Twenty-Five Moves Suffice for Rubik's Cube

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Abstract

How many moves does it take to solve Rubik's Cube? Positions are known that require 20 moves, and it has already been shown that there are no positions that require 27 or more moves; this is a surprisingly large gap. This paper describes a program that is able to find solutions of length 20 or less at a rate of more than 16 million positions a second. We use this program, along with some new ideas and incremental improvements in other techniques, to show that there is no position that requires 26 moves.

1 Introduction

The Rubik's Cube is a simple, inexpensive puzzle with only a handful of moving parts, yet some of its simplest properties remain unknown more than thirty years after its introduction. One of the most fundamental questions remains unsolved: how many moves are required to solve it in the worst case? We consider a single move to be a turn of any face, 90 degrees or 180 degrees in any direction (the 'face turn metric'). In this metric, there are more than 36,000 distinct positions known that require at least twenty moves to solve[8]. No positions are yet known that require twenty-one moves. The best theoretical approaches and computer searches to date have only been able to prove there are no positions that require more than twenty-six moves[3].

In this paper, we prove that all positions can be solved in twenty-five or fewer moves. We prove this new result by separating the cube space into two billion sets, each with 20 billion elements. We then divide our attention between finding an upper bound on the distance of positions in specific sets, and combining those results to calculate an upper bound on the full cube space.

The new contributions of this paper are:

1. We extend Kociemba's near-optimal solving algorithm to consider six transformations of a particular position simultaneously, so it finds near-optimal positions more quickly.

2. We convert his solving algorithm into a set solver that solves billions of positions at a time.

3. We show how to eliminate a large number of the sets from consideration, because the positions in them only occur in conjunction with positions from other sets.

4. We combine the three contributions above with some simple greedy algorithms to pick sets to solve, and with a fair amount of computer power, we actually run the sets, combine the results, and prove that every position in the cube can be solved in 25 moves or less.

2 Colors, Moves, and the Size of Cube Space

The Rubik's cube appears as a stack of 27 smaller cubes (cubies), with each face of the cubies colored one of six colors. Of these 27 cubies, seven form a fixed frame around which the other twenty move. The seven that form the fixed frame are the center cubies on each face and the central cubie.

Each move on the cube consists of grabbing the nine cubies that form a full face of the larger cube, and rotating them as a group 90 or 180 degrees around a central axis shared by the main cube and the nine cubies. Each move maintains the set of fully-visible cubie faces. The eight corner cubies each always have the same set of three faces visible, and the twelve edge cubies each always have the same set of two faces visible. We will frequently use the term 'corner' to mean 'corner cubie', and 'edge' to mean 'edge cubie'.

In the solved position, each face of the main cube has a single color. By convention, we associate these colors with their orientation on the solved cube: U(p), F(ront), R(ight), D(own), B(ack), and L(eft). Each move that uses a a 90 degree clockwise twist is specified by just specifying the face with no suffix; each move that uses a 90 degree counterclockwise twist is specified with the face followed by a prime symbol ($'$), and each move that uses a 180 degree twist is specified with the face followed by the digit 2. So a clockwise quarter turn of the right face is represented by R, and the move sequence R2L2U2D2F2B2 generates a pretty pattern known as *Pons Asinorum*. There are a total of eighteen different moves; we define this set by S. (By convention, a clockwise turn is sometimes written with a '+' or '1' suffix and a counterclockwise turn is sometimes written with a '3' suffix.)

The corner cubies may be permuted arbitrarily, as may the edge cubies, except that the parity of the permutations of the edge and corner permutations must match. This contributes a factor of 12!8!/2 toward the total number of reachable positions.

Every corner cubie has exactly one face with either the U or D color. We define the default orientation for the corner cubies to be that where the U or D face is on the whole-cube u or d face; the corner cubies may also be twisted 120 degrees clockwise or counterclockwise with respect to this default orientation (looking toward the center of the cube). Note that these orientations for each cubie are preserved by the moves U, D, R2, L2, F2, B2, but not by the moves R, L, F, or B. This corner cubie orientation is fully arbitrary, except that the sum of all the twists for all the corner cubies must be a multiple of 360 degrees. These corner orientations contribute an additional $3^8/3$ factor toward the total number of reachable positions.

