Introduction to Economic Growth: Why some countries are poorer than others?

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Lecture 2: The Solow Growth Model

Three big questions that the Solow model helps to answer

Question 1 What can explain striking patterns of stable growth in developed economies over the past 70-100 years?

Question 2 What is behind big differences in living standards across countries: capital accumulation? population growth? technological progress?

Question 3 Will there be convergence in standards of living (i.e., will poorer countries catch up with the richer ones)?

Today's lecture

1. Kaldor Facts, and How we use models to explain facts
   - Kaldor Facts
   - Economic Models
2. The Solow model: theory
   - Basic structure
   - Steady state w/o population and productivity growth
   - Adding population growth
   - Adding productivity growth
3. The Solow model meets the Data

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'Stylized' Kaldor Facts of Growth

1. Output per worker grows at a roughly constant rate over long periods of time Illustration

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Can we make sense of these regularities? Can we explain why we see them? We need a model!

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Economic models...

Solow growth model is the first economic model we cover in this course. But why do we use 'models' anyway?

- to simplify very complex reality, and to keep only necessary elements and relations. What is 'necessary' depends on questions we ask...
  Solar System model
- to build intuition about how various (socioeconomic) variables interact with each other
- to form testable predictions about the real world
- to explain puzzles and surprising regularities we uncover in the data
- ultimately, to expand our understanding of how the world works

The simplest model: market demand and supply

Market demand is given by $Q_d = D(p)$ where D is decreasing in p - quantity demanded Q_d is decreasing in price p

Market supply is given by $Q_s = S(p)$ where S is increasing in p - quantity supplied Q_s is increasing in price p

Market equilibrium is defined as Q^* and p^*, such that $Q^* = D(p^*) = S(p^*)$. Example:

$$\begin{cases} D(p) = 100 \cdot a - 10p \\ S(p) = 15p \end{cases} \tag{1}$$

Together, these demand and supply equations give us $Q^* = 60a, p^* = 4a$ as an equilibrium in this model. We have ***a testable prediction***: an increase in demand (an increase in the value of parameter a) will increase equilibrium price and quantity.

Exogenous and Endogenous Variables

- No model in the world (except the world itself, if we can call it a model of itself) can neither should explain all variables within itself.
- Thus, if we want to better understand a phenomenon, or make sense of relationship between variables that we observe in the data, some of the variables will be set as 'exogenous' - determined outside of the model.
- Those variables, the values of which are determined within the model are called 'endogenous'.
- What the analysis of a model often comes down to is tracing the effect of an exogenous variable (say, a in the previous example) on the equilibrium values of endogenous variables (say, Q^* in the previous example).

Endo(Exo)genous variables: illustration

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The Solow Growth Model: Structure

The model consists of several parts:

- production side (how output is produced and distributed to agents as incomes)
- consumption side (how incomes are spent - on what needs, and according to what rules)
- dynamics of capital accumulation (how does capital increase/decrease from period to period)
- equilibrium (what conditions define the equilibrium in the model)

We will look at these components one by one, then together, and then we'll see what our model can explain, and what testable predictions it has.

Production Side

Final output in period t is produced according to a production function $Y_t = F(K_t, L_t, A_t)$. L_t - total labor input; K_t - capital stock; A_t - productivity or 'efficiency' (sometimes called Total Factor Productivity or TFP; sometimes attributed to 'ideas' or 'technologies'). For simplicity, we will assume the 'Cobb-Douglas' production function:

$$Y_t = A_t \cdot K_t^{\alpha} \cdot L_t^{1-\alpha} \tag{2}$$

We assume that $0 < \alpha < 1$. In this case, we see that:

- Output increases in both capital K_t and L_t
- As K_t grows, the amount of increase in Y_t from each subsequent unit of K_t is ***decreasing*** (decreasing marginal product of capital)
- As L_t grows, the amount of increase in Y_t from each subsequent unit of L_t is ***decreasing*** (decreasing marginal product of labor)

