Complex Numbers:
Concepts, Formulas, etc.

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Concepts

1 Preface

These notes are written as an assignment for AP Calculus AB and are meant to be used by myself as a tool in any future university level mathematics class. If you are finding (or have received) these notes and are wishing to learn/study from them; be aware that they are tailored to myself and may not cover all concepts needed to properly learn/study complex numbers

2 Definition and Basics

To begin, a complex number is one where an "imaginary" part is present, which will be defined as i where $i = \sqrt{-1}$ moving onwards.

The imaginary part is in quotations as i is not "imaginary" but instead on a different plane comapred to our traditional number line. They are better refered to as lateral numbers but they will be referred to as "imaginary" numbers to better reflect modern mathematical vocabulary

2.1 Parts of a Complex Number

A complex number z can be defined by that addition of a real part a and imaginary part bi such that $z = a + bi$ where $a, b \in \mathbb{R}$.

I'll cover this in the next subsection, but it is important to note that 0 is also a real number and that real numbers can also be written in complex form.

2.2 Set Notation

Given z where $z = a + bi$ such that $a, b \in \mathbb{R}$; z can be said to be in the set of complex numbers $\mathbb{C}$ where $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$. This can also be written as $z \in \mathbb{C}$.

It is also important to note that all reals $\mathbb{R}$ are a subset of $\mathbb{C}$. This is inherently true as any real number x can be written as a complex number $a + bi$ where $b = 0$.

3 Arithmetic Operations

All complex numbers abide by their respective arithemtic rules in such a way where any arithmetic performed on a complex number returns another complex number as shown below

- **Addition** $(a + bi) + (c + di) = (a + c) + i(b + d)$
- **Substraction** $(a + bi) - (c + di) = (a - c) + i(b - d)$
- **Multiplication** $(a + bi)(c + di) = (ac - bd) + i(ad + bc)$
- **Division** $\frac{a+bi}{(c+di)} = \frac{(ac+bd)}{c^2+d^2} + \frac{i(bc-ad)}{c^2+d^2}$

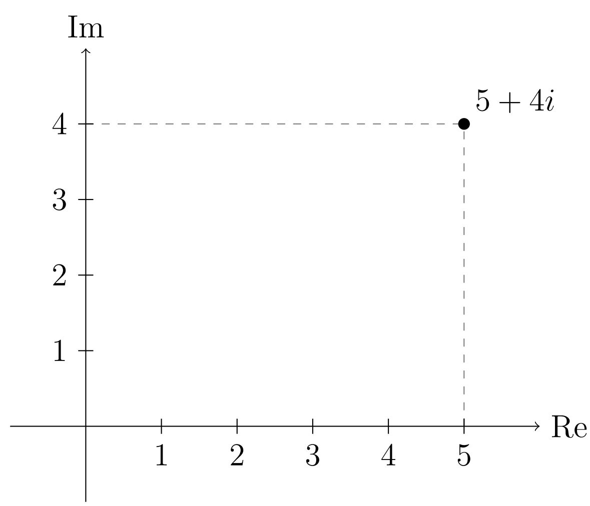
4 Graphical Representation

Although often imagined only in an algebraic context, imaginary numbers can also be visually represented in what can be understood as a modified cartesian plane.

4.1 Complex Plane & Point Representation

Given a complex number $z : z = x + yi$, the y-axis on the complex plane illustrates the imaginary component whilst the x-axis (also referred to as the number line) represents the real component.

Below is a representation of the complex number $4 + 5i$ in point representation, i.e, the real coefficients are treated as coordinates (x, y) on the complex plane.



4.2 Vectors

Complex numbers are not identical to $\mathbb{R}^2$ vectors but in some ways they exhibit similar behaviours to each other. There are also similarities between complex numbers and two dimensional matrices.

There are two things you can do with a pair of complex numbers. You can add (or substract) and you can multiply (or divide) them. Thinking about addition and substraction, suppose you map each complex number to a two dimensional vector as follows $a + bi \mapsto (a, b)$

Then in term of complex numbers $(a+bi)+(c+di) = (a+c)+i(b+d)$ and in terms of vectors $(a, b) + (c, b) = (a + c, b + d)$ it can be observed that $(a + ib) + (c + ib) \mapsto (a, b) + (c, d)$

Now, thinking about multiplication and division, when we multiply a complex number by i we get $i(c + id) = -d + ic$.

