Homework 0

Problem 1 (a)

Let $f(n) = 100n + \log(n)$ and $g(n) = n + \log(n)^2$. Which of the following holds?

- $f(n) = O(g(n))$
- $g(n) = O(f(n))$
- $f(n) = \Theta(g(n))$

Note the following two inequalities

- $\log(n) < n$ for $n > 1$,
- $\log(n)^2 < n$ for $n > 2$,

therefore $100n < f(n) < 101n$ and $n < g(n) < 2n$ giving us $g(n) = \Theta(n)$ and $f(n) = \Theta(n)$ and $f(n) = \Theta(g(n))$.

So the answer is 3.

Problem 1 (b)

Let $f(n) = n\log(n)$ and $g(n) = 10n\log(10n)$. Which of the following holds?

- $f(n) = O(g(n))$
- $g(n) = O(f(n))$
- $f(n) = \Theta(g(n))$

Note that from the rules of logarithms we have

$$g(n) = 10n\log(10n) = 10n(\log(10) + \log(n)) = 10n\log(10) + 10n\log(n).$$

Which for $n > 10$ gives us

$$n\log(n) < g(n) < 20n\log(n)$$

providing the bounds to give $g(n) = \theta(n\log(n)) = \theta(f(n))$.

So the answer is 3.

Problem 1 (c)

Let $f(n) = \sqrt{n}$ and $g(n) = \log(n)^3$. Which of the following holds?

- $f(n) = O(g(n))$
- $g(n) = O(f(n))$
- $f(n) = \Theta(g(n))$

As linear powers of n outgrow logs we have $g(n) = O(f(n))$.

So the answer is 2.

Problem 1 (d)

Let $f(n) = \sqrt{n}$ and $g(n) = 5^{\log_2(n)}$. Which of the following holds?

- $f(n) = O(g(n))$
- $g(n) = O(f(n))$
- $f(n) = \Theta(g(n))$

From base interchange of logs we have that

$$g(n) = 5^{\log_2(n)} = 5^{\log_2(5)\log_5(n)} = 5^{\log_2(5)} 5^{\log_5(n)} = 5^{\log_2(5)} n$$

which gives $f(n) = O(g(n))$.

So the answer is 1.

Problem 2

Show that $g(n) = 1 + a + a^2 + \ldots + a^n$ is $O(a^n)$ when $a > 1$ and $O(1)$ when $a < 1$. (Hint: You may try to prove $g(n) = \frac{a^{n+1}-1}{a-1}$ at first.)

Consider

$$(a-1)g(n) = a\sum_{i=0}^{n} a^i - \sum_{i=0}^{n} a^i = \sum_{i=0}^{n} a^{i+1} - a^i = a^{n+1} - 1$$

giving the hinted inequality

$$g(n) = \frac{a^{n+1} - 1}{a - 1}.$$

Let $a > 1$ then

$$g(n) = \frac{a^{n+1}}{a-1} - \frac{1}{a-1} \leq \frac{a^{n+1}}{a-1} = \frac{a}{a-1}a^n.$$

This gives $g(n) = O(a^n)$.

If $a = 1$ then $g(n) = n$ giving $g(n) = O(n)$.

Let $-1 < a < 1$ then note that

$$|a^{n+1} - 1| < |a - 1|$$

so we have that

$$|g(n)| < 1$$

giving the required $g(n) < 1$ and $g(n) = O(1)$.

If $a = -1$ then $g(n) = \frac{1+(-1)^n}{2}$ so $g(n) = O(1)$.

If $a < -1$ then $g(n)$ is divergent and we can't talk about run time.

> **Problem 3 (a)**
>
> For all parts, $G = (V, E)$ represents an undirected, simple graph (i.e.: no multiple edges and no loops).
>
> Denote by deg(v), the degree of vertex v, the number of edges incident to v. Check that
>
> $$\sum_{v \in V} \deg(v) = 2|E|$$

Lets prove this by induction on the number $|E|$ within a graph.

Suppose a graph has no edges. Therefore $\deg(v) = 0$ for all $v \in V$ as there are no edges to be incident to v. Thus

$$\sum_{v \in V} \deg(v) = 0 = 2|E|.$$

Suppose we have shown the statement true for all graphs where $|E| < k$ and suppose we have a graph with $|E| = k$. Pick any edge $e = (x, y)$ and remove it from the graph to get $G^* = (V, E^*)$. From the induction hypothesis we have

$$\sum_{v \in V} \deg_{G^*}(v) = 2(|E| - 1)$$

Where $\deg_{G^*}$ is the degree in G^*. Note that $\deg_G(v) = \deg_{G^*}(v)$ for all $v \in V \backslash \{x, y\}$. Whereas, $\deg_{G^*}(v) + 1 = \deg_G(v)$ for $v \in \{x, y\}$ (as it is incident to $e = (x, y)$ as well as all the edges in E^*). Therefore

$$
\begin{aligned}
\sum_{v \in V} \deg_G(v) &= \sum_{v \in V \setminus \{x,y\}} \deg_{G^*}(v) + \sum_{v \in \{x,y\}} (\deg_G(v) + 1) \\
&= 2 + \sum_{v \in V} \deg_{G^*}(v) \\
&= 2 + 2(|E| - 1) \\
&= 2|E|.
\end{aligned}
$$

This shows the inductive case and proves the statement.

