Master in High Performance Computing

Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 11 - Introduction to PDEs - Finite Elements in 1D

Luca Heltai <luca.heltai@sissa.it>

International School for Advanced Studies (www.sissa.it)
Mathematical Analysis, Modeling, and Applications (math.sissa.it)
Theoretical and Scientific Data Science (datascience.sissa.it)
Master in High Performance Computing (www.mhpc.it)
SISSA mathLab (mathlab.sissa.it)

FINITE ELEMENT PERSPECTIVE

Given f, find u s.t.

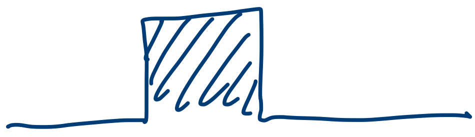
$$(1)\quad \begin{cases} -u'' = f \\ u(0) = u(1) = 0 \end{cases}$$

FD: assume $f \in C^0([0,1])$ & $u \in C^2([0,1])$

approximate $\boxed{-u'' := \dfrac{-u^{i-1} + 2u^{i} - u^{i+1}}{h^2}}$

ISSUES

i) it won't work for

f: 

ii) $u'' = CFD^{II} u + O\left(\|u''''\|_{L^\infty} h^2\right) \Rightarrow u \in C^4$

SOLUTION: WEAK FORM of (1)

1. Define a "smooth enough" vector space V, s.t. $u \in V$,
2. Multiply (1) by $v \in V$ and integrate on $[0,1]$
3. Integrate by parts (using b.c. informations) as many times as possible "minimize the number of derivatives"

INTEGRATION BY PARTS

$$(wv)' = w'v + wv'$$

(2)
$$\int_0^1 (wv)' = \int_0^1 w'v + \int_0^1 wv' = wv\Big|_0^1 := (wv)(1) - (wv)(0)$$

$$\int_0^1 w'v = -\int_0^1 vw' + wv\Big|_0^1$$

$$\int_0^1 -u''v = \int_0^1 f v \qquad u, v \in V$$

"smooth with zero values in $\{0, 1\}$"

$w = -u'$, use (2)

$$\int_0^1 u'v' - \cancel{u'v\Big|_0^1}^{\,0} = \int_0^1 f v \qquad \forall v \in V, \quad v(0) = v(1) = 0$$

$$\boxed{\text{Given } f \in L^2, \text{ find } u \in V = H_0^1 \text{ s.t. } \int_0^1 u'v' = \int_0^1 f v \quad \forall v \in V}$$

WEAK FORM

How do we choose the space V ?

$\int_0^1 u'v'$ must exist $\Rightarrow$ $\underline{\underline{u'}}, \underline{\underline{v'}} \in L^2([0,1])$

$\int_0^1 fv$ must exist $\Rightarrow$ for example, $\underline{\underline{f}}, \underline{\underline{v}} \in L^2([0,1])$

$$V := \{ v \in L^2([0,1]), \text{ s.t. } v' \in L^2([0,1]) \text{ and } v(0) = v(1) = 0 \} \equiv H_0^1([0,1])$$

Approximation of WEAK FORM is obtained by constructing $V_h \subset V$ $\quad V_h := \mathrm{span}\{v_i\}_{i=0}^{N-1}$

$$\boxed{\begin{array}{l}\text{Given } f \in L^2, \text{ find } u_h \in V_h \text{ s.t.} \\ \displaystyle\int_0^1 u_h' v_h' = \int_0^1 f v_h \quad \forall v_h \in V_h \subset V\end{array}}$$ GALERKIN

SIMPLE CASES: V_h is piecewise $\mathbb{P}^k$ and $C^0([0,1])$

Select N points ($N-1$ segments) $\quad x_i = ih \quad i \in (0, N-1)$

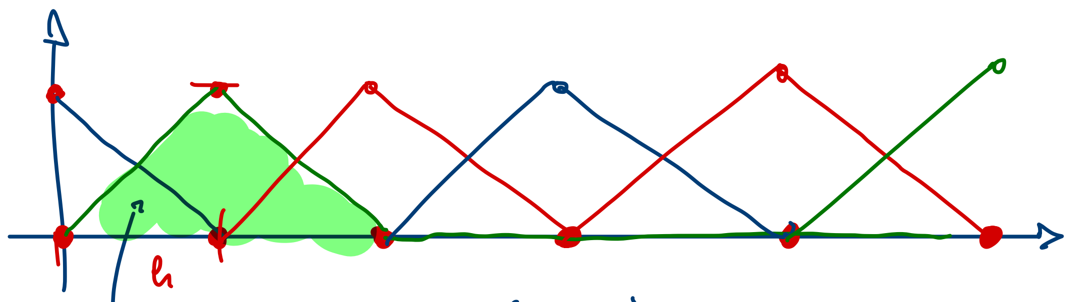
$h = \frac{1}{(N-1)}$

$$V_h^k := \{ v \in \mathbb{P}^k([x_i, x_{i+1}]) \ \forall i \in [0, \dots, N-1), \text{ s.t. } v \in C^0([0,1]) \}$$

