Beraha Numbers as Solutions to Nested Fraction Equations

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1 Problem

Let $y_n = k - \frac{k}{y_{n-1}}$ for some constant k and some initial value $y_0 = x$. Starting with $n = 3$, there are certain values of k for which $y_n = x$ for all $x \neq 0$. Here are the first few examples.

$$k - \cfrac{k}{k - \cfrac{k}{k - \frac{k}{x}}} = x \Leftrightarrow \frac{(k-1)(-k+kx-x^2)}{(k-1)x-k} = 0 \Rightarrow k = 1$$

$$k - \cfrac{k}{k - \cfrac{k}{k - \cfrac{k}{k - \frac{k}{x}}}} = x \Leftrightarrow \frac{(k-2)(-k+kx-x^2)}{(k-2)x-(k-1)} = 0 \Rightarrow k = 2$$

$$k - \cfrac{k}{k - \cfrac{k}{k - \cfrac{k}{k - \cfrac{k}{k - \frac{k}{x}}}}} = x \Leftrightarrow \frac{(k^2-3k+1)(k-kx+x^2)}{(k^2-3k+1)x-k(k-2)} = 0 \Rightarrow k \in \{\phi+1, 2-\phi\}$$

These values of k corresponding to $y_n = x$ seem to be the n^{th} generalized Beraha numbers $B_n^{(m)} = 4\cos^2\left(\frac{m\pi}{n}\right);\ m \in \mathbb{Z}$, excluding 0 and 4. In this paper, we seek to prove this assertion.

Let us define the infinite family of Beraha polynomials $\{p_n(k)\}$ as follows.

$$p_0(k) = 0 \tag{1}$$

$$p_1(k) = 1 \tag{2}$$

$$p_n(k) = \begin{cases} p_{n-1}(k) - p_{n-2}(k) & n \text{ even} \\ kp_{n-1}(k) - p_{n-2}(k) & n \text{ odd} \end{cases}. \tag{3}$$

Here are the first few Beraha polynomials.

$$\begin{aligned}
p_1(k) &= 1 \\
p_2(k) &= 1 \\
p_3(k) &= k - 1 \\
p_4(k) &= k - 2 \\
p_5(k) &= k^2 - 3k + 1 \\
p_6(k) &= k^2 - 4k + 3 \\
p_7(k) &= k^3 - 5k^2 + 6k - 1.
\end{aligned}$$

These polynomials can be seen in the numerators of the examples on the last page. We will show that the appearance of the Beraha polynomials in the numerators is not a coincidence and that the roots of these polynomials are the corresponding generalized Beraha numbers. First, let us show that for $n \geq 3$ the n^{th} generalized Beraha numbers (other than 0 and 4) are indeed the roots of the n^{th} Beraha polynomial.

2 Roots of Beraha Polynomials

Chebyshev Polynomials

Chebyshev polynomials of the second kind are defined as

$$U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}.$$

For example, $\sin(3\theta) = (4\cos^2(\theta) - 1)\sin(\theta)$ so $U_2(x) = 4x^2 + 1$.

They can also be defined recursively. Since

$$\sin((n+1)\theta) + sin((n-1)\theta) = 2\sin(n\theta)\cos(\theta)$$

We get after dividing through by $\sin(\theta)$

$$U_n(x) = 2xU_{n-1}(x) - U_{n-2}(x).$$

Here is a list of $U_n(x/2)$.

$$\begin{aligned}
U_1(x/2) &= x \\
U_2(x/2) &= x^2 - 1 \\
U_3(x/2) &= x^3 - 2x \\
U_4(x/2) &= x^4 - 3x^2 + 1 \\
U_5(x/2) &= x^5 - 4x^3 + 3x \\
U_6(x/2) &= x^6 - 5x^4 + 6x^2 - 1.
\end{aligned}$$

Comparing this with the list of Beraha polynomials, we observe the following.

