Master in High Performance Computing

Applied Mathematics: an introduction to Scientific Computing by Numerical Analysis

Lecture 04 - Properties of Polynomial interpolation

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Polynomial interpolation: $\{v_i\}_{i=0}^n$ belong to $P^n(0,1)$

$\{a_i\}_{i=0}^n$ of $n+1$ points which are the interpolation points

example: 1) $v_i(x) := x^i$

2) $l_i(x) := \prod_{\substack{j\neq i \\ j=0}}^{n} \frac{(x - a_j)}{(a_i - a_j)}$

$$I^n : C^0([0,1]) \longrightarrow \mathbb{P}^n([0,1])$$

$$u \longrightarrow p = \sum_{i=0}^{n} p^i v_i$$

$V_{ij} := v_j(a_i)$ $\quad p^i := (V)^{-1} \{u(a_i)\}_{i=0}^{n}$

$$\underline{p} = (\underline{\underline{V}})^{-1} \underline{u}$$

$$I^n(p) = p \quad \forall\, p \in \mathbb{P}^n$$

$$I^n(p) = I^n(\underbrace{\tilde{p}^J v_J}_{u}) = \underbrace{\Big(\overbrace{V}^{V}{}^{-1} \underbrace{\tilde{p}^J v_J(a_k)}_{u(a_k)}\Big)^\ell}_{p^\ell} v_\ell$$

$$\left(V^{-1} V \underline{p}\right)^i v_i = \underline{p}^i \, v_i$$

For Lagrange:

$$I^n(p) = I^n(p^i e_i) = \Big(p^i \underbrace{e_i(a_k)}_{\delta_{ik}}\Big) e_k$$

$$= p^k e_k$$

$$\boxed{\text{Is interpolation Good to approximate?}}$$

Given $u \in C^0([0,1])$ what can we say about

$$\| u - I^n u \|_{L^\infty} = ?$$

Let's define the Best Approximation (assuming it exists)

p is B.A. $\iff$ $\| u - p \|_{L^\infty} \leq \| u - q \|_{L^\infty} \;\; \forall q \in \mathbb{P}^n$

p is the Best Approximation of u in $\mathbb{P}^n$

$$\|u - I^n(u)\|_{L^\infty} = \|u - p + p - I^n(u)\|_{L^\infty}$$

$$= \| u - p - I^n(u - p) \|_{L^\infty}$$

$$\leq \|u - p\| + \|\underbrace{I^n (u-p)}\|_{L^\infty}$$

$$\leq \|u - p\| + \|I^n\|_* \|u - p\|_{L^\infty}$$

$$\leq (1 + \|I^n\|_*) \|u - p\|_{L^\infty}$$

$$\|I^n\|_* := \sup_{\substack{u \neq 0 \\ u \in C^0([0,1])}} \frac{\|I^n u\|_{L^\infty}}{\|u\|_{L^\infty}} \leq \underbrace{\|V^{-1}\|_{L^\infty}}_{\text{1 for Lagrange}} \underbrace{\|\Sigma |v_i|\|_{L^\infty}}_{\|\Lambda\|_{L^\infty}}$$

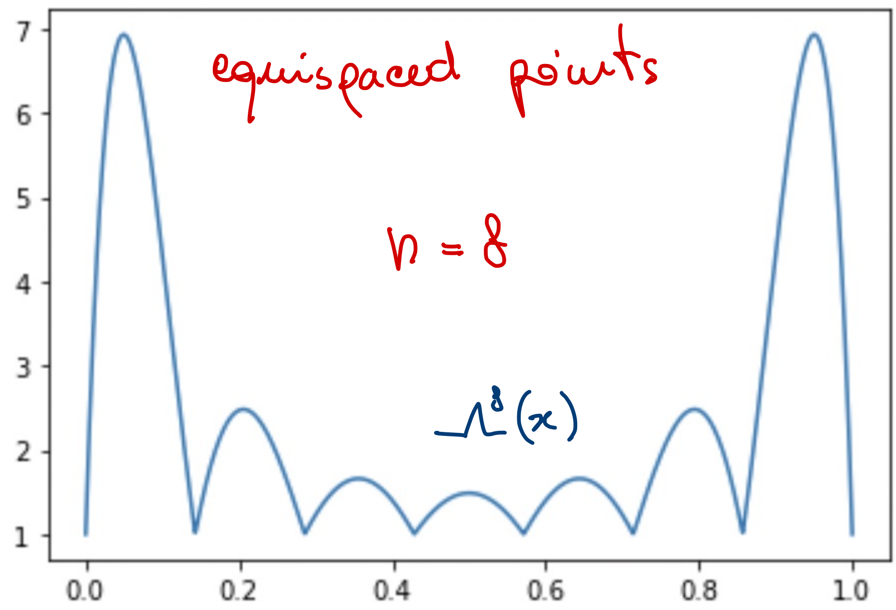
How does $\|\Lambda\|_{L^\infty}$ behave?

