Interpolating values using a tetrahedral mesh

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1 The Math

A 3D tetrahedron, a polyhedron having four triangular faces and four vertices, is defined by its four vertices v_1, v_2, v_3 and v_4, where v_n is a 3 dimensional point in Cartesian space $v_n = \begin{bmatrix} x_n & y_n & z_n \end{bmatrix}^T$. The barycentric coordinates are defined so that the first vertex r_1 maps to barycentric coordinates $\lambda_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$, $r_2 \rightarrow \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$, etc. and that the sum of barycentric parameters $\sum \lambda_n = 1$.

This is a linear transformation and the problem can be written in matrix form so that $v = R\lambda$ with $R = \begin{bmatrix} v_1 | v_2 | v_3 | v_4 \end{bmatrix}$ and $\lambda = \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 \end{bmatrix}^T$. The condition $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 1$ can be augmented into the matrix to form the final equation:

$$\begin{bmatrix} v1_x & v2_x & v3_x & v4_x \\ v1_y & v2_y & v3_y & v4_y \\ v1_z & v2_z & v3_z & v4_z \\ 1 & 1 & 1 & 1 \end{bmatrix} \lambda = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \tag{1}$$

Where x, y, and z define a 3D point in Cartesian space. The barymetric coordinates λ can be solved as the solution of the linear equation 1.

Given a subspace $V = \operatorname{span}(\vec{e_1} | \vec{e_2} | \ldots)$ where $\vec{e_n}$ is the basis for V and $\vec{x}, \vec{e_n} \in \mathbb{R}^N$, the projection of $\vec{x}$ onto V is defined as:

$$\operatorname{proj}_V \vec{x} = A \left(A^\intercal A\right)^{-1} A^\intercal \vec{x} \tag{2}$$

The face of a tetrahedron is defined by the three vertices v_1, v_2 and v_3, where $v_n \in \mathbb{R}^3$. A point $\vec{x} \in \mathbb{R}^3$ is projected into the triangle:

$$\begin{aligned} \vec{a} &= \vec{v_2} - \vec{v_1} \\ \vec{b} &= \vec{v_3} - \vec{v_1} \\ \operatorname{proj}_V \vec{x} &= A \left(A^\intercal A\right)^{-1} A^\intercal \vec{x} \\ &= \begin{bmatrix} a_x & b_x \\ a_y & b_y \\ a_z & b_z \end{bmatrix} \left(\begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \begin{bmatrix} a_x & b_x \\ a_y & b_y \\ a_z & b_z \end{bmatrix} \right)^{-1} \begin{bmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} \end{aligned}$$