We define the default edge orientation to be that orientation in the solved state of the cube that is preserved by the moves U, D, R, L, F2, B2 (but changed by F and B). Each edge is either flipped from this orientation or not; the count of flipped edges must be even. These edge orientations contribute an additional $2^{12}/2$ factor toward the total number of reachable positions.

The total number of reachable positions, and thus, the size, of the cube group, is the product of these factors, which is about 4.33E19. We call the set of reachable positions G. For each of these positions, an infinite number of move sequences obtain that position. We define the *distance* of a position p ($d(p)$) to be the shortest length of any move sequence that obtains that position. We define the distance of a set of positions to be the maximum of the distances of all the positions in that set.

As a convention, we will denote the successive application of two move sequences by concatenation. We will also denote the application of a sequence to a position, or set of positions, by concatenation of the position and the sequence.

3   Symmetry

The Rubik's cube is highly symmetrical. There is no distinction among the faces except for the color; if we were to toss the cube in the air and catch it, the cube itself remains the same; only the color corresponding to the u face, the r face, and so on changes. Indeed, by tossing the cube, catching it, and noting the colors on the various faces in the new orientation, we can enumerate a total of 24 different ways we can orient the cube, each with a distinct mapping of colors to U, F, R, D, B, and L faces. Specifically, there are six different colors the up face can have, and for each of those six colors, there are four colors possible for the front face. These two face colors fully define the orientation of the normal physical cube.

If we peer in a mirror while performing this experiment, we notice that our alter-ego holds a cube with mirror-reversed orientations; these mirror-reversed orientations present an additional 24 possible mappings from colors to oriented faces. We further notice that whenever we do a clockwise move, our alter ego does a counterclockwise move.

If we choose a canonical color representation, then each of these 48 orientations is a permutation of the cube colors. We call this set of color permutations M. If a particular cube position p is obtained by a move sequence s, we can obtain fully corresponding positions by applying one of the 48 color permutations (say, m), performing the sequence s, and then applying the inverse permutation of m. The resulting position shares many properties with the original one (especially, for us, distance). If we repeat this operation for all 48 permutations in M, we will obtain 48 positions. These positions are not always unique, but for the vast majority of cube positions they will be. Using this form of symmetry, we can reduce many explorations of the cube space by a factor of 48.

Each cube position has a single specific inverse position. If a position is reached by a move sequence s, then the inverse position is reached by inverse move sequence s'. To invert a move sequence, you reverse it and invert each move; the face remains the same, but clockwise becomes counterclockwise and vice versa. The set of symmetrical positions of the inverse of position p is the same as the inverses of the symmetrical positions of p. Some properties of a position are

shared by its inverse position (specifically, distance).

We can partition the cube space into symmetry-plus-inverse reduced sets by combining each position with its symmetrical positions and their inverses; there are only 4.51E17 such sets.

4   Calculating the Diameter

We are primarily interested in finding the maximum of the distance for all positions; this is known as the diameter of the group. One technique for this is to enumerate all positions, and optimally solve each. Practical optimal solvers for Rubik's cube have been available for some time[4]; they typically average about 15 minutes for each optimal solution (with great variance both among programs and among positions). If we were to use such a program to solve the reduced set of 4.51E17 positions, with today's hardware, we would require more than three million computers for more than three million years.

We know, however, that some positions require at least twenty moves. The first such position found is called *superflip*; it has every cubie in the correct place, all corners correctly oriented, and all edges flipped[7]. Herbert Kociemba devised an algorithm to quickly find reasonably short but not necessarily optimal solutions to arbitrary positions. That program (slightly improved as we shall describe) can find move sequences of length twenty or less at a rate of about 240 positions per second (subject to the condition that there is such a sequence; no exceptions have been found yet). Even with this kind of speed, proving all 4.51E17 positions would require more than seven thousand computers for more than seven thousand years.

All hope is not lost. Technology marches onward; when we get to the point we can solve nine hundred million positions a second, we will need only four computers for four years to finish the proof. In the meantime, we can come up with better techniques to refine the upper bound, and improve our techniques.