Theo: (Erdos)

$\forall$ collection of points $X_n \in \mathbb{R}^n$ $\quad X_{ni} := a_i^n$

$\exists$ $c > 0$ s.t.

$$\boxed{\|\Lambda^n\|_{L^\infty} \geq \frac{2}{\pi} \log(n-1) - c}$$



Theo (Faber)

$\forall \; \mathcal{X}^n \in \mathbb{R}^n$

$\exists f$ s.t.

$$\boxed{\lim_{n\to\infty} \| I^{\mathcal{X}^n}(f) - f \|_{L^\infty} \to \infty}$$

$$\begin{matrix} a_0^1 & & & \\ a_0^2 & a_1^2 & & \\ a_0^3 & a_1^3 & a_2^3 & \\ a_0^4 & a_1^4 & a_2^4 & a_3^4 \\ \cdot\;\cdot & \cdot\;\cdot & \cdot\;\cdot & \cdot \end{matrix} \sim \mathcal{X}$$

Infinite triang. matrix of points

$\boxed{\text{Can it be used??}}$

Preliminary:

Taylor expansion theorem: if $f \in C^{k+1}([0,1])$

then given $a \in ([0,1])$, $\forall x \in [0,1]$, $\exists \xi \in (a,x)$

s.t.

$$\boxed{f(x) = \sum_{i=0}^{k} \frac{f^{(i)}(a)}{(i)!}(x-a)^i + \frac{f^{(k+1)}(\xi)}{(k+1)!}}$$

$\Rightarrow$ Theo: if $f \in C^{n+1}([0,1])$, $a_i \in (0,1)$, $\forall x \in (0,1)$

$\exists\, \xi$:

$$\left(f - I^n(f)\right)(x) = \frac{1}{(n+1)!} f^{(n+1)}(\xi)\, \underline{\underline{\omega(x)}}$$

where $\boxed{\omega(x) = \prod_{i=0}^{n} (x - a_i)}$ characteristic polynomial

Proof: $\forall x$, define $G(t)$ s.t.

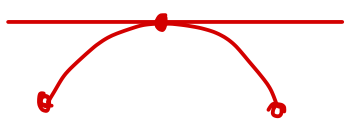
$$G(t) = \left(f(t) - p(t)\right)\omega(x) - \left(f(x) - p(x)\right)\omega(t)$$

$p = I^n(f)$

$$\frac{d^{n+1} G}{dt^{n+1}}(t) = \omega(x)\, f^{(n+1)}(t) - \left(f(x) - p(x)\right)(n+1)!$$

$G(t)$ has $n+2$ zeros, $\longrightarrow$ $G^{(n)}(t)$ will have $\underline{2 \text{ zeros}}$

$\Rightarrow$ $\exists\, \xi$ in the middle s.t. $\frac{d}{dt} G^{(n)} = G^{(n+1)}$ is zero



$$\rightarrow \quad G^{(n+1)}(\xi) = 0 = \omega(x)\, f^{(n+1)}(\xi) - \left(f(x) - p(x)\right)(n+1)!$$