Simplest case: V_h^1 : piecewise linear.

$$V_{h,0}^k := V_h^k \cap V = \{ v \in V_h^k \text{ s.t. } v(0) = v(1) = 0 \}$$

$$\exists \{v_i\}_{i=0}^{N-1} \text{ s.t. } v_i \in V_h^k, \quad v_i(x_j) = v_i(hj) = \delta_{ij}$$

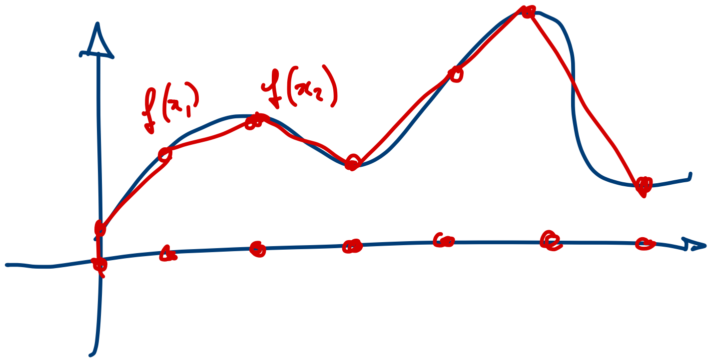


$v_0 := \chi[x_0, x_1] \cdot (1 - x).$ $\quad$ assume $h = 1$

$v_1 := \chi[x_0, x_1]\, x + \chi[x_1, x_2]\,(1 - (x - x_1))$

$v_2 := \dots$

$$(I_{V_h^k} f)(x) := \sum_{i=0}^{N-1} f(x_i)\, v_i(x)$$

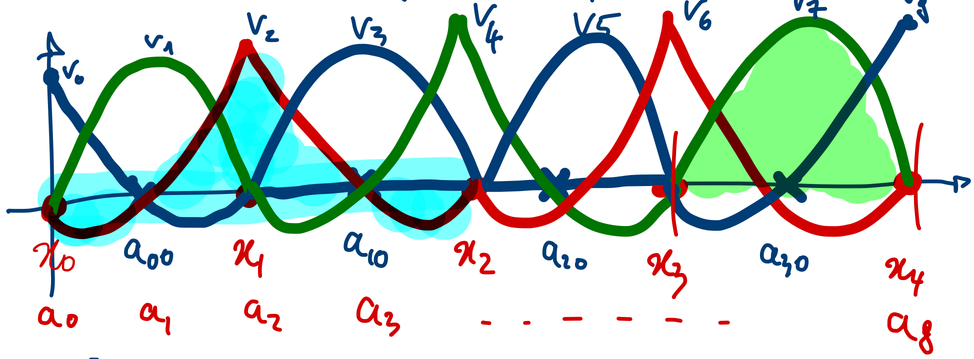


For $k \geq 1$ we define a number of support points.

$M = (k+1) - 2$ interior points a_{ij} (interpolation points)

in $[x_i, x_{i+1}]$ s.t. $x_i, a_{i0}, a_{i1}, a_{i2}, \dots, a_{i(k-2)}, x_{i+1}$

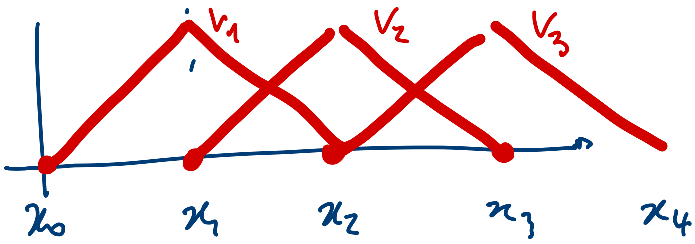
is a collection of interpolation points for $\mathbb{P}^k([x_i, x_{i+1}])$



a_i

$v_i \in V_h^k$ s.t. $v_i(a_j) = \delta_{ij}$

$\mathrm{supp}(v_i) \subset [x_{i-1}, x_{i+1}]$



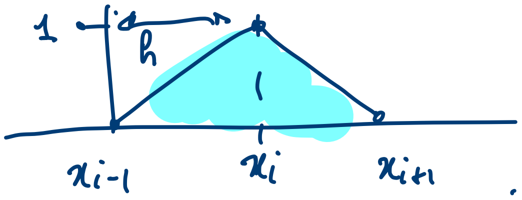
$V_h = \underline{\underline{V^1_{h,0}}} = \mathrm{span}\{v_i\}_{i=1}^{N-2}$

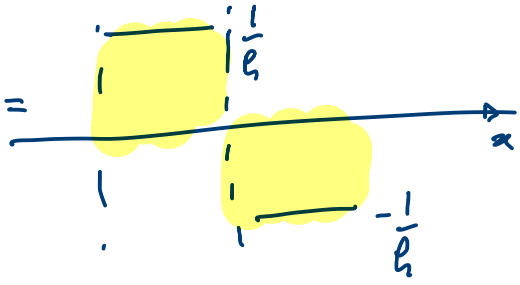
$(I_{V_h} f)(x) := \sum_{i=1}^{N-2} f(x_i)\, v_i(x)$

find $u_h = \sum_{j=1}^{N-2} u^j v_j$ s.t.

