MCV4U

CALCULUS & VECTORS

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Contents

1 Vectors

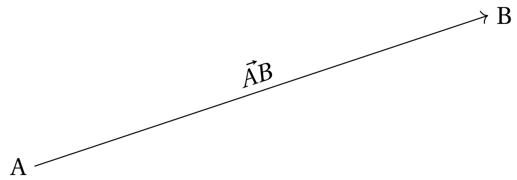
To Introduce, vectors are mathematical entities that extend our understanding beyond the one-dimensional quantities. Unlike scalar values that only have magnitude, vectors incorporate both magnitude and direction, offering a versatile toolkit for describing dynamic sytstems.

Scalar Vs. Vector Quantities

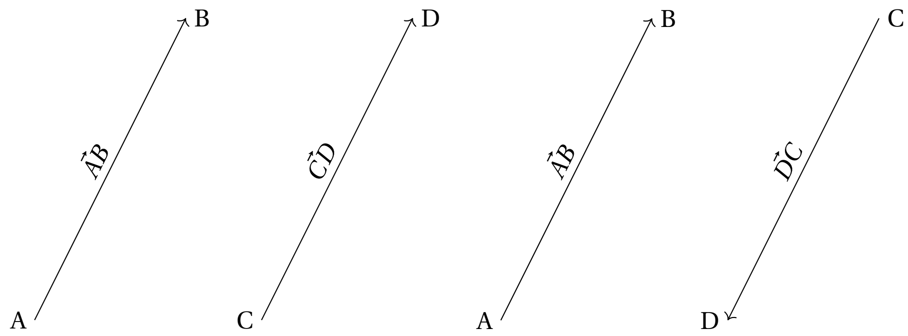
- **Scalar Quantities:** Mass, Temperature, Time, Distance, Speed, Energy, Work, Power, etc.
- **Vector Quantities:** Displacement, Velocity, Acceleration, Force, Momentum, etc.

When written in mathematical equations, vectors are usually represented via a a symbol with a vector indicator (i.e $\vec{v}$) or via a jointery of two points (i.e $\vec{AB}$ is a vector from point A to point B) Vectors can also be represented in many other ways, but the most common ways are, as denoted below, algebraically and geometrically.

- **Algebraically:** $\vec{a} = \langle x, y, z \rangle$ [1]
- **Geometrically:**



Vectors can be equal (or equivalent) to each other. For two vectors to be equal (or equivalent) they must have the same magnitude and direction. Vectors can also be opposite to each other; to be opposing vectors must have the same magnitude but opposite directions (i.e $\vec{AB} = -\vec{CD}$ as shown in **Figure 1b**).



(a) Equivalent Vectors **(b)** Equivalent Vectors

Figure 1: Equivalent and Opposite Vectors

[1] Vectors represented algebraically can also be written in column matrices

Vectors can also be parallel to each other. For two vectors to be considered parallel to each other they must have the same, or opposite direction. Although, they do not have to have the same magnitude. The symbol used to denote parallel vectors is the 'or' symbol (i.e $\vec{v} \parallel \vec{a}$).

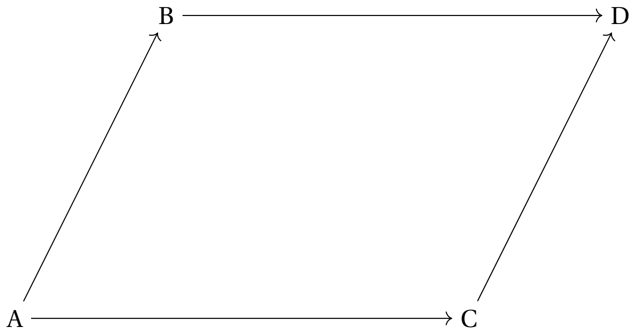


Figure 2: Parallel Vectors

Lastly, when dealing with vectors in any dimension, it is important to note the angle between two vectors. The angle between vectors is often reffered as angle θ where $\theta \leq 180°$ (although it can be denoted as any other variable). It is often in trigonometric laws, such as the cosine law, to solve for the different properties; such as the magnitude of a vector or the angle between differrent vectors.

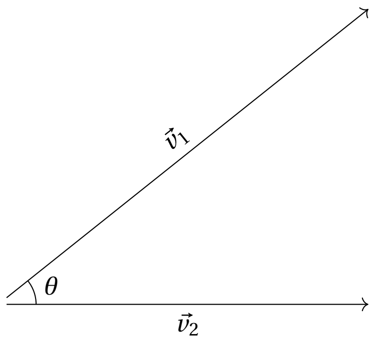


Figure 3: Angle θ Between Vectors $\vec{v}_1$ and $\vec{v}_2$

1.1 Vector Addition and Subtraction

When adding vectors together, a different result is produced compared to that of a scalar addition. When adding vectors together, the resultant vector is the sum of the two vectors. The resultant vector is a vector that begins at the tail-end of the first vector. The resultant is often labeled $\vec{r}$ as depicted in **Fig. 4**

Vector subtraction on the other hand is nothing more than adding the opposite of a vector. For example, given vectors $\vec{AB}$ and $\vec{BC}$, $\vec{AB} - \vec{BC}$ is the same as $\vec{AB} + (-\vec{BC})$. Vector subtraction has all the same properties as vector addition and also produces a resultant vectors as depicted in **Fig. 4**