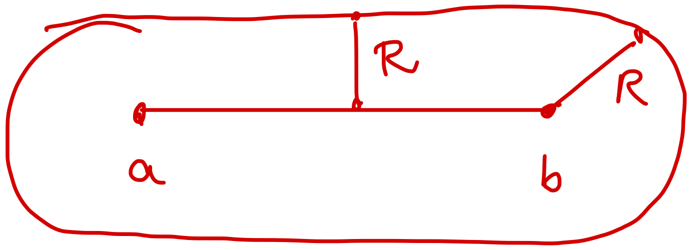
$$\frac{\omega(x)\, f^{(n+1)}(\xi)}{(n+1)!} = f(x) - p(x)$$

Can we apply this?

$$\| f - I^n(f) \|_{L^\infty} \leq \frac{\|\omega\|_{L^\infty}}{(n+1)!} \| f^{(n+1)} \|_{L^\infty}$$

$\boxed{\text{Theo (Runge)}}$ If f analytically extendible on a Oval of radius R, then

$$\| f^{(n+1)} \|_{L^\infty} \leq \frac{(n+1)!}{R^{(n+1)}} \| \tilde{f} \|_{L^\infty(O(a,b,R))}$$



$:= O(a,b,R)$

$z \in \mathbb{C}$ s.t. $\text{dist}(z, [a,b]) \leq R$

In those cases

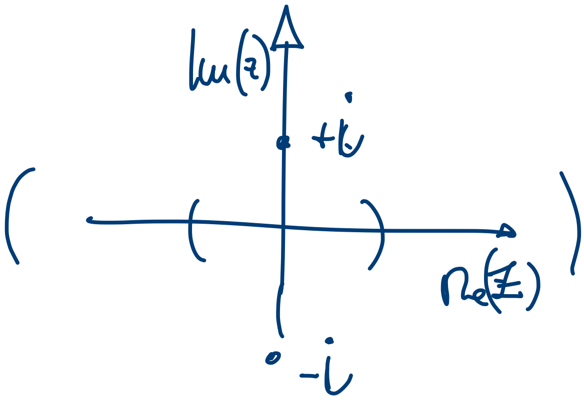
$$\| f - I(f) \|_{L^\infty} \leq \frac{\|\omega\|_{L^\infty}}{\cancel{(n+1)!}} \frac{\cancel{(n+1)!}}{R^{(n+1)}} \| \tilde{f} \|_{L^\infty(O(a,b,R))}$$

$$\| \omega(x) \|_{L^\infty} \leq (b-a)^{n+1}$$

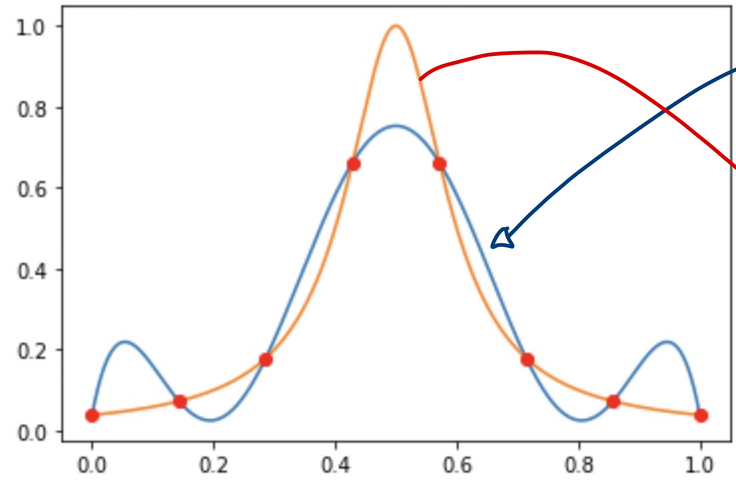
$$\boxed{\| f - I(f) \|_{L^\infty} \leq \left[\frac{(b-a)}{R} \right]^{n+1} \| \tilde{f} \|_{L^\infty(O(a,b,R))}}$$

