Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 02 - Interpolation

Luca Heltai <luca.heltai@sissa.it>

International School for Advanced Studies (www.sissa.it)
Mathematical Analysis, Modeling, and Applications (math.sissa.it)
Theoretical and Scientific Data Science (datascience.sissa.it)
Master in High Performance Computing (www.mhpc.it)
SISSA mathLab (mathlab.sissa.it)

Abstract Problem : $y = F(x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$

$\mathcal{X}$, $\mathcal{Y}$ are Real Vector Spaces, normed.

in $\mathcal{X}$ and $\mathcal{Y}$ we define 2 operations:

$\oplus$ sum of elements (vectors)

$\odot$ scaling with a real number

$(V, \oplus, \odot)$:

$$\boxed{\forall x, y \in V, \quad \alpha \underline{\underline{x}} + \beta \underline{\underline{y}} = z \in V}$$

$\mathbb{R}^n$

$n = 2$ $\quad x = (x_1, x_2)$

$x + y = (x_1 + y_1, x_2 + y_2)$

$\alpha x = (\alpha x_1, \alpha x_2)$

$C^0([a,b])$ $\quad f \in C^0([a,b])$

$z = \alpha \cdot f \in C^0([a,b])$ $\quad z(x) = \alpha f(x)$
$\forall x \in [a,b]$

$f + g = z \Leftrightarrow z(x) = f(x) + g(x)$

Norms on $\mathcal{X}$, $\mathcal{Y}$ ...

$\|\cdot\|_{\mathcal{X}}$, $\|\cdot\|_{\mathcal{Y}}$

Norm : function from $\mathcal{X}$ ($\mathcal{Y}$) $\longrightarrow \mathbb{R}_0^+$

1) $\|x\|_{\mathcal{X}} \geq 0 \quad \forall x \in \mathcal{X}$

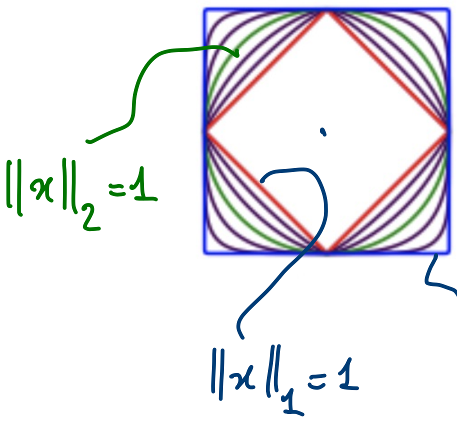
2) $\|x+y\|_{\mathcal{X}} \leq \|x\|_{\mathcal{X}} + \|y\|_{\mathcal{X}} \quad \forall x,y \in \mathcal{X}$

3) $\|\alpha x\|_{\mathcal{X}} = |\alpha| \, \|x\|_{\mathcal{X}}$

optional in many occasions $*$ 4) $\|x\|_{\mathcal{X}} = 0 \iff x = 0_{\mathcal{X}}$

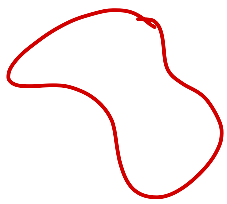
$\mathbb{R}^n$: $\quad \ell_p$-norms $:= \|x\|_{\ell_p}$ or $\|x\|_p := \left(\sum_{i=1}^n |x|^p \right)^{\frac{1}{p}}$

$\|x\|_\infty = \lim_{p\to\infty} \|x\|_p = \max_i |x_i|$



$\mathbb{R}^2$: $\|x\|_p = 1$

$\Omega \in \mathbb{R}^d$



$f \in L^p(\Omega)$, $C^\alpha(\Omega)$

L^p-norm : $\quad \|u\|_{L^p} = \|u\|_p := \left(\int_\Omega |u|^p \right)^{\frac{1}{p}}$

$\lim_{p\to\infty} \|u\|_p = \|u\|_\infty = \operatorname{ess\,sup}_{x\in\Omega} |u(x)|$

$\| \cdot \|_*$ operatorial norm induced by $\mathcal{X}$ and $\mathcal{Y}$.

used to measure norms of functionals:

$$f : \mathcal{X} \longrightarrow \mathcal{Y}$$

$$\boxed{\|f\|_* := \sup_{0 \neq x \in \mathcal{X}} \frac{\|f(x)\|_{\mathcal{Y}}}{\|x\|_{\mathcal{X}}}}$$

Operatorial norm of matrices

$$A : \mathcal{X} \equiv \mathbb{R}^n \longrightarrow \mathcal{Y} \equiv \mathbb{R}^m$$

$$\|A\|_* := \sup_{x \neq 0} \frac{\|Ax\|_m}{\|x\|_m}$$

$$\boxed{\|A\|_p := \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p}}$$

$$\|Ax\|_p \leq \|A\|_p \|x\|_p$$

Example: $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n \equiv \mathcal{X}$, $y \in \mathbb{R}^n \equiv \mathcal{Y}$

use $\|A\|_p$, $\|x\|_p$, $\|y\|_p$ norms. . .

