Beraha Numbers as Solutions to Nested Fraction Equations

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November 2020

1 Problem

Let $y_n = f_k(y_{n-1}) = k - \frac{k}{y_{n-1}}$ be a sequence of rational functions for some constant k and some initial value $y_0 = x$. Starting with $n = 3$, there are certain values of k for which $y_n = x$ for all $x \neq 0$. Here are the first few examples.

$$k - \cfrac{k}{k - \cfrac{k}{k - \frac{k}{x}}} = x \Rightarrow k = 1$$

$$k - \cfrac{k}{k - \cfrac{k}{k - \cfrac{k}{k - \frac{k}{x}}}} = x \Rightarrow k = 2$$

$$k - \cfrac{k}{k - \cfrac{k}{k - \cfrac{k}{k - \cfrac{k}{k - \frac{k}{x}}}}} = x \Rightarrow k \in \{\phi + 1, 2 - \phi\}$$

These values of k corresponding to $y_n = x$ seem to be the n^{th} generalized Beraha numbers $B_n^{(m)} = 4\cos^2\left(\frac{m\pi}{n}\right);\ m \in \mathbb{Z}$, excluding 0 and 4. In this paper, we seek to prove this assertion.

2 f_k as a Linear Transformation

Let us represent y_n, the n^{th} nested fraction, as some ratio $\frac{p_n}{q_n}$ of polynomials $p_n(x), q_n(x)$. Then,

$$\begin{aligned} y_{n+1} &= k\left(1 - \frac{1}{y_n}\right) \\ &= k\left(1 - \frac{1}{\frac{p_n}{q_n}}\right) \\ &= \frac{kp_n - kq_n}{p_n} \end{aligned} \tag{1}$$

$$\Longrightarrow \begin{aligned} p_{n+1} &= kp_n - kq_n \\ q_{n+1} &= p_n. \end{aligned} \tag{2}$$

This is a linear transformation that takes $(p_n, q_n)^T$ to $(p_{n+1}, q_{n+1})^T$, which can be rewritten as the following matrix.

$$M = \begin{bmatrix} k & -k \\ 1 & 0 \end{bmatrix} \tag{3}$$

3 A Matrix Equation

Theorem

The constants k such that $y_n = x$ for all $x \neq 0$ and $n \geq 3$ are the n^{th} generalized Beraha numbers $B_n^{(m)} = 4\cos^2\left(\frac{m\pi}{n}\right);\ m \in \mathbb{Z}$, excluding $B_n^{(m)} = 0$ and $B_n^{(m)} = 4$.

Proof. We rewrite our original problem, $y_n = x$, as a linear algebra problem.

$$M^n \begin{bmatrix} x \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{4}$$

$$\tag{5}$$

Note that the resultant vector can be any scalar multiple ($\lambda \neq 0$) of the original vector since it represents a ratio.

For any $k \neq 0, 4$, we can diagonalize M as

$$M = S\Lambda S^{-1} \tag{6}$$

where

$$
S = \begin{bmatrix} \frac{1}{2}(k - \sqrt{k(k-4)} & \frac{1}{2}(k + \sqrt{k(k-4)} \\ 1 & 1 \end{bmatrix}
$$

$$
\Lambda = \begin{bmatrix} \frac{1}{2}(k - \sqrt{k(k-4)} & 0 \\ 0 & \frac{1}{2}(k + \sqrt{k(k-4)} \end{bmatrix}.
$$

Using this diagonalization, we can solve (4).

$$
S\Lambda^n S^{-1} \begin{bmatrix} x \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} x \\ 1 \end{bmatrix} \tag{7}
$$

$$
\implies (S\Lambda^n S^{-1} - \lambda I) \begin{bmatrix} x \\ 1 \end{bmatrix} = 0 \tag{8}
$$

Since we wish to show that this equation holds regardless of x and vectors of the form $(x, 1)^T$ do not form a subspace (and therefore can't be the nullspace of any matrix), equation (8) leads us to the following.

$$
\begin{aligned}
(8) &\implies S\Lambda^n S^{-1} = \lambda I \\
&\implies \Lambda^n = \lambda I \\
&\implies \left[\frac{1}{2}(k - \sqrt{k(k-4)}\right]^n = \left[\frac{1}{2}(k + \sqrt{k(k-4)}\right]^n.
\end{aligned} \tag{9}
$$

When $k < 0$ or $k > 4$, $\frac{1}{2}(k + \sqrt{k(k-4)})$ and $\frac{1}{2}(k - \sqrt{k(k-4)})$ are two distinct real numbers with different magnitudes. In this case, their nth powers can never be equal.

When $0 < k < 4$, let us perform a change of variables $k = 4\cos^2(\theta)$, for $0 < \theta < \pi/2$. With this substitution, we can compute

$$
\begin{aligned}
\frac{1}{2}(k \pm \sqrt{k(k-4)}) &= \frac{1}{2}(4\cos^2(\theta) \pm 4i\sin(\theta)\cos(\theta)) \\
&= 2(\cos(\theta) \pm i\sin(\theta))\cos(\theta) \\
&= 2e^{\pm\theta i}\cos(\theta).
\end{aligned}
$$

Now, solving (9) reduces to solving

$$
\begin{aligned}
&2e^{-n\theta i}\cos^n(\theta) = 2e^{n\theta i}\cos^n(\theta) \\
\implies &e^{-n\theta i} = e^{n\theta i} \\
\implies &n\theta = m\pi;\ m \in \mathbb{Z}
\end{aligned}
$$

Substituting back in for k, we find the solutions to be

$$
k = 4\cos^2\left(\frac{m\pi}{n}\right) \text{ for } 0 < m < n/2.
$$

Finally, the last remaining cases are when $k = 0$ or $k = 4$. When $k = 0$,

$$M = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

and we see that $M^n = 0$ for all $n \geq 2$.

When $k = 4$,

$$M^n = 2^{n-1} \begin{bmatrix} 2(n+1) & -4n \\ n & -2(n-1) \end{bmatrix}.$$

For $n \geq 1$, it is easy to see that this matrix is also never diagonal.

Therefore, the constants k such that $y_n = x$ for all $x \neq 0$ and $n \geq 3$ are the n^{th} generalized Beraha numbers $B_n^{(m)} = 4\cos^2\left(\frac{m\pi}{n}\right);\ m \in \mathbb{Z}$, excluding $B_n^{(m)} = 0$ and $B_n^{(m)} = 4$.. $\square$