Problem Set 14

Problem 1

(1)

$\because \forall x, y \in \mathbb{Z}, f(x+y) = (-1)^{x+y} = (-1)^x \cdot (-1)^y = f(x) \cdot f(y)$

$\therefore$ 是同态

$\because f(1) = f(3) = -1,$ 不是单同态

$\because$ 不存在x使得$f(x) = \frac{1}{2}$

$\therefore$ 不是满同态, 不是同构

(2)

$$
\begin{aligned}
&\because \forall x, y \in \mathbb{Z}, \\
&\quad f(x+y) = e^{i(x+y)} \\
&= e^{ix} \cdot e^{iy} \\
&= (\cos x + i \sin x) \cdot (\cos y + i \sin y) \\
&= f(x) \cdot f(y)
\end{aligned}
$$

$\therefore$ 是同态

$\because$ 若$f(x) = f(y)$

$\therefore e^{ix} = e^{iy}$

$\therefore x = y$

$\therefore$ 是单同态

$\because$ 不存在x使得$f(x) = \cos\frac{1}{2} + i \sin\frac{1}{2}$

$\therefore$ 不是满同态, 不是同构

Problem 2

设$\langle G,*\rangle$是无限循环群,则$\langle G,*\rangle \simeq \langle \mathbb{Z},+\rangle$

$\therefore$ 存在$f: G \to \mathbb{Z}$,
使得$\forall x,y \in \mathbb{Z}, f(x+y)=f(x)*f(y), f(y+x)=f(y)*f(x)$

$\because \langle \mathbb{Z},+\rangle$是阿贝尔群

$\therefore x+y=y+x$

$\therefore f(x+y)=f(y+x)$

$\therefore f(x)*f(y)=f(y)*f(x)$

$\therefore \langle G,*\rangle$也是阿贝尔群

设$\langle G,*\rangle$是无限循环群,则$\langle G,*\rangle \simeq \langle \mathbb{Z}_n,\oplus_n\rangle$

$\therefore$ 存在$f: G \to \mathbb{Z}_n$,
使得$\forall x,y \in \mathbb{Z}, f(x\oplus_n y)=f(x)*f(y), f(y\oplus_n x)=f(y)*f(x)$

$\because \langle \mathbb{Z}_n,\oplus_n\rangle$是阿贝尔群

$\therefore x\oplus_n y=y\oplus_n x$

$\therefore f(x\oplus_n y)=f(y\oplus_n x)$

$\therefore f(x)*f(y)=f(y)*f(x)$

$\therefore \langle G,*\rangle$也是阿贝尔群

Problem 3

$\therefore \forall x,y \in G_1, f(xy)=f(x)f(y)$

若G_1为有限循环群

设$G_1=\{a^0,a^1,a^2,\cdots,a^{n-1}\}$

$\therefore f(G_1)=\{f(a^0),f(a^1),f(a^2),\cdots,f(a^{n-1})\}$

$\because \forall x \in G_1, f(a^0)f(x)=f(a^0x)=f(x), f(x)f(a^0)=f(xa^0)=f(x)$

$\therefore f(a^0)$是$f(G_1)$的幺

$\because f(a^k)=f(a^1a^{k-1})=f(a^1)f(a^{k-1})=f(a^1)\cdots f(a^1)=f(a^1)^k$

$\therefore f(G_1)=\langle f(a^1)\rangle$

$\therefore f(G_1)$是循环群

若G_1为无限循环群

设$G_1=\{a^0,a^{\pm 1},a^{\pm 2},\cdots\}$

$\therefore f(G_1)=\{f(a^0),f(a^{\pm 1}),f(a^{\pm 2}),\cdots\}$

$\because \forall x\in G_1, f(a^0)f(x)=f(a^0x)=f(x), f(x)f(a^0)=f(xa^0)=f(x)$

$\therefore f(a^0)$是$f(G_1)$的幺

当$k\geq 1$时,

$\therefore f(a^k)=f(a^1a^{k-1})=f(a^1)f(a^{k-1})=f(a^1)\cdots f(a^1)=f(a^1)^k$

当$k\leq -1$时,

$\therefore f(a^k)=f(a^{-1}a^{k+1})=f(a^{-1})f(a^{k+1})=f(a^{-1})\cdots f(a^{-1})=f(a^{-1})^{-k}=(f(a^1)^{-1})^{-k}=f(a^1)^k$