Kabs: the number (if it exists) s.t.

$\forall\ \delta x$ s.t. $x + \delta x \in \mathcal{X}$ $(\forall\ \delta x \in \mathcal{X})$

$y = Ax$ $\quad \|f(x+\delta x) - f(x)\|_p \leq \text{Kabs} \|\delta x\|$

$$\| A(x+\delta x) - Ax \|_p = \| A\,\delta x \|_p \le \|A\|_p \|\delta x\|_p$$

$$\Rightarrow \quad \kappa_{abs} = \|A\|_p$$

$$\sup_{x,\delta x} \frac{\|\delta y\|_p}{\|y\|_p} \cdot \frac{\|x\|_p}{\|\delta x\|_p} = \sup_{x,\delta x} \frac{\|A\,\delta x\|_p \|A^{-1} y\|_p}{\|y\|_p \;\|\delta x\|_p}$$

$$\le \|A\|_p \|A^{-1}\|_p \frac{\cancel{\|\delta x\|_p} \; \cancel{\|y\|_p}}{\cancel{\|\delta x\|_p} \; \cancel{\|y\|_p}}$$

$$\kappa_{rel} = \|A\|_p \|A^{-1}\|_p$$

GENERAL INTERPOLATION

We want to approximate a space V which is "Large" (= infinitely large). Example:

$$V := C^0([0,1]).$$

We construct a finite dimensional subspace (vector space "generated" by a finite number of linearly independent elements of V)

$$V_h \,/\, V^n \qquad V^n = \text{span}\{v_i\}_{i=1}^n$$

$$\forall\, p \in V^n, \; \exists \{p^i\}_{i=1}^n \in \mathbb{R}^n \text{ s.t.}$$

$$p(x) = \sum_{i=1}^n p^i v_i(x) = p^i v_i(x)$$

Einstein notation

General Interpolation problem: Given $V^n = \mathrm{span}\{v_i\}_{i=1}^n$ and a set of n points (called interpolation points)

$$I^n(u) := p \in V^n \text{ s.t.}$$

$$p(x_i) = u(x_i) \quad \forall x_i \text{ interpolation point.}$$

$$I^n : C^0([0,1]) \longrightarrow V^n \subset C^0([0,1])$$

$$I^n(u) : u \longrightarrow p$$

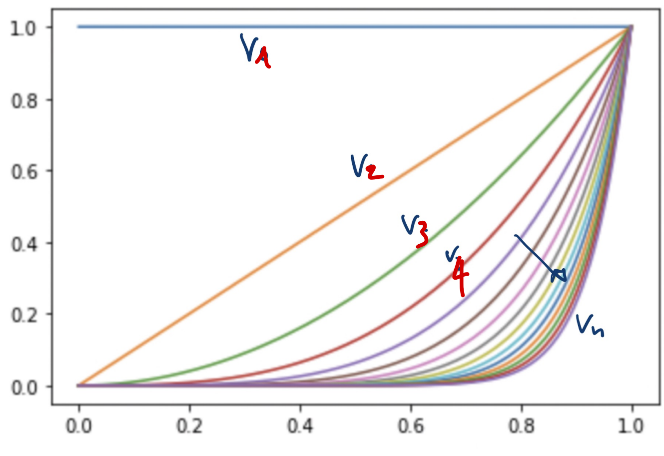
$$\text{s.t. } p(x_i) = u(x_i) \quad \forall x_i$$

$$\Rightarrow \quad p(x_i) = p^j v_j(x_i) = u(x_i)$$

$$\underline{\underline{V}}\, \underline{p} = V_{ij}\, p^j = u_i := u(x_i) = \underline{u}$$

$$\boxed{\underline{\underline{V}} \in \mathbb{R}^{n\times n}, \quad \underline{p} \in \mathbb{R}^n, \quad \underline{u} \in \mathbb{R}^n}$$