Production Side: Illustration

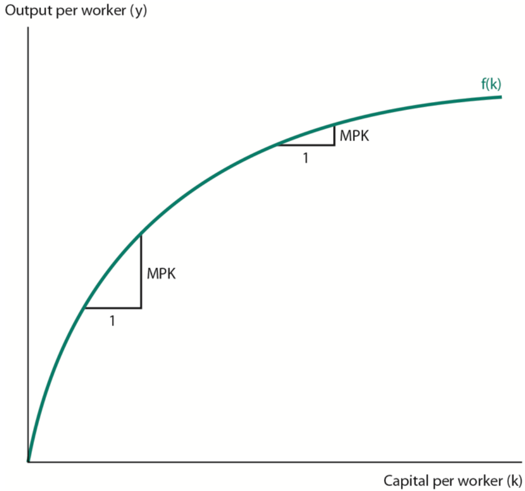

Figure 1: Decreasing marginal product of capital. Source: David Weil, "Economic Growth" (3rd ed.)

Production Side: Per capita terms

Since we are interested in standards of living, not just total GDP, we express everything in per capita terms:

$$Y_t / L_t = A_t \cdot K_t^{\alpha} \cdot L_t^{1-\alpha} / L_t \tag{3}$$

This transforms into

$$y_t = Y_t / L_t = A_t \cdot (K_t / L_t)^{\alpha} = A_t \cdot k_t^{\alpha} \tag{4}$$

Here, $y_t = Y_t / L_t$ is GDP per capita (or incomes per capita), and $k_t = K_t / L_t$ is capital per capita (or capital per worker, since we have no unemployment in this model).

Consumption Side

All output produced is sold on the market for a price = 1 [1]. All this revenue is then distributed to individuals who own capital and labor as their incomes [2].

For simplicity, all agents are identical (i.e., representative agents).

Consumption is very simple: of all income, agents save a fraction of s, and consume the remaining fraction $1 - s$. Thus:

$$C_t = (1 - s) \cdot Y_t \tag{5}$$

$$S_t = s \cdot Y_t \tag{6}$$

All savings are invested on the financial market: $S_t = I_t$. In per capita terms, $i_t = I_t / L_t = s \cdot y_t$

[1] As we have only a single final good, notion of price is redundant here.

[2] Basic macroeconomic identity: GDP = Total Income

Example: Illustration of output and investment functions

Let's consider an example: $Y_t = 2 \cdot K_t^{1/2} \cdot L_t^{1/2}$.

- In this case, do we have our three conditions for production function satisfied?
- Find and graph (i) output per capita function, (ii) investment per capita function

Dynamics of capital accumulation

Capital stock in the following period consists of what is left from the previous period's capital (due to depreciation), plus new capital added from investments:

$$K_{t+1} = K_t - \delta \cdot K_t + I_t \tag{7}$$

δ is the rate of depreciation. Since we know that investment I_t comes from savings, we can update this equation like that:

$$K_{t+1} = K_t - \delta \cdot K_t + s \cdot Y_t \tag{8}$$

Dynamics of capital accumulation: Per capita terms

Assuming (for now) that population does not grow over time, i.e., $L_{t+1} = L_t = L$, it's easy to express capital dynamics in per capita terms as well:

$$k_{t+1} = k_t - \delta \cdot k_t + s \cdot A_t \cdot k_t^{\alpha} \tag{9}$$

Finally, if we assume (again, for now) that productivity does not change over time, meaning that $A_t = A_{t+1} = A$, then equation (9) fully describes the dynamics of our model!