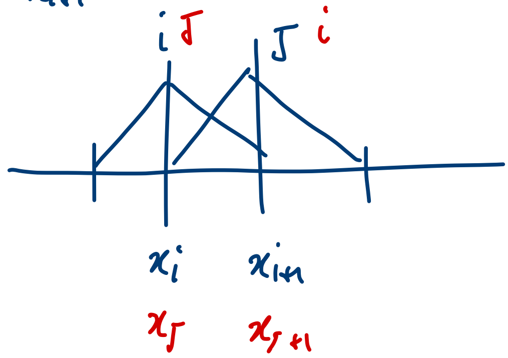
$$\sum_{j=1}^{N-2} \left(\int_0^1 v_j' \, v_i' \right) u^j = \int_0^1 f v_i \qquad i \in [1, N-1)$$

$$A_{ij} u^j = f_i \qquad A_{ij} := \int_0^1 v_j' \, v_i' \quad \Big| \quad f_i := \int_0^1 f v_i$$

$v_i :=$ 

$v_i' :=$ 

$v_i' \, v_j' \neq 0$



$|i-j| \leq 1$

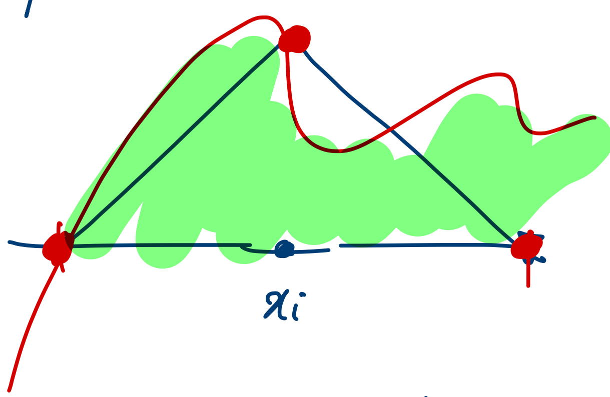
$$\int_0^1 v_i' \, v_j' = \begin{cases} \displaystyle\int_{x_{i-1}}^{x_{i+1}} (v_i')^2 = \frac{2h}{h^2} = \frac{2}{h} & \text{if } i = j \\[2ex] \displaystyle\int_{x_i}^{x_{i+1}} v_i' \, v_j' = -\frac{h}{h^2} = -\frac{1}{h} & \text{if } |i-j| = 1 \\[2ex] 0 & \text{otherwise} \end{cases}$$

$$A_{ij} := \frac{1}{h} \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

Same as FD

except $\frac{1}{h^2}$

$$f_i := \int_{x_{i-1}}^{x_{i+1}} v_i f$$



example rule: $\boxed{\text{Trapez rule}}$ on each segment.

$$f_i := \frac{h}{2}\left[\cancel{(v_i f)(x_{i-1})} + (v_i f)(x_i)\right] + \frac{h}{2}\left[(v_i f)\, x_i + \cancel{(v_i f)(x_{i+1})}\right]$$

$f_i := h f(x_i)$

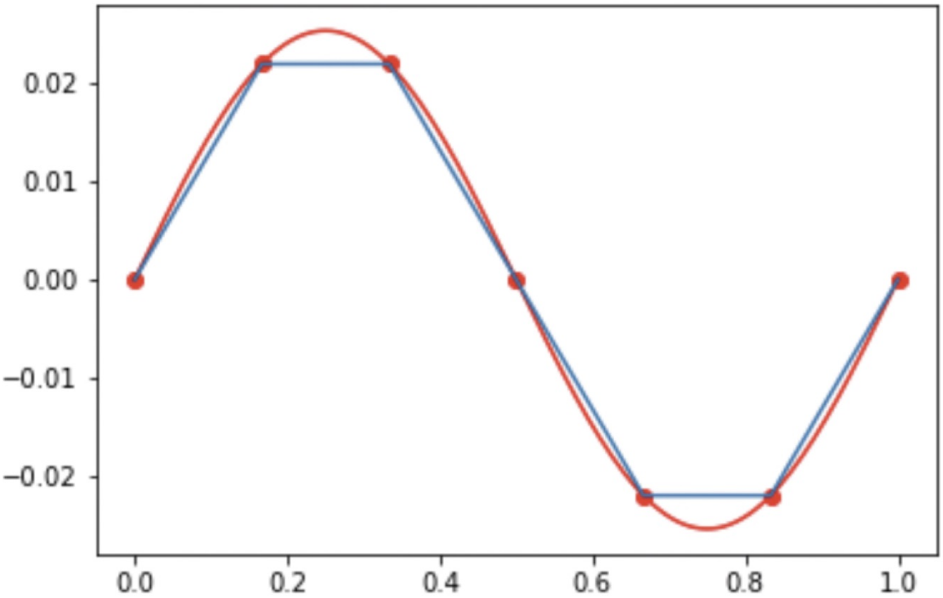
FE in 1D :

