S1. $(\mathbb{Z}, +)$ - The integers under addition is a group, here is how it satisfies the axioms of a group

(a) Given any two integers $x, y \in \mathbb{Z}$, their sum, $x+y$ is bound to be within the set of integers. Therefore the set $\mathbb{Z}$ is clsoed under addition

(b) 0 is the identity element, as, given $e = 0$, $x + e = e + x = x$ where $e = 0$ holds true.

(c) For every $x \in \mathbb{Z}$ there exists $y \in \mathbb{Z}$ such that $x + y = y + x = 0$. In this specific instance $y = -x$. Therefore, each element in the set has an inverse within the set.

(d) Given any $x, y, z \in \mathbb{Z}$. The assoicative statement $x + (y + z) = z + (x + y)$ holds true.

Therefore, as all of the axioms of a group are met, $(\mathbb{Z}, +)$, the set of integers under addition, is a group.

S2. $(\mathbb{Q}, \times)$ - The rations under multiplication, is this a Group?

(a) Given any two $x, y \in \mathbb{Q}$, the product of their multiplication $x \times y$ will always result in a rational number. Therefore, the set $\mathbb{Q}$ is closed under multiplication

(b) The identity element of $\mathbb{Q}$ under multiplication is 1 as, for all $x \in \mathbb{Q}$, where $e = 1$, the identity $x \times e = e \times x = x$ holds true.

(c) Multiplication is associative, same as addition, therefore, for any $x, y, z \in \mathbb{Q}$, the equation $x \times (y \times z) = z \times (x \times y)$ is true.

(d) Now, $(\mathbb{Q}, \times)$ fails to meet the criteria of a group when it comes to inverses. For an element $x \in \mathbb{Q}$ to have an inverse under multiplication there must exist $y \in \mathbb{Q}$ such that $x \times y = y \times x = 1$.

As the set of rational integers $\mathbb{Q}$ includes the element 0, which does not have an inverse, it means that the inverse axiom, stating that each element in the set must have an inverse such that $x \times y = y \times x = 1$ is not met. Therefore, $\mathbb{Q}$ is not a group under multiplication

S3. $(\mathbb{Q}, +, \times)$ is the ring of rationals closed under addition and multiplication, how does it satisfy the axioms of a ring?

(a) It is a group under addition, meaning

- **Closure:** For all $x, y \in \mathbb{Q}$, $x + y \in \mathbb{Q}$ holds true
- **Identity:** For any $x \in \mathbb{Q}$ given $e = 0$, $x + e = e + x = x$ holds true
- **Inverse:** For any $x \in \mathbb{Q}$ there exists $y \in \mathbb{Q}$ such that $y = -x$ and $x + y = y + x = e$ where $e = 0$
- **Associativity:** As addition is associative, for all $x, y, z \in \mathbb{Q}$, the equation $x + (y + z) = z + (x + y)$ holds true

(b) When it comes to the associativity of $\times$, for every $x, y, z \in \mathbb{Q}$, $x \times (y \times z) = z \times (x \times y)$ holds true.

(c) For the distributive properties, it is the exact same scenario. Given $x, y, z \in \mathbb{Q}$, the following properties

$$x \times (y + z) = (x \times y) + (x \times z)$$

and

$$(y + z) \times x = y \times x + z \times x$$

Both hold true, meaning that the axiom of distributive properties holds true in this case.

Therefore, $(\mathbb{Q}, +, \times)$ is a ring.

S4. Using what was just learnt, decide whether $(\mathbb{O}^+, +, \times)$, the set of positive odd integers, is a ring. Why or why not?

(a) This set under addition and multiplication is not a ring as the axiom of being a group under addition is not met, here is why:

- To start, for any $x, y \in \mathbb{O}^+$, their product $x + y$, will not always be in the set of odd positive integers as, for example, $1 + 3 = 4$. Therefore, the set $\mathbb{O}^+$ is not closed under addition
- Given $x \in \mathbb{O}^+$ there must exist an identity element e such that $x + e = e + x = x$. Although 0 is usually the identity element under addition, as $0 \notin \mathbb{O}^+$, there is no element in the set that can satisfy this axiom.
- For any $x \in \mathbb{O}^+$ there must exist $y \in \mathbb{O}^+$ such that $x + y = y + x = 0$. But, as the set $\mathbb{O}^+$ does not include any negative numbers, no elements within the set have an inverse within it.

S5. Is the set of rational numbers $\mathbb{Q}$ a field?

(a) As discussed in S2, $\mathbb{Q}$ is a valid group under addition, therefore the first axiom is satisfied.

(b) This is where it gets interesting. The second axiom to define a field states that a set without the identity set 0 (i.e $\mathbb{Q} - \{0\}$) must be a valid group under multiplication. We know that $\mathbb{Q}$ satisfies all of the conditions of a group under multiplication, save for the fact that 0 does not have inverse within the set.

Therefore, as 0 is no longer within the set, for any $x \in \mathbb{Q}$ there exists $y \in \mathbb{Q}$ such that $x \times y = y \times x = e$ where $e = 1$

Therefore, as both field axioms are obeyed, the set $\mathbb{Q}$ is a field.

S6. Is the set of complex numbers $\mathbb{C}$ a field?

(a) $(\mathbb{C}, +)$ is a valid group

- For all $x, y \in \mathbb{C}$, $x + y \in \mathbb{C}$ holds true
- For any $x \in \mathbb{C}$, $x + 0 = 0 + x = x$, therefore the identity element is 0
- For all $x \in \mathbb{C}$, there exists $y \in \mathbb{C}$ such that $x + y = y + x = 0$
- For all $x, y, z \in \mathbb{C}$, $x + (y + z) = z + (x + y)$ is true

(b) $(\mathbb{S}, \times)$ is also a valid group where $\mathbb{S} = \mathbb{C} - \{0\}$

- For all $x, y \in \mathbb{S}$, $x \times y \in \mathbb{S}$ holds true
- Given any $x \in \mathbb{S}$, $x \times 1 = 1 \times x = x$, therefore the identity element is 1
- For any $x \in \mathbb{S}$ there exists $y \in \mathbb{S}$ such that $x \times y = y \times x = e$ where e is the identity element 1
- For all $x, y, z \in \mathbb{S}$, $x \times (y \times z) = z \times (x \times y)$

Therefore, as all of the field axioms are satisfied, $\mathbb{C}$ is a field.