Master in High Performance Computing

Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 08 - Interpolatory quadrature formulas

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L^2 projection in $L^2([a,b])$, $\mathbb{P}^n([a,b])$

$$\boxed{(\Pi^n u)(x) := M^{ij}(u, v_j)\, v_i(x)}$$

inverse of M

$$\boxed{M^{ij} M_{jk} = \delta^i_k}$$

$$M_{ij} := (v_j, v_i) = \int_a^b v_j\, v_i\, dx$$

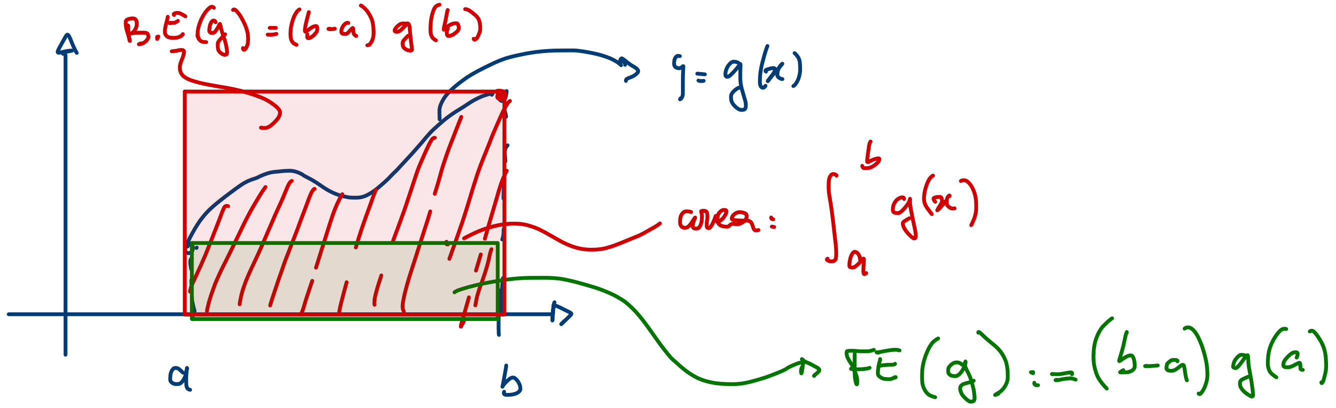
Best choice of Basis Legendre basis: $\{v_i\}_{i=0}^n$

s.t. $(v_i, v_j) = \delta_{ij} \Rightarrow M$ is identity

How do we integrate $\int_a^b g(x)\,dx$ numerically?

Example of approximations:

1. F.E. Forward Euler $\int_a^b g(x)\,dx \simeq (b-a)\, g(a)$
2. B.E Backward Euler $\int_a^b g(x)\,dx \simeq (b-a)\, g(b)$



Mid Point Rule: $MP(g) : (b-a)\, g\left(\frac{a+b}{2}\right)$

Trapezoidal Rule: $T(g) : (b-a)\left[\frac{1}{2} g(a) + \frac{1}{2} g(b)\right]$

Degree of accuracy of a "approximate Integral"

maximum $k \in \mathbb{N}$ s.t. the approximate integrae coincides with the exact integral $\forall\, p \in \mathbb{P}^k$

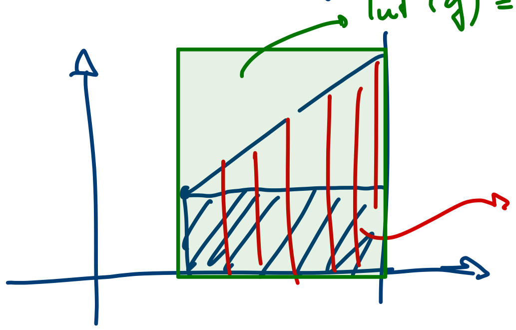
$$Int(g) := \int_a^b g(x)\, dx$$

$Int^{(n)}(g) :=$ approximate integral

$Int^1(g)$ defined as : $Int^{(1)}(g) = (b-a)\, g(a)$ F.E.