In the vector space you start with the vector (c, d) and end up with a vector $(-d, c)$. The vector has been rotated anticlockwise by $90°$, representable via the last matrice below

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We can now come up with a mapping from complex numbers to two dimensional matrices such that

$$a + ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

That gives us $(a + ib)(c + id) = (ac - bd) + i(ad + bc)$ and

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac - bd & -ad - bc \\ ad + bc & ac - bd \end{pmatrix}$$

therefore

$$(a + ib)(c + ib) \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

We have found that the addition/substraction and multplication/division of complex numbers and $\mathbb{R}^2$ vectors are isomorphic, i.e, similar in terms of properties to one another.

5 Complex Conjugate

5.1 Definition

A complex conjugate is one where, given a complex number $z : z = a + bi$ the sign of the imaginary component is flipped. This action returns a complex number with a structure of $a - bi$. The complex conjugate is often noted as $\bar{z}$

5.2 Properties

The following properties apply for all complex numbers z and w unless stated otherwise, and can be proved by writing z and w in the form $a + bi$

For any two complex numbers, conjugation is distributive over addition, substraction, multiplication, and division

$$\overline{z + w} = \overline{z} + \overline{w}$$

$$\overline{z - w} = \overline{z} - \overline{w}$$

$$\overline{zw} = \overline{zw}$$

$$\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}, \text{if} \neq 0$$

A complex number is equal to its complex conjugate if its imaginary part, b, is equal to 0. In other words, real numbers are the only fixed points of conjugation Conjugation does not change the modulus of a complex nmumber: $|\overline{z}| = |z|$. Conjugation is also an involution, that is, the conjugate of the conjugate of a complex number z is z, also written as $\overline{\overline{z}} = z$

6 Modulus

Given a complex number z such that $z = a + bi$ the modulues (also known as the magnitude) of the complex number can be found by square rooting the sum of the squares of the number's real parts, i.e

$$|z| = \sqrt{a^2 + b^2}$$

The modulus is often written as r where $r = |z|$

7 Argument

7.1 Definition

For a complex number $z = a + bi$ where $a, b \in \mathbb{R}$ the argument θ is the angle formed between the positive real axis (defined as $1 + 0i$) and the line representing the complex number z.

7.2 Mathematical Expression

The argument can be found using the arctangent function such that $\theta = \arctan\left(\frac{b}{a}\right)$

However, since arctangent alone only returns results in the range $(-\frac{\pi}{2}, \frac{\pi}{2})$, adjustements are needed to place θ correctly in all four quadrants. This leads to the use of the $\arctan2(b, a)$ function, which returns the angle in the correct quadrant

7.3 Range of Values

The argument θ can theoretically take on a real value, but it is often restricted to a principal value to ensure that it is unique. The principal value of the argument is typically within the interval $-\pi < \theta \leq \pi$

Note that, $Arg(z)$ restricts the angle in the above interval while $\arg(z)$ does not and can be defined as $\arg(z) = \{Arg(z) + 2\pi n \mid n \in \mathbb{N}\}$ where $Arg(z) = \arctan2(z)$

7.4 Principal Argument

The principle argument, denoted $Arg(z)$, is the unique value of the argument within the specified interval $Arg(z) = arg(z) \mod 2\pi$ with $-\pi < Arg(z) \leq \pi$

8 Polar Form

The polar form of a complex number is a different way to represent a plex number apart from the traditional rectangular form. Usually a complex numer is represented in the form $a + bi$, but in polar form, it is representated as a combination of the modulus (magnitude) and argument.

8.1 Conversion

Converting a complex number from rectangular form to polar form involved finding the modulus and the argument. Polar form is represented as $z = r(\cos\theta + i\sin\theta)$ or more commonly as $z = re^{i\theta}$. Below are steps to follow to perform the Conversion

1. Calculate the modulus

$$r = \sqrt{a^2 + b^2}$$

2. Calculate the argument

8.2 Formula

8.3 Euler's Formula

8.4 Inverse Conversion

8.5 Properties

9 De Moivre's Theorem

9.1 Definition

9.2 Application

10 Roots of Complex Numbers

10.1 nth Roots

11 Exponential Form and Logarithms

11.1 Exponential Form

11.2 Euler's Formula (with Logarithms)

11.3 Logarithm

11.4 Principal Value (with Logarithms)