Question 1

It is known that the difference in two normal distributions is another normal distribution. We can calculate the expected value and variance as follows:

$$E(\bar{y}_1 - \bar{y}_2) = E\left(\sum_j^{n_1} y_{1j}/n_1\right) - E\left(\sum_k^{n_2} y_{2k}/n_2\right)$$

$$= \sum_j^{n_1} E(y_{1j})/n_1 - \sum_k^{n_2} E(y_{2k})/n_2$$

$$= \frac{n_1 \mu_1}{n_1} - \frac{n_2 \mu_2}{n_2} = \mu_1 - \mu_2$$

$$Var(\bar{y}_1 - \bar{y}_2) = Var(\bar{y}_1) + Var(\bar{y}_2) \because \text{Independence}$$

$$= Var\left(\sum_j^{n_1} y_{1j}/n_1\right) + Var\left(\sum_k^{n_2} y_{2k}/n_2\right)$$

$$= \frac{1}{n_2^2} V(y_{11} + y_{12} + \dots + y_{1n_1}) + \frac{1}{n_2^2} V(y_{21} + y_{22} + \dots + y_{n_2})$$

$$= \frac{1}{n_1^2}(V(y_{11}) + \dots + V(y_{1n_1})) + \frac{1}{n_2^2}(V(y_{21}) + \dots + V(y_{2n_2}))$$

$$\because \text{samples are i.i.d.} = \frac{n_1}{n_1^2}\sigma^2 + \frac{n_2}{n_2^2}\sigma^2$$

$$= \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)$$

Thus, $\bar{y}_1 - \bar{y}_2 \sim N\left(\mu_1 - \mu_2, \sigma^2\left(\frac{1}{n_1} + \frac{1}{n_2}\right)\right)$

Question 2

In [ ]:

```
library(tidyverse)
```

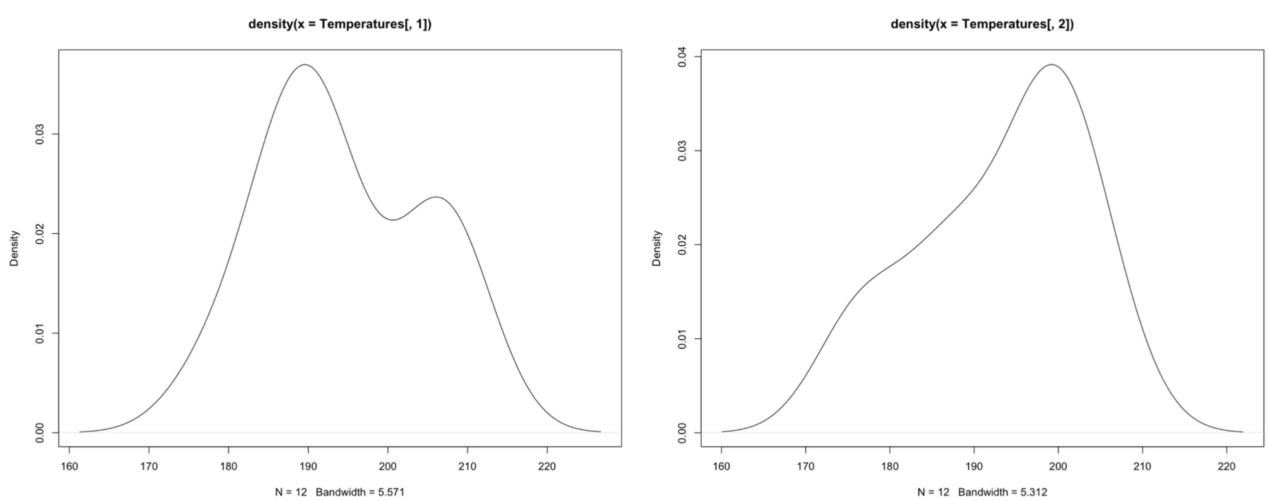
The deflection temperature under load for two different formulations of ABS plastic pipe is being studied. Two samples of 12 observations are prepared using each formulation (F1 and F2), and the deflection temperatures (in F) are reported below:

In [11]:

```
Temperatures <- matrix(
    c(206, 188, 205, 187, 193, 207, 185, 189,
      192, 210, 194, 178, 177, 197, 206, 201,
      176, 185, 200, 197, 198, 188, 189, 203),
      ncol = 2)
colnames(Temperatures) <- c("F1", "F2")
t(Temperatures)
n <- length(Temperatures[,1])
par(mfrow = c(1, 2))
options(repr.plot.width = 20, repr.plot.height = 8)
plot(density(Temperatures[,1]))
plot(density(Temperatures[,2]))
```

A matrix: 2 × 12 of type dbl

F1	206	188	205	187	193	207	185	189	192	210	194	178
F2	177	197	206	201	176	185	200	197	198	188	189	203

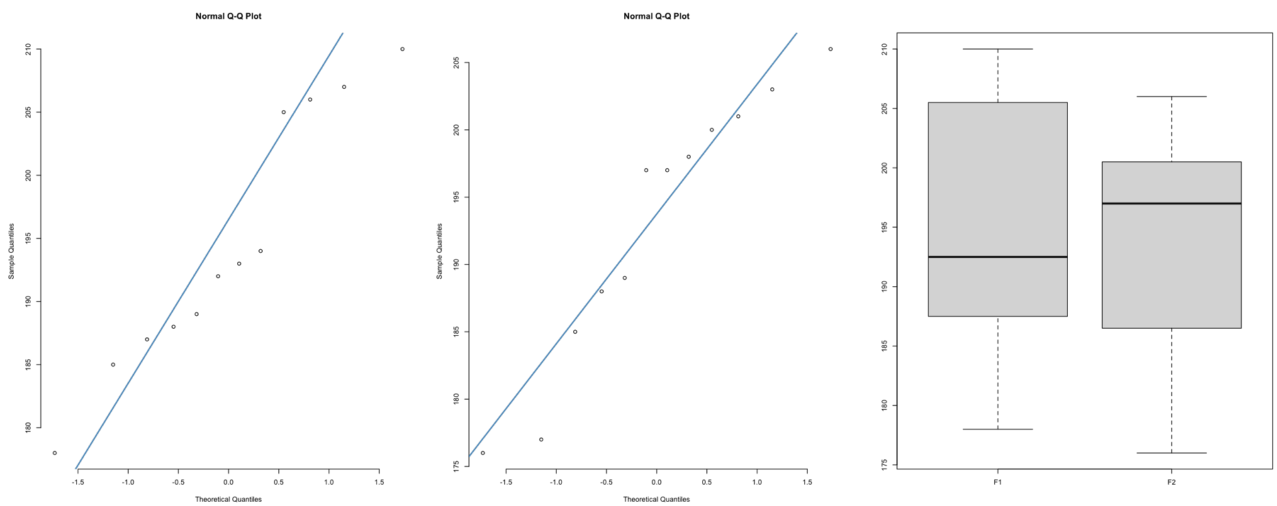


The goal is to investigate whether the data support the claim that the mean deflection temperature under load for formulation 2 exceeds that of formulation 1.

a) Use $\alpha = 0.05$ to perform a complete analysis in R, including normality check and the appropriate test. Use the rejection region method to test your hypothesis.

Before conducting our hypothesis test it is necessary to check for normality and to test if the variances are equal. We can get an idea of normality by creating QQ plots and we can compare variances using a box plot.

In [44]:
```
par(mfrow = c(1, 3))
options(repr.plot.width = 20, repr.plot.height = 8)
# QQ plot of first formulation.
qqnorm(Temperatures[,1], pch = 1, frame = FALSE)
qqline(Temperatures[,1], col = "steelblue", lwd = 2)
# QQ plot of second formulation.
qqnorm(Temperatures[,2], pch = 1, frame = FALSE)
qqline(Temperatures[,2], col = "steelblue", lwd = 2)
# Box Plot of formulations.
boxplot(Temperatures)
```



The QQplots follow the lines fairly closely, it appears that the normality assumptions are justified. The boxplot shows similar variances for the two distributions. We will verify by conducting a test.