Relating Beraha and Chebyshev Polynomials

$$p_n(k) = \begin{cases} U_{n-1}(\frac{\sqrt{k}}{2}) & n \text{ odd,} \\ \frac{1}{\sqrt{k}} U_{n-1}(\frac{\sqrt{k}}{2}) & n \text{ even.} \end{cases}$$

Proof. We show this inductively. The base cases are listed above. Assume this formula holds true for p_{n-2}, p_{n-1}.
If n odd, then

$$\begin{aligned} p_n(k) &= kp_{n-1}(k) - p_{n-2}(k) \\ &= \sqrt{k}U_{n-2}(\frac{\sqrt{k}}{2}) - U_{n-3}(\frac{\sqrt{k}}{2}) \\ &= 2(\frac{\sqrt{k}}{2})U_{n-2}(\frac{\sqrt{k}}{2}) - U_{n-3}(\frac{\sqrt{k}}{2}) \\ &= U_{n-1}(\frac{\sqrt{k}}{2}) \text{ By the recursive definition of } U. \end{aligned}$$

If n even, then

$$\begin{aligned} p_n(k) &= p_{n-1}(k) - p_{n-2}(k) \\ &= U_{n-2}(\frac{\sqrt{k}}{2}) - \frac{1}{\sqrt{k}}U_{n-3}(\frac{\sqrt{k}}{2}) \\ &= \frac{1}{\sqrt{k}} \left(2(\frac{\sqrt{k}}{2})U_{n-2}(\frac{\sqrt{k}}{2}) - U_{n-3}(\frac{\sqrt{k}}{2}) \right) \\ &= \frac{1}{\sqrt{k}}U_{n-1}(\frac{\sqrt{k}}{2}). \end{aligned}$$

$\square$

Now, from the definition

$$U_n(\cos(\theta)) = \frac{\sin((n+1)\theta)}{\sin(\theta)}$$

We find that $\cos\left(\frac{m\pi}{n+1}\right)$ is a root of U_n where $m \in \mathbb{Z}$, except when $n+1$ divides m. Plugging this into the above formula for p_n, we get $4\cos^2\left(\frac{m\pi}{n}\right)$ as a root of $p_n(k)$, unless n is even and $4\cos^2\left(\frac{m\pi}{n}\right) = 0$. Thus, the roots of the n^{th} Beraha polynomial are n^{th} generalized Beraha numbers $B_n^{(m)} = 4\cos^2\left(\frac{m\pi}{n}\right)$. We also know that they are the unique roots since there are $\left\lfloor \frac{n-1}{2} \right\rfloor$ unique values of $B_n^{(m)}$ that satisfy our conditions, which matches the degree of the n^{th} Beraha polynomial.

3 Solutions to the Nested Fraction Problem

Proposition 1

We give the following formula for y_n in terms of the Beraha polynomials:
For even n,

$$y_n - x = \frac{p_n(k)(-k + kx - x^2)}{xp_n(k) - p_{n-1}(k)}.$$

For odd n,

$$y_n - x = \frac{p_n(k)(-k + kx - x^2)}{xp_n(k) - kp_{n-1}(k)}.$$

Proof. We show this by induction. The base cases are listed at the top of the main document, and we leave the inductive step to the appendix. □

Theorem 2

The constants k such that $y_n = x$ for all $x \neq 0$ and $n \geq 3$ are roots of the n^{th} Beraha polynomial $p_n(k)$.

Proof of Theorem 2

Proof. From the formulas in Proposition 1, we see for each y_n, all such constants k must be roots of the Beraha polynomial $p_n(x)$. To show that the Beraha numbers are always solutions to $y_n - x = 0$, we must show that they are not roots of the denominators in Proposition 1.

Recall that the roots of $p_n(x)$ are of the form $B_n^{(m)} = 4\cos^2(\frac{m\pi}{n})$. For all $n \geq 3$, it is clear that $p_n(x)$ and $p_{n-1}(x)$ share no roots. This means that if we plug $B_n^{(m)}$ into the denominators of the fractions, we'll always get something nonzero:

$$xp_n(B_n^{(m)}) - p_{n-1}(B_n^{(m)}) = 0 - p_{n-1}(B_n^{(m)}) \neq 0.$$

Similarly,

$$xp_n(B_n^{(m)}) - B_n^{(m)}p_{n-1}(B_n^{(m)}) = 0 - B_n^{(m)}p_{n-1}(B_n^{(m)}) \neq 0.$$

Thus, $B_n^{(m)}$ is never a root of the denominator of y_n. We have now shown that the solutions to the nth nested fraction problem are exactly the roots of the nth Beraha polynomial. □