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Dynamic equilibrium: steady state of the model

Definition: Steady state of the model

A ***steady state*** is reached when there is no further change in the dynamic variables from period to period, unless a system is externally 'shocked'. In our case, the steady state is given by:

$$\Delta k_t = k_{t+1} - k_t = 0 \tag{10}$$

Example: water dynamics

Using (9) we can express the steady state condition as:

$$\Delta k_t = -\delta \cdot k_t + s \cdot A \cdot k_t^{\alpha} = 0 \tag{11}$$

Or alternatively, as the equality of capital inflow (investment per capita) and capital outflow (depreciation per capita):

$$\delta \cdot k_t = s \cdot A \cdot k_t^{\alpha} \tag{12}$$

Dynamic equilibrium: steady state of the model

We can solve equation (12) for the value of k_t. First divide by k_t on both sides:

$$\delta \cdot k_t / k_t = s \cdot A \cdot k_t^{\alpha} / k_t \tag{13}$$

Which gives us:

$$\delta = s \cdot A \cdot k_t^{\alpha - 1} \tag{14}$$

And finally, denoting the st.state capital by k^*:

$$k^* = \left(\frac{s \cdot A}{\delta} \right)^{1/1-\alpha} \tag{15}$$

Steady state of the model: illustration

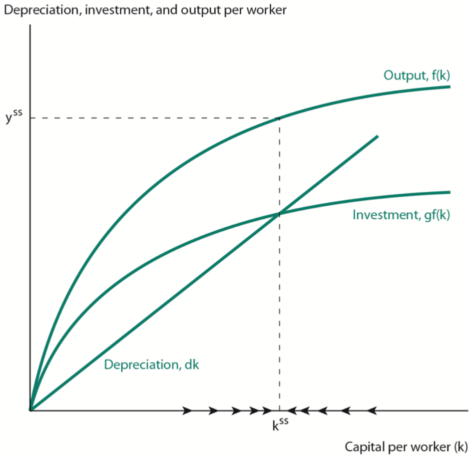

Figure 2: Steady state in a model w/o population growth and productivity growth. Source: David Weil, "Economic Growth" (3rd ed.)

Predictions and explanations the model gives so far

1. Steady state income per capita: increases in s, and increases in A (also decreases in δ, and increases in α but that's not so interesting)
2. Growth rate at the steady state is zero (but we haven't introduced dynamics of productivity just yet!), unaffected by any of the parameters
3. However! Growth rate in transition to steady state (so-called 'catch-up' growth) increases in s, increases in A, and decreases in k (law of diminishing returns)

Think of the examples of post-war Germany and Japan as illustrations for the last prediction Example: post-war recoveries, Germany and Japan

There will be an exercise in your homework assignment where you'll prove result #3

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Population growth in the Solow model

Now that we understand the basic dynamics of the Solow model, it will be easy to incorporate missing elements:

- population growth (before we had $L_{t+1} = L_t = L$)
- productivity growth (before we had $A_{t+1} = A_t = A$)

Starting with population growth, let the rate of population growth be denoted by n, i.e., $L_{t+1} = (1 + n) \cdot L_t$

The only thing that changes from the previous version of the model is the dynamics of capital per capita (or per worker).

Dynamics of capital accumulation with population growth: Per capita terms

Dynamics of the aggregate capital accumulation is as before:

$$K_{t+1} = K_t - \delta \cdot K_t + s \cdot Y_t \tag{16}$$

We want to get to capital per capita, i.e., K/L in each period, so divide the equation above by $L_{t+1} = L_t \cdot (1+n)$ on both sides:

$$K_{t+1}/L_{t+1} = \frac{K_t - \delta \cdot K_t + s \cdot Y_t}{L_t \cdot (1+n)} \tag{17}$$

Simplifying, we get:

$$k_{t+1} = \frac{k_t - \delta \cdot k_t + s \cdot A \cdot k_t^{\alpha}}{1+n} \tag{18}$$

Steady state for the model with population growth

Recall the definition of the steady state: such level of k^*, for which $\Delta k_t = 0$ (or, in other words, $k_{t+1} = k_t = k^*$).

Apply this definition to the new dynamics of capital per worker we got in (18), and derive the new value of the steady state capital per worker k^*.

What do you get?

Additional prediction from this version of the model

If you got the formula for the steady state correctly, you see that higher population growth rate decreases output per capita!

