THERMAL DIFFUSIVITY:
TORTURED RUBBER AND BESSEL FUNCTIONS!

Experimental

Time or equivalently phase delays are a characteristic of the physical response of diffusive systems to stimuli. The objective of the experiment is to determine the thermal diffusivity of rubber tubing by applying a variable temperature to its outer surface and recording the impressed temperature at its inner surface. The difference in phase between these input and output signals is a measure of the diffusivity.

You are provided with a thermometer, a length of rubber tubing, a plug and a small quantity of sealant. You also should obtain a beaker of crushed ice (MP125) and a beaker of boiling water. **Please be careful - boiling water is a hazard!**

Assemble the rubber tubing around the bulb of the thermometer and seal the tubing so no water can penetrate it. Place the tubing alternately in boiling water and freezing ice with a period of several minutes and record the measured temperature. Plot the outer surface temperature, a square wave, and the measured temperature at the inner surface, a low-pass filtered square wave, on the same graph and estimate the phase delay between the two signals at the principal period.

Repeat the experiment for a range of suitable periods of the square wave.

The Thermal Diffusion Equation

The differential equation governing the flow of heat by conduction through a solid is based on the law of conservation of energy. Let V be an arbitrary volume of the solid and S its bounding surface. The thermal energy within V is the volume integral

$$\int_V \rho e \, dv, \tag{1}$$

where ρ is the density of the solid, assumed independent of time t, and e the energy per unit mass. The thermal energy will decrease with time if there is a flow of heat from V through S. If $\vec{q}$ is the heat flux vector and $\vec{n}$ the outward unit vector normal to S, then

$$\frac{d}{dt} \int_V \rho e \, dv = - \int_S \vec{q}.\vec{n} \, ds. \tag{2}$$

Gauss' divergence theorem may be used to convert the surface integral to a volume integral. Also, the time derivative may be taken under the integral, so that

$$\int_V \rho \frac{de}{dt}\, dv = - \int_V \vec{\nabla}.\vec{q}\, dv. \tag{3}$$

The relationship can only be true for any volume V if the integrands are equal, giving

$$\rho \frac{de}{dt} = -\vec{\nabla}.\vec{q}. \tag{4}$$

We must now introduce additional information of an experimental nature. For many materials, there is a linear relationship between a *small* change dT in the temperature of a solid and a *small* change de in its thermal energy. The constant of proportionality is called the *specific heat* γ, so that

$$\rho\gamma \frac{dT}{dt} = -\vec{\nabla}.\vec{q}. \tag{4}$$

Also, Fourier showed that there was a simple experimental relationship between the rate of flow of heat through a slab of material and the temperature gradient across it where the direction of the flow is from a high temperature to a low temperature. The constant of proportionality is the *thermal conductivity* κ. His law may be written

$$\vec{q} = -\kappa \vec{\nabla} T. \tag{5}$$

The law is an analogue to Ohm's law in electrical current flow. If equations (4) and (5) are combined, we obtain the thermal diffusion equation

$$\rho\gamma \frac{dT}{dt} = \kappa \nabla^2 T. \tag{6}$$

The combination of parameters $\kappa/\rho\gamma$ is called the *thermal diffusivity* m.

<u>The Solution of the Diffusion Equation in One Dimension</u>

In one cartesian dimension x, the diffusion equation for $T(x,t)$ takes on the simple form

$$m \frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}. \tag{7}$$

In our problem, we have axial symmetry and the only space variable is the polar coordinate r. In this case, there is an extra first derivative term in the Laplacian, so that

$$m \left[\frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} \right] = \frac{\partial T}{\partial t}. \tag{8}$$

The equation is conveniently solved by the method of separation of variables. We write $T(r,t) = R(r)\exp(i\omega t)$, where ω is the angular frequency of the temperature applied to the surface of the rubber. If we substitute this trail solution in equation (8), we obtain an equation for $R(r)$ as

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \lambda^2 R = 0, \tag{9}$$

where $\lambda^2 = -i\omega/m$. The differential equation is known as *Bessel's Equation* of order zero. It was first studied by Daniel Bernouilli in 1738 when he looked at the oscillation of heavy chains! Fourier followed the development above for heat flow in 1822 and obtained Bessel's equation too.

A solution of the equation suitable for our problem is $AJ_0(\lambda r)$, where J_0 is a Bessel function of order zero and A is a constant. Much like a sine or cosine or exponential function, the Bessel function can be written as a convergent series in ascending powers of $z = \lambda r$ and

$$J_0(z) = 1 - \frac{z^2}{2^2} + \frac{z^4}{2^2 4^2} - \frac{z^6}{2^2 4^2 6^2} \cdots . \tag{10}$$

The argument z is the complex number $(-i\omega/m)^{1/2}r = [(\omega/m)^{1/2}r]\exp(-i\pi/4)$. Lord Kelvin wrote

$$J_0[(-i\omega/m)^{1/2}r)] = \mathrm{ber}_0[(\omega/m)^{1/2}r] + i\ \mathrm{bei}_0[(\omega/m)^{1/2}r], \tag{11}$$

where ber and bei are know as Kelvin functions. They are obtained also from power series

$$\mathrm{ber}_0(z) = 1 - \frac{z^4}{2^2 4^2} \cdots \tag{12}$$

and

$$\mathrm{bei}_0(z) = \frac{z^2}{2^2} - \frac{z^6}{2^2 4^2 6^2} \cdots . \tag{13}$$

Now, it possible to compute the *phase* ϕ of the Bessel function as

$$tan(\phi) = \mathrm{bei}_0[(\omega/m)^{1/2}r]/\mathrm{ber}_0[(\omega/m)^{1/2}r]. \tag{14}$$

The phase is zero at $r = 0$ but has a finite value for all other values of r. Clearly, *a phase difference* in the temperature for two values r for a known ω can be used to find the diffusivity m!

Analysis

In the experiment, you *estimated* the phase difference between an applied square wave temperature function at the outer surface of the rubber and the impressed modified square wave at the inner surface. You can now convert this phase difference into a value for the diffusivity m.

Start by plotting from tabulated or computed data the functions $\mathrm{ber}(x)$ and $\mathrm{bei}(x)$ for real, positive values of x. Also, compute and plot the phase $\phi(x)$ of the function $\mathrm{ber}(x) + i\,\mathrm{bei}(x)$.

Use the phase curves to find values $(\omega/m)^{1/2}\,r$ and hence determine m. What are the systematic and random errors associated with your results?

References

Carlslaw and Jaeger; Heat Conduction in Solids (Some Copies in the Engineering Library).

Abramowitz and Stegun; Handbook of Mathematical Functions, Dover.