In [45]:
```
var.test(Temperatures[,1], Temperatures[,2])
```

```
        F test to compare two variances

data:  Temperatures[, 1] and Temperatures[, 2]
F = 1.046, num df = 11, denom df = 11, p-value = 0.9419
alternative hypothesis: true ratio of variances is not equal to 1
95 percent confidence interval:
 0.3011181 3.6334674
sample estimates:
ratio of variances
          1.045994
```

The F-statistic of 1.046 gives a p-value of 0.9419, suggesting there is no evidence that the variances are different. We will assume equal variances under our hypothesis test.

Hypothesis Test Using Rejection Rejoin Method

We are interested in whether the deflection temperature of formulation 2 exceeds formulation 1, thus we can state the null hypothesis as:

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 < \mu_2$$

where μ_1 and μ_2 are the means of the two formulations respectively. Thus we have a one-sided, two sample t-test, where we assume equal variances because of our work above. We can conduct the test in R as follows:

In [6]:
```
test <- t.test(Temperatures[,1], Temperatures[,2], alternative = "less", 
test
```

```
        Two Sample t-test

data:  Temperatures[, 1] and Temperatures[, 2]
t = 0.34483, df = 22, p-value = 0.6333
alternative hypothesis: true difference in means is less than 0
95 percent confidence interval:
     -Inf 8.471217
sample estimates:
mean of x mean of y 
 194.5000  193.0833 
```

The t-statistic of 0.34483 gives a p-value of 0.6333. We do not have sufficient evidence to reject the null hypothesis in favour of the null.

b) Does the confidence interval support your conclusion on part a? Justify.

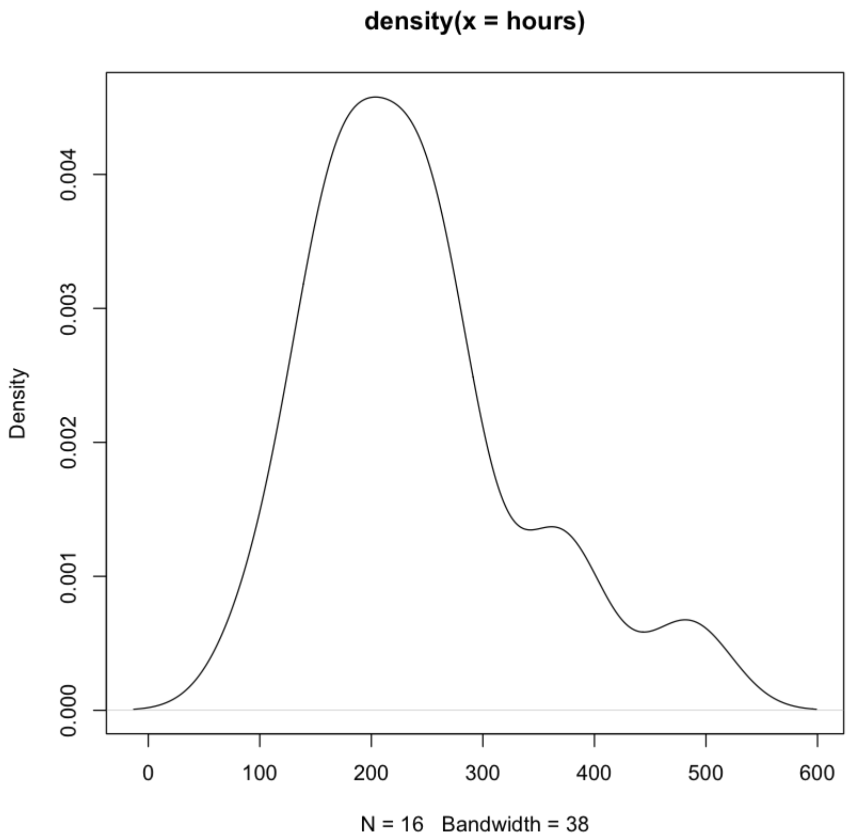
The output of our test gives a 95 percent confidence interval of $(-\infty, 8.4712)$, which contains the value 0. This supports our conclusion above, as we would expect a confidence interval that fails to reject a difference in the two samples to contain the value 0.

Question 3

The time to repair an electronic intrsument is a normally distributed random variable measured in hours. The repair time (in hours) for 16 such intruments chosen at random are as follows:

In [2]:

```
hours <- c(159, 224, 222, 149, 280, 379, 362, 260, 101, 179, 168, 485, 21
n <- length(hours)
plot(density(hours))
```



a) You wish to know if the mean repair time exceeds 225 hours. Set up appropriate hypotheses for investigating this issue.