Moreover, it is possible to show that higher population growth rate lowers GDP per capita growth rate in transition to the steady state ('catch-up' growth).

Additional prediction from this version of the model: increase in population growth rate

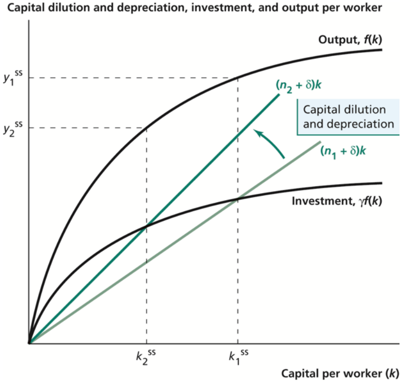

Figure 3: Increase in population growth rate. Source: David Weil, "Economic Growth" (3rd ed.)

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Productivity growth in the Solow model

Now we add our final piece to the puzzle: productivity (TFP) growth. So far, we had $A_{t+1} = A_t = A$, i.e., constant productivity parameter.

Let the rate of productivity growth be denoted by g: $A_{t+1} = (1 + g) \cdot A_t$

Formally, we would have to adjust our dynamics of capital per worker once again... But let's just assume we did :) [3]. What is the effect of a continuing increase in productivity?

- First, it will be increasing incomes directly, as the same amount of capital and labor now produce more output.
- Second, as incomes per capita grow, it will also increase savings, and hence, investment per capita.
- Thus, while with $g = 0$, capital and incomes per worker converged to a steady state with zero growth rates, this is no longer the case with $g > 0$.

[3] It's a bit more tedious, but, for curious folks: check out the Mathematical Appendix to Chapter 8 in David Weil's "Economic Growth" (3rd ed.).

Productivity growth in the Solow model

- Recall that before, we had $y_t = A \cdot k_t^{\alpha}$. We have shown that when productivity A is constant, $k_t = \frac{K_t}{L_t}$ converges to a steady-state.
- Thus, the growth rate of y_t denoted by g_y could have been calculated as $g_y = g_A + \alpha \cdot g_k$, which equaled zero in a steady state, because $g_A = 0$ and $g_k = 0$ in a steady state.
- With TFP growth, because g_A now equals $g > 0$, we have $g_y = g_A + \alpha \cdot g_k > 0$, due to positive g_A, and because k_t will be increasing further after each advance in A_t.
- Thus, with TFP growth, Solow model predicts that the growth rate of output per capita is proportional to the growth rate of TFP.
- Note an important difference with an increase in s: while A_t can grow without bound (at least in the Solow model), savings rate cannot exceed 1, and cannot grow forever to maintain positive growth rates.

Productivity growth in the Solow model: illustration

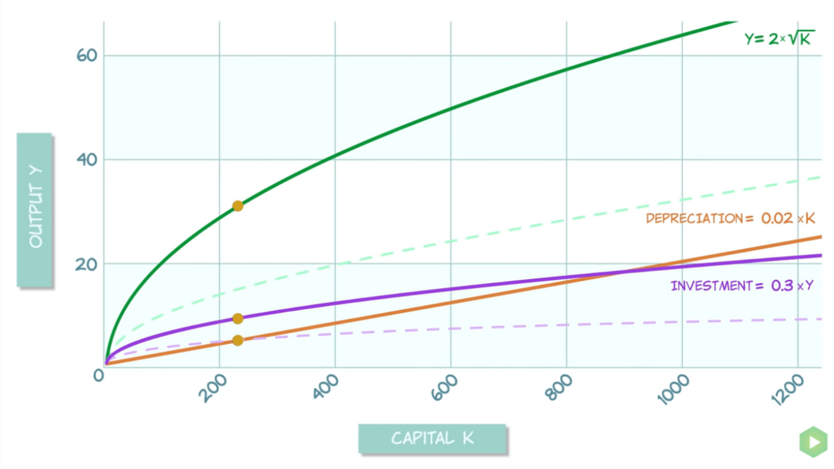


Figure 4: An increase in productivity (TFP), and a new steady state. Source: Tabarrok and Cowen, MRU

