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Examples of unknotted curves which bound only surfaces of high genus within their convex hulls*

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1. Introduction

For each positive integer g we show how to construct a smooth simple closed unknotted (rather complicated) curve C in $\mathbf{R}^3$ which bounds within its convex hull only surfaces having at least g handles. Similarly, we show the construction of an oriented smooth simple closed unknotted (somewhat less complicated) curve B which among oriented surfaces lying within its convex hull can bound only those having at least g handles.

In saying that a surface S *has at least g handles* we mean that S can be represented topologically as the connected sum

$$S = S_0 \# \underbrace{T^2 \# T^2 \# \cdots \# T^2}_{g \text{ times}}$$

where S_0 is some 2-dimensional manifold, T^2 denotes the 2-dimensional torus, and $\#$ denotes formation of a connected sum.

Surfaces lying within the convex hulls of their boundaries include surfaces of nonpositive gaussian curvature which, for example, arise as solutions to constant coefficient elliptic variational problems in various settings, e.g. the problems of least oriented and unoriented areas. In cases in which these solution surfaces are not known to be regular at their boundaries our estimates in terms of numbers of handles guarantees that interior rather than boundary topological complexity is being measured. We discuss briefly several such problems.

2. The oriented case

We will construct an oriented smooth simple closed unknotted curve B which, among oriented surfaces M lying within its convex hull, can bound only those having at least three handles. The example readily generalizes to yield any desired minimum number of handles. The orientation requirement on M is necessary since B bounds a Moebius band within its convex hull.

On the boundary of the ellipsoidal solid

$$E = \mathbf{R}^3 \cap \{(x, y, z): x^2 + y^2 + (z/4)^2 \leqq 1\},$$

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we construct a spiral curve B_1 having endpoints p and q lying in the yz plane as illustrated in Figure 1 and winding around E seven times (increasing the number of windings here increases the number of handles required by oriented surfaces bounded within the convex hull of the final curve). A curve B_2 is constructed as illustrated in Figure 2 beginning at q and continuing in the yz plane P in a counterclockwise manner such that (i) for $-3<z<7/2$ and $y>0$, having just started at q, B_2 runs in P just outside $E \cap P$ and nearly parallel with $\partial E \cap P$; (ii) for $z \geqq 7/2$, B_2 coincides with $\partial E \cap P$; (iii) for $-7/2<z<7/2$, B_2 runs in P just outside $E \cap P$ and nearly parallel with $\partial E \cap P$; (iv) for $z \leqq -7/2$, B_2 coincides with $\partial E \cap P$; (v) then for $-7/2<z \leqq 5/2$,

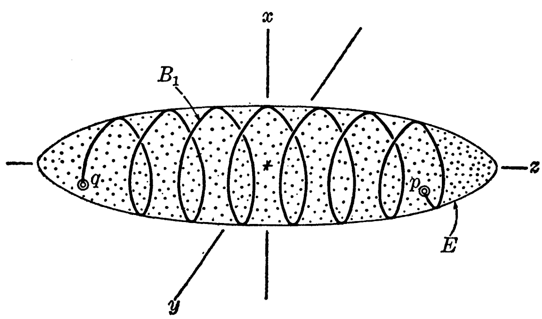

FIGURE 1

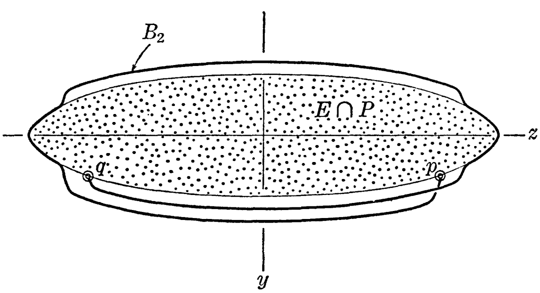

FIGURE 2

B_2 continues by running within P a bit farther outside $E \cap P$ to avoid intersecting itself and still nearly parallel with $\partial E \cap P$; (vi) for $5/2<z<3$, B_2 then continues on to the point p with a slightly negative x coordinate as necessary to avoid self intersection. The final curve B pictured in Figure 3 is $B_1 \cup B_2$ with smoothing near p and q to yield a single smooth simple closed curve. One sees by inspection that B is unknotted. We let K denote the convex hull of B.