Runge function $f(x) = \dfrac{1}{1+x^2}$

$\rightsquigarrow f(z) = \dfrac{1}{1+z^2}$

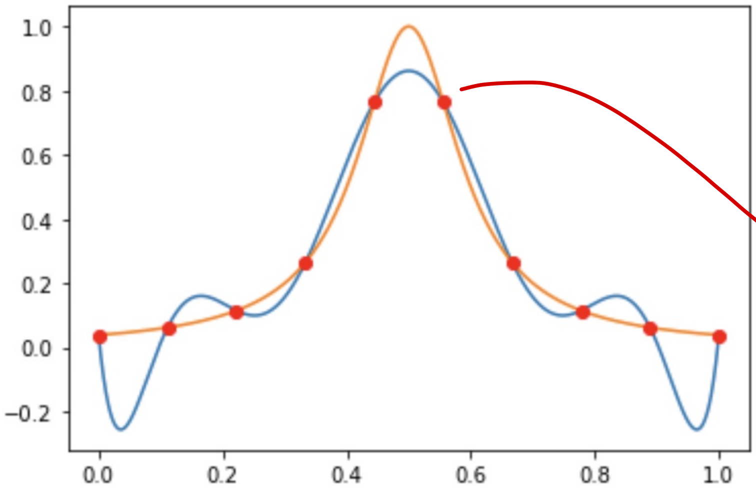


$z \to \pm i \quad |f| \to \infty$



$I^8(f)$

$f(x) = \dfrac{1}{1+100(x-\frac{1}{2})^2}$



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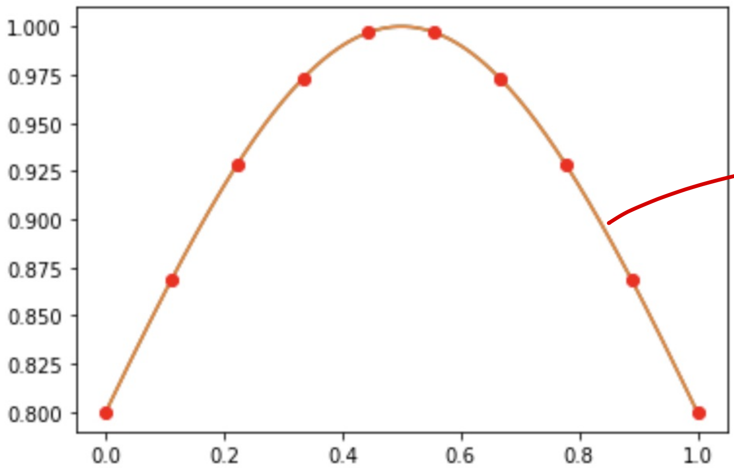
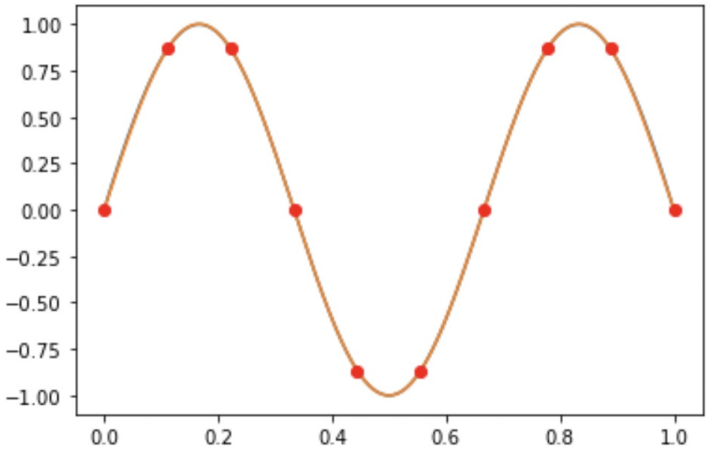
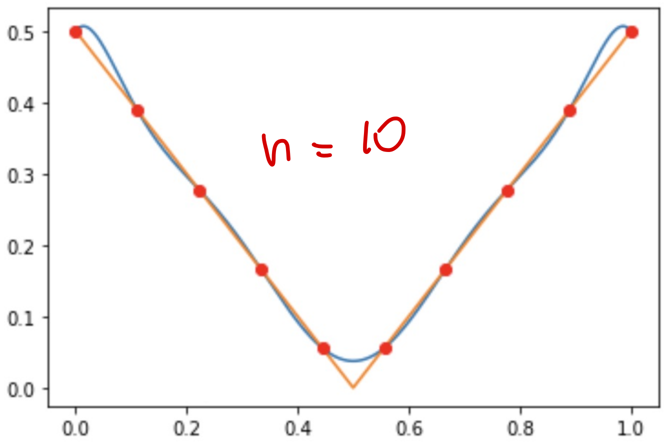


