Master in High Performance Computing

Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 06 - Weierstrass approximation theorem and L2 projections

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Interpolation: $I : C^0([a,b]) \longrightarrow \mathbb{P}^n([a,b])$

I: a set interpolation points $\{a_i\}_{i=0}^{n}$ n+1 points

$\mathbb{P}^n$: span $\{v_i\}_{i=0}^{n}$ (n+1) dimensional space

$$I : u \longrightarrow p(x) := \left[(\underline{V}^{-1})\,\underline{u}\right]^i v_i(x)$$

$$:= \underline{\underline{V}}^{ij}\, \underline{u}(a_j)\, v_i(x)$$

$$V^{ij} V_{jk} = \delta^i_k \quad (\equiv (V^{-1})_{ij})$$

$V_{ij} := v_j(a_i)$ $\rightarrow$ Lagrange $V_{ij} = \delta_{ij}$

Is interpolation good for approximating funcions?

Can we bound $\| u - I^n u \|_{L^\infty([a,b])}$ when $n \to \infty$?

In general <u>No</u>.

<u>Weierstrass</u>: $\forall f \in C^0([a,b])$, $\forall \varepsilon > 0$
$\exists p \in \mathbb{P}^n$ s.t. $\|f - p\|_{L^\infty([a,b])} \leq \varepsilon$

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proof: Let $B_n : C^0([a,b]) \longrightarrow \mathbb{P}^n$ be s.t.

1) B_n is linear and positive ($f \geq 0 \Rightarrow B_n f \geq 0$)

2) $\|B_n x^i - x^i\|_{L^\infty([a,b])} \to 0$ $\boxed{\forall\, i = 0,1,2}$

Then $\|B_n f - f\|_{L^\infty([a,b])} \longrightarrow 0$

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1. $\forall f \in C^0([a,b])$, $\forall x_0 \in [a,b]$

- construct $q^{\pm} \in \mathbb{P}^2$ s.t.