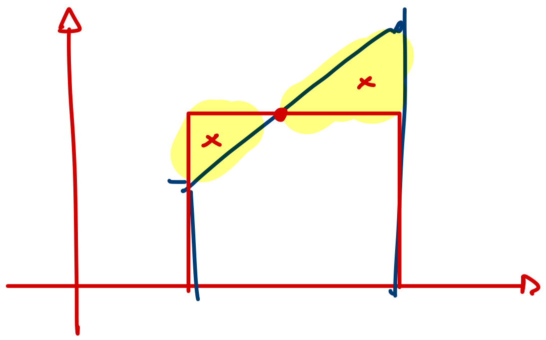
What is k for $F.E = Int^1(g) = (b-a)\, g(a)$

$Int'(g) = B.E(g) \rightsquigarrow k = 0$



$\boxed{k = 0}$

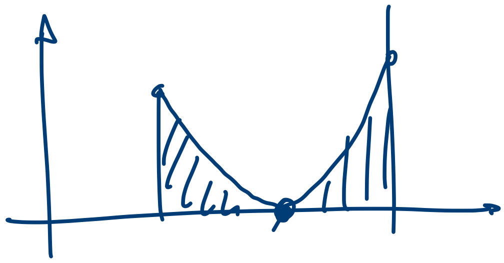
MID POINT:



$Int^1(g) := (b-a)\, g\left(\frac{b+a}{2}\right)$

has k (degree of accuracy) equal to 1

$Int^1(g+f) = Int^1(g) + Int^1(f)$



$k \geq 1 \qquad k < 2 \quad \Rightarrow \quad k = 1$

$\boxed{\text{Interpolatory quadrature formulas}}$

- Take "g", interpolate it on some points
- Integrate the resulting polynomial

$$Int^n_{\{a_i\}_{i=0}^{n-1}} g := \int_a^b I^{n-1}_{\{a_i\}_{i=0}^{n-1}} g \, dx$$

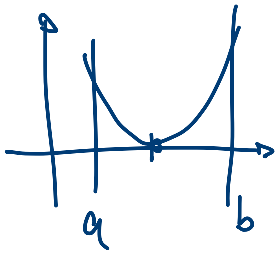
consequence 1 : $Int^n_{\{a_i\}_{i=0}^{n-1}}$ $(n \geq 1)$ are exact for polynomials of order <u>at least $(n-1)$</u>

$$I^{n-1} p = p \qquad \forall p \in \mathbb{P}^{n-1}$$

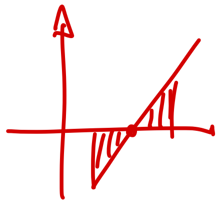
$$\Rightarrow \quad Int(p) = \int_a^b p = \int_a^b I^{n-1} p = Int^n(p) \qquad \forall p \in \mathbb{P}^{n-1}$$

what is the upper limit for k given $\{a_i\}_{i=0}^{n-1}$?