4 A Second Approach to $p_n(k)|y_n - x$.

Below, we present a more generalizable approach to solving the problem of showing $p_n(k)|y_n - x$. Again, we approach this by induction. Assume

$$y_n = x \Leftrightarrow \frac{p_n(k)A}{D_n} = 0 \tag{4}$$

where D_n is a polynomial that *may* share factors with $p_n(k)$ and A is an unimportant polynomial that has no monic factors in k. This factorization tells us that all the solutions of $y_n = x$ are the roots of $p_n(k)$ that are not also roots of D_n. (4) is equivalent to

$$y_n = \frac{p_n(k)A + xD_n}{D_n} \quad \left(= \frac{N_n}{D_n}\right). \tag{5}$$

Also assume

$$k \text{ divides } N_n \text{ iff } n \text{ is odd.} \tag{6}$$

Lemma 3

$$D_n = \begin{cases} \frac{p_{n-1}(k)A + xD_{n-1}}{k} & n \text{ even} \\ p_{n-1}(k)A + xD_{n-1} & n \text{ odd} \end{cases} \tag{7}$$

for all $n \geq 3$.

Proof of Lemma 3

We start by expanding y_{n+1}.

$$y_{n+1} = k(1 - \frac{1}{y_n}) \tag{8}$$

$$\text{substituting in (5)} \qquad = k\left(\frac{p_n(k)A + xD_n - D_n}{p_n(k)A + xD_n}\right) \tag{9}$$

We can see from (9) that $D_{n+1} = p_n(k)A + xD_n$ as long as this expression does not have a factor of k. In the case where n is odd, $p_n(k)A + xD_n = N_n$ must have a factor of k because of (6), which will cancel out with the k in the numerator of (9), implying that $D_{n+1} = \frac{p_n(k)A + xD_n}{k}$ and N_{n+1} does not have a factor of k if n is odd. If n is even, $p_n(k)A + xD_n = N_n$ does not have a factor of k by (6). This implies that (6) holds for $n+1$ and that (7) holds for $n+1$. Because (6) and (7) hold for $n = 3$,

$$k \text{ divides } N_n \text{ iff } n \text{ is odd} \tag{10}$$

$$D_n = \begin{cases} \frac{p_{n-1}(k)A + xD_{n-1}}{k} & n \text{ even} \\ p_{n-1}(k)A + xD_{n-1} & n \text{ odd} \end{cases}$$

for all $n \geq 3$. This proves Lemma 1 and simultaneously (6) of our inductive hypothesis.

Lemma 4

The denominators follow the same recursion as the Beraha polynomials. That is, for all $n \geq 3$

$$D_n(k) = \begin{cases} D_{n-1}(k) - D_{n-2}(k) & n \text{ even} \\ kD_{n-1}(k) - D_{n-2}(k) & n \text{ odd} \end{cases} \tag{11}$$

The proof is algebra and is left to the appendix.

Inductive Case for Odd n

Let us look at the case where n is odd first, and use (7) to simplify (9).

$$\begin{aligned} y_{n+1} &= \frac{p_n(k)A + xD_n - p_{n-1}(k)A - xD_{n-1}}{D_{n+1}} \\ &= \frac{(p_n(k) - p_{n-1}(k))A + x(D_n - D_{n-1})}{D_{n+1}} \\ \text{substituting in (3) and (11)} \qquad &= \frac{p_{n+1}(k)A + xD_{n+1}}{D_{n+1}} \end{aligned}$$

$$\iff \left(y_{n+1} = x \Leftrightarrow \frac{p_{n+1}(k)A}{D_{n+1}} = 0 \right) \quad \text{for odd } n \tag{12}$$

Inductive Case for Even n

Use (7) to simplify (9).

$$\begin{aligned} y_{n+1} &= k\left(\frac{p_n(k)A + xD_n - \frac{1}{k}(p_{n-1}(k)A + xD_{n-1})}{D_{n+1}} \right) \\ &= \frac{(kp_n(k) - p_{n-1}(k))A + x(kD_n - D_{n-1})}{D_{n+1}} \\ \text{substituting in (11)} \qquad &= \frac{p_{n+1}(k)A + xD_{n+1}}{D_{n+1}} \end{aligned}$$

$$\iff \left(y_{n+1} = x \Leftrightarrow \frac{p_{n+1}(k)A}{D_{n+1}} = 0 \right) \quad \text{for even } n \tag{13}$$

Since every case is either even or odd the inductive hypothesis (4) holds for $n+1$. Since we have shown that our inductive hypotheses imply their successors, and each is true for $n = 3$,

$$y_n = x \Leftrightarrow \frac{p_n(k)A}{D_n} = 0$$

for all $n \geq 3$. QED.