$\therefore f(G_1)=\langle f(a^1)\rangle$

$\therefore f(G_1)$是循环群

Problem 4

(1)

$\therefore G=\{a^0,a^1,a^2,\cdots,a^{14}\}$

设a^r是G的生成元

$\therefore \gcd(15,r)=1$

$\therefore r$的值只能取$1,2,4,7,8,11,13,14$

$\therefore G$的生成元为$a^1,a^2,a^4,a^7,a^8,a^{11},a^{13},a^{14}$

(2)

$\because$ 15的因子为$1, 3, 5, 15$

$\therefore G$的子群分别是：

$$
\begin{aligned}
&G\text{自身} \\
&G_3 = \{a^0, a^3, a^6, a^9, a^{12}\} \\
&G_5 = \{a^0, a^5, a^{10}\} \\
&G_{15} = \{a^0\}
\end{aligned}
$$

Problem 5

$\because$ 3阶群在同构意义下只有一个

*	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

$\therefore$ 设$G = \{e, a, b\}$

$\therefore a^0 = e, a^1 = a, a^2 = b$

$\therefore G$为循环群, 设另一个3阶群为S

$\therefore \exists$双射函数$f: G \to S, \forall x, y \in G, f(xy) = f(x)f(y)$

$\because \forall x \in G, f(e)f(x) = f(ex) = f(x), f(x)f(e) = f(xe) = f(x)$

$\therefore f(e)$是S的单位元

$\therefore e \neq a \neq a^2, f$为双射

$\therefore f(e) \neq f(a) \neq f(a^2) = f(a)f(a) = f(a)^2$

$\therefore S$的生成元为$f(a)$

$\therefore$ 三阶群必为循环群

Problem 6

$\therefore \forall x \in G, x^2 = e$

$\therefore \forall x, y \in G, (xy)^2 = xyxy = e$

$\therefore xyxyy = xyx(yy) = xyx = y$

$\therefore xyxx = xy(xx) = xy = yx$

$\therefore \langle G, * \rangle$是阿贝尔群

Problem 7

设$G = \{a^0, a^{\pm 1}, a^{\pm 2}, \cdots\}, G' = \{b^0, b^1, b^2, \cdots, b^{n-1}\}$

令$f : G \to G', f(a^k) = b^{(k \mod n)}$

$\therefore \forall a^x, a^y \in G,$
$f(a^x a^y) = f(a^{x+y}) = b^{(x+y \mod n)} = b^{(x \mod n)+(y \mod n)} = b^{(x \mod n)} + b^{(y \mod n)} = f(a^x)f(a^y)$

$\therefore f$是G到G'的同态

$\because f(a^r) = b^{(r \mod n)} = b^r, 0 \le r \le n-1$

$\therefore f$是满射

$\therefore f$是G到G'的满同态映射

Problem 8

假设有理数加群Q为循环群

$\therefore \exists q \in Q,$使得$Q = \langle q \rangle = \{q^0, q^{\pm 1}, q^{\pm 2}, \cdots\}$

$\because$ 无限循环群的生成元q只有两个

不妨设$q > 0,$易知Q的幺为$0,$设Q中的元素为$q^n, n \in \mathbb{Z}$

当$n \le 0$时$, q^n = -\sum_{i=1}^{-n} q \le 0$

当$n \ge 1$时$, q^n = \sum_{i=1}^{n} q \ge q$

$\because \frac{q}{2} \in Q,$且$0 < \frac{q}{2} < q$

$\therefore$ 不存在$n \in \mathbb{Z}$使得$q^n = \frac{q}{2}$

$\therefore$ 与假设矛盾,Q不是循环群

$\because$ 整数加群Z是循环群

$\therefore$ 整数加群Z喝有理数加群Q不同构