One observes that the existence of $\delta>0$ such that $(x, y, z) \in B$ with $|y|<\delta$ implies (x, y, z) is an extreme point of K; in particular, K intersects the

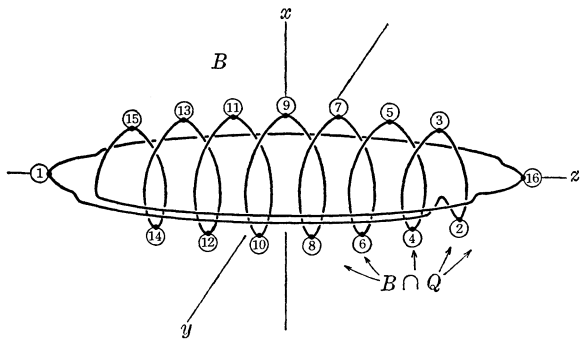

FIGURE 3

tangent plane to ∂E only at the point (x, y, z).

Now suppose M is a smooth compact oriented surface lying in K and having boundary B. We will show M contains at least three handles. To do this we choose $-\delta<t<\delta$ such that the oriented plane $Q = \{(x, y, z): y = t\}$ meets both M and B transversally. The oriented intersection $Q \cap B$ consists of sixteen points numbered 1 through 16 in Figure 3, this numbering corresponding to the order of these points on B. The odd numbered points are marked $(+)$ to record the fact that B passes through Q there with increasing y coordinate; the even numbered points are marked $(-)$ since B passes through Q there with decreasing y coordinate. Clearly each of these sixteen points is an extreme point of $Q \cap K$.

The intersection $Q \cap M$ consists of eight oriented arcs connecting the eight $(+)$ points to the eight $(-)$ points with possibly several additional simple closed curves. We label our arcs $\gamma_1, \gamma_3, \gamma_5, \cdots, \gamma_{15}$ with the subscript indicating the number of $(+)$ endpoints of the arc. Since the arcs must connect $(+)$ to $(-)$ points while staying inside $Q \cap K$ and also avoiding each other, the negative endpoints are determined as follows.

Curve	γ_1	γ_3	γ_5	γ_7	γ_9	γ_{11}	γ_{13}	γ_{15}
$(-)$ Endpoint	14	16	2	4	6	8	10	12

For example, if the $(-)$ endpoint of γ_1 were any point other than 14, there would be no way for γ_i, $i = 3, 5, \cdots, 15$ to have $(-)$ endpoint 14 while remaining within $Q \cap K$ and avoiding γ_1.

In order to estimate the number of handles of M we form a new surface M^* by attaching an annulus A to M by identifying one of the boundary circles of A with B. Clearly M is homeomorphic with M^*. In M^* we extend

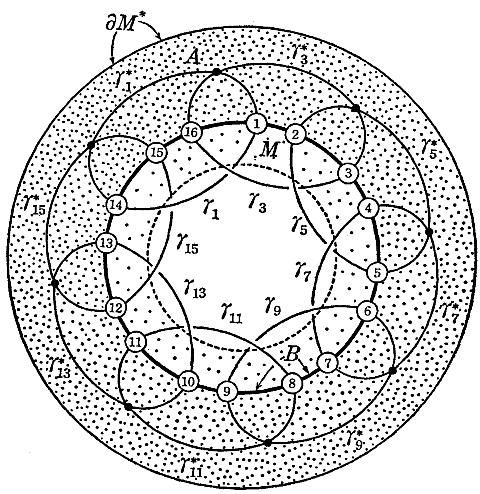

FIGURE 4

our curves $\gamma_1, \cdots, \gamma_{15}$ to be simple closed curves $\gamma_1^*, \cdots, \gamma_{15}^*$ as illustrated in Figure 4. Letting $[\gamma_i^*] \in H_1(M^*; \mathbf{Z}_2)$ denote the homology class of γ_i^* with coefficients in the integers modulo 2 for each i one has by inspection the intersection matrix,

