Master in High Performance Computing

Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 13 - Lax Milgram and Cea's Lemma

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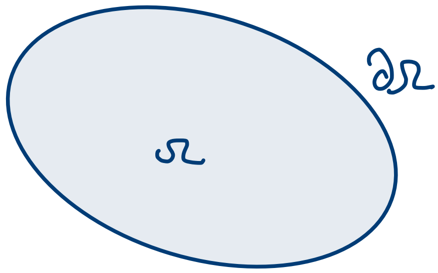
GENERAL GALERKIN :

PROTOTYPICAL PROBLEM



Given a source term $f : \Omega \to \mathbb{R}$

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

open Lipschitz subset of $\mathbb{R}^d$
$d = 1,2,3$

Restrict to a space $\underline{\underline{V}}$ ($H^1_0(\Omega) := \{ v \in L^2(\Omega) \mid \nabla v \in L^2(\Omega) \}$
$v|_{\partial\Omega} = 0$)

1 multiply by arbitrary $v \in V$
2 integrate by parts on Ω
3 use B.C.

(1) $-\Delta u \, v = f v$

(2) $\int_\Omega -\Delta u \, v = \int_\Omega f v$

$\int_\Omega \nabla u \cdot \nabla v - \int_{\partial\Omega} \frac{\partial u}{\partial n} v = \int_\Omega f v$

(3) : $-\int_{\partial\Omega} \frac{\partial u}{\partial n} v = 0$; $v = 0$ on $\partial\Omega$

We are left with a bilinear form

$$a : V \times V \longrightarrow \mathbb{R}$$

$$a(u,v) := \int_\Omega \nabla u \, \nabla v \, dx$$

$${}_{V^*}\langle f, v \rangle_V := \int_\Omega f v \, dx$$ duality between V^* and V

General case : Start from V, $a : V \times V \to \mathbb{R}$, $f \in V^*$

V is Hilbert ($\Rightarrow$ it admits a scalar product $(\cdot,\cdot)_V$)

Theorem $\boxed{\text{Lax-Milgram's Lemma}}$

Given V Hilbert, a : bilinear, coercive, then

$\forall f \in V^*$, $\exists!\, u$ s.t.

$$a(u,v) = \langle f, v \rangle \quad \forall v \in V$$

1) a is bilinear : $|a(u,v)| \leq \|A\| \, \|u\|_V \|v\|_V \quad \forall u,v \in V$

2) a is coercive : $a(u,u) \geq \alpha \|u\|_V^2 \quad \forall u \in V$

$$\alpha \|u\|_V^2 \leq a(u,u) \leq \|A\| \, \|u\|_V^2$$

$$\boxed{\|f\|_* := \sup_{0 \neq v \in V} \frac{\langle f, v \rangle}{\|v\|_V}}$$

Moreover we have

$$\boxed{\|u\|_V \leq \frac{1}{\alpha} \|f\|_*}$$

$$a(u,v) = \langle f, v \rangle \qquad \forall v \in V$$

Valid for $v \equiv u$

$$\alpha \|u\|_2^V \leq a(u,u) = \langle f, u \rangle \leq \|f\|_* \|u\|_V$$

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_*$$

Rietz Theorem

Given an Hilbert space V, $\exists$ Rietz operator $\tau : V^* \to V$

$\forall f \in V^*$, $\exists! \tilde{f} \in V$ s.t.

1) $(\tilde{f}, v)_V = \langle f, v \rangle = f(v) \quad \forall v \in V$

2) $\|\tilde{f}\|_V = \|f\|_* = \|\tau f\|_V$

$$\boxed{\tilde{f} = \tau f}$$

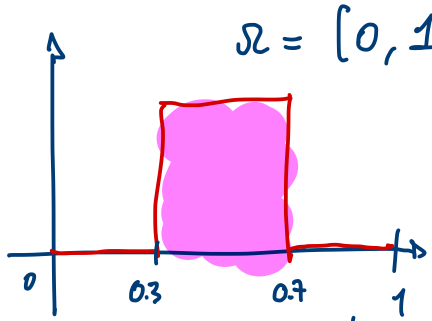
$V = H_0^1(\Omega)$

$$\boxed{(u,v)_V := \int_\Omega u v + \int_\Omega \nabla u \nabla v}$$

Example:

$f := \chi[0.3, 0.7]$

$\Omega = [0, 1]$



$\exists! \tilde{f} \in H_0^1(\Omega)$ continuous function s.t.

$$(\tilde{f}, v)_V = \int_\Omega f v \qquad \forall v \in H_0^1(\Omega)$$

$$(u,v) + (\nabla u, \nabla v) = \int_\Omega f v \qquad \forall v \in H_0^1(\Omega)$$

Prove Lax Milgram using a contraction theorem.

$T : V \longrightarrow V$ is a contraction if

$\exists L < 1$ s.t. $\| T(u) - T(v) \|_V \leq L \| u - v \|_V$

then $\exists! \, \varphi \in V$ s.t. $T(\varphi) = \varphi$

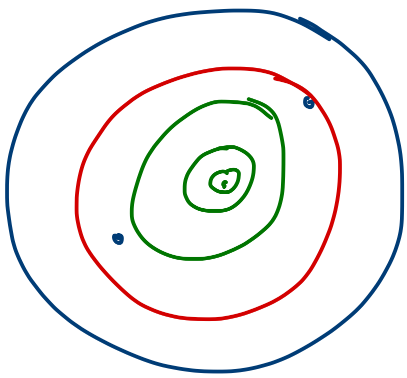
Let $u^0 \in V$ arbitrary, define $u^{k+1} = T(u^k)$

$$\| u^{k+2} - u^{k+1} \|_V = \| T(u^{k+1}) - T(u^k) \|_V \leq L \| u^{k+1} - u^k \|_V$$

$$\| u^{k+2} - u^{k+1} \|_V \leq L^k \| \overset{u^1}{T(u^0)} - u^0 \|_V$$

$k \to \infty$, $L < 1 \Rightarrow L^k \to 0 \Rightarrow \| u^{k+2} - u^{k+1} \| \to 0$

u^k is a cauchy sequence $\left(\exists \varphi \text{ s.t } \lim_{k \to \infty} u^k = \varphi \right)$



Proof of Lax Milgram: 1) Rietz theorem 2) Contraction

Write $Au = f$ as a contraction

$\forall$ every bilinear form Π can write an operator:

$a : V \times V \longrightarrow \mathbb{R}$

$a(u,v) = \langle Au, v \rangle \qquad A : V \longrightarrow V^*$

at fixed u, $a(u,v)$ is a linear operator, i.e.:

$$a(u,v) = (Au)(v) = {}_{V^*}\langle Au, v \rangle_V$$

$\boxed{Au = f}$ in V^* is the same as $\boxed{a(u,v) = \langle f, v \rangle \ \forall v \in V}$

$T : V \longrightarrow V \qquad \tau(Au - f) = 0$ in V

$$T(v) = v - \rho\tau(Av - f) \qquad \text{for some } \rho \in \mathbb{R}$$

$$T(u) = u - \rho\tau(Au - f) = u$$

$$\|T(u) - T(v)\|_V^2 = \|u - \rho\tau(Au - f) - (v - \rho\tau(Av - f))\|_V^2$$

$$= \|u - v - \rho\tau A(u - v)\|_V^2$$

$$= \|u - v\|^2 + \rho^2 \|\tau A(u - v)\|_V^2 - 2\rho(\tau A(u-v), (u-v))_V$$

$$= \|u - v\|^2 + \rho^2 \|A(u - v)\|_{V^*}^2 - 2\rho\, \underbrace{{}_{V^*}\langle A(u-v), u - v \rangle_V}_{a(u-v,\, u-v)}$$

$$\leq \|u - v\|^2 + \rho^2 \|A\|_*^2 \|u - v\|_V^2 - 2\alpha\rho \|u - v\|_V^2$$