for $n=1$ $\rightsquigarrow$ $\left(x-\frac{(b+a)}{2}\right)^2$



$$\omega(x) = \prod_{i=0}^{n-1} (x-a_i) \in \mathbb{P}^n$$

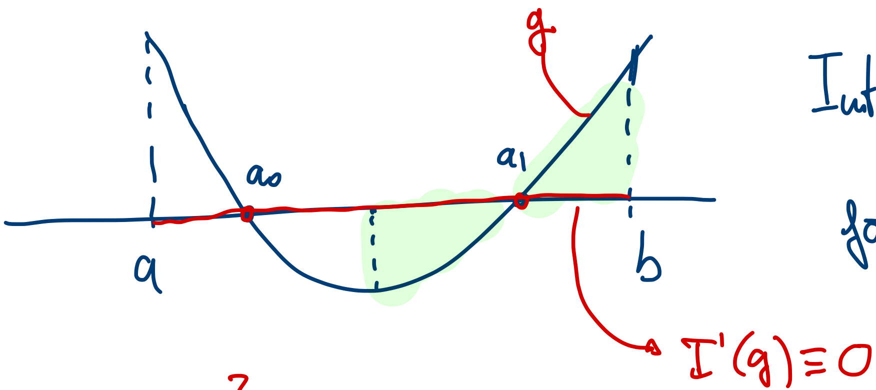


$\mathrm{Int}^n(\omega) = 0$ by construction

$\mathrm{Int}^n(\omega^2) = 0$ but $\int_a^b \omega^2 > 0$

$\Rightarrow n-1 \leq k(\mathrm{Int}^n) < 2n$

Mid point $n-1=0$ but $k=1 = 2n-1$



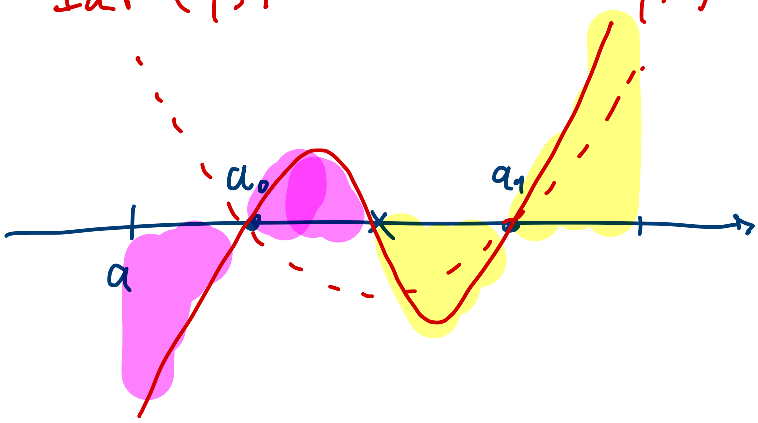
$\mathrm{Int}^2(g) = \int I^1(g) = \int g$

for $g \in \mathbb{P}^2$

$\forall q_2 \in \mathbb{P}^2$ $q_2 = p + \alpha g$ for some $\alpha \in \mathbb{R}$ and $p \in \mathbb{P}^1$

How about $\mathbb{P}^3$? is there a $q_3 \in \mathbb{P}^3$, s.t.

$\mathrm{Int}^2(q_3) = 0 = \mathrm{Int}(q_3)$?



$$\boxed{\begin{array}{l} k \geq 3 \\ k < 2n = 4 \end{array}}$$

$\boxed{\text{Theorem}}$ Let $u \in \mathbb{P}^{n-1+m}$, $\boxed{0 \leq m \leq n}$

$$\mathrm{Int}^n(u) = I(u)$$

$$\Longleftrightarrow$$

$$\int_a^b \omega(x) \cdot p = 0 \quad \forall p \in \mathbb{P}^{m-1}$$

$$\omega(x) := \prod_{i=0}^{n-1} (x - a_i)$$

$\boxed{\text{Proof}}$ Any polynomial p of order $(n-1+m)$ can be written as:

$$p = \underbrace{\omega}_{n} \underbrace{\Pi}_{m-1} + \overbrace{q}^{n-1}$$

$q \in \mathbb{P}^{n-1}$

$$\mathrm{Int}(p) = \mathrm{Int}^n(p)$$

$$\mathrm{Int}(\omega\Pi) + \mathrm{Int}(q) = \cancel{\mathrm{Int}^n(\omega\Pi)} \to 0 + \mathrm{Int}^n(q) \quad \in \mathbb{P}^{n-1}$$

by construction

$\omega(a_i) \cdot \Pi(a_i) = 0 \quad \forall i$

$$\underbrace{\mathrm{Int}(\omega\Pi)}_{\equiv 0} + \mathrm{Int}(q) = \mathrm{Int}^n(q) = \mathrm{Int}(q) \quad q \in \mathbb{P}^{n-1}$$

$$\Longleftrightarrow$$

$$\int_a^b \omega\Pi = 0$$

Take $\boxed{a_i}$ as the roots of Legendre basis of order n

This is known as Gauss-Legendre quadrature rules. K is $\boxed{2n-1}$

$$\mathrm{Int}^n(g) = \sum_{i=0}^{n-1} g(a_i)\, w_i = \int_a^b I^{n-1} g$$

$$= \sum_{i=0}^{n-1} g(a_i) \underbrace{\int_a^b \ell_i(x)\, dx}$$

$\{\ell_i\}_{i=0}^{n-1}$ are the n Lagrange basis functions (polynomials) of order $(n-1)$ for the roots of the Legendre polynomial of order n