$$\begin{array}{c|cccccccc}
 & [\gamma_1^*] & [\gamma_3^*] & [\gamma_5^*] & [\gamma_7^*] & [\gamma_9^*] & [\gamma_{11}^*] & [\gamma_{13}^*] & [\gamma_{15}^*] \\
\hline
[\gamma_1^*] & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
[\gamma_3^*] & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
[\gamma_5^*] & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
[\gamma_7^*] & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
[\gamma_9^*] & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
[\gamma_{11}^*] & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
[\gamma_{13}^*] & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\
[\gamma_{15}^*] & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array},$$

which one checks to have rank 6. Since intersection with a fixed homology class is a $\mathbf{Z}_2$ linear function on the $\mathbf{Z}_2$ vectorspace $H_1(M^*; \mathbf{Z}_2)$, one concludes that $H_1(M^*; \mathbf{Z}_2) = H_1(M; \mathbf{Z}_2)$ has dimension at least 6 over $\mathbf{Z}_2$. One knows M can be represented topologically

$$M = S^2 \# \underbrace{T^2 \# \cdots \# T^2}_{g \text{ times}} \# D^2$$

where S^2 denotes a topological 2 sphere, T^2 denotes a topological 2 torus, D^2 denotes a topological 2 disk, and $\#$ denotes formation of connected sum. One concludes $g \geqq 3$ and readily sees that M contains at least three handles.

A Moebius band which B bounds can be constructed as follows. Figure

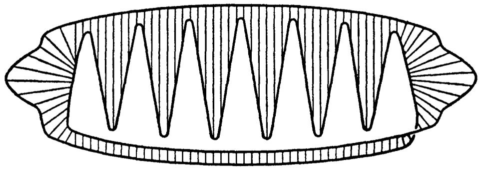

FIGURE 5

5 illustrates the orthogonal projection of B onto the yz plane. Each straight line segment shown is the unique projection of a straight line in $\mathbf{R}^3$ connecting two points of B. The Moebius band is the union of such straight line segments in $\mathbf{R}^3$ together with one vertical segment above the crossing point of the projection of B.

3. The unoriented case

We will construct a smooth simple closed unknotted curve C which, among all (not necessarily orientable) surfaces T lying within its convex hull, can bound only those having at least three handles. The example readily generalizes to yield any desired minimum number of handles.

We fix the ellipsoidal solid

$$F = \mathbf{R}^3 \cap \{(x, y, z): x^2 + y^2 + (z/40)^2 \leqq 1\}$$

within which C will lie. C is made by fitting together modules of types a, b, c, d, e in the sequence

$$dcacbcacbcacbcacbce$$

along the z axis within blocks of z length 4. In particular, the module d lies within $F \cap \{(x, y, z): -38 \leqq z \leqq -34\}$, the first c module within $F \cap \{(x, y, z): -34 \leqq z \leqq -30\}$, the first a module within $F \cap \{(x, y, z): -30 \leqq z \leqq -26\}$, etc. These various module types are pictured in Figure 6. The various modules of type c are made of varying heights and curvatures so that the part of C within the middle half of each module of type c lies on ∂F, i.e., setting

$$G = \{(x, y, z): |z-8i| \leqq 1 \text{ for some } i = -4, -3, -2, -1, 0, 1, 2, 3, 4\}$$

we have $C \cap G \subset \partial F$ as illustrated in Figure 7.

Assume then C has been constructed as indicated. Since the braiding in each module of type a is unraveled by the mirror image braiding in the following module of type b, one checks by inspection that C is unknotted. We let L denote the convex hull of C. Clearly $p \in C \cap G$ implies p is an extreme point of L.