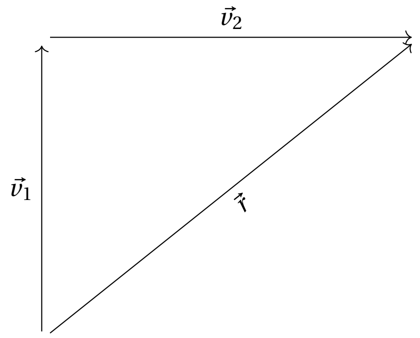
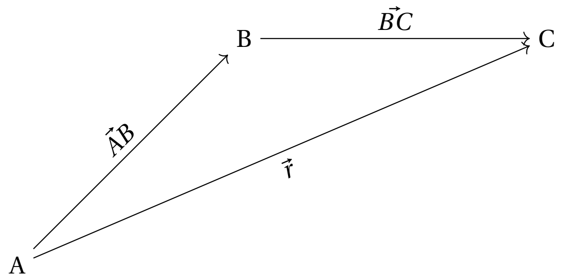
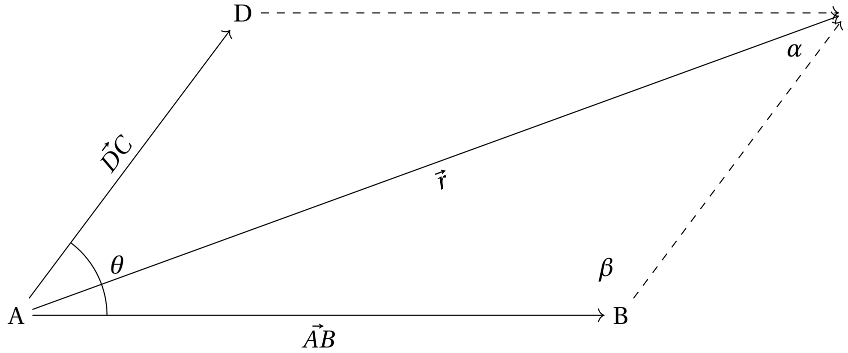


Figure 4: Resultant Vector

Consider the following vectors $\vec{AB}$, and $\vec{BC}$ respectively. To be added they must be positioned tip to tail as indicated below in such a way that $\vec{AB} = \vec{r}$



They can also be positioned tail to tail, so that a when built a pallalelogram is formed with the resultant vector splitting the pallalelogram in half diagonally as depicted in the diagram below.



When adding, or subtracing, vectors the magnitude of the resultant vector $\vec{r}$ can be found using the cosine law with adapted variables. The direction of the vector can also be defined by angles α, β, and γ using the sine law.

$$\|\vec{r}\| = \sqrt{\|\vec{a}\|^2 + \left\|\vec{b}\right\|^2 - 2\|\vec{a}\|\left\|\vec{b}\right\|\cos C}$$

(a) Magnitude using Cosine Law

$$\frac{\sin\alpha}{\|\vec{a}\|} = \frac{\sin\beta}{\left\|\vec{b}\right\|} = \frac{\sin\gamma}{\|\vec{r}\|}$$

(b) Direction using Sine Law

Figure 5: Vector magnitude and angle relationships using sine and cosine law

When it comes to vector addition, the angle between the vectors added matter when determining the magnitude and direction of the resultant vector. There are special cases, that although can be worked out, are simpler to memories and recognise when solving problems.

Special cases

1. $\theta = 0$ then $\left\|\vec{a} + \vec{b}\right\| = \|\vec{a}\| + \left\|\vec{b}\right\|$
2. $\theta = 90$ then $\left\|\vec{a} + \vec{b}\right\| = \sqrt{\|\vec{a}\|^2 + \left\|\vec{b}\right\|^2}$
3. $\theta = 180$ then $\left\|\vec{a} + \vec{b}\right\| = \|\vec{a}\| - \left\|\vec{b}\right\|$

1.2 Scalar Multiplication

When multiplying a vector by a scalar, some assumptions must be made. $\vec{v}$ must be a vector with magnitude and direction and a scalar k, where $k \in \mathbb{R}$. Assuming that those conditions are fufilled then a set of rules can be applied to determined the result of the multiplication as listed below.

1. If $k > 0$ then $k\vec{v}$ has the same direction as $\vec{v}$
2. If $k < 0$ then $k\vec{v}$ has the opposite direction as $\vec{v}$
3. If $k = 0$ then $k\vec{v} = \vec{0}$

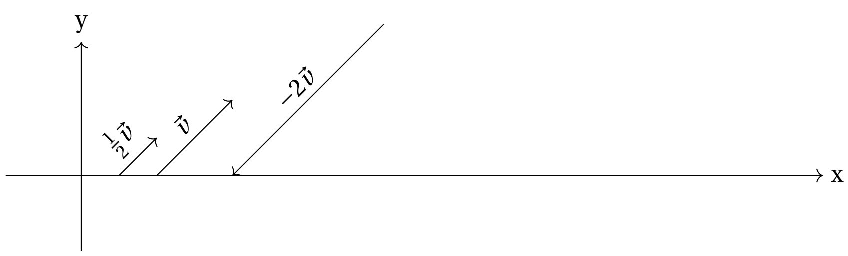


Figure 6: Examples of Scalar Multiplication

1.3 Properties of Vectors

1.4 Vectors as Forces

1.5 Vectors as Velocity

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