5 Appendix

Proof of Proposition 1

Proof. The following is the inductive step of Proposition 1. Assume the theorem holds for $y_n - x$. Then, we write

$$\begin{aligned} y_{n+1} - x &= \left(k - \frac{k}{y_n}\right) - x \\ &= \frac{(k-x)y_n - k}{y_n} \\ &= \frac{(k-x)((y_n - x) + x) - k}{(y_n - x) + x} \\ &= \frac{(k-x)(y_n - x) + (-k + kx - x^2)}{(y_n - x) + x}. \end{aligned}$$

If n is odd, we can substitute and multiply both top and bottom by $xp_n(k) - kp_{n-1}(k)$ to get

$$\begin{aligned} &\frac{(-k + kx - x^2)[(k-x)p_n(k) + xp_n(k) - kp_{n-1}(k)]}{p_n(k)(-k + kx - x^2) + x(xp_n(k) - kp_{n-1}(k))} \\ &= \frac{(-k + kx - x^2)(k)(p_n(k) - p_{n-1}(k))}{kx(p_n(k) - p_{n-1}(k)) - kp_n(k)} \\ &= \frac{(-k + kx - x^2)(p_n(k) - p_{n-1}(k))}{x(p_n(k) - p_{n-1}(k)) - p_n(k)}. \end{aligned}$$

Substituting $p_{n+1}(k) = p_n(k) - p_{n-1}(k)$, we get as desired

$$y_{n+1} - x = \frac{p_{n+1}(k)(-k + kx - x^2)}{xp_{n+1}(k) - p_n(k)}.$$

The even case is basically the same. Substitute and multiply through by $xp_n(k) - p_{n-1}(k)$ to get something which simplifies to

$$\frac{(-k + kx - x^2)(kp_n(x) - p_{n-1}(k))}{x(kp_n(x) - p_{n-1}(k)) - kp_n(k)}.$$

Then substituting $p_{n+1}(k) = kp_n(k) - p_{n-1}(k)$, we get as desired

$$y_{n+1} - x = \frac{p_{n+1}(k)(-k + kx - x^2)}{xp_{n+1}(k) - kp_n(k)}.$$

$\square$

Proof of Lemma 4

Proof. By induction. In the case where n is even:

$$\begin{aligned}
\text{Assume}\quad D_{n-1} &= kD_{n-2} - D_{n-3} && (14)\\
D_{n-1} - D_{n-2} &= p_{n-2}(k)A + xD_{n-2} - \frac{p_{n-3}(k)A + xD_{n-3}}{k} \\
&= \frac{(kp_{n-2}(k) + p_{n-3}(k))A + x(kD_{n-2} + D_{n-3})}{k} \\
\text{using (3) and (14)}\qquad &= \frac{p_{n-1}(k)A + xD_{n-1}}{k} \\
\text{by (7)}\qquad D_{n-1} - D_{n-2} &= D_n && (15)
\end{aligned}$$

The odd case is proven by doing the same process on D_{n+1}:

$$\begin{aligned}
kD_n - D_{n-1} &= p_{n-1}(k)A + xD_{n-1} - p_{n-2}(k)A - xD_{n-2} \\
&= (p_{n-1}(k) - p_{n-2}(k))A + x(D_{n-1} - D_{n-2}) \\
\text{using (3) and (15)}\qquad &= p_n(k)A + xD_n && (16)\\
\text{by (7)}\qquad kD_n - D_{n-1} &= D_{n+1}
\end{aligned}$$

(16) is equivalent to (14) for $n + 2$. Since (14) is true for the case $n - 1 = 3$, Lemma 2 holds for all n. $\square$