  $\underline{q^-(x) \leq f(x) \leq q^+(x)}$ $\forall x \in [a,b]$

- use $B_n q^{\pm} \to q^{\pm}$
- use $B_n(f - q^-)(x) \geq 0$

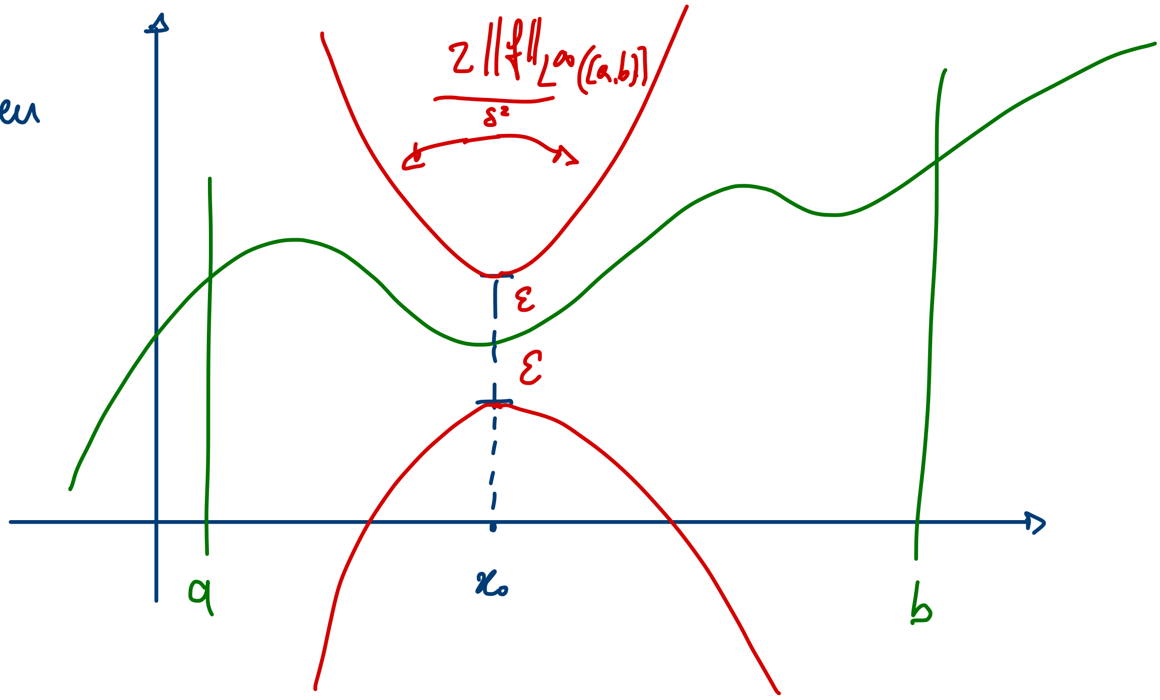
  $B_n(q^+ - f)(x) \geq 0$

$f \in C^0([a,b])$, fix $x_0 \in [a,b]$

$\forall \varepsilon > 0$, $\exists \delta_\varepsilon > 0$ s.t. $\boxed{|x_1 - x_2| < \delta_\varepsilon \Rightarrow |f(x_1) - f(x_2)| \leq \varepsilon}$

$$q^{\pm} := f(x_0) \pm \left(\frac{\varepsilon}{2} + \frac{2\|f\|_{L^\infty([a,b])}}{\delta^2} (x - x_0)^2 \right)$$

ε is given
$\Downarrow$
δ_ε



For varying x_0, $\quad q^{\pm} = a^{\pm}_{x_0} x^2 + b^{\pm}_{x_0} x + c^{\pm}_{x_0}$

$\boxed{a^{\pm}, b^{\pm}, c^{\pm} \text{ depend on } x_0, \|f\|_{L^\infty([a,b])}, \delta_\varepsilon, \varepsilon}$

$$M := \max_{x_0 \in [a,b]} \left(|a^{\pm}|, |b^{\pm}|, |c^{\pm}| \right)$$

$\boxed{\begin{array}{l} M \text{ will depend on } \|f\|_{L^\infty([a,b])}, \delta_\varepsilon, \varepsilon \\ \text{but NOT on } x_0 \end{array}}$

Choose N large enough s.t.

$*\quad \|B_n x^i - x^i\|_{L^\infty} \leq \frac{\varepsilon}{6M} \qquad \forall n \geq N \quad$ for $i = 0, 1, 2$

in x_0: $\quad \underbrace{f(x_0) - \varepsilon \leq q^-(x_0)}_{\text{Definition of } q^-(x_0)} \overbrace{- \frac{\varepsilon}{2} \leq B_n q^-(x)}^{*} \underbrace{\leq B_n f(x_0)}_{\text{positivity}}$

$\exists N$ s.t., $\forall n \geq N$ $\quad B_n f \leq B_n q^+ \leq q^+ + \frac{\varepsilon}{2} \leq f(x) + \varepsilon$

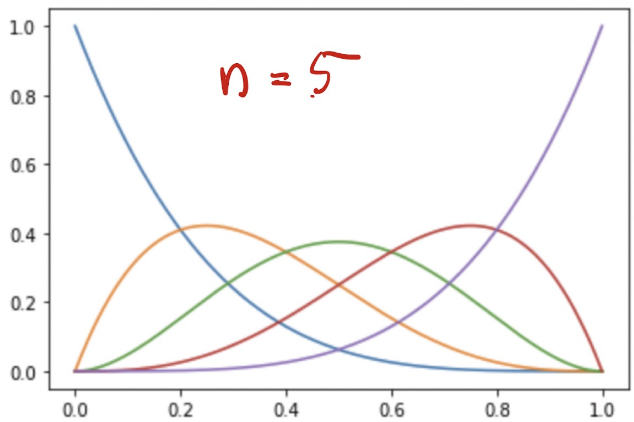
$$\boxed{\| B_n f - f \| \leq \varepsilon}$$