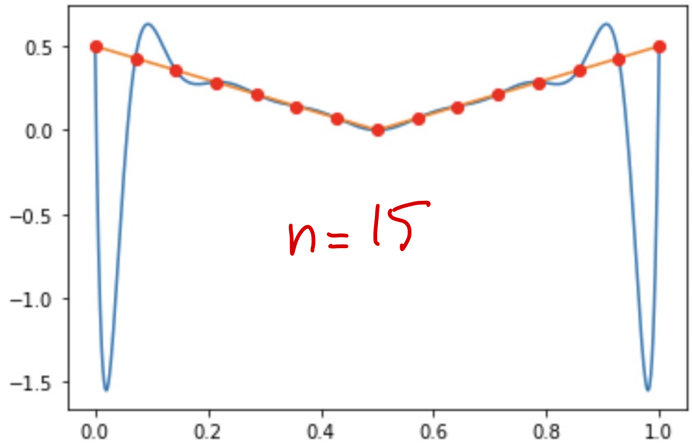
$\dfrac{1}{1+(x-\frac{1}{2})^2}$



$\sin(3\pi x)$

$f(x) = |x - 0{,}5|$

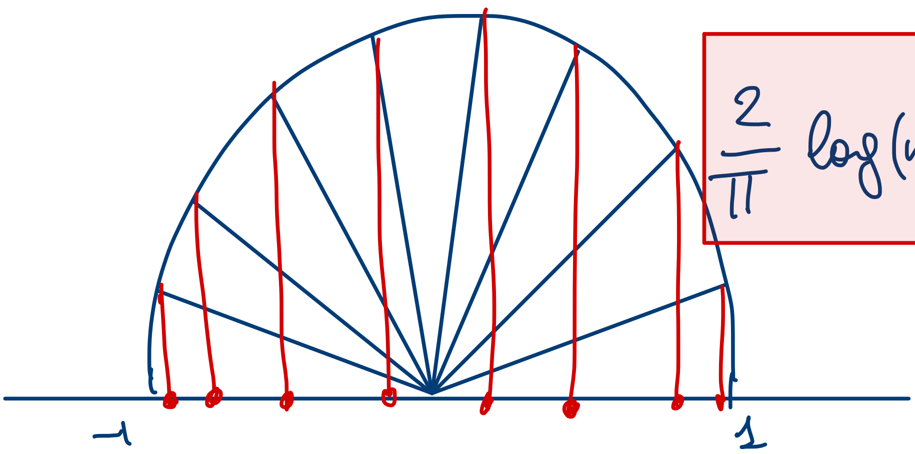




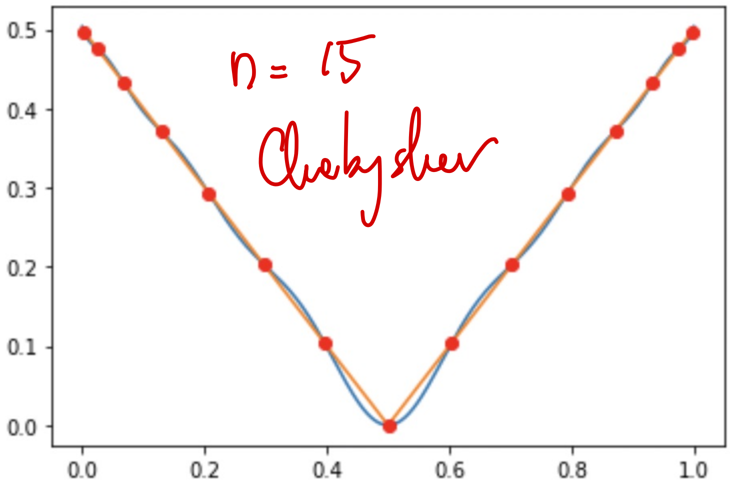
Everything depends on $\|\Lambda\|_{L^\infty}$.