Problem 3 (b)

Review the concepts of path, cycle, connectivity.

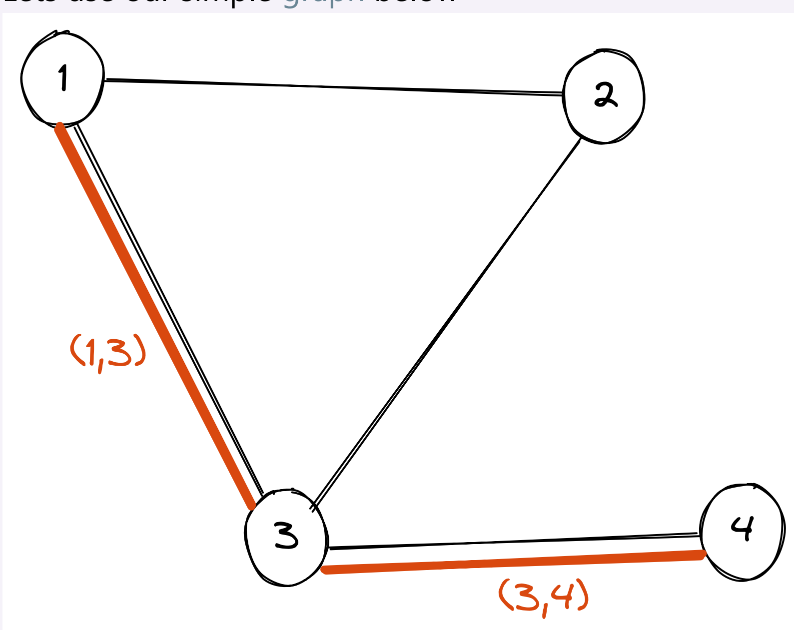
Path (graph)

Path (graph)

Given a graph $G = (V, E)$ a path is a sequence of edges $\{e_i = (x_i, y_i)\}_{i=1}^k$ such that $x_i = y_{i+1}$ for all $1 \leq i < k$. We say the *length* of the path is k.

Visual representation

Lets use our simple graph below



There is a path that goes $(1, 3), (3, 4)$ from vertex 1 to vertex 4. Note there is an implicit directionality when we say this.

The path graph

In graph theory we define the Path graph P_k to be a graph as a graph with

$$V = \{1, 2, \ldots, k\} \text{ and } E = \{(i, i+1) \mid 1 \leq i < k\}.$$

There is also an infinite analogy, both one sided and two sided.

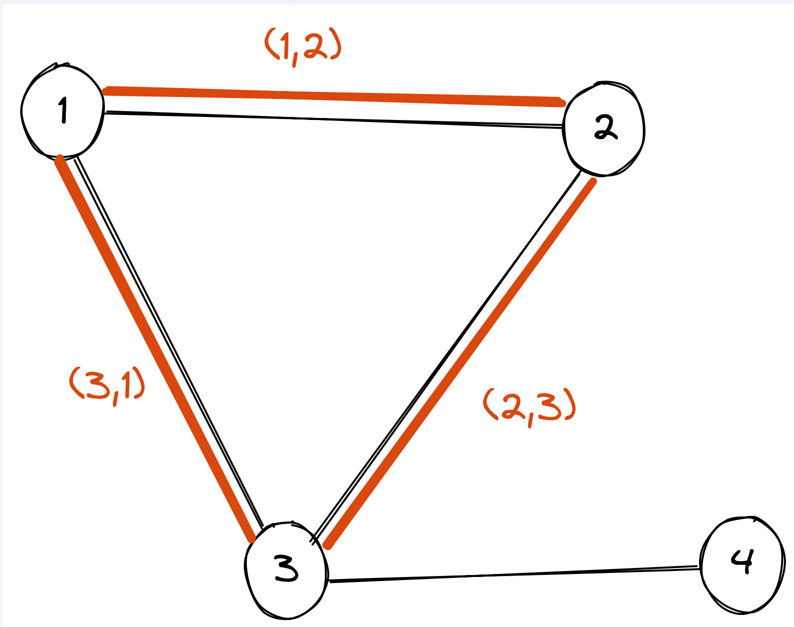
Cycle (graph)

Cycle (graph)

A *cycle* in a graph is path $\{e_i = (x_i, y_i)\}_{i=1}^k$ such that $x_1 = y_k$ and $e_i = e_j \Rightarrow i = j$.

Visual representation

Lets use our simple graph below



In this graph we have a cycle using the vertices $\{1, 2, 3\}$.

Connected (graph)

Connected (graph)

A graph $G = (V, E)$ is considered connected if for all vertices $x, y \in V$ there exists a path $\{v_i\}_{i=1}^k$ such that $v_1 = x$ and $v_k = y$.