Let N be a smooth compact surface lying within L and having C as

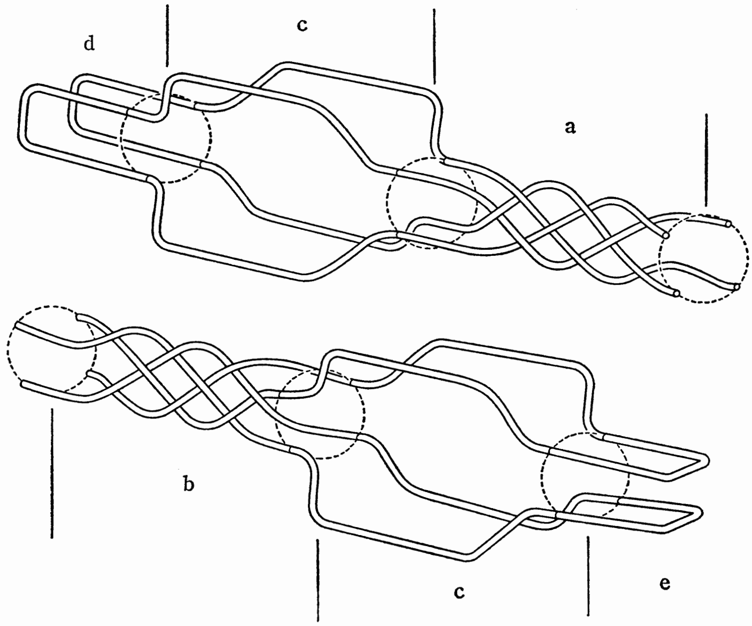

FIGURE 6

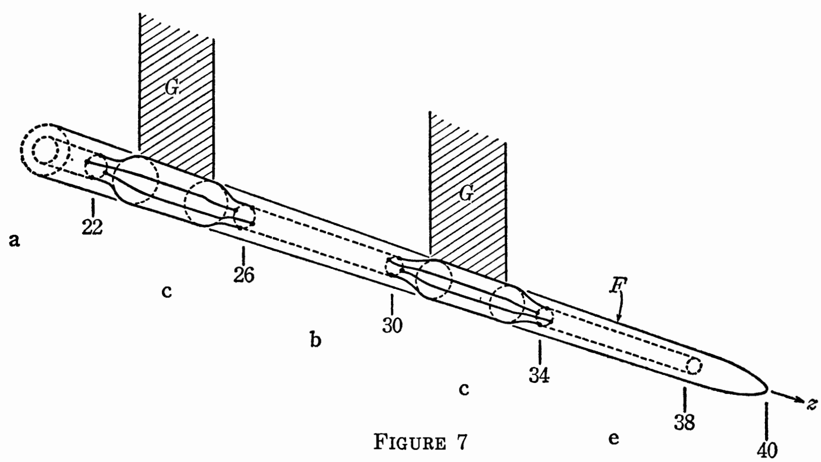

FIGURE 7

boundary.

We now choose $-1 < t < 1$ such that each of the planes

$$P_i = \{(x, y, z): z = 8i + t\}$$

corresponding to $i = -4, -3, -2, -1, 0, 1, 2, 3, 4$ intersects both C and N transversally. For each $i = -4, -3, -2, -1, 0, 1, 2, 3$ we further set

$$U_i = \{(x, y, z): 8i + t < z < 8(i + 1) + t\} .$$

We will show for each $-4 \leqq i \leqq 3$ the existence of a simple closed curve α_i lying in $N \cap U_i \sim C$ together with a simple curve β_i lying in $N \cap U_i$ such that β_i intersets α_i exactly once and β_i is either a simple closed curve in $N \cap U_i \sim C$ or is an arc in $N \cap U_i \sim C$ having its endpoints on $U_i \cap C$. Assuming such α_i and β_i exist, we let $[\alpha_i] \in H_1(N; \mathbf{Z}_2)$ denote the homology class of α_i in N with coefficients in the integers modulo 2. Because of the intersections of the $\{\alpha_i\}_i$ with the $\{\beta_j\}_j$ one concludes that the $\{[\alpha_i]\}_i$ are linearly independent over $\mathbf{Z}_2$ in $H_1(N; \mathbf{Z}_2)$ and hence $H_1(N; \mathbf{Z}_2)$ has dimension at least 8 over $\mathbf{Z}_2$. One knows N can be represented topologically

$$M = S^2 \# \underbrace{T^2 \# \cdots \# T^2}_{g \text{ times}} \# \underbrace{P^2 \# \cdots \# P^2}_{c \text{ times}} \# D^2$$

where S^2, T^2, D^2, $\#$ are as in Section 2, P^2 denotes a 2-dimensional real projective space, and $c \in \{0, 1, 2\}$. One concludes that $g \geqq 3$ because $c \leqq 2$ and hence that N contains at least three handles.

