Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 10 - Introduction to PDEs - Finite Differences in 1D

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1D - Poisson Problem

Given f, Find the function u s.t.

$$(1) \begin{cases} -u'' = f & \text{in } (0,1) \\ u(0) = u(1) = 0 \end{cases}$$

- elasticity
- heat
- maxwell
- ....

Strong formulation (1): $f \in C^0([0,1])$, seek for $u \in C^2([0,1])$

FINITE DIFFERENCES

FD: Approximate differential operators with "finite diff."
= difference between the value of the function at points

simplest case: choose a step size $h = \frac{b-a}{(N-1)} = \frac{1}{(N-1)}$

for $u \in C^0([0,1])$ define $\boxed{u^i := u(x_i) = u(ih)}$ $\quad i \in [0, N)$

range (N)

Replace $\left(\frac{\partial u}{\partial x}\right)(x_i) := D_u^{F/B/C}$

$$[D^F u]^i := \frac{u^i - u^{i-1}}{h} \qquad i \in [1, N)$$

$$[D^B u]^i := \frac{u^{i+1} - u^i}{h} \qquad i \in (0, N-1)$$

$$[CFD^I_u]^i = [D^C u]^i := \frac{u^{i+1} - u^{i-1}}{2h} \qquad i \in [1, N-1)$$

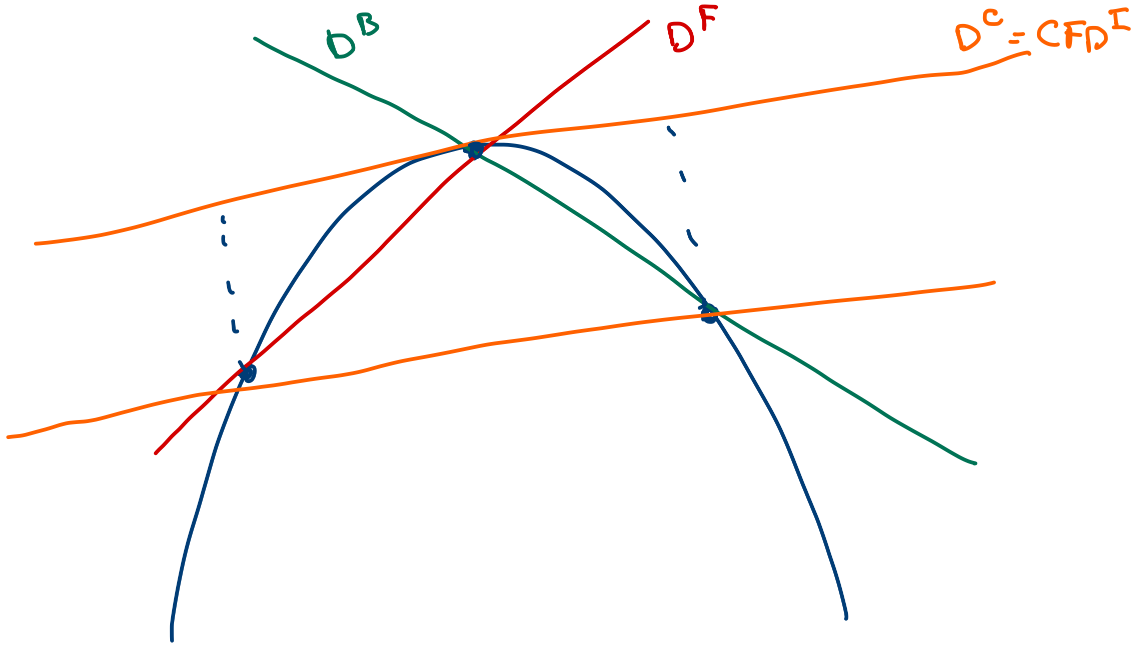
For second order derivatives:

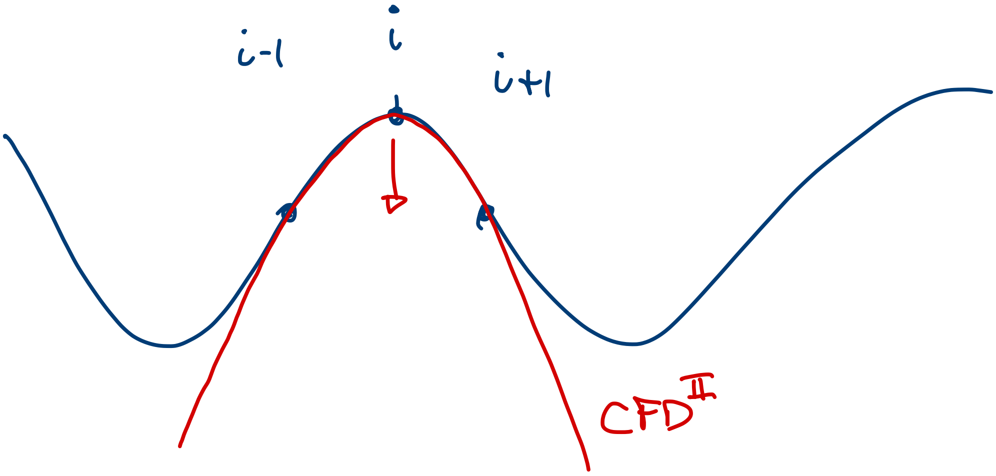
$$D^F D^B \;/\; D^B D^F \;/\; D^F D^F \;/\; D^B D^B \;/\; \ldots$$

$$[CFD^{II} u]^i := \frac{u^{i-1} - 2u^i + u^{i+1}}{h^2} = \frac{(u^{i+1} - u^i) - (u^i - u^{i-1})}{h^2}$$

$$[CFD^{II} u]^i = [D^F [D^B] u]^i = [D^B [D^F] u]^i$$

$$[D^F]^i = [D^B]^{i-1} \qquad [D^B]^i = [D^F]^{i+1}$$





$$-u''(x) = f(x) \qquad \text{in} \quad (0,1) \equiv \Omega$$

$$u(0) = u(1) = 0 \qquad \text{on} \quad \partial\Omega$$

— CFD^{II}

$$-\left[CFD^{II} u\right]^i = f_i \qquad i \in [1, N-1)$$

$$u^0 = u^{N-1} = 0$$

$$-CFD^{II} u^i = \frac{-u^{i-1} + 2u^i - u^{i+1}}{h^2} = f_i$$

$$\sum_{J=0}^{N-1} \boxed{A_{iJ} u^J = f_i} \qquad A_{iJ} = \begin{cases} 2/h^2 & i = J, \quad J \neq 0, N-1 \\ -1/h^2 & |i-J| = 1 \quad J \neq 0, N-1 \end{cases}$$

$$u^0 = u^{N-1} = 0 \longrightarrow A_{ii} = 1 \qquad i=0, \; i = N-1 \qquad f_i = 0$$

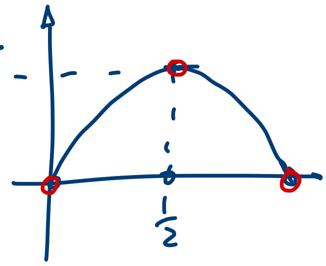
$$\mathbb{A} := \frac{1}{h^2}\begin{pmatrix} 1 & 0 & \cdots & & & 0 \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & & \ddots & & \\ & & & 1 & -2 & 1 \\ 0 & \cdots & & & 0 & 1 \end{pmatrix} \qquad f := \begin{pmatrix} u_0 = 0/h^2 \\ f(x_1) \\ f(x_2) \\ \vdots \\ f(x_{N-2}) \\ u^{N-1} = 0/h^2 \end{pmatrix}$$

$$u^i := \mathbb{A}^{ij} f_j \qquad A^{ij}\mathbb{A}_{jk} = \delta^i_k$$

3 points $\quad 0, \frac{1}{2}, 1 \quad f = 1$

$$-u'' = 1$$

$$u(0) = u(1) = 0 \implies u(x) = \frac{1}{2}x(1-x)$$



$$\frac{u^0}{h^2} = \frac{0}{h^2}$$

$$\left(\frac{-u^0 + 2u^1 - u^2}{h^2}\right) = 1$$

$$\frac{u^2}{h^2} = \frac{0}{h^2}$$

$$u^1 = \frac{h^2}{2} = \frac{1}{8}$$

$$\mathbb{A} = \frac{1}{h^2}\begin{pmatrix} 1 & 0 & 0 \\ -1 & +2 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} u^0 \\ u^1 \\ u^2 \end{pmatrix} = \begin{pmatrix} 0/h^2 \\ 1 \\ 0/h^2 \end{pmatrix}$$