Problem Set 9

Problem 1

(1) $|A| = 3$

(2) $|B| = \aleph_0$

(3) $|C| = \aleph_0$

(4) $|B \cap C| = \aleph_0$

(5) $|B \cap C| = \aleph_0$

(6) $|D| = \aleph_1$

Problem 2

(1)

令$C = B - A$,易知C也是可数集

$\therefore A \cup B = A \cup (B - A) = A \cup C, A \cap C = \emptyset$

不妨假设$|A| \le |C|$,否则将A与C对调

由A, C均为可数集可定义$f_A : |A| \xrightarrow[onto]{1-1} A, f_C : |C| \xrightarrow[onto]{1-1} C$

定义函数$f : |A \cup B| \to A \cup B$

当$x < 2|A|$时,

$$f(x) = \begin{cases} f_A(\dfrac{x}{2}), & x = 2k, k \in \mathbb{N} \\ f_C(\dfrac{x-1}{2}), & x = 2k+1, k \in \mathbb{N} \end{cases}$$

当$x \ge 2|A|$时,

$f(x) = f_C(x - |A|)$

$\therefore$ 易知f为双射函数, $|A \cup B| \in \mathbb{N}$或$|A \cup B| = \aleph_0$

$\therefore A \cup B$是可数集

(2)

不妨定义$f_A : \mathbb{N} \xrightarrow{1-1, onto} A, f_B : \mathbb{N} \xrightarrow{1-1, onto} B$

令$f : A \times B \to \mathbb{N}, f((f_A(m), f_B(n))) = \sum_{i=1}^{m+n} i + m = \frac{(m+n)(m+n+1)}{2} + m$

$\because$ 易知f是双射函数

$\therefore A \times B$是可列集

Problem 3

a) 可数无限的, 取$f(x) = x + 11$

b) 可数无限的, 取$f(x) = -2x - 1$

c) 有限的

d) 不可数的

e)可数无限的, 取$f(x) = \begin{cases} (2, \frac{x}{2} + 1), & x = 2k, k \in \mathbb{N} \\ (3, \frac{x+1}{2}), & x = 2k+1, k \in \mathbb{N} \end{cases}$

f)可数无限的, 取$f(x) = \begin{cases} 5x, & x = 2k, k \in \mathbb{N} \\ -5x - 5, & x = 2k+1, k \in \mathbb{N} \end{cases}$

Problem 4

假设$A - B$是可数的

$\because B$是可数集合

$\therefore A \cap B$是可数集合

$\because A - (A \cap B) = A - B$也为可数集合

$\therefore A = A - (A \cap B) + (A \cap B)$

由Problem 2.(1)可知两个可数集合的并集也是可数集合

$\therefore A$是可数集合

$\because A$是不可数集合,产生矛盾,假设不成立

$\therefore A - B$是不可数的

Problem 5

$\because A$是可数集合

当A是有穷集时,不妨记$A \approx n$,则有$g : n \xrightarrow[onto]{1-1} A$

$\because$ 存在$f : A \xrightarrow[onto]{1-1} B$

$\therefore f \circ g : n \xrightarrow[onto]{1-1} B$

$\therefore B \approx n$

$\therefore B$是有穷集

$\therefore B$是可数的

当A是无穷可列集时,$A \approx \mathbb{N}$

$\therefore$ 同理可知$B \approx \mathbb{N}$

$\therefore B$是可数的

Problem 6

取$f_A : |A| \xrightarrow[onto]{1-1} A, f_B : |B| \xrightarrow[onto]{1-1} B$

$\because A \subset B$

$\therefore |A| < |B|$

$\therefore$ 令$f : A \to B, f(f_A(n)) = f_B(n)$

易知f是单射函数

$\therefore A \preceq \cdot B$

Problem 7

$\because A = \{a, b, c\}$

$\therefore \mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, \{a, b, c\}\}$

$$
\begin{aligned}
\therefore B = \{ & \\
& \{(a,0),(b,0),(c,0)\}, \\
& \{(a,0),(b,0),(c,1)\}, \\
& \{(a,0),(b,1),(c,0)\}, \\
& \{(a,0),(b,1),(c,1)\}, \\
& \{(a,1),(b,0),(c,0)\}, \\
& \{(a,1),(b,0),(c,1)\}, \\
& \{(a,1),(b,1),(c,0)\}, \\
& \{(a,1),(b,1),(c,1)\} \\
\} &
\end{aligned}
$$

