Problem Set 13

Problem 1

A B

Problem 2

$\because ea = ae = a$

$\therefore e \in N(a)$

$\therefore N(a) \neq \emptyset$

对于$\forall x \in N(a)$,

$\because xa = ax$

$\therefore a^{-1}x^{-1} = x^{-1}a^{-1}$

$\therefore aa^{-1}x^{-1}a = ax^{-1}a^{-1}a$

$\therefore x^{-1}a = ax^{-1}$

$\therefore x^{-1} \in N(a)$

对于$\forall x, y \in N(a)$

$\because xa = ax, ya = ay$

$\therefore xay = axy, xya = xay$

$\therefore xya = axy$

$\therefore xy \in N(a)$

$\therefore N(a)$是G的子群

Problem 3

$\because$ 单位元$e \in H, xex^{-1} = xx^{-1} = e$

$\therefore e \in xHx^{-1}, xHx^{-1} \neq \emptyset$

对于$\forall a, b \in H$, 有$xax^{-1}, xbx^{-1} \in xHx^{-1}$

$\therefore xax^{-1}(xbx^{-1})^{-1} = xax^{-1}xb^{-1}x^{-1} = xab^{-1}x^{-1}, ab^{-1} \in H$

$\therefore xab^{-1}x^{-1} \in xHx^{-1}$

$\therefore xHx^{-1}$是G的子群

Problem 4

$\because e \in H, e \in K$

$\therefore e \in H \cap K$

对于$\forall a, b \in H \cap K$

$\because H, K$都是群

$\therefore ab^{-1} \in H, ab^{-1} \in K$

$\therefore ab^{-1} H \cap K$

$\therefore H \cap K$也是一个群, 且是H和K的子群

设$H \cap K$的阶为p

$\therefore$ 由$Lagrange$定理可知, $p|r, p|s$

$\because r$和s互素

$\therefore p = 1$

$\therefore H \cap K$为平凡群

$\therefore H \cap K = \{e\}$

Problem 5

$\because G$中只有一个2阶元, 设为a

对$\forall x \in G$

若$xax^{-1}=e$,则$xa=x \Rightarrow a=e$矛盾

$\therefore xax^{-1} \neq e$

$\because (xax^{-1})^2=xax^{-1}xax^{-1}=xaax^{-1}=xx^{-1}=e$

$\therefore xax^{-1}$也是二阶元

$\therefore xax^{-1}=a$

$\therefore xa=ax$

Problem 6

即需证,若$aH \cap bH \neq \emptyset$,则$aH=bH$

$\because aH \cap bH \neq \emptyset$

$\therefore$ 有$h_1,h_2 \in H$,使得$ah_1=bh_2$

$\therefore a=bh_2h_1^{-1}$

$\therefore$ 对于任意$h \in H, ah=bh_2h_1^{-1}h \in bH$

$\therefore aH \subseteq bH$

同理有$bH \subseteq aH$

$\therefore aH=bH$

$\therefore$ 原命题得证

Problem 7

令$f:H \rightarrow Ha, f(h)=ha$

若f非单射,即存在$h_1,h_2 \in H, h_1 \neq h_2$,使得$f(h_1)=h_1a=f(h_2)=h_2a$

$\therefore h_1a=h_2a$

$\therefore$ 由消去律可知$h_1=h_2$,与$h_1 \neq h_2$矛盾

$\therefore f$为单射函数

$\because |Ha| \leq |H|$

$\therefore$ 易知f为满射函数

$\therefore H \approx Ha$

同理可知$H \approx aH$

$\therefore H \approx Ha \approx aH$

$\therefore H$的任意陪集的大小是相等的

Problem 8

对于充分性：

$\because b \in aH$

$\therefore \exists h \in H, b = ah$

$\therefore a^{-1}b = a^{-1}ah = h \in H$

对于必要性：

$\because a^{-1}b \in H$

$\therefore$ 假设$a^{-1}b = h$

$\therefore aa^{-1}b = ah$

$\therefore \exists h \in H, b = ah$

$\therefore b \in aH$