HW2

1a) $x_t = \beta_0 + \beta_1 t + w_t + w_{t-1}$

$E[x_t] = E[\beta_0 + \beta_1 t + w_t + w_{t-1}]$

$= E[\beta_0] + E[\beta_1 t] + E[w_t] + E[w_{t-1}]$

$= 0 + \beta_1 E[t] + 0 + 0$

Since $E[x_t]$ is dependent on t, $E[x_t]$ is not stationary.

b) $\{y_t\}$ is stationary because ∇x_t removes the linearity of x_t, making it not dependent on t and $\{w_t\}$ is stationary, so $\{w_t\}$ plus a constant is still stationary.

$y_t = \nabla x_t = x_t - B(x_t) = \beta_0 + \beta_1 t + w_t + w_{t-1}$
$- \beta_0 - \beta_1 (t-1) - w_{t-1} - w_{t-2}$
$= \beta_1 t + w_t - \beta_1 t + \beta_1 - w_{t-2}$
$= w_t + \beta_1 - w_{t-2}$

$E[y_t] = E[w_t] + 0 - E[w_{t-2}] = 0$

$\gamma(y_{t+h}, y_t) = E[(y_{t+h} - E[y_{t+h}])(y_t - E[y_t])]$
$= E[(y_{t+h})(y_t)]$
$= E[(w_{t+h} + \beta_1 - w_{t+h-2})(w_t + \beta_1 - w_{t-2})]$
$= E[w_{t+h} w_t + \beta_1 w_{t+h} - w_{t+h} w_{t-2} + \beta_1 w_t + \beta_1^2$
$- \beta_1 w_{t-2} - w_{t+h-2} w_t - w_{t+h-2} \beta_1 + w_{t+h-2} w_{t-2}]$
$= E[w_{t+h} w_t - w_{t+h} w_{t-2} - w_{t+h-2} w_t + w_{t+h-2} w_{t-2}]$

$h=0: \sigma^2 - 0 - 0 + \sigma^2 = 2\sigma^2$
$h=1: 0 - 0 - 0 - 0 = 0$
$h=2: 0 - 0 - \sigma^2 - 0 = -\sigma^2$
$h=3: 0 - 0 - 0 - 0 = 0$
$h=4: 0 - 0 - 0 + \sigma^2 = \sigma^2$

$$\gamma(h) = \begin{cases} 2\sigma^2, & h=0 \\ 0, & 0<h<2 \\ -\sigma^2, & h=2 \\ 0, & 2<h<4 \\ \sigma^2, & h=4 \\ 0, & h>4 \end{cases}$$

$$\rho(h) = \gamma(h)/\gamma(0)$$

$$\rho(h) = \begin{cases} 1, & h=0 \\ 0, & 0<h<2 \\ -1/2, & h=2 \\ 0, & 2<h<4 \\ 1/2, & h=4 \\ 0, & h>4 \end{cases}$$

3) $x_t = 0.6x_{t-1} + 0.08x_{t-2} + 0.03w_{t-2} + 0.4w_{t-1} + w_t$

$$\phi(z) = 1 - 0.6z - 0.08z^2$$

$$z = \frac{0.6 \pm \sqrt{0.36 + 0.32}}{-0.16}$$

$$= 1.404 \text{ or } -8.904$$

$$|z| \geq 1$$

Therefore, the ARMA model is causal.

$$\theta(z) = 1 + 0.4z + 0.03z^2$$

$$= (1 + 0.3z)(1 + 0.1z)$$

$$z = -10/3 \text{ or } -10$$

$$|z| > 1$$

Therefore, the ARMA model is invertible.