METHODS OF CONSTRUCTING MULTIVARIATE POWER SERIES DISTRIBUTIONS

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The paper proposes methods for constructing multidimensional power series distributions (PSD) with *natural parameterization*, which generalize the one-dimensional case ([1] and [2]) to the m-dimensional space.

Let $k = (k_1, \dots, k_m)$ be a multi-index with non-negative integer coordinates, and let the series

$$\omega(y_1, \dots, y_m) = \sum_{k_1 \geqslant 0, \dots, k_m \geqslant 0} a_{k_1} \dots a_{k_m}\, y_1^{k_1} \dots y_m^{k_m},$$

where $a_{k_1} \geqslant 0, \dots, a_{k_m} \geqslant 0$, converge in the polycylinder

$$Y := \{y_1 \mid 0 \leqslant y_1 < R\} \times \dots \times \{y_m \mid 0 \leqslant y_m < R\}.$$

The distribution of a random vector $\xi = (\xi_1, \dots, \xi_m)$ with coordinate probabilities

$$P\{\xi_1 = k_1, \dots, \xi_m = k_m\} = \frac{a_{k_1} \dots a_{k_m} y_1^{k_1} \dots y_m^{k_m}}{\omega(y_1, \dots, y_m)}, \quad k_1 \geqslant 0, \dots, k_m \geqslant 0$$

is called the *power series distribution* (PSD) of the function ω.

The Generating Function is:

$$P(\mathbf{z}) = \sum_{\mathbf{k}} \frac{(z_1 y_1)^{k_1} \dots (z_m y_m)^{k_m}}{\omega(y_1, \dots, y_m)} a_{k_1} \dots a_{k_m} = \frac{\omega(z_1 y_1, \dots, z_m y_m)}{\omega(y_1, \dots, y_m)}.$$

For such a distribution

$$\mathbb{E}[\xi] = \frac{\partial \log \omega}{\partial \log y}, \qquad \mathrm{Cov}(\xi) = \frac{\partial^2 \log \omega}{(\partial \log y)^2}.$$

On the convex domain Y, the mapping $y \mapsto x$ is one-to-one and has an inverse function $y = f(x)$.

Theorem 1. *The following relation holds:*

$$\frac{\partial P}{\partial x} V(x) - \frac{\partial P}{\partial z} \tilde{z} + xP = 0, \quad P(1, \dots, 1, x_1, \dots, x_m) = 1,$$

$$P = P(z_1, \dots, z_m, f_1(x_1, \dots, x_m), \dots, f_m(x_1, \dots, x_m)),$$

$$\frac{\partial P}{\partial x} = \left(\frac{\partial P}{\partial x_1}, \dots, \frac{\partial P}{\partial x_m}\right), \quad \frac{\partial P}{\partial z} = \left(\frac{\partial P}{\partial z_1}, \dots, \frac{\partial P}{\partial z_m}\right), \quad \tilde{z} = \begin{pmatrix} z_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & z_m \end{pmatrix},$$

where $P(x_1, \dots, x_m)$ is the probability function constructed in the natural parameterization, and $V(x)$ is the covariance matrix of ξ, expressed in terms of x.

In the correctness of the theorem one can be convinced by direct verification.

To construct specific PSD functions, we shall make use of power series in the one-dimensional case.

Example 1. $\omega(y) = (1 + y)^n,\ n \in \mathbb{N}$.
We have

$$f(x) = \frac{x}{n - x}, \quad \frac{\omega(y)}{\omega'(y)} = \frac{1 + y}{n}, \quad \left(\frac{\omega(y)}{\omega'(y)}\right)' = \frac{1}{n},$$

hence, for the elements of the covariance matrix of the PCP function $\omega(y_1 + \dots + y_m)$ we get:

$$\nu_{ij} = \delta_{ij} x_i - x_i x_j n^{-1}, \quad \det V(x) = x_1 \dots x_m (1 - |x| n^{-1}).$$

If we denote $p_1 = x_1/n, \dots, p_m = x_m/n$, then

$$P\{\xi_1 = k_1, \dots, \xi_m = k_m\} =$$

$$\frac{n!}{k_1! \dots k_m! (n - |k|)!} p_1^{k_1} \dots p_m^{k_m} (1 - p_1 - \dots - p_m)^{n - |k|}$$

Example 2. $\omega(y) = \exp y$.

In this case, $f(x) = x$, and for the elements of the covariance matrix of the PSD function $\omega(y_1)\omega(y_1 y_2) \cdots \omega(y_1 y_2 \cdots y_m)$ we obtain $\nu_{ij} = x_i$, if $i \geqslant j$ and $\nu_{ij} = x_j$, if $i < j$, $\det V(x) = (x_1 - x_2)(x_2 - x_3) \dots (x_{m-1} - x_m) x_m$.

References

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