It remains therefore to show the existence of the α_i and β_i. It is clearly sufficient to show the existence of α_1, β_1. We let $N_1 = \text{clos}(N \cap U_1)$. Since P_1, P_2 intersect both N and C transversally we conclude that $P_1 \cap N$ consists of two arcs γ_1', γ_1'' connecting the four points in $P_1 \cap C$ together possibly with a finite number of simple closed curves. Similarly $P_2 \cap N$ consists of two arcs γ_2', γ_2'' connecting the four points in $P_2 \cap C$ together possibly with a finite number of simple closed curves. We now construct a new surface N^* in $\mathbf{R}^3$ by attaching topological disks lying in $\mathbf{R}^3 \sim U_1$ to the various simple closed curves which may lie in $P_1 \cap N \cup P_2 \cap N$ in such a way that N^* is a compact topological two dimensional submanifold of $\mathbf{R}^3$ with

$$\partial N^* = U_1 \cap C \cup \gamma_1' \cup \gamma_1'' \cup \gamma_2' \cup \gamma_2'' .$$

By construction $P_1 \cap C \subset \partial F$ so that $P_1 \cap C$ contains only extreme points of L and hence the four points of $P_1 \cap C$ are extreme points of $P_1 \cap L$. Since $N \subset L$, $\gamma_1' \cup \gamma_1'' \subset P_1 \cap N \subset P_1 \cap L$. It follows that γ_1' can only connect points of $P_1 \cap C$ which are adjacent in $\partial(P_1 \cap L)$; indeed if γ_1' connected opposite points then γ_1'' could not connect the remaining two points without crossing γ_1'. Similarly γ_1'' must connect points of $P_1 \cap C$ which are adjacent in $\partial(P_1 \cap L)$, and both γ_2' and γ_2'' must connect points of $P_2 \cap C$ which are adjacent in $\partial(P_2 \cap L)$. One concludes therefore that there are a total of four ways in which the points of $P_1 \cap C \cup P_2 \cap C$ can be joined combinatorially by the arcs of $P_1 \cap N \cup P_2 \cap N$. These four possible joinings are indicated schematically in Figure 8 as possibilities 1, 2, 3, 4. We consider the structure of N^* and ∂N^* in each of these possible cases.

Possibility 1. Here ∂N^* has two components which are nontrivially linked. There are two relevant subpossibilities.