5   Kociemba's Algorithm

Several techniques have been used to find an upper bound on the diameter of the cube group. Thistlewaite gave a four-stage algorithm that requires a maximum of 52 moves. Herbert Kociemba improved this to an algorithm that requires a maximum of 29 moves (as shown by Michael Reid[6]). Our work is based on Kociemba's algorithm, so we will describe it a bit further here. Kociemba himself has a much more detailed explanation on his web site[2]. In 2006, Silviu Radu reduced the upper bound to 27[5], and in 2007 Kunkle and Cooperman reduced it to 26[3].

Kociemba's algorithm identifies a subset of 20 billion positions, called H. Reid showed that every position in this subset is solvable in at most 18 moves, and further that every cube position is at most 12 moves from this subset. Phase one finds a move sequence that takes an arbitrary cube position to some position in the subset H, and phase two finds a move sequence that takes this new position to the fully solved state.

To describe this subset, we will introduce some new terminology. A cubie *belongs* in a particular place, if it is in that place in the solved cube. Thus, all cubies that have some face colored u belong in one of the top nine cubies. The middle layer consists of the nine cubies between the top and bottom layers; only four of these cubies (edges all) move.

The subset H is composed of all patterns that have the following characteristics:

1. All corners and edges are properly oriented.

2. The edge cubies that belong in the middle layer are located in the middle layer.

The number of positions for which these conditions hold are the permissible permutations of the corners, the top and bottom edges, and the middle edges, with the condition that the parity between the edge permutation and the corner permutation must match. This is thus 8!8!4!/2 or 19.5 billion positions.

These characteristics are preserved by the moves U, U2, U′, D, D2, D′, R2, L2, F2, B2, which we call the set A. Further, these moves suffice to transform every position in H to the solved state. (This is a nontrivial result, but it can easily be shown by brute force enumeration.) For almost all positions in H, the shortest move sequence consisting only of moves from A is the same length as the shortest move sequence consisting only of moves from S, as shown in Table 1. Further, the worst case is 18 in both cases.

Because the defining characteristics of this set treat the U and D faces differently than the L, R, F, and B faces, all 48 symmetries of the cube cannot be used; however, 16 can be used. Thus, after reducing by symmetry, and using two bits per entry, it is possible to store a distance table for the entire set H in only about 300MB of memory. To determine a sequence that takes a position in H to the solved position, simply look up the distance for the current position in the large table. If it is not zero, try each of the 10 moves in A to find a position that is closer, and make that move. Repeat until the cube is solved.

d	moves in S	moves in A
0	1	1
1	10	10
2	67	67
3	456	456
4	3,079	3,079
5	20,076	19,948
6	125,218	123,074
7	756,092	736,850
8	4,331,124	4,185,118
9	23,639,531	22,630,733
10	122,749,840	116,767,872
11	582,017,108	552,538,680
12	2,278,215,506	2,176,344,160
13	5,790,841,966	5,627,785,188
14	7,240,785,011	7,172,925,794
15	3,319,565,322	3,608,731,814
16	145,107,245	224,058,996
17	271,112	1,575,608
18	36	1,352
	19,508,428,800	19,508,428,800

Table 1: The number of positions in H at a given distance using moves from S and moves from A; the numbers are strikingly similar.

The remaining problem is how we can transform an arbitrary cube position into a position in H in 12 or fewer moves. To illustrate how this can be done, we describe a way to relabel the cube so that all positions in H have the same appearance, and all positions not in H have a different appearance.

Consider an arbitrary position p. To be in H, the permutations of the corners are irrelevant; only the orientation matters. To represent this, we remove all colored stickers from the corners, replacing the stickers colored U or D with U and leaving the other faces, say, the underlying black plastic. (To make it easy to follow, we also replace the D sticker in the center of the bottom face with U.) All corner cubies are now interchangeable, but we have sufficient information to note the orientation of the corners.

The permutation of the middle edges does not matter either, but they must lie in the middle layer and be oriented correctly. We thus remove the colored stickers from four edge cubies that belong in the middle layer, replacing the F and B colors with F and leaving the L and R colors as black. (We also replace the B center sticker with F for convenience.)