Steady state of the model: illustration

You can play around with the Solow model, altering parameters, and checking how the st.state changes in response: check out the interactive tool here: https://demonstrations.wolfram.com/SimpleSolowModel/

And here, you can also see the dynamic adjustments of various variables (like, GDP per capita, growth rates, consumption per capita, etc.): https://demonstrations.wolfram.com/DynamicsInTheSolowSwanGrowthMode

Testable predictions of the Solow Model: income levels

Prediction 1: incomes per capita

The Solow model predicts that the economy's GDP per capita in a steady state depends on:

1. the savings rate (+)
2. the population growth rate (-)

Does the data support these predictions?

GDP per capita and savings rates

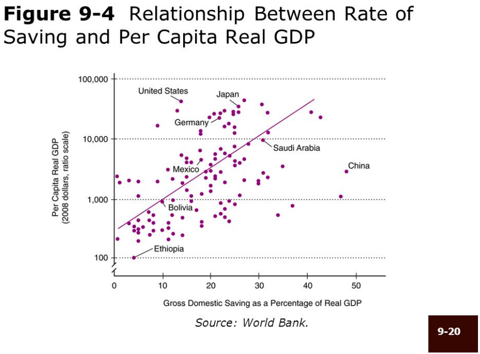


Figure 5: Correlation between savings rate and GDP per capita

But are we sure that this reflects *the causal effect* of investment on GDP per capita (not the other way around? not some omitted factor affecting both?)

GDP per capita and poplation growth rates

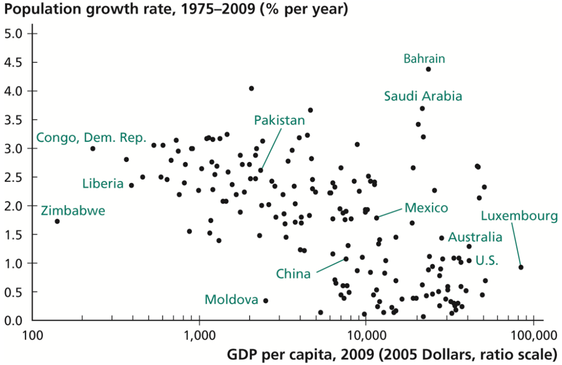


Figure 6: Correlation population growth rate and GDP per capita

But are we sure that this reflects *the causal effect* of population growth on GDP per capita (not the other way around? not some omitted factor affecting both?)

Testable predictions of the Solow Model: income growth rates

Prediction 2: 'Catch-up' growth

The Solow model predicts that the economy's growth rates when the economy is catching up (in transition to a steady state) depend on (i) capital per worker (-); (ii) savings rate (+); (iii) population growth rate (-); (iv) rate of TFP growth (+)

Prediction 3: 'Cutting edge' growth

The Solow model predicts that the economy's growth rate in a steady state ('cutting edge' growth) only depends on the rate of TFP (productivity) growth (+)

Testable predictions of the Solow Model: Convergence

Prediction 4: Conditional Convergence

The Solow model predicts ***conditional*** convergence. Namely, among countries that share similar structural parameters (population growth rate, production technologies, savings rate, etc.), the lower the level of income per capita, the higher is the growth rate of income per capita

Important note!!! The Solow model DOES NOT predict absolute (or, unconditional) convergence. Namely, the model DOES NOT predict that, irrespective of structural parameters, poorer countries will grow faster.

Does data support these predictions about growth rates and convergence?