Example : $v_i(x) := x^{(i-1)}$ monomial basis



$\underline{\underline{V}}$ van-der-monde matrix

$$\boxed{V_{ij} := x_i^{(j-1)}}$$

Condition number

$$p = \underline{\underline{V}}^{-1} \underline{u}$$

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n = 1 : 1.0
n = 2 : 2.618033988749895
n = 3 : 15.099657722502098
n = 4 : 98.86773850722759
n = 5 : 686.434941818595
n = 6 : 4924.371056611224
n = 7 : 36061.16088021232
n = 8 : 267816.7009075794
n = 9 : 2009396.3800421846
n = 10 : 15193229.677753646
n = 11 : 115575244.54431371
n = 12 : 883478687.0721825
n = 13 : 6780588492.454725
n = 14 : 52214926084.1525
n = 15 : 403234616528.72504
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$$I : C^0([a,b]) \equiv \mathcal{X} \longrightarrow \mathbb{R}^n$$

$$\|\cdot\|_{\mathcal{X}} := \|\cdot\|_{L^\infty} \qquad \|\cdot\|_{\ell^\infty}$$

$$k_{abs} : \sup_{x, \delta x} \frac{\|I^n(x+\delta x) - I^n(x)\|_{\ell^\infty}}{\|\delta x\|_{L^\infty}}$$

$$I^n(f+g) = I^n(f) + I^n(g)$$

$$\Longrightarrow \sup_{\delta x} \frac{\|I^n(\delta x)\|_{\ell^\infty}}{\|\delta x\|_{L^\infty}}$$

$$I^n(\delta x)_J = (V^{-1})_{Ji} \, \delta x(a_i)$$

a_i = interp. points

$$\underline{\delta x} := \{\delta x(a_i)\}_{i=1}^n$$

$$\| I^n(\delta x)\|_{\ell^\infty} \le \| V^{-1}\|_\infty \, \| \underline{\delta x} \|_{\ell^\infty} \le \| V^{-1}\|_\infty \, \| \delta x\|_{L^\infty}$$

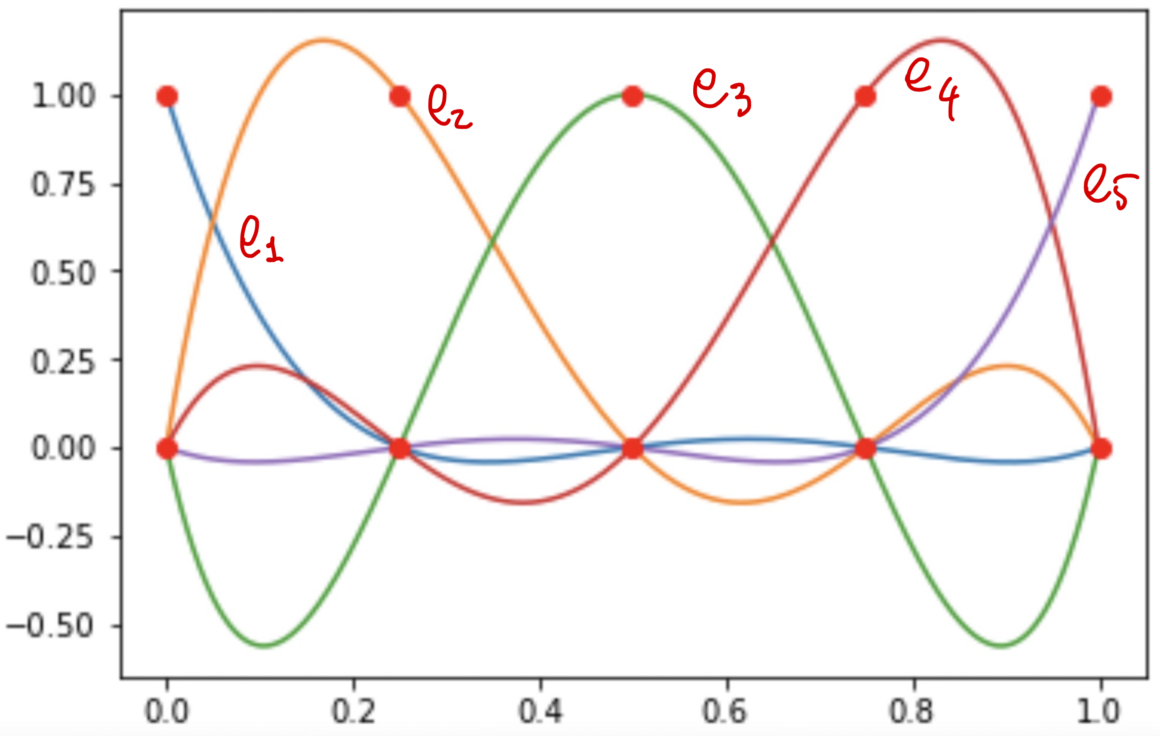
$$\Rightarrow \quad K_{abs} := \| V^{-1}\|_\infty$$