An approriate null and alternative hypthesis for the statement above can be stated as

$$H_0 : \mu = 225$$

$$H_1 : \mu > 225$$

where μ is the average number of hours for repair.

b) Using R and assuming $\alpha = 0.05$ test the hypotheses you formulated in part (a). What are your conclusions?

We have a one sided test and are estimating the variance from the data. A one sided t-test is appropriate. We can calculate this is R as follows:

In [3]:
```
test <- t.test(hours, mu = 225, alternative = "greater")
test
```

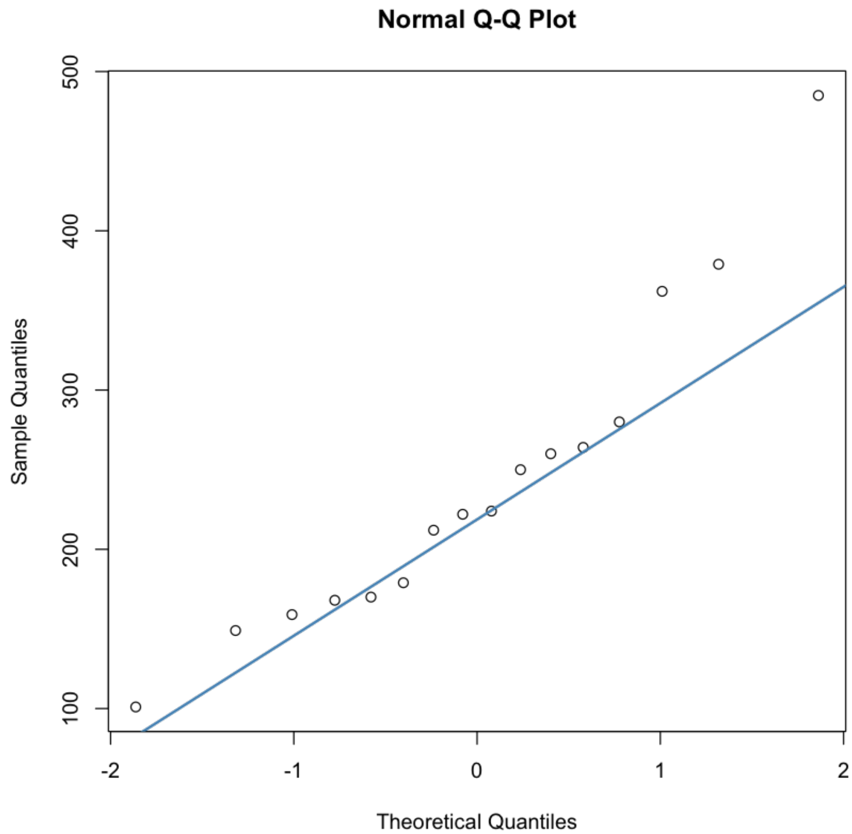
```
	One Sample t-test

data:  hours
t = 0.66852, df = 15, p-value = 0.257
alternative hypothesis: true mean is greater than 225
95 percent confidence interval:
 198.2321      Inf
sample estimates:
mean of x
    241.5
```

Our test has calcualted a p-value of 0.257, which is greater than 0.05. We fail to reject the null hypothesis in favour of the alternative.

c) Is the normality assumption satisfied? Justify.

We can test the normality assumption by plotting a normal Q-Q plot and tracking how closely the data falls onto the line.

In [5]:
```
qqnorm(hours)
qqline(hours, col = "steelblue", lwd =2)
```



The data is roughly normal, with the exception that the higher values at the top are slightly off. The normality assumption seems to be justified.

d) By hand, construct a 95 percent confidence interval on mean repair time to test your hypothesis. in (a). Shour your work.