Testing Solow's predictions on growth rates and convergence

Mankiw, Romer, and Weil (1992) is one of the most widely acknowledged empirical tests of the Solow model (augmented with human capital)

- The authors propose the following Cobb-Douglas production function with human capital: $Y_t = K_t^{\alpha} \cdot H_t^{\beta} \cdot (A_t L_t)^{1-\alpha-\beta}$
- Dynamics of human capital is similar to that of physical capital: each period, it increases with investment (education, measured as the share of people aged 15 to 19 attending school), and depreciates with a given rate
- The authors estimate several specifications of the Solow model using regression analysis Primer on regressions

$$g_{y_{60-85}} = a + \beta_1 \cdot ln(y_{60}) + \beta_2 \cdot ln(s_k) + \beta_3 \cdot ln(s_h) + \beta_4 \cdot ln(n + \delta + g)$$

Testing Solow's predictions on growth rates and convergence

Mankiw, Romer, and Weil (1992) show that the data supports all of the main results of the Solow model:

TABLE V
TESTS FOR CONDITIONAL CONVERGENCE

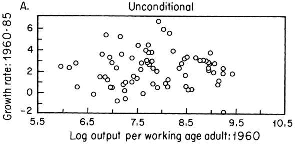
Dependent variable: log difference GDP per working-age person 1960–1985

Sample:	Non-oil	Intermediate	OECD
Observations:	98	75	22
CONSTANT	3.04	3.69	2.81
	(0.83)	(0.91)	(1.19)
ln(Y60)	−0.289	−0.366	−0.398
	(0.062)	(0.067)	(0.070)
ln(I/GDP)	0.524	0.538	0.335
	(0.087)	(0.102)	(0.174)
$\ln(n + g + \delta)$	−0.505	−0.551	−0.844
	(0.288)	(0.288)	(0.334)
ln(SCHOOL)	0.233	0.271	0.223
	(0.060)	(0.081)	(0.144)
$\overline{R}^2$	0.46	0.43	0.65
s.e.e.	0.33	0.30	0.15
Implied λ	0.0137	0.0182	0.0203
	(0.0019)	(0.0020)	(0.0020)

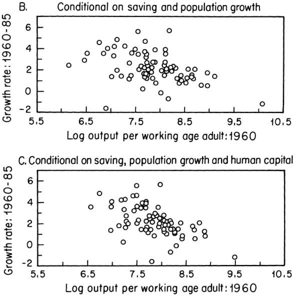
Note. Standard errors are in parentheses. Y60 is GDP per working-age person in 1960. The investment and population growth rates are averages for the period 1960–1985. $(g + \delta)$ is assumed to be 0.05. SCHOOL is the average percentage of the working-age population in secondary school for the period 1960–1985.

Figure 7: Regression analysis results for the model augmented with human capital. Source: Mankiw, Romer, and Weil (1992).

Testing Solow's predictions on growth rates and convergence



(a) Test for unconditional convergence



(b) Test for conditional convergence, with and w/o human capital

Figure 8: Testing for convergence visually. Source: Mankiw, Romer, and Weil (1992).

Limitations of the Solow model

1. Explains the recent dynamics of developed countries well; performs worse in explaining growth in the developing world
2. The model does poorly in matching income differences quantitatively. More on that in the problem set!
3. The main driver of the 'cutting edge' growth - the rate of TFP growth, technological progress - is exogenous. The model is not designed to explain where it comes from.
4. The model is not designed to explain initial growth take-offs, industrialization, demographic change, and other crucial dynamics before the 2nd half of the 20th century.
5. The model predicts (conditional) convergence, while for most of the human history we saw either stagnation or divergence.
    - More recent evidence (post-MRW 1992) also finds evidence for 'club convergence', see Galor (1996) and Quah (1996) among others.