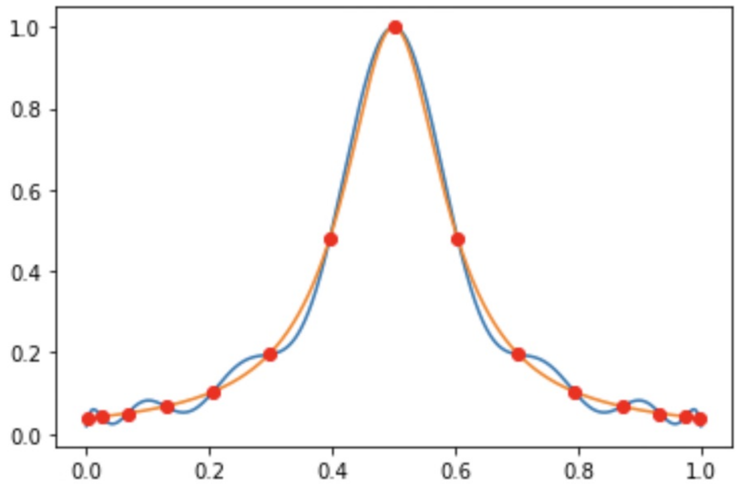
Chebyshev: take $\{a_i\}_{i=0}^{n} = \underset{a \in \mathbb{R}^{n+1}}{\arg\min} \|\Lambda\|_{L^\infty}$

$\rightsquigarrow$ you get in $[-1, 1]$: $a_i = \cos\left(\frac{(2i+1)\pi}{2n+2}\right)$



$$\boxed{\frac{2}{\pi}\log(n+1) - c \leq \|\Lambda\|_{L^\infty} \leq \frac{2}{\pi}\log(n+1)+1}$$





If interpolation is not good to approximate in L^∞, how do we get good results in L^∞?

Weierstrass Approximation Theorem

$\forall f \in C^0([a,b])$, $\forall \varepsilon >0$, $\exists n$, $p \in \mathbb{P}^n$ s.t. $\|f-p\|_{L^\infty} \leq \varepsilon$

<u>Lemma</u>: Let B_n be a sequence of linear operators s.t.

1) B_n is positive
2) $B_n q \to q \quad \forall q \in \mathbb{P}^2([a,b])$

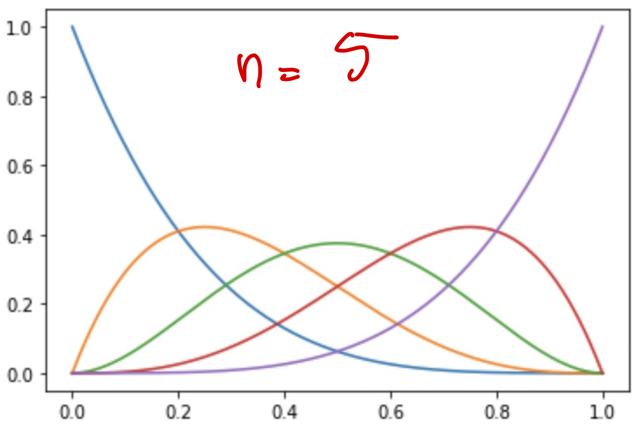
Then $\boxed{\forall \varepsilon >0, \ \exists n \ \text{s.t.} \ \forall \bar{n} \geq n, \ \|B_{\bar{n}} f - f\|_{L^\infty} \leq \varepsilon}$