First subpossibility. We first suppose that ∂N^* lies in a single com-

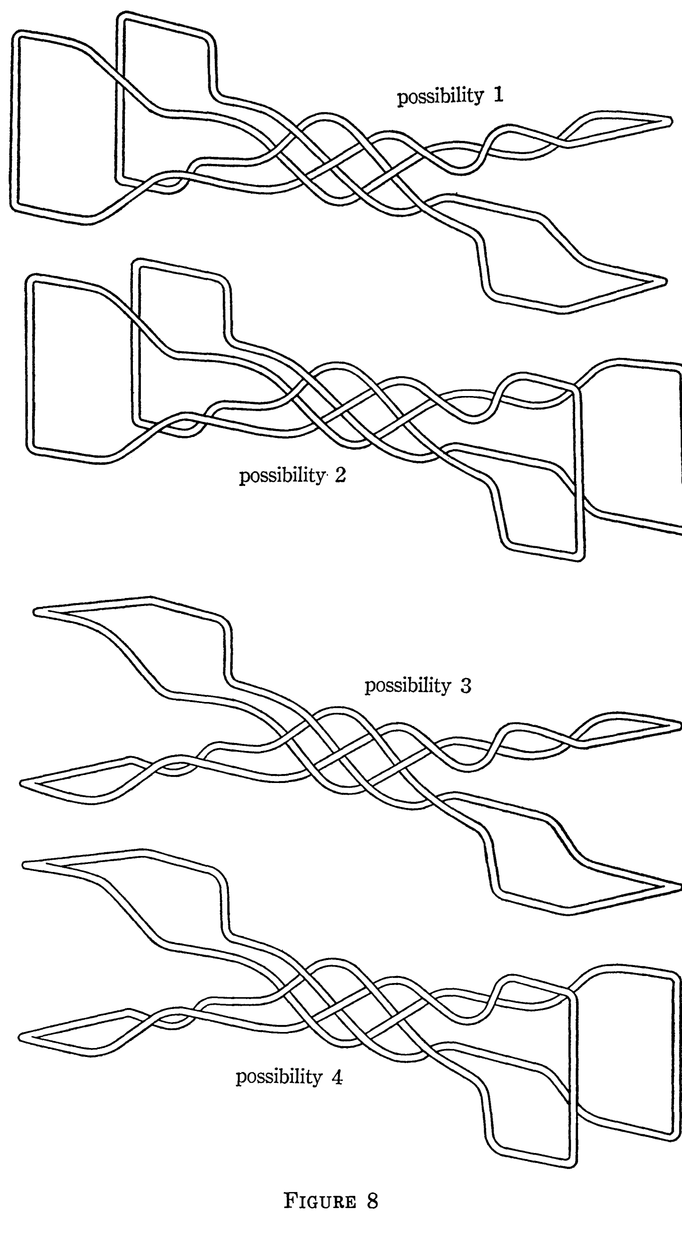

FIGURE 8

ponent of N^*. In this case we choose α_1 to be a curve running just inside one of the components of ∂N^* and β_1 to be an arc joining the two components of ∂N^* and crossing α_1 once.

Second subpossibility. Secondly, we suppose that some one component N^0 of N^* contains exactly one component of ∂N^*. Since the two components of ∂N^* are nontrivially linked, N^0 cannot be a topological 2 disk. Hence N^0

contains either a punctured torus or a punctured projective plane (a Moebius band) and α_1 and β_1 can be found.

Possibility 2. Here ∂N^* is a nontrivial knot and N^* cannot be a topological 2 disk. This possibility is similar to the second subpossibility above.

Possibility 3. Here ∂N^* is a nontrivial knot as in Possibility 2.

Possibility 4. Here ∂N^* has two components which are nontrivially linked as in Possibility 1.

We have thus shown how to find α_1 and β_1 in each possible case.

4. Some elliptic variational problems

4.1. Mass minimizing flat chains modulo 2. Each smooth closed curve C in $\mathbf{R}^3$ is the boundary of a 2-dimensional flat chain modulo 2 N having least mass among all such chains having C as boundary [1, 4.2.26]. It follows from [2, 8.16], [3, 4.5], [4, 5.4] that N is a smooth 2-dimensional minimal submanifold of $\mathbf{R}^3$ with boundary C. Minimal surfaces are well known to lie within the convex hulls of their boundaries. Hence, for C as in Section 3, N must have at least three handles. Note also, for C as in [5], a version of which is pictured in Figure 1-2 of [6], any flat chain modulo 2 of least area having C as boundary, as in Figure 1-3 of [6], must have infinitely many handles.

4.2. Mass minimizing integral currents. Each smooth closed oriented curve B in $\mathbf{R}^3$ is the boundary of a 2-dimensional integral current M of least mass among all integral currents having B as oriented boundary [1, 4.2.17]. It follows from [1, 5.4.15] that $M \sim B$ is a smooth 2-dimensional minimal submanifold of $\mathbf{R}^3$. General boundary regularity of such mass minimizing currents M is not known so that somewhat more work is required to show the necessity of interior handles. Suppose then B is as in Section 2.

Boundary regularity at the points of $\partial E \cap B$ is implied by [3, 4.5.2] so that the various smooth curves $\{\gamma_i\}_i$ of Section 2 are readily obtained and meet B transversally. One then chooses a very small positive number r such that the surface $T = \{x\colon \operatorname{dist}(x, B) = r\}$ intersects M and the $\{\gamma_i\}_i$ transversally. One checks that $T \cap M$ consists of a simple closed curve isotopic with B within the closed solid torus K bounded by T together possibly with a finite number of simple closed curves which bound disjoint topological disks within the interior of K and avoiding B. The remainder of the argument to show the necessity of three interior handles is a straightforward adaptation of Section 2.

