Complex Numbers: Concepts, Formulas, etc.

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Concepts

1 Preface

These notes are written as an assignment for AP Calculus AB and are meant to be used by myself as a tool in any future university level mathematics class. If you are finding (or have received) these notes and are wishing to learn/study from them; be aware that they are tailored to myself and may not cover all concepts needed to properly learn/study complex numbers

2 Definition and Basics

To begin, a complex number is one where an "imaginary" part is present, which will be defined as i where $i = \sqrt{-1}$ moving onwards.

The imaginary part is in quotations as i is not "imaginary" but instead on a different plane comapred to our traditional number line. They are better refered to as lateral numbers but they will be referred to as "imaginary" numbers to better reflect modern mathematical vocabulary

2.1 Parts of a Complex Number

A complex number z can be defined by that addition of a real part a and imaginary part bi such that $z = a + bi$ where $a, b \in \mathbb{R}$.

I'll cover this in the next subsection, but it is important to note that 0 is also a real number and that real numbers can also be written in complex form.

2.2 Set Notation

Given z where $z = a + bi$ such that $a, b \in \mathbb{R}$; z can be said to be in the set of complex numbers $\mathbb{C}$ where $\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}, i^2 = -1\}$. This can also be written as $z \in \mathbb{C}$.

It is also important to note that all reals $\mathbb{R}$ are a subset of $\mathbb{C}$. This is inherently true as any real number x can be written as a complex number $a + bi$ where $b = 0$.

3 Arithmetic Operations

All complex numbers abide by their respective arithemtic rules in such a way where any arithmetic performed on a complex number returns another complex number as shown below

- **Addition** $(a + bi) + (c + di) = (a + c) + i(b + d)$
- **Substraction** $(a + bi) - (c + di) = (a - c) + i(b - d)$
- **Multiplication** $(a + bi)(c + di) = (ac - bd) + i(ad + bc)$
- **Division** $\frac{a+bi}{(c+di)} = \frac{(ac+bd)}{c^2+d^2} + \frac{i(bc-ad)}{c^2+d^2}$

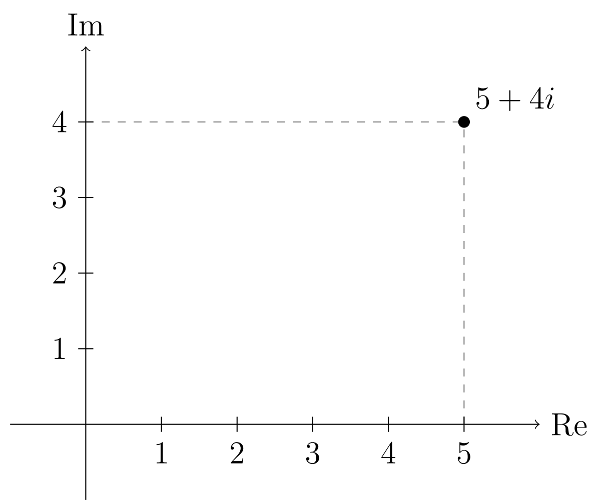
4 Graphical Representation

Although often imagined only in an algebraic context, imaginary numbers can also be visually represented in what can be understood as a modified cartesian plane.

4.1 Complex Plane & Point Representation

Given a complex number $z : z = x + yi$, the y-axis on the complex plane illustrates the imaginary component whilst the x-axis (also referred to as the number line) represents the real component.

Below is a representation of the complex number $4 + 5i$ in point representation, i.e, the real coefficients are treated as coordinates (x, y) on the complex plane.



4.2 Magnitude and Angle

Given a complex number z such that $z = a + bi$ the magnitude of the complex number can be found by square rooting the sum of the squares of the number's real parts, i.e

$$|z| = \sqrt{a^2 + b^2}$$

To find the angle of a complex number, arctangent can not be used due to the fact that if $\tan z = i$ and $\sin z = i \cos z$,

$$\sin^2 z + \cos^2 z$$

$$= (i \cos z)^2 + \cos^2 z$$

$$= i^2 \cos^2 z + \cos^2 z$$

$$= -\cos^2 z + \cos^2 z$$

$$= 0$$

but, $\sin^2 z + \cos^2 z = 1$ therefore there exists no solution to $\tan z = \pm i$. So instead the angle is findable using the real terms in the number such that

$$\tan\theta = \frac{b}{a}$$

4.3 Vectors

Complex numbers are not identical to $\mathbb{R}^2$ but in some ways they exhibit similar behaviours to each other. There are also similarities between complex number and two dimensional matrices.

There are two things you can do with a pair of complex numbers. You can add (or substract) and you can multiply (or divide) them. Thinking about addition and substraction, supposed you map each complex number to a two dimensional vector as follow $a + bi \mapsto (a, b)$

Then in term of complex numbers $(a+bi)+(c+di) = (a+c)+i(b+d)$ and in terms of vectors $(a, b) + (c, b) = (a + c, b + d)$ it can be observed that $(a + ib) + (c + ib) \mapsto (a, b) + (c, d)$

Now, thinking about multiplication and division, when we multiply a complex number by i we get $i(c+id) = -d+ic$.

In the vector space you start with the vector (c, d) and end up with a vector $(-d, c)$. The vector has been rotated anticlockwise by $90°$, representable via the last matrice below

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

We can now come up with a mapping from complex numbers to two dimensional matrices such that

$$a+ib \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

That gives us $(a+ib)(c+id) = (ac-bd)+i(ad+bc)$ and

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix} = \begin{pmatrix} ac-bd & -ad-bc \\ ad+bc & ac-bd \end{pmatrix}$$

therefore

$$(a+ib)(c+ib) \mapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} c & -d \\ d & c \end{pmatrix}$$

We have found that the addition/substraction and multplication/division of complex numbers and $\mathbb{R}^2$ vectors are isomorphic, i.e, similar in terms of properties to one another.

5 Complex Conjugate

5.1 Definition

5.2 Properties

6 Magnitude (Modulus)

6.1 Formula

6.2 Properties

7 Argument (Phase)

7.1 Definition

7.2 Principal Value

7.3 Argument Function

7.4 Properties

8 Polar Form

8.1 Conversion

8.2 Formula