Stable growth rates: developed world

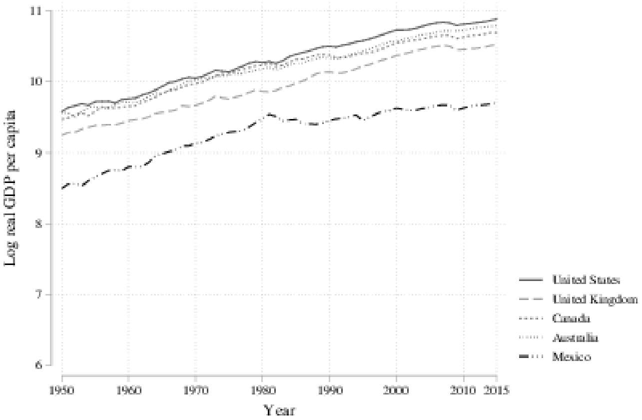


Figure 9: Stable growth rates for developed world

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Stable growth rates in developing world?

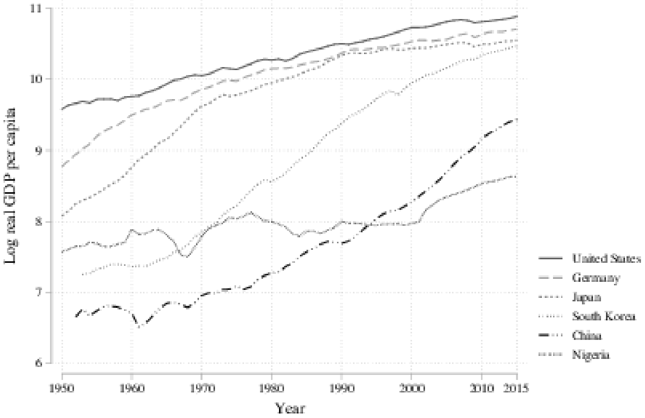

Figure 10: Unstable growth rates for developing world

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Stable labor share in GDP

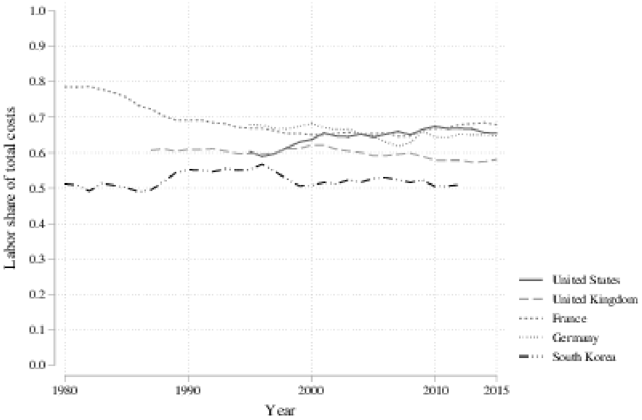

Figure 11: Stable share of labor income in total income for developed countries

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Stable capital-to-output ratio

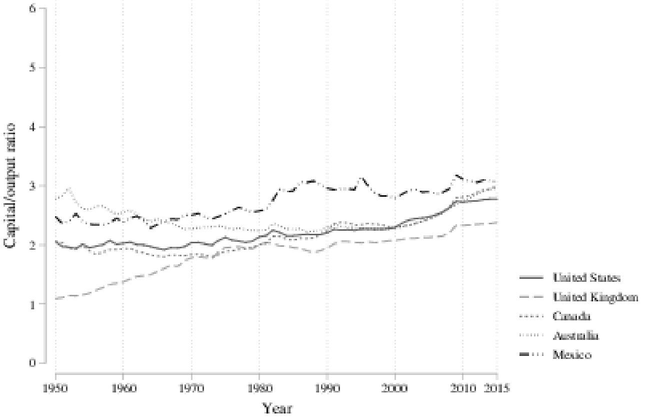

Figure 12: Stable capital-to-output ratio in developed countries

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The Solar System Model

Figure 13: The model of our Solar System

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Exogenous and endogenous variables

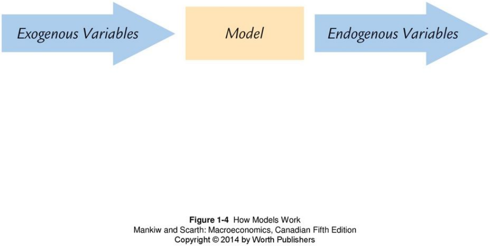

Figure 14: Exogenous and endogenous variables in a model. Source: Gregory Mankiw, "Macroeconomics"