$\therefore |\mathcal{P}(A)| = |B| = 8$

$\therefore \mathcal{P}(A) \approx B$

Problem 8

(1) 是, 取$f(x) = \begin{cases} x, & x = 2k, k \in \mathbb{N} \\ -x - 1, & x = 2k + 1, k \in \mathbb{N} \end{cases}$

(2) 不是, $(0, 0.5) \approx \mathbb{R} \not\approx \mathbb{N}$

(3) 是, 取$f(x) = \begin{cases} \dfrac{7x}{2}, & x = 2k, k \in \mathbb{N} \\ \dfrac{-7x - 7}{2}, & x = 2k + 1, k \in \mathbb{N} \end{cases}$

(4) 是, 取$f(x) = \begin{cases} \dfrac{3x}{2} + 1, & x = 2k, k \in \mathbb{N} \\ \dfrac{x - 1}{2} + 2, & x = 2k + 1, k \in \mathbb{N} \end{cases}$

Problem 9

a) $A=\mathbb{R}, B=\mathbb{R}$

b) $A=\mathbb{R}, B=\mathbb{R}-\mathbb{N}$

c) $A=P(\mathbb{R}), B=\mathbb{R}$

Problem 10

设这些可数集合为$C_i, \quad (i=1,2,\cdots,m)$

令$S_1=C_1, S_i=C_j-(C_1\cup\cdots\cup C_{j-1}), \quad (j=2,3,\cdots,m)$
易知S_i也是可数集,且互不相同的两个S_i的交集为空

不妨假定$|S_1|\leq|S_2|\leq\cdots\leq|S_m|$,否则交换它们的位置

$\therefore C_1\cup C_2\cup\cdots\cup C_i=S_1\cup S_2\cup\cdots\cup S_i, \quad (i=1,2,\cdots,m)$

由S_i均为可数集可定义$f_{S_i}:|S_i|\xrightarrow[onto]{1-1}A$

定义函数$f:|C_1\cup C_2\cup\cdots\cup C_i|\to C_1\cup C_2\cup\cdots\cup C_i$

当$0\leq x<m|S_1|$时,

$$f(x)=\begin{cases} f_{S_1}(\dfrac{x}{m}), & x=mk, k\in\mathbb{N} \\ f_{S_2}(\dfrac{x-1}{m}), & x=mk+1, k\in\mathbb{N} \\ \cdots \\ f_{S_m}(\dfrac{x-m+1}{m}), & x=mk+m-1, k\in\mathbb{N} \end{cases}$$

当$\displaystyle\sum_{i=1}^{n}(m-i+1)|S_i|\leq x<\sum_{i=1}^{n+1}(m-i+1)|S_i|$时,

$$f(x)=$$
$$\begin{cases} f_{S_n}(\dfrac{x}{m-n}), & x=(m-n)(\displaystyle\sum_{i=1}^{n}(m-i+1)|S_i|+k), k\in\mathbb{N} \\ f_{S_{n+1}}(\dfrac{x-1}{m-n}), & x=(m-n)(\displaystyle\sum_{i=1}^{n}(m-i+1)|S_i|+k)+1, k\in\mathbb{N} \\ \cdots \\ f_{S_m}(\dfrac{x-m+n+1}{m-n}), & x=(m-n)(\displaystyle\sum_{i=1}^{n}(m-i+1)|S_i|+k)+m-n-1, k\in\mathbb{N} \end{cases}$$

其中$n=1,2,\cdots,m-1$

$\therefore$ 易知f为双射函数, $|C_1 \cup C_2 \cup \cdots \cup C_i| \in \mathbb{N}$或$|C_1 \cup C_2 \cup \cdots \cup C_i| = \aleph_0$

$\therefore C_1 \cup C_2 \cup \cdots \cup C_i$是可数集