May $X(t)$ and $Y(t)$ be two nonnegative functions defined on the same segment of t: $[0, s]$. May p_X be the total count of local maxima of $X(t)$. The vector $a(X) = (a^1(X), a^2(X), \ldots, a^{p_X-1}(X), a^{p_X}(X))^T$ is defined as follows. $a^1(X)$ is the t-coordinate of the first local maximum. $a^2(X), \ldots, a^{p_X-1}(X)$ are the distances between sequential local maxima of $X(t)$ along the t-axis. Finally, $a^{p_X}(X)$ is the distance between the last local maximum and the right boundary of the segment (s). The superscripts denote the index of the vector's elements. Obviously (1) holds.

$$a^1(X) + \ldots + a^{p_X}(X) = s \tag{1}$$

How to compare statistically the sequence of local maxima between the two functions? Assume for now that $Y(t)$ has the same number of local maxima as $X(t)$. Later we will remove this constraint. A. T. Fomenko offers the following approach. Since both vectors $a(X)$ and $a(Y)$ satisfy (1), they both belong to the $(p_X - 1)$-dimensional simplex S in $\mathbb{R}^{p_X}$, defined by the condition (2).

$$\begin{cases} x^1 + \ldots + x^{p_X} = s \\ x^n \geq 0, \quad n = 1, \ldots, p_X \end{cases} \tag{2}$$

We shall construct the $(p_X - 1)$-dimensional ball centered at $a(X)$ with the radius λ equal to the distance between twe two vectors (3). Given that S lies entirely in a linear subspace of $\mathbb{R}^{p_X}$, the distance (3) can be viewed as the distance in $\mathbb{R}^{p_X}$ as well as the distance in S.

$$\lambda = \sqrt{\sum_{n=1}^{p_X} (a^n(X) - a^n(Y))^2} \tag{3}$$

May $D(a(X), a(Y))$ be the $(p_X - 1)$-dimensional ball with radius λ centered at $a(X)$. The volume of its intersection with S is $v(a(X), a(Y))$ (4).

$$v(a(X), a(Y)) = \mathrm{Vol}(D(a(X), a(Y)) \cap S) \leq \mathrm{Vol}(D(a(X), a(Y))) = \frac{\pi^{\frac{p_X-1}{2}} \lambda^{p_X-1}}{\Gamma(\frac{p_X-1}{2} + 1)} \tag{4}$$

If $D(a(X), a(Y))$ lies entirely in S, then $v(a(X), a(Y)) = \mathrm{Vol}(D(a(X), a(Y)))$.

Now examine what happens if $Y(t)$ has fewer local maxima than $X(t)$: $a(Y) = (a^1(Y), a^2(Y), \ldots, a^{p_Y-1}(Y), a^{p_Y}(Y))^T$. We can consider all possible ways to add the zero elements to $a(Y)$, compute v for each of the variants and choose their minimal value for further work (5).

$$\begin{cases} w(X, Y) = \min\limits_{\widetilde{a}(Y)} \{v(a(X), \widetilde{a}(Y))\} \\ \widetilde{a}(Y) = (a^1(Y), 0, \ldots 0, a^2(Y), 0, \ldots, 0, \ldots, a^{p_Y-1}(Y), 0, \ldots 0, a^{p_Y}(Y))^T \\ \qquad\qquad \text{Every possible way to insert zeros into } a(Y) \end{cases} \tag{5}$$

Now symmetrize $v(X, Y)$ with respect to X and Y (6).

$$u(X, Y) = u(Y, X) = \frac{1}{2}(v(X, Y) + v(Y, X)) \tag{6}$$

Finally, we introduce the function $\widetilde{d}(X, Y)$ equal to $u(X, Y)$ normalized by the total volume of S. Given the explicit formula for the volume of S (7), the variable $\widetilde{d}(X, Y)$ is provided by (8).

$$\mathrm{Vol}(S) = \frac{s^{p_X-1}\sqrt{p_X}}{(p_X-1)!} \tag{7}$$

$$\widetilde{d}(X,Y) = \frac{u(X,Y)}{\mathrm{Vol}(S)} = \frac{u(X,Y)(p_X-1)!}{s^{p_X-1}\sqrt{p_X}} \tag{8}$$

Exact evaluation of $\widetilde{d}(X,Y)$ via (8) is quite challenging, therefore, the author of the test offered to use an upper estimate of $\widetilde{d}(X,Y)$, denoted $d(X,Y)$, by combining (4), (8), and the fact that $\mathrm{Vol}(D(a(X),a(Y))) = \mathrm{Vol}(D(a(Y),a(X)))$ (9).

$$d(X,Y) = \frac{\mathrm{Vol}(D(a(X),a(Y))) + \mathrm{Vol}(D(a(Y),a(X)))}{2\mathrm{Vol}(S)} = \frac{\pi^{\frac{p_X-1}{2}}\lambda^{p_X-1}(p_X-1)!}{\Gamma(\frac{p_X-1}{2}+1)s^{p_X-1}\sqrt{p_X}} \tag{9}$$

$d(X,Y)$ can be interpreted as an upper estimate of the probability that the sequences of local maxima coordinates of the two input functions $X(t)$ and $Y(t)$ belong to the λ-neighborhood of each other by chance. **Therefore, one can use** $\min(d(X,Y),1)$ **as the p-value of the test.**

If the functions bear experimental noise that produces many small peaks and wells, one can smooth them first and then apply the test mentioned above. The author offers to smooth the functions to various degree and select the minimal value $d(X,Y)$. If the input functions are measured in the discrete points $t_1,\ldots,t_m$, the smoothing of degree 1 is defined in (10), and the smoothing of degree k is defined in (11)

$$\begin{cases} R^1[X](t_1) = \dfrac{X(t_1)+X(t_2)}{2} \\ R^1[X](t_n) = \dfrac{X(t_{n-1})+X(t_n)+X(t_{n+1})}{3} \\ R^1[X](t_m) = \dfrac{X(t_{m-1})+X(t_m)}{2} \end{cases} \tag{10}$$

$$R^k[X] = R^1[R^{k-1}[X]] \tag{11}$$

For the original publication, see **A. T. Fomenko "Empirico-Statistical Analysis of Narrative Material and its Applications to Historical Dating: Volume I: The Development of the Statistical Tools." 1994 edition. Publisher: Springer; 1993. 234 p.**, chapter 4.4.