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Dynamics of water inflows and outflows: the steady state

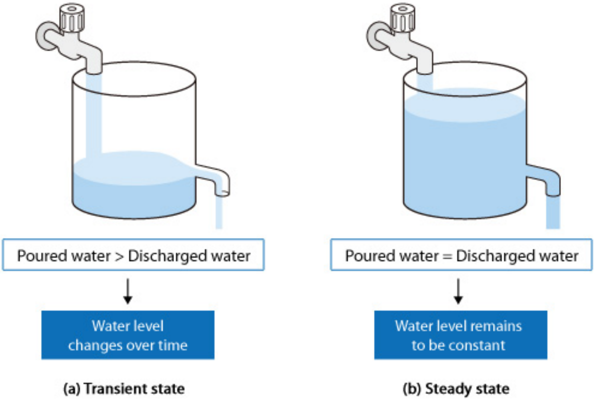

Figure 15: The dynamics of water, and the concept of a steady state

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Post-WWII recoveries in Germany and Japan, low-K rapid growth

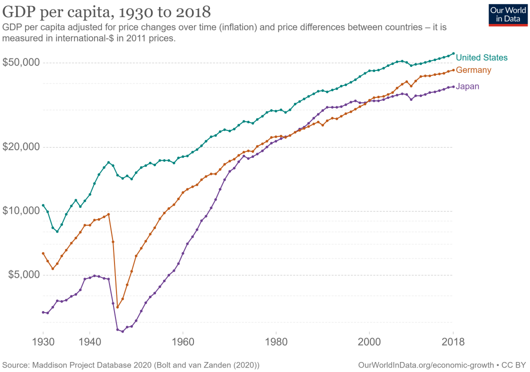

Figure 16: Post-war recoveries in Germany and Japan illustrating rapid growth from a 'low base'

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Basic idea behind (linear) regression analysis

- We assume that relationship between our dependent variable y, and independent variable(s) x is linear, and that some of the data is generated by random 'noise' ε.
- Namely, for the bivariate model: $y_i = \alpha + \beta \cdot x_i + \varepsilon_i$
- We want to pick such values for parameters α and β, that this line would give the best fit for the data
- In the multivariate model, we have multiple explanatory variables (x_1, x_2, x_3, etc.), and hence multiple coefficients to estimate: β_1, β_2, β_3, etc.
- Intuitively, estimated parameters tell us, what is the expected value of y, given values of x. If we change x, what change should we expect from y.

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Basic idea behind (linear) regression analysis

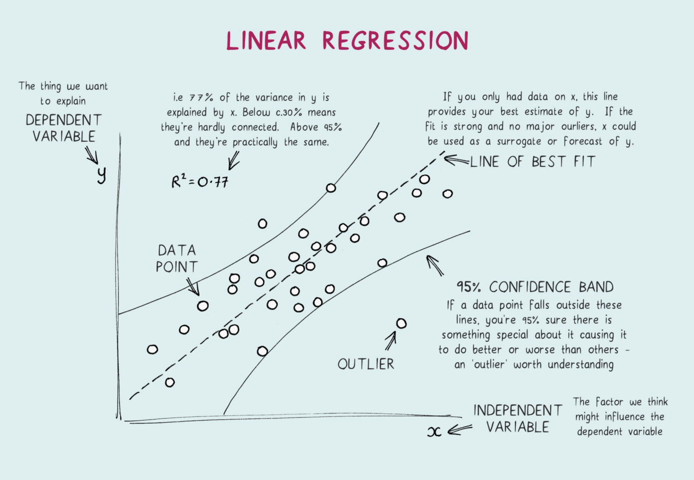

Figure 17: Basic linear regression example. Source: https://towardsdatascience.com/linear-regression-explained-1b36f97b7572