The permutations of the top and bottom edges also does not matter; for these we do the same color change we did for the corners (U and D get turned

into U, and the other four colors get removed).

With this transformation, all positions in H get turned into the same solved cube: eight corners, each with a U sticker on either the up or down face; four middle edges, each with a F sticker on either the front or back face; eight top/bottom edges, each with a U sticker on the top or bottom face. Every position not in H has a different appearance.

This relabeled puzzle has a much smaller state space than the full cube space. Specifically, the space consists of $3^8/3$ corner orientations multiplied by $2^{12}/2$ edge orientations multiplied by $\binom{12}{4}$ ways to distribute four middle edges among twelve edge positions, for a total of 2.22E9 positions. We call this set of positions R. With 16 ways to reduce this by symmetry and using only two bits per state, a full distance table is easy to fit in memory, and the full state space can be explored easily. We shall call this relabeling process r; it takes a position in G and transforms it into a position in R.

Kociemba's algorithm, then, is to take the original position, call it p, compute $r(p)$, the relabeling; solve the relabeled puzzle with some sequence $a \in S^*$, apply those moves to an original cube yielding pa which lies in H, and then finish the solution with another sequence $b \in A^*$ such that pab is the solved cube. The final solution sequence is ab.

Kociemba's algorithm splits the problem into two roughly equal subproblems, each of which is easy to exhaustively explore, using a lookup table that fits in memory, yielding a fairly good solution to the much larger problem. This algorithm can find a solution of distance 29 or less almost instantaneously (in well under a millisecond). This defines an upper bound on the worst-case position distance.

Kociemba extended this algorithm for another purpose: to quickly find near-optimal solutions for a given position. He proposed finding many phase 1 solutions, starting with the shortest and increasing in length, and for each finding the shortest phase 2 solution. By considering dozens, thousands, or even millions of such sequences, he has found that in practice nearly optimal solutions are found very quickly. Given an input which is the initial cube position denoted by p, his algorithm is given as Algorithm 1. The algorithm can either run to completion, or it can be terminated by the user or when a solution of a desired length is attained.

In Kociemba's algorithm, d_2 is a table lookup that takes a position in H and returns the distance to the identity element (e) using moves in S. (Kociemba actually uses a smaller, faster table that gives a bound on this value; see [2] for details.) The for loop is implemented by a depth-first recursive routine that

Algorithm 1 Kociemba's Algorithm.

1: $d \leftarrow 0$
2: $b \leftarrow \infty$
3: **while** $d < b$ **do**
4:    **for** $s \in S^d, r(ps) = e$ **do**
5:      **if** $d + d_2(ps) < b$ **then**
6:        Solve phase 2; report new better solution
7:        $b = d + d_2(ps)$
8:      **end if**
9:    **end for**
10:    $d \leftarrow d + 1$
11: **end while**

maintains ps incrementally and has a number of further refinements, such as not permitting s to end in a move in A. The phase two solution process is omitted both because it is straightforward and because it takes much less time than enumerating phase one solutions.

This algorithm is extremely effective. Some reasons are:

1. Phase one solutions are found very fast, and mostly access the portions of the phase 1 lookup table near the solved position; this locality enhances the utility of caches significantly.

2. When searching for a phase 2 solution, almost always the very first lookup shows that the distance to the solved position would make the total solution longer than the best found so far; thus, almost all phase 1 solutions are rejected with a single lookup in the phase 2 table.

3. Kociemba has found that in practice the algorithm runs considerably faster if he does not consider phase 1 solutions that contain a strict prefix that is also a phase 1 solution. This is motivated by the fact that we had already explored that prefix earlier (since we consider phase 1 solutions by increasing length).

4. The last move at the end of phase 1 is always a quarter turn of F, B, R, or L; the inverse move is also a solution of phase 1, so candidate solutions are always found in pairs at the leaves of the phase 1 search tree.

5. There are a number of optimizations that can be performed for the phase 1 search when the distance to H is small, such as storing specifically which moves decrease the distance from that point.

