Problem Set 17

Problem 1

(1) $b \wedge (a \vee c)$

(2) $b \vee (a \wedge c)$

Problem 2

对于$a \preceq b \Rightarrow a \wedge b' = 0$:

$\because a \preceq b$

$\therefore a \wedge b = a$

$\therefore a \wedge b' = a \wedge b \wedge b' = a \wedge 0 = 0$

对于$a \wedge b' = 0 \Rightarrow a' \vee b = 1$:

由对偶式可知$a \preceq b \Leftrightarrow a' \vee b = 1$

对于$a' \vee b = 1 \Rightarrow a \preceq b$:

$\because a' \vee b = 1 = a' \vee a \vee b$

$\therefore b = a \vee b$或$a \vee b \preceq a'$且$a \preceq a'$(舍去)

$\therefore a \preceq b$

Problem 3

易见$x \oplus y$对B封闭

$\therefore \langle B, \oplus \rangle$构成代数系统

$$\begin{aligned}
\because (x \oplus y) \oplus z &= ([(x \wedge y') \vee (x' \wedge y)] \wedge z') \vee ([(x \wedge y') \vee (x' \wedge y)]' \wedge z) \\
&= (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee [(x' \vee y) \wedge (x \vee y') \wedge z] \\
&= (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee [(x' \vee y) \wedge (x \wedge z)] \vee [(x' \vee y) \wedge (y' \wedge z)] \\
&= (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee [(x' \wedge x \wedge z) \vee (y \wedge x \wedge z)] \vee [(x' \wedge y' \wedge z) \vee (y \wedge y' \wedge z)] \\
&= (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x \wedge y \wedge z)
\end{aligned}$$

$$\begin{aligned}
x \oplus (y \oplus z) &= (x \wedge [(y \wedge z') \vee (y' \wedge z)]') \vee (x' \wedge [(y \wedge z') \vee (y' \wedge z)]) \\
&= [x \wedge (y' \vee z) \wedge (y \vee z')] \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \\
&= (x \wedge y' \wedge z') \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x \wedge y \wedge z)
\end{aligned}$$

$\therefore \langle B, \oplus \rangle$满足结合律,构成半群

$\because x \oplus 0 = (x \wedge 1) \vee (x' \wedge 0) = x, 0 \oplus x = (0 \wedge x') \vee (1 \wedge x) = x$

$\therefore 0$是$\langle B, \oplus \rangle$的幺,$\langle B, \oplus \rangle$形成幺半群

$\because x \oplus x = (x \wedge x') \vee (x' \wedge x) = 0$

$\therefore x$的逆元是它自身,逆元均存在,$\langle B, \oplus \rangle$形成群

$\because x \oplus y = (x \wedge y') \vee (x' \wedge y) = (y \wedge x') \vee (y' \wedge x) = y \oplus x$

$\therefore \langle B, \oplus \rangle$形成阿贝尔群

Problem 4

$\because a \preceq c$

$\therefore a \wedge c = a$

$\therefore a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c$

Problem 5

(1)

当$n = 2$时,

由德摩根律可知$(a_1 \vee a_2) = a_1' \wedge a_2'$成立

假设当$n = k$时,有$(a_1 \vee a_2 \vee \cdots \vee a_k)' = a_1' \wedge a_2' \wedge \cdots \wedge a_k'$

当$n = k + 1$时,

$$\begin{aligned}(a_1 \vee a_2 \vee \cdots \vee a_k \vee a_{k+1})' &= (a_1 \vee a_2 \vee \cdots \vee a_k)' \wedge a_{k+1}' \\ &= a_1' \wedge a_2' \wedge \cdots \wedge a_k' \wedge a_{k+1}'\end{aligned}$$

$\therefore (a_1 \vee a_2 \vee \cdots \vee a_n)' = a_1' \wedge a_2' \wedge \cdots \wedge a_n'$成立

(2)

当$n = 2$时,

由德摩根律可知$(a_1 \wedge a_2) = a_1' \vee a_2'$成立

假设当$n = k$时,有$(a_1 \wedge a_2 \wedge \cdots \wedge a_k)' = a_1' \vee a_2' \vee \cdots \vee a_k'$

当$n = k + 1$时,

$$\begin{aligned}(a_1 \wedge a_2 \wedge \cdots \wedge a_k \wedge a_{k+1})' &= (a_1 \wedge a_2 \wedge \cdots \wedge a_k)' \vee a_{k+1}' \\ &= a_1' \vee a_2' \vee \cdots \vee a_k' \vee a_{k+1}'\end{aligned}$$

$\therefore (a_1 \wedge a_2 \wedge \cdots \wedge a_n)' = a_1' \vee a_2' \vee \cdots \vee a_n'$成立

Problem 6

对于$\forall a, b \in B$,

$$
\begin{aligned}
a \preceq b &\Leftrightarrow a \wedge b = a \\
&\Leftrightarrow a' \vee b' = a' \\
&\Leftrightarrow b' \vee a' = a' \\
&\Leftrightarrow b' \preceq a'
\end{aligned}
$$

Problem 7

$\because B_1 \simeq B_2, B_2 \simeq B_3$

$\therefore$ 存在双射函数$f : B_1 \to B_2$,双射函数$g : B_2 \to B_3$
使得$(\forall x, y \in B_1)(x \preceq y \leftrightarrow f(x) \preceq f(y))$
$(\forall x, y \in B_2)(x \preceq y \leftrightarrow g(x) \preceq g(y))$

$\therefore$ 双射函数$F = g \circ f$可以满足
$(\forall x, y \in B_1)(x \preceq y \leftrightarrow f(x) \preceq f(y) \leftrightarrow g(f(x)) \preceq g(f(y)) \leftrightarrow F(x) \preceq F(y))$

$\therefore B_1 \simeq B_3$

Problem 8

(1)

x	000	001	010	011	100	101	110	111
F(x)	0	1	1	1	0	1	0	0

(2)

$F(x, y, z) = (x' \wedge y' \wedge z) \vee (x' \wedge y \wedge z') \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x \wedge y \wedge z)$

(3)

$F(x, y, z) = (x' \wedge y) \vee (x' \wedge z) \vee (x \wedge y' \wedge z)$

Problem 9

对于充分性：

当$a=b$时,易见$(a \wedge b') \vee (a' \wedge b) = (a \wedge a') \vee (a' \wedge a) = 0$

对于必要性：

$\because (a \wedge b') \vee (a' \wedge b) = 0$

$\therefore a \wedge b' = 0, a' \wedge b = 0$

$\therefore a \wedge b' = 0, a \vee b' = 1$

$\therefore b'$是a的补元

$\therefore$ 由补元唯一性定理可知,$b' = a'$

$\therefore a = b$