Does polynomial interpolation converge?

L^∞ space

"converge" = $\max_x |f(x) - I_k(f)(x)| \xrightarrow{k} 0$

Interpolation of f with k nodes

	Equally spaced nodes	Chebyshev nodes
$\mathcal{C}^0$ functions	✗	✗
$\mathcal{C}^1$ functions	✗	✓
$\mathcal{C}^\infty$ functions	✗	✓

Why not?

~~Because polynomials can not approximate continuous functions~~

Wrong!

?

Because interpolation does not choose a "good approximation" for big degrees

Indeed, ***Berstein polynomials*** can approximate any continuous function!

...but they are too slow to be used in practice (and they do not interpolate the function on the nodes)

$(q_1, \ldots, q_n)$ interpolation nodes;
$(x_1, \ldots, x_d)$ evaluation points

Lagrangian polynomials: $\ell_i(x) \stackrel{\text{def}}{=} \prod_{\substack{1 \le j \le n \\ j \neq i}} \frac{x - q_j}{q_i - q_j}$

$$I_n(f)(x) \stackrel{\text{def}}{=} \sum_{i=1}^{n} f(q_i)\ell_i(x)$$

$I_n(f)$ can be evaluated on $(x_1, \ldots, x_d)$ using a matrix product

$$\begin{pmatrix} \ell_1(x_1) & \ldots & \ell_n(x_1) \\ \vdots & \ddots & \vdots \\ \ell_1(x_d) & \ldots & \ell_n(x_d) \end{pmatrix} \begin{pmatrix} f(q_1) \\ \vdots \\ f(q_n) \end{pmatrix} = \begin{pmatrix} I_n(f)(x_1) \\ \vdots \\ I_n(f)(x_d) \end{pmatrix}$$

Chebyshev nodes on $[-1, 1]$:

$$\cos\left(\frac{2i-1}{2n}\pi\right) \quad \text{for } i \in \{1, \ldots, n\}$$

Berstein polynomial: $b_{i,n}(x) \stackrel{\text{def}}{=} \binom{n}{i} x^i (1-x)^{n-i}$

$$\begin{pmatrix} b_{0,n}(x_1) & \ldots & b_{n,n}(x_1) \\ \vdots & \ddots & \vdots \\ b_{0,n}(x_d) & \ldots & b_{n,n}(x_d) \end{pmatrix} \begin{pmatrix} f\left(^0/_n\right) \\ f\left(^1/_n\right) \\ \vdots \\ f\left(^n/_n\right) \end{pmatrix}$$

(WARNING: we are using $n+1$ nodes!)