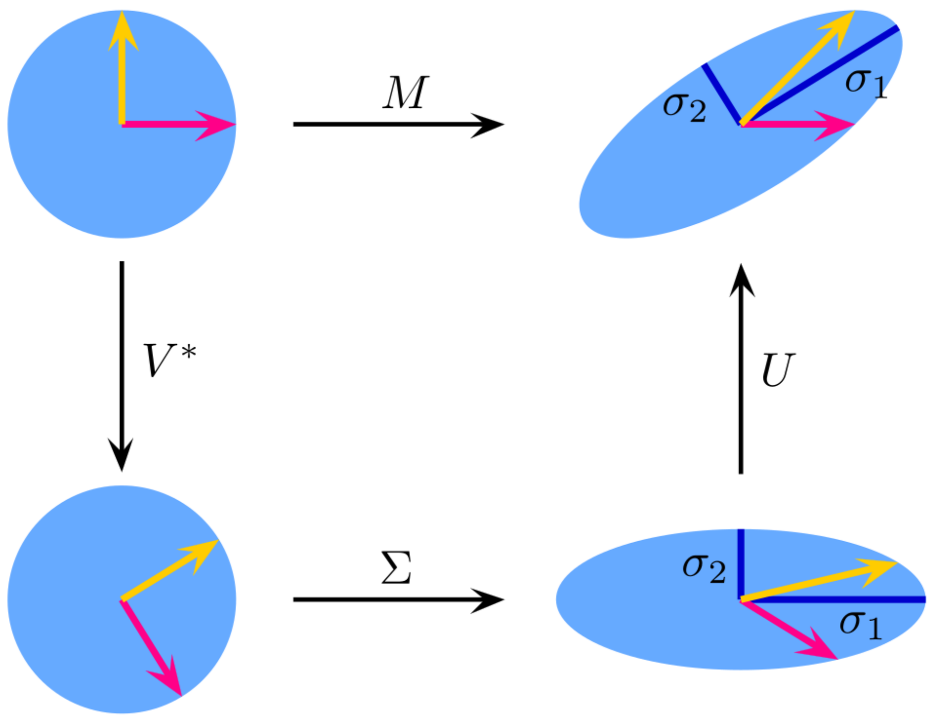
Suppose M is a $m \times n$ matrix whose entries come from the field of real numbers or the field of complex numbers. Then there exists a factorization, called a **singular value decomposition** of M, of the form

$$\mathbf{M} = \mathbf{U\Sigma V}^*$$

where

- U is an $m \times m$ unitary ($U^*U = I$) matrix,
- Σ is a diagonal $m \times n$ matrix with non-negative real numbers, the **singular values**, on the diagonal,
- V is an $n \times n$ unitary matrix, and V^* is the conjugate transpose of V.

Intuition: M is an $m\times$ m real square matrix with positive determinant: U, V^*, and Σ are real $m \times m$ matrices as well. Σ can be regarded as a **scaling** matrix, and U, V^* can be viewed as **rotation** matrices [wikipedia]:



$$M = U \cdot \Sigma \cdot V^*$$

In [16]:
```
#example
M = np.array([ [1, 0, 0, 0], [0,0,0,2], [0,3,0,0], [0,0,0,0], [2,0,0,0] ])
print(M)
U,S,V = np.linalg.svd(M)
```

```
[[1 0 0 0]
 [0 0 0 2]
 [0 3 0 0]
 [0 0 0 0]
 [2 0 0 0]]
```

In [16]:
```
#example
M = np.array([ [1, 0, 0, 0], [0,0,0,2], [0,3,0,0], [0,0,0,0], [2,0,0,0] ])
print(M)
U,S,V = np.linalg.svd(M)
```

```
[[1 0 0 0]
 [0 0 0 2]
 [0 3 0 0]
 [0 0 0 0]
 [2 0 0 0]]
```

In [17]:
```
U
```

Out[17]:
```
array([[ 0.        , -0.4472136 ,  0.        ,  0.        , -0.89442719],
       [ 0.        ,  0.        , -1.        ,  0.        ,  0.        ],
       [-1.        ,  0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  1.        ,  0.        ],
       [ 0.        , -0.89442719,  0.        ,  0.        ,  0.4472136 ]])
```

In [16]:
```
#example
M = np.array([ [1, 0, 0, 0], [0,0,0,2], [0,3,0,0], [0,0,0,0], [2,0,0,0] ])
print(M)
U,S,V = np.linalg.svd(M)
```

```
[[1 0 0 0]
 [0 0 0 2]
 [0 3 0 0]
 [0 0 0 0]
 [2 0 0 0]]
```

In [17]:
```
U
```

Out[17]:
```
array([[ 0.        , -0.4472136 ,  0.        ,  0.        , -0.89442719],
       [ 0.        ,  0.        , -1.        ,  0.        ,  0.        ],
       [-1.        ,  0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  1.        ,  0.        ],
       [ 0.        , -0.89442719,  0.        ,  0.        ,  0.4472136 ]])
```

In [18]:
```
S
```

Out[18]:
```
array([3.        , 2.23606798, 2.        , 0.        ])
```

In [16]:
```
#example
M = np.array([ [1, 0, 0, 0], [0,0,0,2], [0,3,0,0], [0,0,0,0], [2,0,0,0] ])
print(M)
U,S,V = np.linalg.svd(M)
```

```
[[1 0 0 0]
 [0 0 0 2]
 [0 3 0 0]
 [0 0 0 0]
 [2 0 0 0]]
```

In [17]:
```
U
```

Out[17]:
```
array([[ 0.        , -0.4472136 ,  0.        ,  0.        , -0.89442719],
       [ 0.        ,  0.        , -1.        ,  0.        ,  0.        ],
       [-1.        ,  0.        ,  0.        ,  0.        ,  0.        ],
       [ 0.        ,  0.        ,  0.        ,  1.        ,  0.        ],
       [ 0.        , -0.89442719,  0.        ,  0.        ,  0.4472136 ]])
```

In [18]:
```
S
```

Out[18]:
```
array([3.        , 2.23606798, 2.        , 0.        ])
```

In [19]:
```
V
```

Out[19]:
```
array([[-0., -1., -0., -0.],
       [-1., -0., -0., -0.],
       [-0., -0., -0., -1.],
       [-0., -0., -1., -0.]])
```

```
In [20]: #now: reconstruct M
         np.dot(U[:,:4]*S,V)

Out[20]: array([[1., 0., 0., 0.],
                [0., 0., 0., 2.],
                [0., 3., 0., 0.],
                [0., 0., 0., 0.],
                [2., 0., 0., 0.]])
```

In [20]:
```
#now: reconstruct M
np.dot(U[:,:4]*S,V)
```

Out[20]:
```
array([[1., 0., 0., 0.],
       [0., 0., 0., 2.],
       [0., 3., 0., 0.],
       [0., 0., 0., 0.],
       [2., 0., 0., 0.]])
```

In [21]:
```
#now reconstruct with loss, using only th first 2 of 4 singular values
np.dot(U[:,:2]*S[:2],V[:2,:])
```

Out[21]:
```
array([[1., 0., 0., 0.],
       [0., 0., 0., 0.],
       [0., 3., 0., 0.],
       [0., 0., 0., 0.],
       [2., 0., 0., 0.]])
```

Let's apply SVD to our recommender matrix R ...