# Resampling Methods Exercises

Slides on *Introduction to Statistical Learning*, Chapter 5

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## Exercise 1

Using basic statistical properties of the variance, as well as single- variable calculus, derive (5.6). In other words, prove that $\alpha$ given by (5.6) does indeed minimize $\mathrm{Var}(\alpha X + (1-\alpha)Y)$. (5.6) is:

$$\alpha = \frac{\sigma_Y^2 - \sigma_{XY}}{\sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}}.$$

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**Answer** Simply note that

$$\begin{aligned}
\mathrm{Var}(\alpha X + (1-\alpha)Y) &= \alpha^2\sigma_X^2 + (1-\alpha)^2\sigma_Y^2 + 2\alpha(1-\alpha)\sigma_{XY} \\
&= A\alpha^2 + B\alpha + C \\
&= A\left(\alpha + \frac{B}{2A}\right)^2 + \tilde{C}
\end{aligned}$$

where $A = \sigma_X^2 + \sigma_Y^2 - 2\sigma_{XY}$ and $B = -2\sigma_Y^2 + 2\sigma_{XY}$, QED.

## Exercise 2

We will now derive the probability that a given observation is part of a bootstrap sample. Suppose that we obtain a bootstrap sample from a set of n observations.

(a) What is the probability that the first bootstrap observation is not the $j$th observation from the original sample? Justify your answer.

(b) What is the probability that the second bootstrap observation is not the $j$th observation from the original sample?

(c) Argue that the probability that the $j$th observation is not in the bootstrap sample is $\left(1 - \frac{1}{n}\right)^n$.

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**Answer**

(a) $1 - \frac{1}{n}$ since the bootstrap observation is uniformly randomly chosen.

(b) Its sampling with replacement, so $1 - \frac{1}{n}$.

(c) $\mathbb{P}(j\text{th obs not in boot sample})$ is the product as each bootstrap sample is independent.

## Exercise 2 cont. 1

(d) When $n = 5$, what is the probability that the $j$th observation is in the bootstrap sample?

(e) When $n = 100$, what is the probability that the $j$th observation is in the bootstrap sample?

(f) When $n = 10\,000$, what is the probability that the $j$th observation is in the bootstrap sample?

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**Answer**

(d) $\left(1 - \frac{1}{5}\right)^5 \approx 0.3276800000000002.$

(e) $\left(1 - \frac{1}{100}\right)^{100} \approx 0.3660323412732289.$

(f) $\left(1 - \frac{1}{10000}\right)^{10000} \approx 0.3678610464330241 4.$

## Exercise 2 cont. 2

(g) Create a plot that displays, for each integer value of $n$ from $1$ to $100\,000$, the probability that the $j$th observation is in the bootstrap sample. Comment on what you observe.

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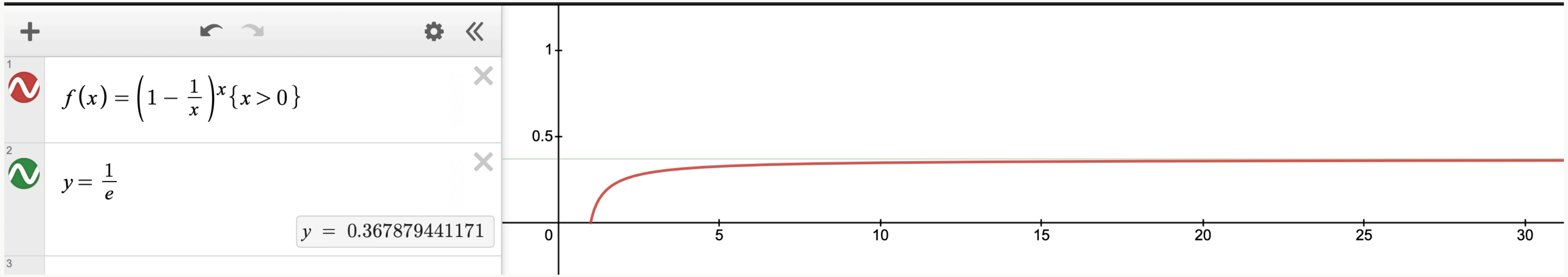
### Answer

(g) 


(see Desmos plot.) I observe convergence to $\frac{1}{e}$. Of course, this is because

$$\left(1 - \frac{1}{n}\right)^n = e^{n \log\left(1 - \frac{1}{n}\right)} = e^{n\left(-\frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)\right)} = e^{-1 - \frac{1}{2n} + O\left(\frac{1}{n^2}\right)} = \frac{1}{e} - \frac{1}{2en} + O\left(\frac{1}{n^2}\right).$$

## Exercise 2 cont. 3

(h) We will now investigate numerically the probability that a bootstrap sample of size $n = 100$ contains the $j$th observation. Here $j = 4$. We first create an array store with values that will subsequently be overwritten using the function `np.empty()`. We then repeatedly create bootstrap samples, and each time we record whether or not the fifth observation is contained in the bootstrap sample.

```
rng = np.random.default_rng(10)
store = np.empty(10000)
for i in range(10000):
    store[i] = np.sum(rng.choice(100, size=100, replace=True) == 4) > 0
np.mean(store)
```

(NB typo corrected) Comment on the results obtained.

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```

(NB typo corrected) Comment on the results obtained.

**Answer** We get `0.6362`. The bootstrap sample size is $10\,000$, so the true probability is $1 - 0.3678610464330241 4 = 0.6321389535669759$ which is consistent.

## Exercise 3

We now review $k$-fold cross-validation.

(a) Explain how $k$-fold cross-validation is implemented.

(b) What are the advantages and disadvantages of $k$-fold cross-validation relative to:

  i. The validation set approach?

  ii. LOOCV?

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### Answer

(a) Split the data randomly into $k$ bins. For each bin, train on the other $k-1$ bins and then test on the $k$th bin. Then report the average test error across bins $\mathrm{CV}_{(k)} = \frac{1}{k}\sum_{i=1}^{k} \mathrm{MSE}_i$.

(b)

  i. **Pros** Helps prevent fitting to the particular split

  **Cons** not repeatable (unless seed fixed), more compute needed

  ii. **Pros** usually less compute needed, less variance (for small $k$)

  **Cons** not repeatable, higher bias

## Exercise 4

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**Answer** We use the bootstrap. First we use our method to make some small number $n$ of predictions $Y_i$. Then we use these to create $N \gg n$ bootstrap samples $Y_i^* = (Y_{i1}^*, ..., Y_{in}^*)$. Finally, our estimate for the standard deviation is the average of the bootstrap standard deviations,

$$\sigma_Y \approx \frac{1}{N}\sum_{i=1}^{N}\sigma_{Y_i^*} = \frac{1}{N}\sum_{i=1}^{N}\sqrt{\frac{1}{n}\sum_{j=1}^{n}Y_{ij}^{*2} - \left(\frac{1}{n}\sum_{j=1}^{n}Y_{ij}^*\right)^2}.$$