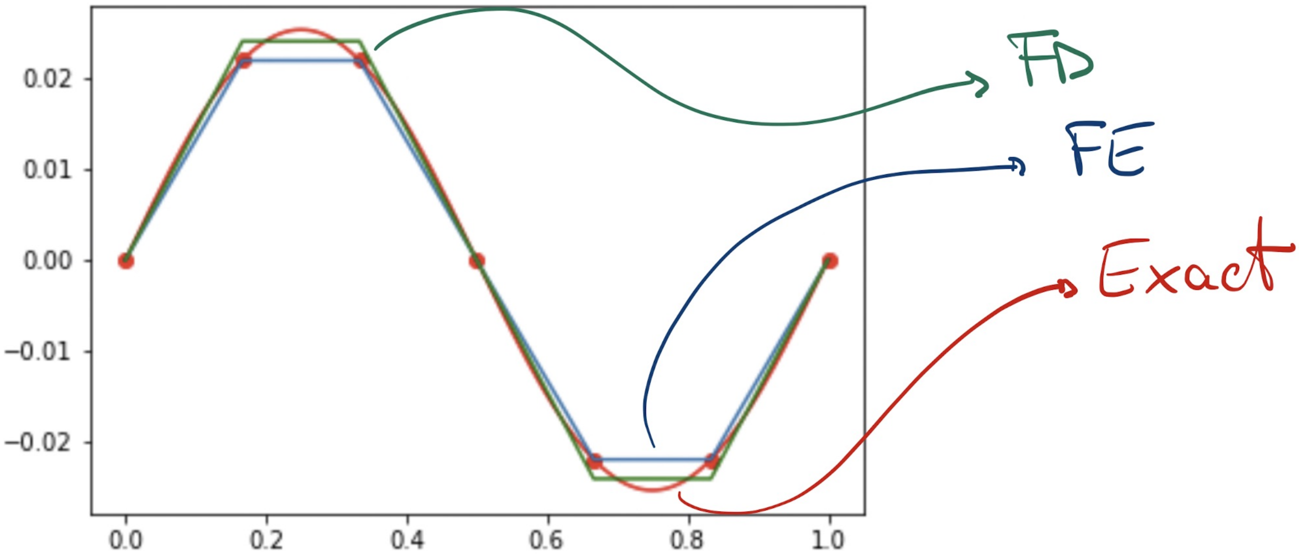
if f_i is computed $\boxed{\text{exactly}}$

then

$u^i = u(x_i)$

is exact in 1D





$$\int_0^1 u' v' = \int_0^1 f v \qquad \forall\, v \in V$$

$$\int_0^1 u_h' v_h' = \int_0^1 f v_h \qquad \forall\, v_h \in V_h \subset V$$

$$\otimes \quad \int_0^1 (u - u_h)' v_h' = 0 \qquad \forall\, v_h \in V_h$$

$$\boxed{\| u - u_h \|_V \leq C \inf_{v_h \in V_h} \| u - v_h \|_V}$$

Cea's Lemma

$$\| u \|_V := \left(\int_0^1 (u')^2 \right)^{\frac{1}{2}} \qquad a(u,v) := \int_0^1 u' v'$$

$$a : V \times V \longrightarrow \mathbb{R}$$

$$a(u,v) \leq C \|u\|_V \|v\|_V$$

$$\Rightarrow \left| \int_0^1 u' v' \right| \leq C \left(\int_0^1 (u')^2 \int_0^1 (v')^2 \right)^{\frac{1}{2}} \leq C \|u\|_V \|v\|_V$$

$$\text{if} \quad a(u,u) \geq \alpha \| u \|_V^2 \qquad \text{then}$$

$$a(u - u_h, v_h) = 0 \quad \forall v_h \in V_h$$

$$\alpha \| u - u_h \|_V^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h)$$

$$\leq C \| u - u_h \| \, \| u - v_h \|_V \qquad \forall\, v_h \in V_h$$

$$\| u - u_h \|_V \leq \frac{C}{\alpha} \| u - v_h \|_V \qquad \forall\, v_h \in V_h$$