Best choice: $V = Id$

$$V = Id \iff V_j(a_i) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$$

$$\iff V_j \equiv \ell_j \quad \text{Lagrange basis functions.}$$

$$\ell_j := \prod_{\substack{i \ne j \\ i=1}}^{n} \frac{(x-a_i)}{(a_j-a_i)} \quad \Rightarrow \quad I^n(u) := \sum_{i=1}^{n} u(a_i)\,\ell_i$$



Lets look at $I^n : C^0([a,b]) \longrightarrow C^0([a,b])$

$$\|\cdot\|_{\mathcal{X}} = \|\cdot\|_{L^\infty} = \|\cdot\|_{\mathcal{Y}}$$

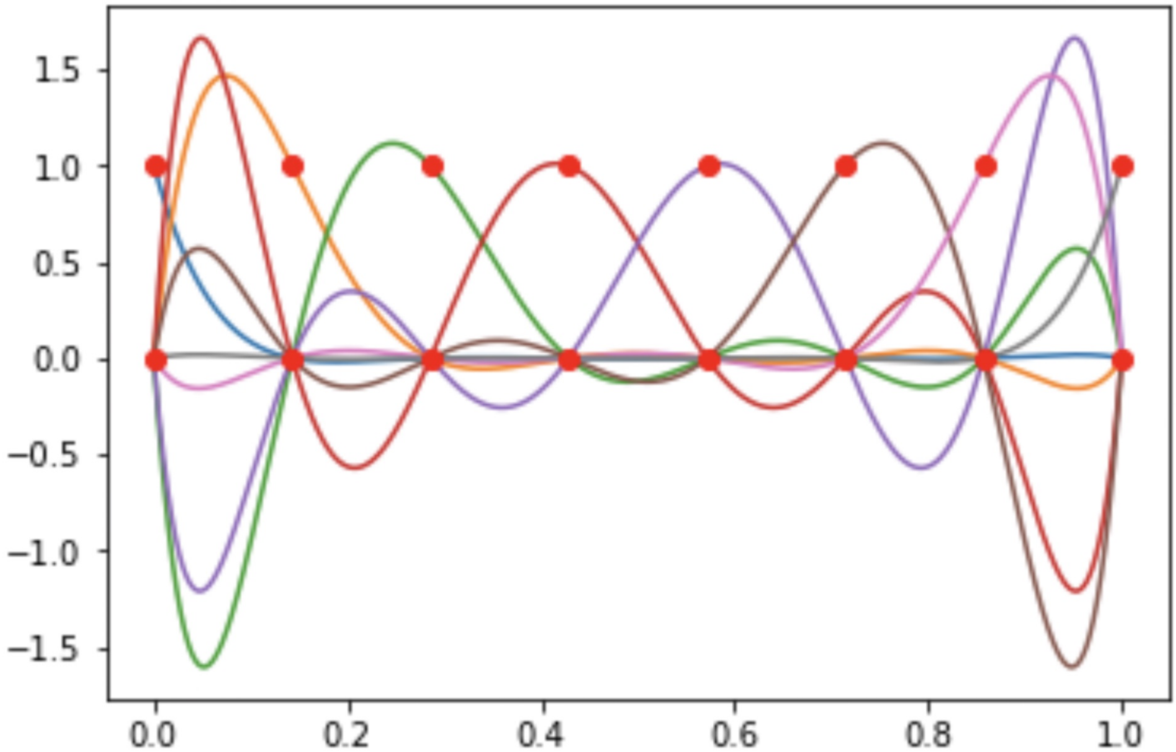
$$I^n(f)(x) = \sum_{i=1}^{n} \ell_i(x)\, u(a_i)$$

$$\sup_{\delta f \in C^0([a,b])} \frac{\|I^n(\delta f)\|_{L^\infty}}{\|\delta f\|_{L^\infty}}$$