We are intersted in the C.I.

$$P(\mu \geq L) = 1 - \alpha = 1 - 0.05 = 0.95$$

where L is a lower bound value for the stated α. We can manipulate the formula for a t-stat to get L like so

$$t_{\alpha,n-1} = \frac{\bar{y} - L}{\sqrt{S^2/n}} \implies \bar{y} - t_{\alpha,n-1} \cdot \sqrt{S^2/n} = L$$

In [10]:
```
tcalculated <- qt(0.05, n-1, lower.tail = FALSE)
ybar <- mean(hours)
var <- var(hours)
tcalculated
ybar
var
```

1.75305035569257

241.5

9746.8

Plugging the calcualted values in to get the confidence interval we have

In [11]:
```
L = ybar - tcalculated * sqrt( var / n)
L
```

198.232138865275

Thus the 95% CI for our test hypothesis is

$$(198.2321, \infty)$$

The value 225 is included in this test which matches our previous work in failing to reject the null hypothesis in favour of the alternative.

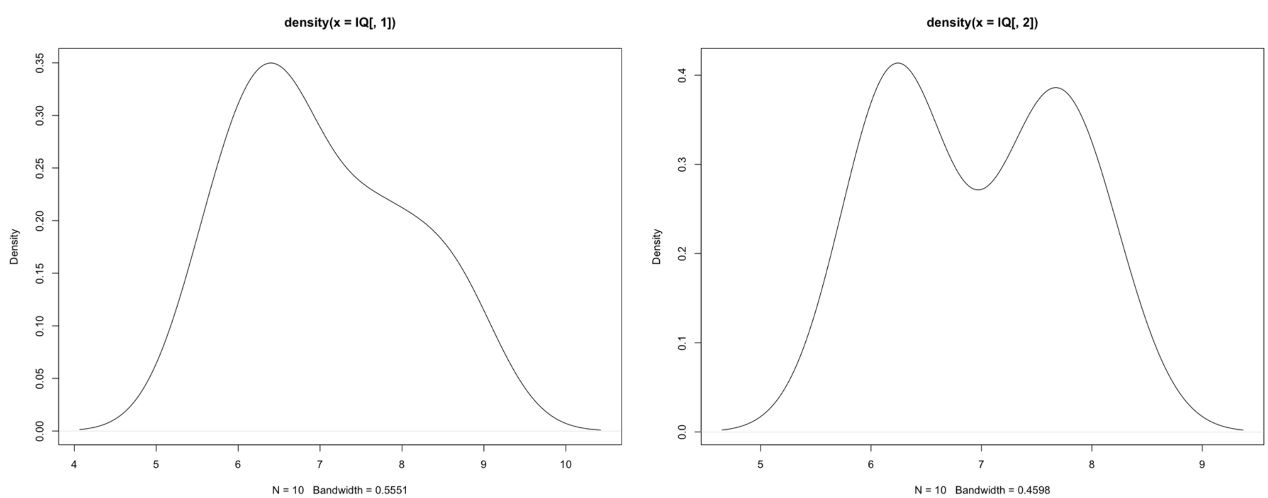
Question 4

An articale in the journal of Neurology (1998, vol. 50, pp.1246-1252) observed that monozygotic twins share numerous phyisical, psychological and pathological traits. The investigators measured an intelligence score of 10 pairs of twins. The data are obtained as follows:

In [32]:
```
IQ <- matrix(
    c(5.73,5.80,8.42,6.84,6.43,8.76,6.32,7.62,6.59,7.67,
      6.08,6.22,7.99,7.44,6.48,7.99,6.32,7.60,6.03,7.52),
    ncol = 2 )
colnames(IQ) <- c("Birth Order:1", "Birth Order:2")
n <- length(IQ[,1])
t(IQ)
par(mfrow = c(1, 2))
options(repr.plot.width = 20, repr.plot.height = 8)
plot(density(IQ[,1]))
plot(density(IQ[,2]))
```

A matrix: 2 × 10 of type dbl

Birth Order:1	5.73	5.80	8.42	6.84	6.43	8.76	6.32	7.62	6.59	7.67
Birth Order:2	6.08	6.22	7.99	7.44	6.48	7.99	6.32	7.60	6.03	7.52



a) By hand, find a 95% confidence interval on the difference in the mean score. Based on this interval, is there any evidence that mean score depdends on birth order?

This is a paired test because the two samples are not independent of one another. We start by calcualting a vector $\mathbf{d}$ which represents the difference between the two samples.

