Master in High Performance Computing

Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 12 - Introduction to PDEs - Finite Elements in 1D - Implementation

Luca Heltai <luca.heltai@sissa.it>

International School for Advanced Studies (www.sissa.it)
Mathematical Analysis, Modeling, and Applications (math.sissa.it)
Theoretical and Scientific Data Science (datascience.sissa.it)
Master in High Performance Computing (www.mhpc.it)
SISSA mathLab (mathlab.sissa.it)

IMPLEMENTATION FEM (1D)

Given $a: V \times V \longrightarrow \mathbb{R}$, and $f \in V^*$

Find $u \in V$ s.t.

$$a(u,v) = \langle f, v \rangle$$

Prototypical:

$$\boxed{\begin{aligned} V &:= H^1_0(\Omega) \\ a(u,v) &:= \int_\Omega \nabla u \cdot \nabla v \, dx \\ \langle f, v \rangle &:= \int_\Omega v f \, dx \end{aligned}}$$

Discretization: $V_h \subset V$

Find $u_h \in V_h$ s.t.

$$a(u_h, v_h) = \langle f, v_h \rangle \quad \forall v_h \in V_h \subset V$$

$$V_h = \operatorname{span}\{v_i\}_{i=0}^{N-1} \longrightarrow A_{ij} u^j = f_i$$

$$A_{ij} := a(v_j, v_i) := \int_\Omega \nabla v_j \nabla v_i$$

1) Construction of V_h

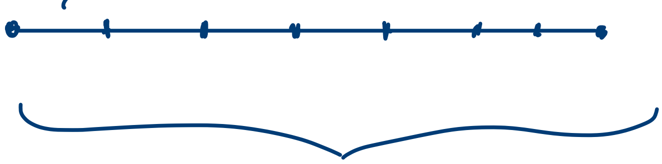
2) Computation of A_{ij} , f_i

Finite Elements: split Ω into "simple" subdomains

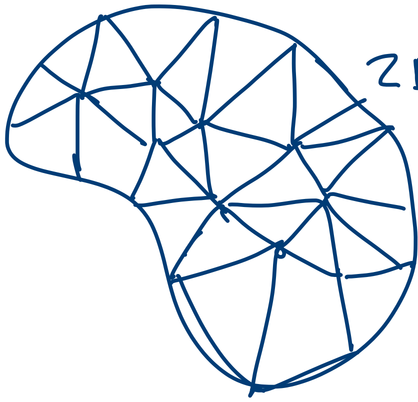
$$\overline{\Omega} = \bigcup_{k=0}^{M-1} \overline{T_k} \qquad \mathcal{T}_h := \{T_k\}_{k=0}^{M-1}$$

M elements:

1D $\quad T_k = (a_k, a_{k+1})$



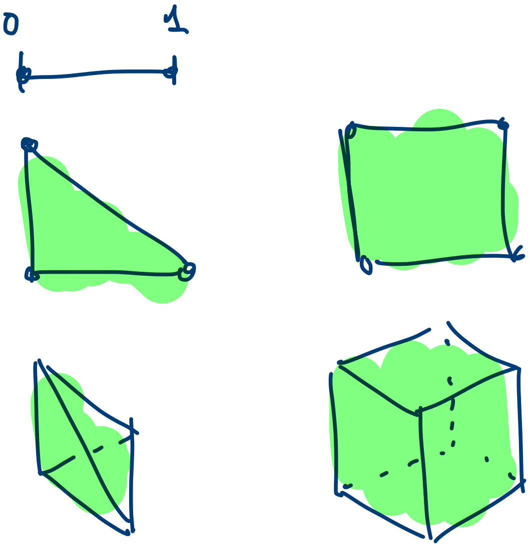
$\Omega = [a,b]$



2D

We can always write $T_k = F_k(\hat{T})$

$\hat{K}$ is a Reference element.



1D $\qquad F_k : \hat{T} \longrightarrow T_k$

2d

3D

$$V_h\Big|_{T_k} = \operatorname{span}\left\{ \hat{V}_\alpha \circ F_k^{-1} \right\}_{\alpha=0}^{N_{loc}-1}$$

$$V_h \subset V \longrightarrow V_h \subset C^0(\overline{\Omega}) \qquad \overline{\Omega} = \bigcup_{k=0}^{M-1} \overline{T_k}$$

$$\exists \quad P_{ki\alpha} = \begin{cases} 1 & \text{if } v_i(F_k(\hat{x})) = \hat{v}_\alpha(\hat{x}) \quad \forall \hat{x} \in \hat{T} \\ 0 & \text{otherwise} \end{cases}$$

$i \in [0, N)$ $\hookrightarrow (M \cdot d + 1)$

$v_i(F_k(\hat{x})) = \hat{v}_i \equiv \hat{v}_\alpha$

shape: (M, N, N_{loc})

$\alpha \in [0, N_{loc})$ $\hookrightarrow d+1$

$$\int_\Omega g(x)\,dx = \sum_{k=0}^{M-1} \int_{T_k} g(x)\,dx = \sum_{k=0}^{M-1} \int_{\hat{T}} g \circ F_k \, J \, d\hat{x}$$

$$J := \det\left(\frac{\partial F_k}{\partial \hat{x}}\right) \qquad = \sum_{k=0}^{M-1} \sum_{q=0}^{N_q - 1} (g \circ F_k)(\hat{x}_q) \, J_q \, \hat{w}_q$$

$\boxed{\hat{x}_q, \hat{w}_q}$: quadrature rule on $\hat{T}$

$$\frac{\partial}{\partial \hat{x}}\left(\underbrace{v_i \circ F_k}_{= \hat{v}_\alpha}\right) = \left[\left(\frac{\partial}{\partial x} v_i\right) \circ F_k\right] \cdot \overbrace{\left(\frac{\partial F_k}{\partial \hat{x}}\right)^T}^{\mathbb{F}^T}$$

$$(\nabla v_i) \circ F_k = \left[\left(\hat{\nabla} \hat{v}_\alpha\right)\right] \cdot \mathbb{F}^{-T}$$

$$f_i := \int_\Omega f v_i \, dx = \sum_k \int_{T_k} f v_i \, dx = \sum_k \int_{\hat{T}} f \circ T_k \, \hat{v}_\alpha \, J \, d\hat{x} \, P_{ki\alpha}$$

$$= \sum_k \sum_q (f \circ F_k)(\hat{x}_q) \; \hat{v}_\alpha(\hat{x}_q) \; J_q w_q \; P_{ki\alpha}$$

$$A_{ij} := \sum_k \sum_q \underbrace{P_{ki\alpha}\left(\hat{\nabla}\hat{v}_\alpha\right)(\hat{x}_q)\, \mathbb{F}^{-T}(x_q)}_{(\nabla v_i) \circ F_k} \; \underbrace{P_{kj\beta}\left(\hat{\nabla}\hat{v}_\beta\right)(\hat{x}_q)\, \mathbb{F}^{-T}(x_q)}_{(\nabla v_j) \circ F_k} \; J_q w_q$$

$$A_{ij} := \sum_K \boxed{P_{Ki\alpha} \; \boxed{a_{\alpha\beta}} \; P_{Kj\beta}}$$

global renumbering

local contribution

$$a_{\alpha\beta} := \sum_q \left(\hat{\nabla}\hat{v}_\alpha\right)(\hat{x}_q)\, \mathbb{F}^{-T}(x_q)\, P_{Kj\beta} \left(\hat{\nabla}\hat{v}_\beta\right)(\hat{x}_q)\, \mathbb{F}^{-T}(x_q)\, J_q w_q$$