Kociemba's algorithm can be run as described above, or it can be run in triple-axis mode. Note how the algorithm treats the u and d faces differently than the other four. Instead of just exploring a single given position p, in triple-axis mode we explore three rotated positions, one with the cube rotated such that the r and l faces correspond to u and d, one such that the b and f faces correspond to u and d, and the original unrotated position. We try each rotation for a given phase 1 depth before moving on to the next phase 1 depth. Our tests show that this finds smaller positions much faster than the standard single-axis mode; when trying to find solutions of length 20 or less, this works approximately six times faster on average than single-axis search.

We have taken this idea one step further; we also consider the inverse position in three orientations for a new six-axis mode. We find this gives on average a further factor of two speed increase when trying to find positions of twenty moves or less.

6   Our Set Solver

Reid showed a bound of 30 by proving it takes no more than 12 moves to bring an arbitrary cube position to the H set (by solving the restickered cube), and then showing that every cube position in H can be solved in 18 moves. (He then reduced that to 29 with a clever insight we omit for brevity[6].) Our proof of 25 is similar, but instead of using just the H set, we use a union of thousands of sets all related to H.

Consider how Kociemba's solver solves an arbitrary position to find a near-optimal solution. It first brings the position into H, using some sequence of moves, and then calculates within H how close to solved it is. It then finds another way to bring the position into H, and checks how close it is to solved at that point. It does this dozens, or hundreds, or thousands, millions, or even billions of times, each time checking for a shorter solution.

We turn this technique inside out. Instead of using a big table containing, for each position within H, how close it is to solved, instead we use a table that indicates whether we've gotten to that particular position in H already. We then enumerate all sequences that take that initial position into H, for longer and longer positions, until that table is completely marked; that every position has been seen. At this point, the length of the longest sequence we considered is the maximum distance from solved for an entire large set of positions; specifically, all those positions that, when phase-one relabeled, give the same

labeling as the initial position we started with.

With this technique, we are essentially solving 20 billion cube positions, rather than one; if we were to write out each sequence every time we set a new bit in the large table, we would eventually write out an optimal sequence for every position in that set.

The main input to our set solver is a sequence $a \in S^*$, which takes the solved cube into some position; the set that will be solved is Ha. Another input is the maximum depth m to run the phase one search; we have found the value $m = 16$ is usually sufficient to prove an upper bound for the distance of the set to be 20. To find the exact distance, m should be set to ∞. Our algorithm is given as Algorithm 2. In line

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Algorithm 2 Set Solver

1: $f \leftarrow \emptyset$
2: $d \leftarrow 0$
3: **loop**
4:      $f = f \cup Af$ {—prepass}
5:      **if** $f = Ha$ **then**
6:          **return** d
7:      **end if**
8:      **if** $d \leq m$ **then**
9:          **for** $s \in S^d, r(as) = e$ **do** {–search}
10:             $f \leftarrow f \cup s^{-1}$
11:         **end for**
12:     **end if**
13:     **if** $f = Ha$ **then**
14:         **return** d
15:     **end if**
16:     $d \leftarrow d + 1$
17: **end loop**

4, we use left multiplication of a set by a move; this is unconventional but still quite fast. In line 9, s is the solution to the position reached by s^{-1}. Unlike Kociemba's search, we do permit our phase 1 search to enter and then leave the H group; we do this in order to compute the precise set bound (should we want to). We have not yet explored the performance impact of this on our running time.

The set f is represented by a bitmap, one bit per position. For the prepass (line 4), we need to have both a source and destination set, so we need to have two of these bitmaps in memory at once. Our memory requirements are completely dominated by these bitmaps, requiring a total of 4.7GB of memory. For positions in R that have symmetry, we take advantage of that symmetry to run the positions on older machines without that much memory.

The indexing of f is done by splitting the cube position into independent coordinates, representing

the permutation of the corners, the permutation of the up/down edges, and finally the permutation of the middle edges.

The time spent in the code is split between the prepass and the search phases. The prepass is a simple scan over the entire f set, multiplying by the ten moves in A; this can be done efficiently by handling the coordinates from most significant to least significant in a recursive algorithm so that the inner loop only need deal with the permutation of the middle edges, and the more expensive corner coordinate computation is performed early in the recursion and thus substantially fewer times.

