Problem Set 12

Problem 1

(1) 构成半群, 独异点, 群

(2) 构成半群, 独异点, 群

(3) 构成半群, 不构成独异点, 群

(4) 构成半群, 不构成独异点, 群

(5) 构成半群, 独异点, 不构成群

(6) 构成半群, 独异点, 群

Problem 2

(1)

易知$x * y = x \in S$,即$(S, *)$满足封闭性,构成一个代数系统

$\because$ 对于$\forall x, y, z \in S$,都有$(x * y) * z = x * z = x, x * (y * z) = x * y = x$

$\therefore (x * y) * z = x * (y * z)$

$\therefore S$关于$*$运算满足结合性

$\therefore S$关于$*$运算构成半群

(2)

$\because (S, *)$是半群,要成为独异点,则需有单位元

假设$\exists e \in S$对于$\forall x \in S$满足$e * x = x * e = x$

不妨设$e = a$,若单位元e为b或c同理.

$\therefore a * a = a, a$为单位元对a满足

$a * b = a = b, b * a = b, a$为单位元对$b$满足, 则需$a = b$

$a * c = a = c, c * a = c, a$为单位元对$c$满足, 则需$a = c$

$\therefore$ 称为独异点的条件是$a = b = c$

Problem 3

易知$a \circ b = a \in A$, 即$\langle A, \circ \rangle$满足封闭性, 构成一个代数系统

$\because$ 对于$\forall a, b, c \in A$, 都有$(a \circ b) \circ c = a \circ c = a, a \circ (b \circ c) = a \circ b = a$

$\therefore (a \circ b) \circ c = a \circ (b \circ c)$

$\therefore A$关于$\circ$运算满足结合性

$\therefore \langle A, \circ \rangle$是一个半群

Problem 4

$\because xaxba = xbc$

$\therefore$ 由左消去律可知$axba = bc$

$\therefore a^{-1}(axba) = a^{-1}(bc)$

$\therefore (a^{-1}a)xba = a^{-1}bc$

$\therefore x(ba) = a^{-1}bc$

$\therefore$ 该方程在群G中仅有一个解

Problem 5

$\because a$是群$\langle G, \circ \rangle$的幂等元

$\therefore a \circ a = a$

对于$\forall x \in G$,

$\because a \circ x = (a \circ a) \circ x = a \circ (a \circ x)$

$\therefore$ 由左消去律得$x = a \circ x$

$\therefore a$是x的左单位元

$\because x \circ a = x \circ (a \circ a) = (x \circ a) \circ a$

$\therefore$ 由右消去律得$x = x \circ a$

$\therefore a$也是x的右单位元

$\therefore a$是单位元

Problem 6

$(abc)^{n+1} = abcabc \cdots abc$

当abc的阶有穷, 设为r

$\therefore (abc)^{n+1} = a(bca)^n bc$

$\therefore abc = a(bca)^n bc$

$\therefore (bca)^n = e$

$\therefore bca$阶有穷, 设为r', 可知$r' | r$

$\therefore$ 同理$|bca| = r'$时$|abc|$有穷为r, 有$r | r'$

$\therefore |abc| = |bca|$

同理可知$|abc| = |cab|$

$\therefore |abc| = |bca| = |cab| = r$

当abc的阶无穷,

假设$|bca|$有穷, 由前面的论述可知会使$|abc|$有穷, 与$|abc|$无穷矛盾

$\therefore |bca|$无穷

同理可知$|cab|$无穷

$\therefore |abc| = |bca| = |cab| = \infty$

综上$|abc| = |bca| = |cab|$

Problem 7

法一：

已知G为偶数阶群,设阶数为$2k$

可知存在$a \in G, a \neq e,$有$a^{2k} = e$

$\because a^k \in G$

$\therefore (a^k)^2 = a^{2k} = e$

$\therefore$ 存在二阶元a^k

法二：

已知G为偶数阶群,设阶数为$2k$

对于$a \in G,$若$|a| > 2,$则$a \neq a^{-1}$

若不然,则$a = a^{-1},$从而$a^2 = e, |a| \leq 2$与$|a| > 2$矛盾

$\therefore G$中阶大于2的元素a与其逆a^{-1}成对出现,所以个数是偶数个

$\therefore G$中阶小于等于2的元素a个数也是偶数个,最少也有两个

$\because$ 只有$|e| = 1$

$\therefore$ 必定存在一个元素$a,$使得$|a| = 2$

Problem 8

对于一个不等于单位元e的元$a,$可知它有逆元$a^{-1},$
且易知a^{-1}也不是单位元

当$a \neq a^{-1}$时,令$b = a^{-1},$则满足$ab = ba = e$

当$a = a^{-1}$时,

则此时我们可知对于$\forall a \in G,$都有$a = a^{-1}$

$\therefore ab = (ab)^{-1} = b^{-1}a^{-1} = ba,$其中$a, b \in G, a \neq b$

$\therefore G$中存在非单位元a和$b, a \neq b,$且$ab = ba$

Problem 9

	$\mathbf{1}$	$\mathbf{-1}$	$\boldsymbol{i}$	$\boldsymbol{-i}$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

通过观察易知$\langle S, * \rangle$与$\langle \mathbb{Z}_4, \oplus_4 \rangle$同构

$\therefore \langle S, * \rangle$构成群

Problem 10

对于充分性：

$\because G$为交换群

$\therefore \forall a, b \in G, ab = ba$

$\therefore (ab)^2 = abab = a(ba)b = a(ab)b = aabb = a^2b^2$

对于必要性：

$\therefore a^2b^2 = aabb = (ab)^2 = abab$

$\therefore a(ab)b = a(ba)b$

$\therefore ab = ba$

$\therefore G$为交换群