In [8]:
```
d = as.matrix(IQ[,1] - IQ[,2])
```

We are interested in the difference in intelligence between monozygotic twins based off of their birth order. The samples are related in this case so we will set up a paired t-test hypothesis on the difference between the two samples $d_i = y_{1i} - y_{2i}$. We calculate the difference between the samples as:

$$H_0 : \mu_d = 0$$

$$H_1 : \mu_d \neq 0$$

where $\mu_d = \mu_1 - \mu_2$.

The proper test statistic is

$$t_0 = \bar{d}/\sqrt{(S_d^2/n)}$$

where $\bar{d} = \sum_{i=1}^{n}(y_{1i} - y_{2i})/n$ and $S_d^2 = \sum_{i=1}^{n} \frac{(d_i - \bar{d})^2}{n-1}$.

Thus, we can construct out Confidence interval like so

$$\bar{y}_d - t_{\alpha/2,n-1} \cdot \sqrt{S^2/n} \leq \mu_d \leq \bar{y}_d + t_{\alpha/2,n-1} \cdot \sqrt{S^2/n}$$

In [29]:
```
# Preliminary values
dbar <- 0
S2d <- 0
for (i in 1:n) { dbar = dbar + d[i,1] / n}
for (i in 1:n) { S2d = S2d + (d[i,1] - dbar)^2 / (n-1)}
# Test Statistic
t <- qt(0.025, n-1, lower.tail = TRUE)
# Upper and Lower Bounds
Lower <- dbar - t * sqrt(S2d / n)
Upper <- dbar + t * sqrt(S2d / n)
Lower
Upper
```

0.366414804856497
-0.264414804856497

The calculated 95% confidence interval is

$$(-0.2644 \leq \mu_d \leq 0.3664)$$

The confidence interval contains 0, thus we have not found evidence that $\mu_d \neq 0$, we fail to reject the null hypothesis in favour of the alternative.

b) Using R, test the hypothesis that the mean score does not depend on birth order considering $\alpha = 0.05$.

We can calculate the test in R using the following.

In [30]:
```
# Paired t-test
test <- t.test(IQ[,1], IQ[,2], alternative = "two.sided", paired = TRUE, 
test
```

```
        Paired t-test

data:  IQ[, 1] and IQ[, 2]
t = 0.36577, df = 9, p-value = 0.723
alternative hypothesis: true mean difference is not equal to 0
95 percent confidence interval:
 -0.2644148  0.3664148
sample estimates:
mean difference
          0.051
```

The test matches are work above. We have a calcualted p-vlaue of 0.723 which is greater than 0.05. Thus, like above, we fail to reject the null hypothesis in favour of the alternative.

Question 5

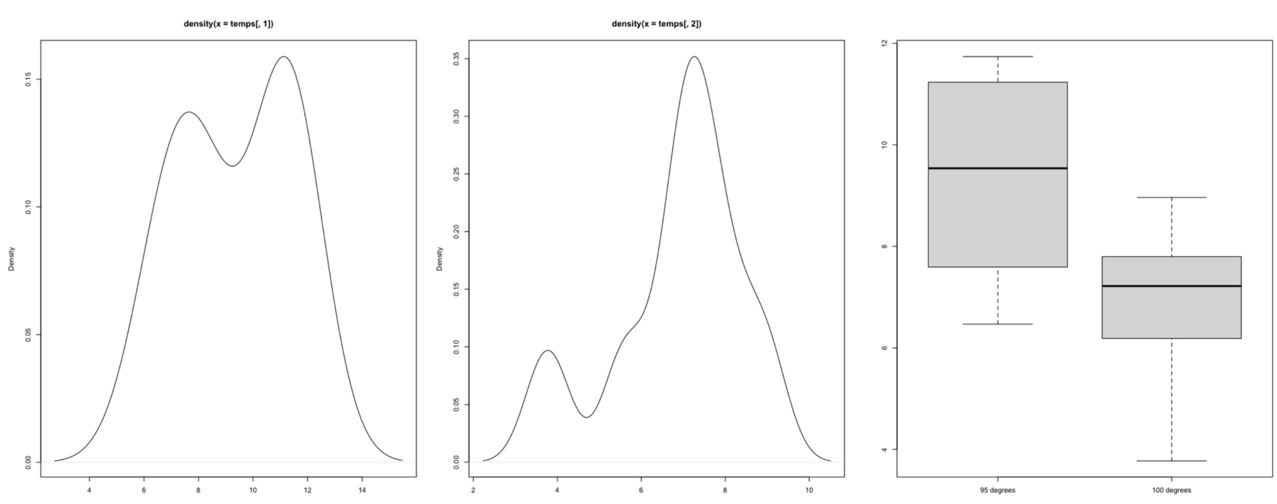
Photoresist is a light-sensitive material applied to semiconductor wafers so that the circuit patter can be imaged on to the wafer. After application, the coated wafers are baked to remove the solvent in the photoresist mixture and to harden the resist. Here are measurements of photoresist thinkness (in $K\dot{A}$) for eight wafers baked at two different temperatures. Assume that all of the 16 runs were made in random order. Note: a wafer cannot be baked twice.