The time in the search phase (lines 9–11) is very small for low d, because there are few sequences s that satisfy the conditions, but as d grows, so does the time for the search phase, exponentially. Typically a search at level $d + 1$ will require ten times as much time as a search at level d. By limiting m to 16 in the typical case, we limit the total time in the search phase, and the whole program runs fast. For values of m of 17 or higher, the search phase will dominate the total runtime.

Our current implementation of this set solver has a major restriction; at the moment it only solves those sets for which the middle four edges are placed in the middle. This restriction simplifies some of the move calculations, but it is not essential to our basic ideas. In the future we plan to lift this restriction.

7   Improving the Bound

Some sets we solve have relatively few positions in the furthest distance. Since for lower values of m our set solver only gives us an upper bound on the set distance, in many cases the true distance of all these positions is less than the calculated upper bound. By solving these explicitly using a single-cube solver, and proving they do not require as many moves as our set solver found, we can frequently reduce our bound on the distance for the set by 1. To facilitate this, if the count of unsolved positions in one of the sets falls below 65,536 at the top of the loop, we print each of these positions to a log file.

To solve these positions, we first use our six-axis implementation of Kociemba's solution algorithm. Since the solution distance we seek is almost always 19 or 20, this algorithm finds solutions very quickly, usually in a fraction of a second. For those positions that resist Kociemba's solver, we solve them using our optimal solver.

8 The Set Graph

The set R of relabeled positions of G has about two billion elements. Consider a position $a \in R$; we can define the set of a to be all elements $g \in G$ such that $r(g) = a$. Let us pick a single one of the elements i in the set of a; the entire set can be represented by Hi. Each set has precisely the same number of elements, about 20 billion; every pair of sets is either identical or disjoint; and the union of all of the sets is G, the full cube space. (This can be shown with elementary group theory because H is a subgroup of G and each set Hi is a coset.)

These sets are all related by the moves in S. Consider a cube position a and its set Ha. The set Hab for $b \in S$ is adjacent to the set Ha. We can consider R as a graph, where the vertices are the sets represented by the positions of R, and the edges are moves in S. Clearly for any given position $|d(ab) - d(a)| \le 1$, and therefore the same is true for sets as a whole: $|d(Hab) - d(Ha)| \le 1$. If we have shown that $d(Ha) \le c$ for some value of c, we have also shown that $d(Has) \le c + |s|$ where $s \in S^*$ and $|s|$ is the length of s. This allows us to find an upper bound for one set, and use it to infer constraints on upper bounds of neighboring sets in the graph of R.

The relabeled puzzle shows 16-way symmetry, so there are really only about 139 million relabeled positions when reduced by this symmetry. This reduced graph easily fits into memory, and operations on this graph can be performed reasonably quickly. For each vertex, we maintain a value which is the least upper bound we have proved to date. These values are initialized to 30, since we know every position and thus every set has a distance of no more than that. As we solve new sets, we update the value for the vertex associated with that set, and update adjacent vertices recursively with the new upper bound implied by this value.

9 Eliminating Unneeded Sets

Just as the triple-axis search improves the performance of Kociemba's algorithm, we can use remappings to decrease the amount of work needed to prove a new bound on the diameter.

The R-relabeled cube's space can be defined by three coordinates: the placement of the four middle edges, the orientation of the eight corners, and the orientation of the twelve edges. Note that the middle edges are those between the up and down faces. If you reorient the cube so the R and L faces are up and down, then the four edges that were between R and L are now the middle edges. Similarly, if you reorient the cube so the F and B faces are up and down, the the four edges that were between F and B are now middle edges. Thus, the same original cube position has three relabels depending on orientation, and the middle edge positioning for these three orientations cover all twelve edges of the cube.

Each corner cubie in the graph is adjacent to three edges. Therefore, there are eight corners adjacent to an odd number of edges. Consider any subset a of edges, and count the number of corners adjacent to an odd number of edges in that subset; call that $f(a)$. For any two subsets a and b, $f(a + b) \le f(a) + f(b)$. If we partition the twelve edges into three subsets a, b, and c, and we know that $f(a) + f(b) + f(c) = 8$, then at least one of $f(a)$, $f(b)$, or $f(c)$ must be three or more.