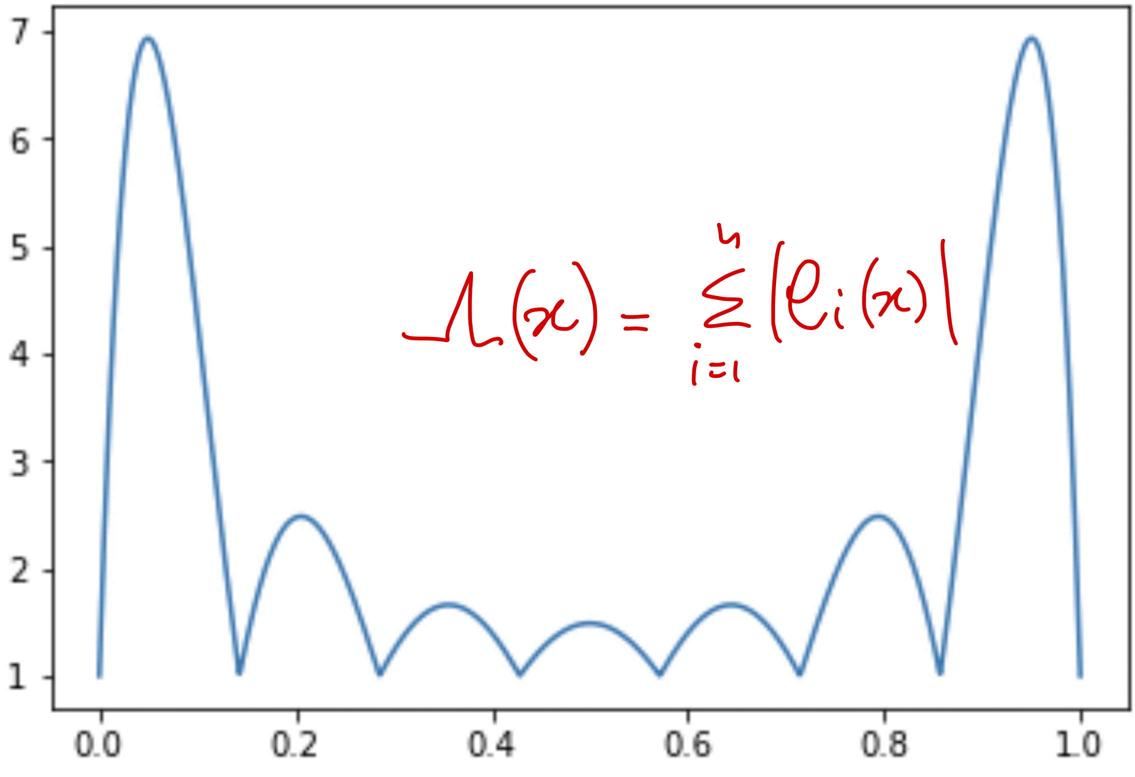
$$\sup_{\delta f} \frac{\left\| \sum_{i=1}^{n} \ell_i(x)\, \delta f(a_i) \right\|_{L^\infty}}{\|\delta f\|_{L^\infty}}$$

$$\leq \frac{\boxed{\left\| \sum_{i=1}^{n} |\ell_i(x)| \right\|_{L^\infty}} \cancel{\|\delta f\|_{L^\infty}}}{\cancel{\|\delta f\|_{L^\infty}}}$$