Problem 3 (c)

G is said to be a tree if it is connected and have no cycles. Think why the following three conditions are equivalent:

1. G is a tree.
2. G is connected and $|E| = |V| - 1$.

3. G has no cycles and $|E| = |V| - 1$.

Proof of $(1) \Rightarrow (2)$.

We prove this by induction on the number of vertices.

Suppose $|V| = 1$ and set $V = \{v\}$ then if G had an edge it must be (v, v). However, we then have a cycle (v, v) so it would not be a tree. So $|E| = 0 = |V| - 1$.

Suppose the induction hypothesis is correct on all graphs with $|V| < k$ and let G be a graph with $|V| = k$.

As a finite tree that has more than one vertex must have at least two leaf vertices let $v \in V$ be such a leaf vertex with single edge $e \in E$. Remove v to form $G^* = (V \backslash \{v\}, E \backslash \{e\})$. Note that G^* has to be a tree as if a cycle existed in G^* it would exist in G and as G was connected so is G^* as no path would need to use e.

By the induction hypothesis $|E \backslash \{e\}| = |V \backslash \{v\}| - 1$. Giving

$$|E| = |E \backslash \{e\}| + |\{e\}| = |V \backslash \{v\}| - 1 + 1 = (|V \backslash \{v\}| + 1) - 1 = |V| - 1.$$

Thus proving our statement by induction.

Proof of $(2) \Rightarrow (3)$.

Suppose we have a graph G that is connected and $|E| = |V| - 1$. Though G has a cycle in it.

Take a minimal, in terms of $|V|$, counter example.

Note that from problem 3(a) we have

$$\sum_{v \in V} \deg(v) = 2|E| = 2|V| - 2$$

that says one vertex in V must have $\deg(v) = 1$ (it has to be at least 1 as it is connected).

As v can't be in the cycle (this would require degree 2), it must be outside. We can now remove this vertex and the remaining graph will still satisfy (2) with the same cycle. The new graph will be a smaller example and contradict its minimality.

This proves the claim.

Proof of $(3) \Rightarrow (1)$.

Note all we need to show is that G is connected.

Lets use proof be contradiction. Suppose G satisfies (3) but is not a tree, let G be a minimal such example.

If G has a vertex of degree 1, we can remove it and find a smaller counter example. Therefore G must not have any vertices of degree 1.

Take a connected component of G, as it has no vertices of degree 1 they must have vertices of degree 2 or more. Repeat the argument in a finite tree that has more than one vertex must have at least two leaf vertices, this shows that this connected component has a cycle it in. This contradicts (3) so no such graph exists.

This proves this claim and the equivalence.

> **Problem 3 (d)**
>
> A vertex is called a leaf if it has degree one. Show that every tree has at least two leaves. Think of an example of a tree with exactly two leaves.

> A finite tree that has more than one vertex must have at least two leaf vertices
>
> > **Proposition**
> >
> > A finite graph who has more than one vertex that is a tree must have at least two leaf vertex.
>
> Proof by contradiction.
>
> Suppose this is not the case, then if it has a single vertex of degree one start a path at this vertex. Otherwise pick a random vertex to start the path at. As the tree is connected and has more than one vertex the degree of the chosen vertex is at least 1. It then has an edge to another vertex or itself.
>
> If it has an edge to itself it is not a tree.
>
> Then it must have an edge to another vertex, include this edge in our path.
>
> Our path now contains 2 vertices.
>
> We will now repeat the following argument for proceeding vertices.
>
> The next vertex then must have another edge incident to it as it has degree at least 2. This edge is either incident to a vertex we have already visited or lead to a vertex we have not visited.
>
> If it is incident to a vertex we have already visited we have a cycle in our graph and this is not a tree.
>
> If it is incident to a vertex we have not already visited, we include this edge in our path and increase the set of vertices we have visited in our path by 1.

> As the graph is finite, the set has a maximum size and therefore we must visit a vertex we have already visited before, creating a cycle.
>
> Therefore there must be a vertex of degree 1.

The examples of graphs with exactly two leaf vertices are the path graphs.

Problem 4

For each example below, decide which functions are in CNF and find an assignment of the variables such that the corresponding function evaluates to true, if such assignment exists.

- $(x \lor y \lor z) \land (x \lor w) \land (y \lor \neg w)$
- $(\neg x \lor \neg y) \land (x) \land (z \lor \neg z)$
- $x \land (y \land (z \lor \neg w))$

For $(x \lor y \lor z) \land (x \lor w) \land (y \lor \neg w)$.

This is in CNF and $x = y = z = w = 1$ evaluates to true.

For $(\neg x \lor \neg y) \land (x) \land (z \lor \neg z)$.

This is in CNF and $x = z = 1$, and $y = 0$ evaluates to true.

For $x \land (y \land (z \lor \neg w))$.

This in not in CNF and $x = y = z = w = 1$ equates to true.