Therefore, every cube position has some orientation where the middle edges in that orientation have an odd corner count (f) of three or more. If we find an orientation-independent property that applies to all of these subsets, we can extend that property to the full set of cube positions, since there is some orientation of every cube position that lies within one of these subsets.

This idea is easily generalized. The relabeled cube position information can be described by three independent coordinates: the middle edge positions, the edge orientations, and the corner orientations. For each of coordinates, the full set of possible combination three-tuples can be explicitly enumerated. Given these three-tuples, a covering set based on any particular weighting can be chosen. Using the middle edge position or the corner orientation, it is not difficult to eliminate a third of the sets; with the edge orientation, it is not difficult to eliminate half of the sets. For the results in this paper, we have eliminated sets to solve based on the edge positioning, because the restrictions of our set solver make this the most advantageous coordinate to use. We have eliminated 94 possible middle edge positions out of the 495 possible, including ones that are the furthest from the middle-edges-in-middle restriction of our set solver. This eliminated 25.7 million of the 138.6 million sets, especially those furthest from the ones that our set solver could solve.

10 Choosing Sets to Solve

This work grew out of a search for distance 21 positions[8] that involved solving a number of these sets exactly. We thus started this work with a few thousand cosets already solved; we used those as our

base set. At every point during this exploration we maintained the symmetry-reduced graph R on disk annotated with the best upper bound we had proven for each corresponding set. To select a new set to solve, we used a simple greedy strategy. We chose a vertex that, when we pushed its bound down to 20, and propogated its implications over the graph R, it would reduce the maximum number of vertices from above 25 to 25 or less; we call this value the 'impact' of the vertex. We annotated each node with this value as well. The candidate set was small, since restrictions in our set solver mean that it can only solve approximately 288,000 of the 139 million sets.

Once we had selected a vertex, we added it to the list of sets to solve, updated the relevant vertices on the in-memory copy of the graph (not the persistent one on disk), and repeated this process to select another vertex. Since the potential impact of solving a particular vertex only decreased as other vertices were solved, we skipped vertices that had a prior impact score less than the current best; this allowed subsequent vertices to be selected relatively quickly.

We typically generated lists of a few hundred sets to solve in this manner. Since some of the sets actually came in with a bound of 19 or even 18, and this changed the graph in ways differently than our above algorithm assumed, we generated a brand new list of vertices to solve every few days based on the updated R graph.

11 Correctness

For computer searches such as this one, it is imperative that the programs be correct. In order to ensure this, we have taken a number of measures:

1. We developed an independent, simple set solver that uses the simplest possible code with a simple hashtable structure to store the found positions. We compared the results of our fast set solver against this simpler, slower program, to the search depth that the simpler program was able to handle. We also compared the full set of positions found between the two programs to the depth possible.

2. All set solutions were solved only on server-class machines containing error-checked-and-corrected memory.

3. All separately solved cube positions were taken as sequences, and a separate program was run to aggregate these positions into a simple database of solved positions. Where upper bounds were reduced by calculating solutions for small sets of positions, these sequences were queried and verified that they did, indeed, yield the appropriate position.

4. Every optimization, including symmetry reductions of various kinds, search restrictions, parallel multicore operation, and so on, were switchable on and off, and the results of the program validated.

5. As a sanity check, from the 21 found distance-18 sets, a random 3,000,000 cube positions were selected and for each of these, using the six-axis Kociemba solver, a solution of length 18 or better was found.

12 Results

Solving sets with symmetry can be done with machines that do not have much memory. Initially our largest machine had 3.5GB of RAM, not enough to solve sets lacking symmetry. So we started with the symmetrical sets. The symmetrical sets are also faster to solve, but the results from the symmetrical sets were insufficient to prove a bound of 25. We then purchased a machine with 8GB of memory and set it to solving sets lacking symmetry. We selected sets to solve based on how many neighbors would be reduced to below 25 by solving that set, assuming an upper bound of 20. After solving approximately two thousand additional sets, we were able to bring the upper bound for each vertex in the set graph below 26, proving that every position of Rubik's cube can be solved in 25 or fewer moves.