$a(u,u) \geq \alpha \|u\|_V^2$

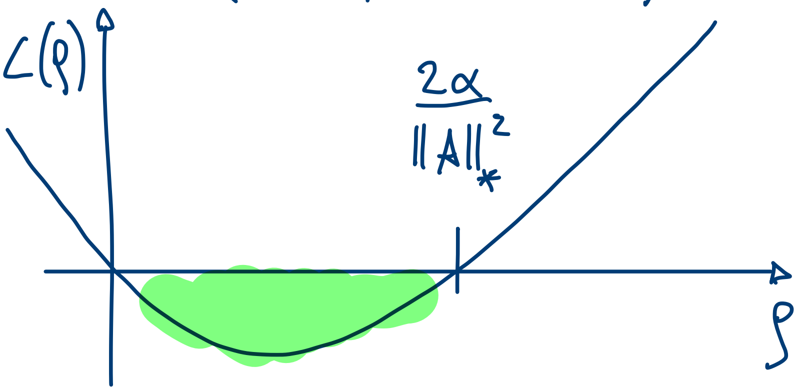
$-a(u,u) \leq -\alpha \|u\|_V^2$

$$\|T(u)-T(v)\|_V^2 \leq \underbrace{\left(\|A\|_*^2\rho^2 - 2\alpha\rho + 1\right)}_{L} \|u-v\|^2$$

$L < 1$

$$0 \leq \|A\|_*^2 \rho^2 - 2\alpha\rho + 1 \leq 1$$

$$\rho\left(\|A\|_*^2\rho - 2\alpha\right) \leq 0$$



$\Rightarrow$

if we choose ρ s.t.

$$0 < \rho < \frac{2\alpha}{\|A\|_*^2}$$

then T is a contraction

$\exists!$ u s.t. $T(u) = \cancel{u} = \cancel{u} - \rho\tau(Au - f) = 0$

$\Rightarrow Au = f$

Take a bilinear, coercive, $f \in V^*$, $V_h = \mathrm{span}\{v_i\}_{i=0}^{N-1}$,

$v_i \in V \Rightarrow V_h \subset V$ (V_h has dimension N)

Then

$Au = f$ in V^* because $Au_h = f$ in V_h^*

$\langle Au, v\rangle \overset{1}{=} \langle f, v\rangle \ \forall v \in V$ $\qquad$ $\langle Au_h, v_h\rangle \overset{2}{=} \langle f, v_h\rangle \ \forall v_h \in V_h$

$\boxed{\text{Galerkin Orthogonality:}}$ $\qquad$ 1 − 2 gives 3

3 $\langle A(u-u_h), v_h\rangle = a(u-u_h, v_h) = 0 \quad \forall v_h \in V_h$

$$\alpha \|u-u_h\|_V^2 \leq \langle A(u-u_h),(u-u_h)\rangle$$

$$\leq \langle A(u-u_h),(u-v_h)\rangle \qquad \forall v_h \in V_h$$

$$\leq \|A\|_* \|u-u_h\|_V \|u-v_h\|_V \qquad \forall v_h \in V_h$$

$$\boxed{\|u-u_h\|_V \leq \frac{\|A\|_*}{\alpha} \inf_{v_h \in V_h} \|u-v_h\|_V}$$

A priori: if dist(V,V_h) is small, we win!!

$$\text{dis}(V,V_h) := \sup_{u \in V} \inf_{v_h \in V_h} \|u-v_h\|$$

For Finite elements of order k we have

$$\boxed{\|u-I(u)\|_{m,T} \leq C\, h_T^{k+1-m} |u|_{k+1,T}}$$

on each element T of the triangulation

$$|u|_{m,T}^2 := \sum_{|\alpha|=m} \int_T (D_\alpha u)^2$$

$$|u|_{0,T}^2 = \|u\|_{L^2(T)}^2$$

$$|u|_{1,T}^2 := \int_T |\nabla u|^2$$

$$|u|_{2,T}^2 := \int_T |Hu|^2$$

$$\|u\|_{m,T}^2 := \sum_{\alpha=0}^{m} |u|_{\alpha,T}^2$$

$$\|u\|_{1,T}^2 := \|u\|_{L^2(T)}^2 + \|\nabla u\|_{L^2(T)}^2$$