4.3. Flat chains modulo 2 and integral currents minimizing the inte-

grals of constant-coefficient elliptic integrands. For each smooth constant-coefficient elliptic integrand $\Phi\colon G(3, 2) \to \mathbf{R}^+$ as in [7, I.1(6), IV.1(7)] [1, 5.1.2] and each boundary curve C as in Section 3 or oriented boundary curve B as in Section 2, there will exist a 2-dimensional flat chain modulo 2 N having boundary C and a 2-dimensional integral current M having boundary B each of which minimizes the integral of Φ with respect to its boundary. It is shown in [8] that both $M \sim B$ and $N \sim C$ are smooth Φ minimal submanifolds of $\mathbf{R}^3$. The maximum principle for elliptic partial differential equations implies M and N must be within the convex hulls of their boundaries. As in 4.3, general boundary regularity is not known. However, boundary regularity at the points of $\partial E \cap B$ and $\partial F \cap C$ is proved in [9]. An argument similar to that of 4.2 then shows M and N each contain at least three handles.

4.4. Mappings of least area. One context in which the problem of least area has been extensively studied is that of mappings from a fixed compact 2-dimensional domain into $\mathbf{R}^3$ satisfying appropriate conditions on the boundary. In case this domain is a standard 2 disk, mappings of least area exist for arbitrary smooth simple closed boundary curves B and are immersions on their interiors (see [10]). For boundary curves such as constructed in Sections 2 or 3 there can be no minimal imbeddings of domains having too few handles, in particular, no minimal imbeddings of disks.

4.5. Soap-film-like minimal surfaces. In [11; 21, 22] and [12] a procedure, modeled on the physical formation of soap films, is discussed for obtaining minimal surfaces which are usually different from those already discussed. Heuristically one imagines a water filled balloon containing a wire and collapses the balloon over the wire frame by sucking out the water. Such a procedure done mathematically yields a surface of least area in comparison with deformation images. For curves such as B or C in Sections 2 and 3 above, the resulting soap-film-like minimal surface cannot be free of singularities; the possible kinds of singularities which may be forced on the surface are classified in [13].

4.6. Convex hull genus and handle equivalence. The genus of a knotted curve B usually refers to the smallest possible genus of a compact imbedded oriented surface having B as boundary. By analogy, the examples of this paper suggest the notion of a "convex hull genus" of a curve. Such a genus is of course invariant under affine homeomorphisms and imbeddings of $\mathbf{R}^3$ and, as discussed in 4.4, 4.5, is associated with the necessity of self-intersections or other singular behavior in solutions to various variational problems. This association further suggests the notion of "handle equivalence" for singularities so defined that the sum of the actual handles together with the handle equivalences of singularities cannot be less than the convex hull genus.

4.7. Curvature estimates. It is stated in [14] that a smooth curve which lies on the boundary of its convex hull and has total curvature not exceeding 4π bounds an imbedded minimal disk; it is further conjectured that the curvature estimate alone is sufficient for the conclusion. One might speculate that a large curvature integral is necessary in order that a curve bound within its convex hull only surfaces with many handles. This is not the case, and there seems no general relationship between curvature and "convex hull genus". For example, let B and C be the curves constructed in Sections 2 and 3 above and note that for each $\varepsilon > 0$ there exist sufficiently large $m < \infty$ and corresponding linear function $f\colon \mathbf{R}^3 \to \mathbf{R}^3$, $f(x, y, z) = (x, y, mz)$ for $(x, y, z) \in \mathbf{R}^3$, such that $f(B)$ and $f(C)$ each have total curvature less than $4\pi + \varepsilon$.

4.8. Convex hulls and minimal surface hulls. It is well-known that minimal surfaces are barriers for other minimal surfaces, and, for a given smooth boundary curve Γ, typically it is possible to show that any minimal surface having Γ as boundary must lie well inside the convex hull of Γ by use of well chosen minimal surface barriers such as pieces of catenoids. In particular, one can replace the ellipsoid used in the construction of Section 3 by a much shorter piece of a catenoid, say

$$\{(x, y, z)\colon x^2 + y^2 = \cosh^2 z,\ -1 \leqq z \leqq 1\},$$

having the z axis as axis of rotation, to obtain a curve Γ analogous to C such that Γ bounds an imbedded disk within its convex hull although each imbedded minimal surface having Γ as boundary has at least 3 handles. A corresponding, but slightly more complicated, modification of the curve B of Section 2 is also possible with similar results.

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