We have to date found 21 sets that have a distance of 18 or less, 948 that have a distance of 19 or less, and 7,990 that have a distance of 20 or less. Because we initially focused on symmetrical sets (due to memory constraints), many of these sets are neighbors, and some are redundant (for instance, a set with a lower bound of 19, that is adjacent to a set with a lower bound of 18, is redundant because the latter implies the former.) The 21 sets we have found that have a distance of 18 are listed in Table 2. These, and all other sets, are given by a representative element; ϵ means the representative element is the empty sequence so that set is just H.

Any single set that we've shown to be 18, with the exception of the ϵ set immediately proves a bound of 29 on G because every node in the graph of R is within 11 edges of some orientation of each of these sets.

d	sequence
18	ϵ
18	R′B2UB2U′R
18	R′DL2D′R
18	R′U′R2F2R′F2UR′
18	B′U′R2UR2B′
18	F′R2UR2U′F′R2U′
18	F′DFR2U′FU2F′
18	F′D′BR2BDF
18	B′L2R2F′D2U2F′U
18	B′U′F2UB′U′
18	F′D′BR2BDF′
18	B′D′L2DL2B′R2
18	R′F2U2RF2R2U2R′
18	R′DL2D′RU′
18	B′UF2U2F2UB′
18	L′D′U′L′DUL′U′
18	F′UFUF′U2F′
18	F′DUF′D′U′F′U′
18	B′DUB′D′U′B′
18	F′D′F′R2B′D′B′
18	B′D2R2U2L2F

Table 2: All known sets (by representative) that have a distance of exactly 18.

d	sequence
20	R′B2U′L′F′D2R2D′LB′U′
20	R′D2B′L′ULF′R′F2D′B
19	F′L′DF′U′R′B2R′
19	L′BF′DU′B′RF2L′F′U′

Table 3: Four sets, with the given upper bounds on distances, that can be added to the set of 18's to prove a distance of 27 for the set G.

Proving these sets to be 18 took four or five hours each; in some cases, we ran our solver with $m = 17$, and in other cases we simply solved the remaining positions at the top level.

No single set we have solved by itself shows a bound of 28 for the diameter of G, but many pairs of the ones listed above do. One example such pair is B′UF2U2F2UB′ and B′U′R2UR2B′. That is, just solving these two sets proves a bound of 28, because every set in the graph of R is within 10 moves of some orientation of one of these two sets.

To prove a bound of 27 for G, we include all 21 of the sets we list above, plus an additional four sets listed in Table 3. These 25 sets are sufficient to bring every necessary node of R down to a bound of 27. This list was determined with a greedy search. We started with our set of 21 distance-18 sets, and propogated their implications over the graph of R. We then iteratively chose the solved set that had the greatest impact on R (reduced the maximum number of vertices from 28 or more to below 28, in this case), until all vertices were below 28. All subsequent set counts mentioned were generated by this sort of post-pass over our collection of solved sets.

To reproduce the result of Cooperman and Kunkle, all of the above sets plus an additional 127 set bounds that we have found are needed, for a total of 152 sets.

Our new result requires a total of 3,868 set bounds (about half of the total number we have solved). This includes 621 of the sets for which we've shown a bound of 19, and 3,226 of the sets for which we've shown a bound of 20.

It is difficult to say what the total execution time has been, because most of the sets were run slowly, using an older, slower version of the set solver, on older, slower hardware. But given that the current version of the program, on a modern Q6600 CPU running at 1.6GHz, using a value of $m = 16$, and proving a set to a depth of 20, can be done in 18 minutes on average, and taking into account the individual positions we solved to lower some of the set bounds, we believe the total execution time would be on the order of 1,500 CPU hours to reproduce our proof of 25.

We continue to execute sets, and we are making progress toward proving a bound of 24. In order to complete this proof without requiring an inordinate number of sets, we need to lift the restriction on our set solver that it only solve sets that have the middle edges in place. Once this is done, we believe that with only a few more CPU months, we can show a new bound of 24 on the diameter of the cube group.

13 Acknowledgements

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