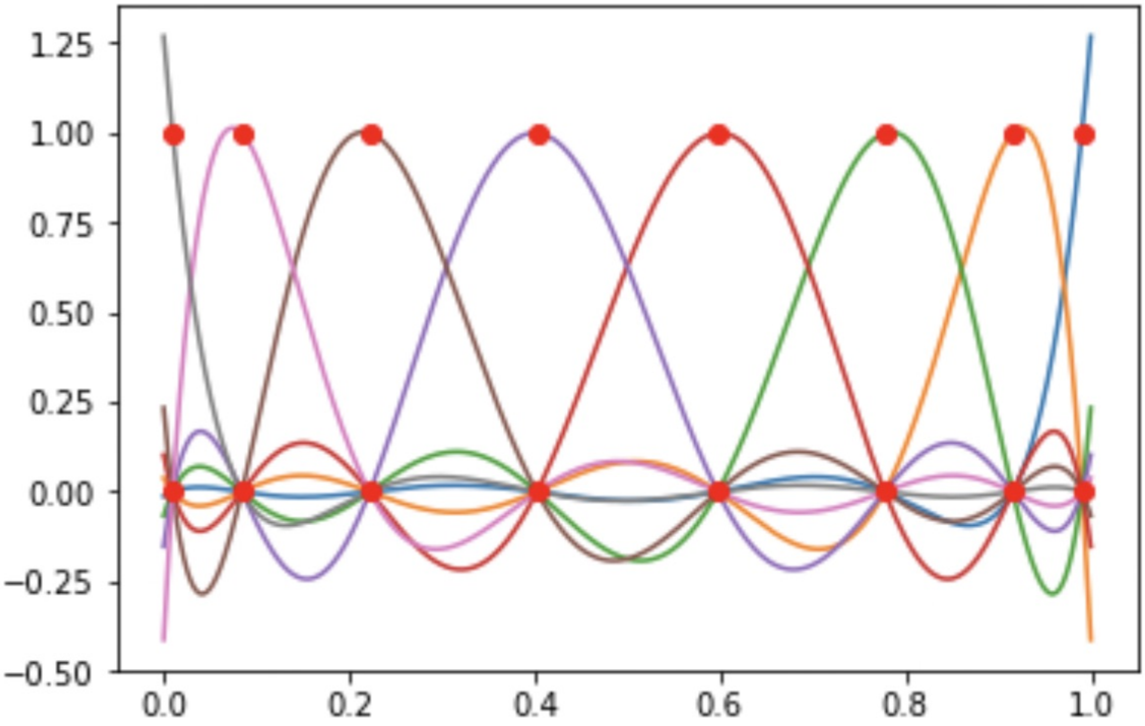
Lebesgue function $\boxed{\sum_{i=1}^{n} |\ell_i(x)| := \Lambda(x)}$ $\boxed{K_{rel} := \|\Lambda\|_{L^\infty}}$



$n = 10$

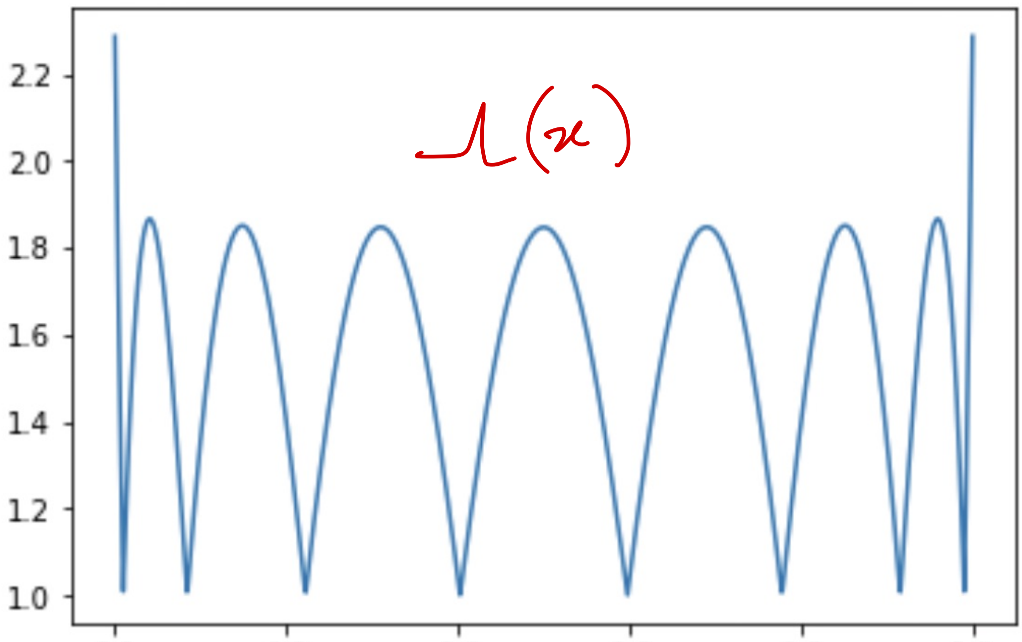


$$\Lambda(x) = \sum_{i=1}^{n} |\ell_i(x)|$$



1.25
1.00
0.75
0.50
0.25
0.00
−0.25
−0.50
0.0
0.2
0.4
0.6
0.8
1.0

Chebyshev



2.2
2.0
1.8
1.6
1.4
1.2
1.0
0.0
0.2
0.4
0.6
0.8
1.0
$\Lambda(x)$