One power series distribution with parameterization by mean

O. Volkov

Taras Shevchenko National University of Kyiv, Kyiv, Ukraine

volkoval2704@gmail.com

In the report, the distribution of the power series of the function $w(y) = (1+\sqrt{1-y})^{-\frac{1}{2}}$ is studied.

The distributions of power series have been studied since 1950 by Noack [1]

Definition 1. Let

$$w(y) = \sum_{k=0}^{\infty} a_k y^k, 0 < y < R,$$

and all the coefficients of this series be non-negative numbers. The distribution of an integer-valued random variable ξ, which is given by the formula

$$p_k(y) = \Pr(\xi = k) = \frac{a_k y^k}{w(y)}, k = 0, 1, 2, \dots,$$

is called the power series distribution of the function $w(y)$ with parameter y.

For obtaining the numerical characteristics of the distribution, the generating function is used:

$$P(z) := \sum_{k=0}^{\infty} p_k(y) z^k = \frac{1}{w(y)} \sum_{k=0}^{\infty} a_k y^k z^k = \frac{w(yz)}{w(y)}.$$

Then the expected value is $E\xi = \frac{yw'(y)}{w(y)}$, and the variance is $D\xi = y\frac{dE\xi}{dy}$.

Let us denote $E\xi$ by x. Then the function $x = \frac{yw'(y)}{w(y)}$ on the interval $(0, R)$ has an inverse $y = y(x)$, since $\frac{dx}{dy} = \frac{D\xi}{y} > 0$. Therefore, one can study the power series distribution with another parameterization, namely to characterize the distribution by the parameter x. This parameterization is called the mean parameterization (see, for example, [2, p. 670]). With this parameterization

$$\Pr(\xi = k) = p(k, x) = p_k(y(x)) = \frac{(y(x))^k a_k}{w(y(x))}, k = 0, 1, 2, \dots,$$

$$x \in X = \left(0, \frac{Rw'(R)}{w(R)}\right), D\xi = \frac{y}{\frac{dy}{dx}} = \frac{y(x)}{y'(x)}.$$

The function $v(x) = \frac{y(x)}{y'(x)}$ is called the variance function of the distribution.

The aim of this work is to study the power series distribution of the function $w(y) = (1+\sqrt{1-y})^{-\frac{1}{2}}$.

Theorem 1. *The initial moments α_m, central moments μ_m, cumulants χ_m, $m = 1, 2, \dots$, satisfy the following recurrence relations:*

$$\alpha_{m+1} = x\alpha_m + v(x)\frac{d\alpha_m}{dx}, \alpha_0 = 1, \alpha_1 = x, \tag{1}$$

$$\mu_{m+1} = m\mu_{m-1} + v(x)\frac{d\mu_m}{dx}, \mu_0 = 1, \mu_1 = 0, \tag{2}$$

$$\chi_{m+1} = v(x)\frac{d\chi_m}{dx}, \chi_1 = x. \tag{3}$$

In particular, from (1), (2), (3) we find:

$$\alpha_2 = x^2 + x(2x+1)(4x+1), \quad \alpha_3 = x^3 + 3x^2(2x+1)(4x+1) + v(x)v'(x),$$
$$\chi_2 = v(x), \quad \chi_3 = v(x)v'(x).$$

1. Noack A. A. Class of Random Variables with Discrete Distribution. Ann. Math. Statist., 1950, Vol. 21, No. 1, 127 − 132.
2. Kotz S., Balakrishnan N., Johnson N. Continuous Multivariate Distribution. — New York: John Wiley & Sons Inc, 2000, 722 pp.