In [11]:

```
temps <- matrix(c(
    11.176, 07.089, 08.097, 11.739, 11.291, 10.759, 06.467, 08.315,
    05.623, 06.748, 07.461, 07.015, 08.133, 07.418, 03.772, 08.963
), ncol = 2)
colnames(temps) <- (c("95 degrees", "100 degrees"))
t(temps)
par(mfrow = c(1, 3))
options(repr.plot.width = 20, repr.plot.height = 8)
plot(density(temps[,1]))
plot(density(temps[,2]))
boxplot(temps)
```

A matrix: 2 × 8 of type dbl

95 degrees	11.176	7.089	8.097	11.739	11.291	10.759	6.467	8.315
100 degrees	5.623	6.748	7.461	7.015	8.133	7.418	3.772	8.963



(a) Is there evidence to support the claim that the higher baking temperature results in wafers with a lower mean photoresis thickness? Use $\alpha = 0.05$ and justify your answer

First we should test if the variances are different so we know what assumptions we can make when we do our hypothesis test.

In [5]:

```
var.test(temps[,1], temps[,2])
```

```
	F test to compare two variances

data:  temps[, 1] and temps[, 2]
F = 1.7326, num df = 7, denom df = 7, p-value = 0.4855
alternative hypothesis: true ratio of variances is not equal to 1
95 percent confidence interval:
 0.3468649 8.6539742
sample estimates:
ratio of variances 
          1.732559 
```

With a p-value of 0.4855 we do not have sufficient evidence that the variances between the samples are different. We will assume the populations have equal variances in our hypothesis test.

We can attempt to find evidence that the higher baking temperature results in wafers with a lower mean photoresis thickness by setting up the following hypothesis test:

$$\mu_1 = \mu_2$$

$$\mu_1 > \mu_2$$

We can test this hypothesis in R like so

In [8]:

```
test <- t.test(temps[,1], temps[,2], alternative = "greater", var.equal = 
test
```

```
	Two Sample t-test

data:  temps[, 1] and temps[, 2]
t = 2.6549, df = 14, p-value = 0.009424
alternative hypothesis: true difference in means is greater than 0
95 percent confidence interval:
 0.8330468       Inf
sample estimates:
mean of x mean of y 
 9.366625  6.891625 
```

The calculated p-value is $0.0094 < \alpha = 0.05$, thus we can conclude that there is strong evidence to reject the null hypothesis in favour of the alternative.

(b) Find a 95% confidence interval on the difference in means. Provide a practical interpretation of this interval.

The output above gives a 95% confidence interval of $(0.8330468, \infty)$. This matches our prior work using the p-value approach to reject the null hypothesis in favour of the alternative. If the range of the confidence interval had included 0 it wouldn't make sense to reject the null hypothesis. A practical interpretation of the interval is that if you were to repeatedly draw samples from the given populations, the calculated bounds of the confidence interval will include the true difference in the mean values about 95% of the time.