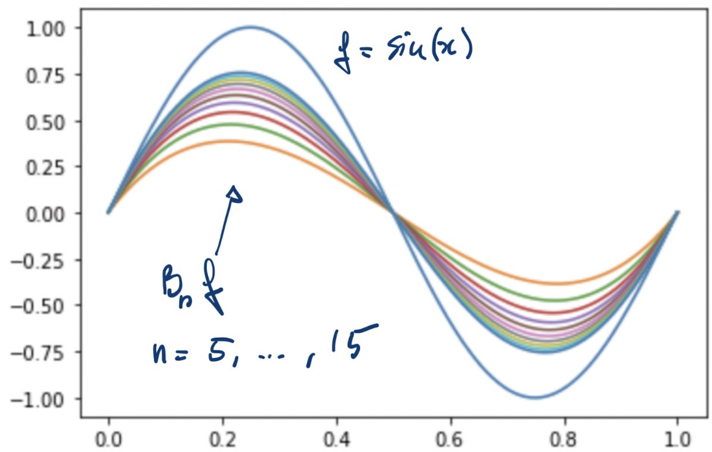
Construction of B_n $\quad$ (set $[a,b] = [0,1]$)

$$\boxed{\left((1-x) + x\right)^n = \underbrace{\sum_{i=0}^{n} \binom{n}{i} x^i (1-x)^{n-i}}_{\sum b_i^n(x)}}$$

$$\boxed{B_n f := \sum_{i=0}^{n} b_i^n(x) \, f\left(\frac{i}{n}\right)}$$

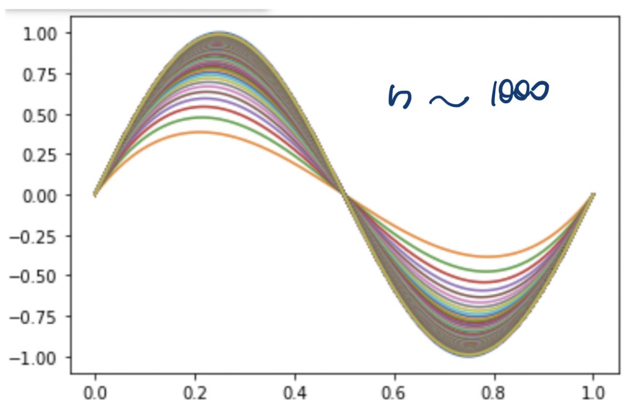
similar to interpolation, but different!





$$\boxed{B_n 1 = 1} \qquad \boxed{B_n x = x} \qquad \boxed{B_n x^2 = \left(\frac{n-1}{n}\right) x^2 + \frac{1}{n} x}$$

$$\neq x^2$$



Robust, but slow

Recall definition of Best Approximation.

V : Banach

M : is finite dimensional subspace of V

$p \in M$ is best approximation of $f \in V$

$$\|p-f\|_V \leq \|q-f\|_V \quad \forall q \in M$$

V is strictly convex, then B.A. $\exists$ and it is unique

V is strictly convex if given f, g s.t.

$$\|f\| = \|g\| = 1 \qquad \forall \alpha \in (0,1)$$

$$\|\alpha f + (1-\alpha) g\| < 1$$

V Hilbert $\Rightarrow$ $\exists$ $(\cdot,\cdot)$ scalar product.

and $\|u\|_V := \sqrt{(u,u)} \quad \forall u \in V$

Constructively building B.A.:

Given $u \in V$, find p s.t.

$$(p,q) = (u,q) \quad \forall q \in M \subset V$$

$$* \quad (u-p, q) = 0 \quad \forall q \in M \subset V$$

$\Leftrightarrow$ $p \in M$ is b.a. of $u \in V$

$\|u-p\|_V \leq \|u-q\|_V \quad \forall\, q \in M$

Think of $M = \text{span}\{v_i\}_{i=0}^{n}$ $\quad (u,v) := \int_0^1 uv$

$$V = L^2([0,1]) = \{v \text{ s.t. } \int_0^1 v^2 dx < +\infty\}$$

$$\|u\|_V^2 := \int_0^1 u^2 dx = (u,u)$$

$$p \in M \Rightarrow p = p^i v_i$$

$$(p-u,q) = 0 \quad \forall\, q \in M$$ (same as asking $\forall\, v_i$ in basis)

$\Longleftrightarrow$

$$(p^J v_J - u, v_i) = 0 \quad \text{for } i = 0, \dots, n$$

$$M_{iJ}\, p^J = (u, v_i) = u_i = \int_0^1 u v_i\, dx$$

$$M_{iJ} := (v_J, v_i) \Rightarrow p^J = M^{Ji} u_i$$

$$p^J = (M^{-1})^{Ji} u_i$$

$(u-p,q) = 0 \quad \forall q \in M \Longleftrightarrow \|u-p\| \leq \|u-q\| \quad \forall q \in M$