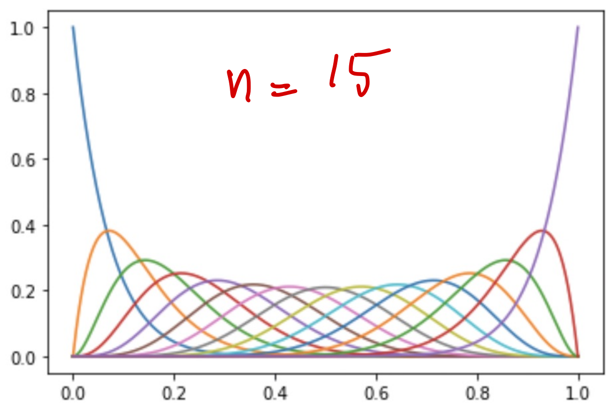
$$f(x) \geq 0 \Rightarrow (B_n f)(x) \geq 0 \qquad (B_n \text{ is positive})$$

$$B_n : C^0([a,b]) \longrightarrow \mathbb{P}^n \quad \text{linear}$$

Bernstein polynomials: built from

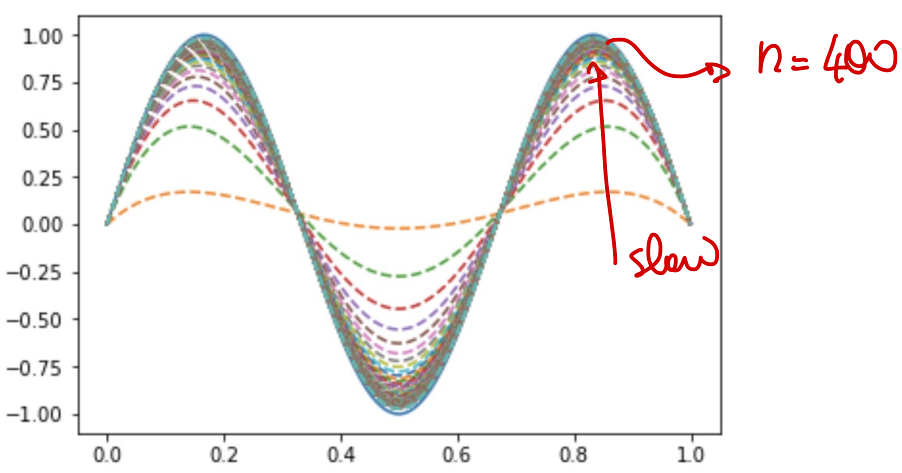
$$1 = 1^n = \left((1-x)+x\right)^n = \sum_{i=0}^{n} \underbrace{\binom{n}{i} x^i (1-x)^{n-i}}_{:= b_i^n(x)}$$

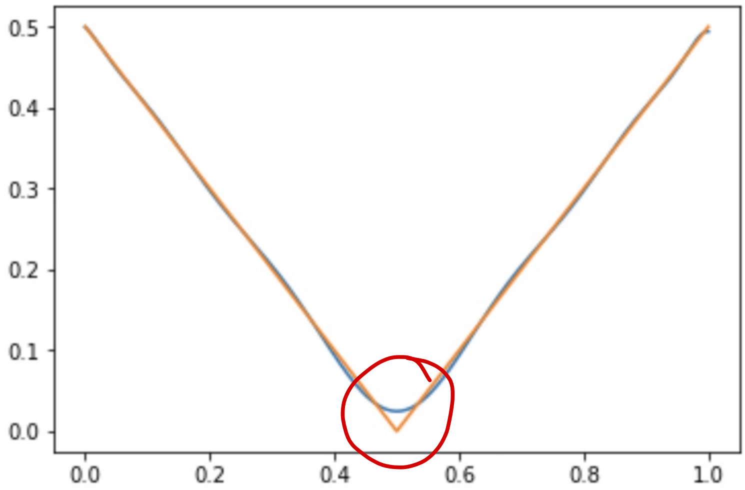




$\{b_i^n\}_{i=0}^{n}$ form a basis for $\mathbb{P}^n$

$$\boxed{B_n f := \sum_{i=0}^{n} b_i^n(x)\, f(i/n)}$$





0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.2
0.4
0.6
0.8
1.0