Question 6

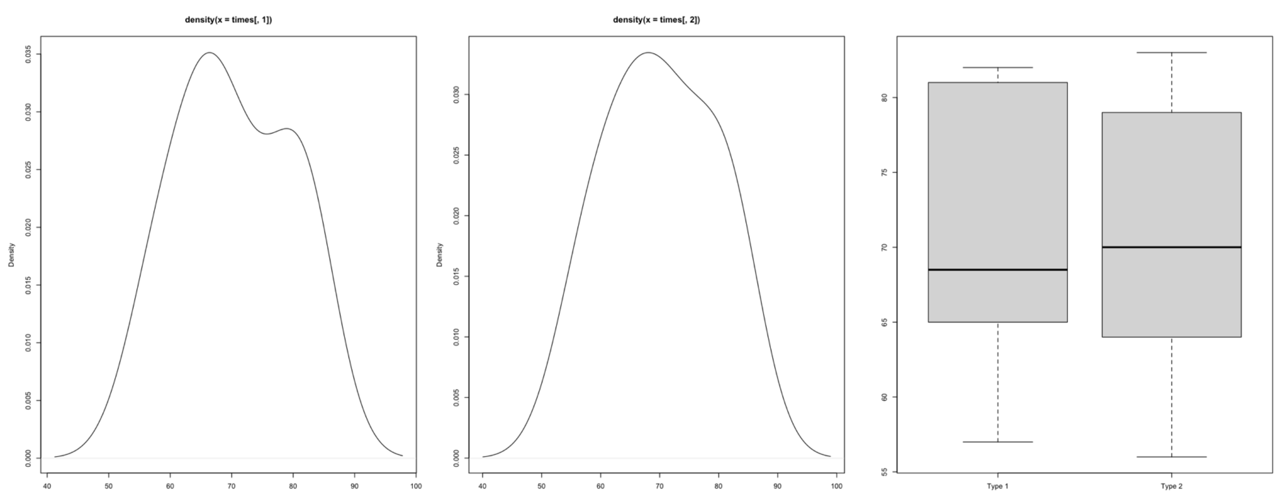
The following are the burning times (in minutes) of chemical flares of two different formulations. The design engineers are interested in both the means and variance of the burning times.

In [5]:

```r
times <- matrix(c(
    65, 81, 57, 66, 82, 82, 67, 59, 75, 70,
    64, 71, 83, 59, 65, 56, 69, 74, 82, 79
), ncol = 2)
colnames(times) <- c("Type 1", "Type 2")
t(times)
par(mfrow = c(1, 3))
options(repr.plot.width = 20, repr.plot.height = 8)
plot(density(times[,1]))
plot(density(times[,2]))
boxplot(times)
```

A matrix: 2 × 10 of type dbl

Type 1	65	81	57	66	82	82	67	59	75	70
Type 2	64	71	83	59	65	56	69	74	82	79



(a) Test the hypotheses that the two variances are equal. Use $\alpha = 0.05$.

Setting up the null and alternative hypothesis we have

$$H_0 : \sigma_1^2 = \sigma_2^2$$

$$H_1 : \sigma_2^2 \neq \sigma_2^2$$

We can test this hypothesis using R with the following.

In [3]:
```
var.test(times[,1], times[,2])
```

```
	F test to compare two variances

data:  times[, 1] and times[, 2]
F = 0.97822, num df = 9, denom df = 9, p-value = 0.9744
alternative hypothesis: true ratio of variances is not equal to 1
95 percent confidence interval:
 0.2429752 3.9382952
sample estimates:
ratio of variances
         0.9782168
```

The p-value is $0.9744 > \alpha = 0.05$, thus we do not reject the null hypothesis in favour of the alternative that the two population variances are different.

(b) Using the results of (a), test the hypotheses that the mean burning times are equal. Use $\alpha = 0.01$. What is the p-value for this test?

Assuming equal variances because of our results in (a), we set up the following null and alternative hypothesis:

$$H_0 : \mu_1 = \mu_2$$

$$H_1 : \mu_1 \neq \mu_2$$

We can test this in R using the following

In [4]:
```
test <- t.test(times[,1], times[,2], conf.level = 0.99)
test
```

```
	Welch Two Sample t-test

data:  times[, 1] and times[, 2]
t = 0.048008, df = 17.998, p-value = 0.9622
alternative hypothesis: true difference in means is not equal to 0
99 percent confidence interval:
 -11.79175  12.19175
sample estimates:
mean of x mean of y
     70.4      70.2
```

The cacluated p-value is $0.9622 > \alpha = 0.01$, so we don't not have sufficient evidence to reject the null hypothesis in favour of the alternative at the $\alpha = 0.01$ level.