"$\Leftarrow$" $\quad$ p. is b.a. $\Rightarrow (u-p,q) = 0 \quad \forall\, q \in M$

$$\|u-p\|^2 \leq \|u-p+tq\|^2 \quad \forall\, t>0, \forall\, q \in M$$

$$\|\underbrace{u-p+\tfrac{t}{2}q}_{(a} + \underbrace{\tfrac{t}{2}q}_{b)}\|^2 - \|u-p\|^2 \geq 0$$

$$(a+b)^2 - (a-b)^2 = 4ab \geq 0$$

$$4\left(u-p+\tfrac{t}{2}q, \tfrac{t}{2}q\right) \geq 0$$

$$2t(u-p,q) + t^2(q,q) \geq 0 \qquad t>0$$

$$\underbrace{-t\|q\|^2 \leq (u-p,q)}_{t>0} \leq \frac{t}{2}\|q\|^2 \qquad \forall t, \forall q$$

$t<0$

$$\Rightarrow \quad (u-p,q) = 0$$

$(u-p,q) = 0 \ \forall q \quad \Longrightarrow \quad p$ is b.a. of u

$$\|u-q\|^2 = \|u-p+p-q\|^2 = \|u-p\|^2 + \|p-q\|^2 + \cancel{2(u-p,p-q)} \to 0$$

$$\|u-p\|^2 \leq \|u-q\|^2 \quad \forall q \in M$$

$\boxed{\text{Apply to polynomials}}$ $\qquad \mathbb{P}^n := \operatorname{span}\{v_i\}_{i=0}^n$

$$v_i := x^i = \text{pow}(x,i)$$

$$M_{ij} := (v_j, v_i) = \int_0^1 x^j x^i = \boxed{\frac{1}{1+j+i}}$$

Hilbert Matrix. (H)

$$\text{cond}(H\ \text{matrix}) \simeq \frac{(1+\sqrt{2})^{4n}}{\sqrt{n}}$$

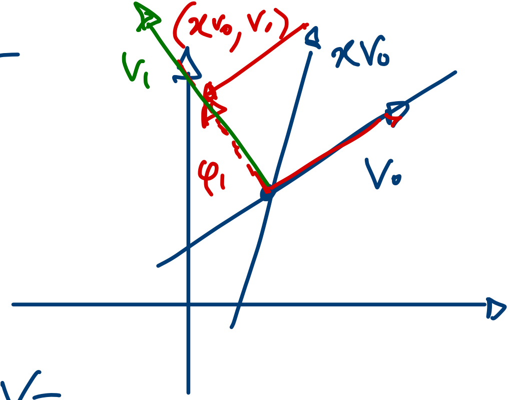
What's the best basis for M?

Identity!

$$(v_j, v_i) = \delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Legendre basis $(v_j, v_i) = \delta_{ij}$

Gram - Smidt procedure



set $v_0 = 1$

$$\varphi_{i+1} = x v_i - \sum_{j=0}^{i} (x v_i, v_j) v_j$$

$$v_{i+1} = \frac{\varphi_{i+1}}{\|\varphi_{i+1}\|}$$

Bonnet Recursion

$$\boxed{e_0 := 1 \quad e_1 = x \quad [a,b] = [-1,1]}$$

$$\boxed{(n+1) e_{n+1}(x) = (2n+1) x\, e_n(x) - n e_{n-1}(x)}$$

$e_n(1) = 1 \quad \forall n \qquad \|e_n\| \neq 1$

$$\Pi^n : L^2([a,b]) \longrightarrow \mathbb{P}^n([a,b])$$

$$u \longrightarrow M^{ij} (u, v_j)\, v_i$$

$M_{ij} = (e_i, e_j)$ Legendre : $\displaystyle\sum_{i=0}^{n} \frac{e_i}{\|e_i\|^2} (u, e_i)$

$= \|e_i\|^2 \delta_{ij}$ no sum on i

computation? (u, e_i)

$M^{ii} = \dfrac{1}{\|e_i\|^2} \qquad M^{ij} = 0 \quad i \neq j$