PHY408

Lecture 4: Fourier Transforms

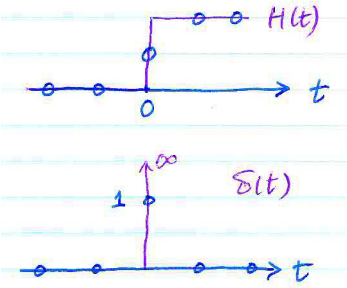
February 1, 2023

From Lecture 3: Discretize step function $H(t)$ and $\delta(t)$

1. Heaviside function (step function) $H(t)$

$$H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases} \tag{1}$$

   step response: $g(t) = H(t) * w(t)$
2. $\delta(t)$ function
3. Discretization:



$$\delta_0 = 1, \quad H_0 = 0.5 \tag{2}$$

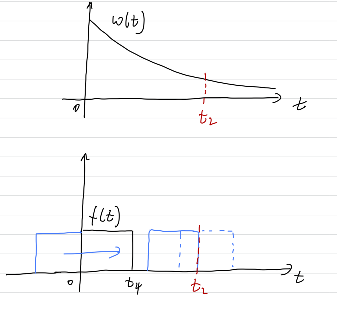
From Lecture 3: A special case

$$g(t) = \int_{-\infty}^{\infty} w(\tau) f(t - \tau)\, d\tau \tag{3}$$

In many applications, both $w(t)$ and $f(t)$ are causal, and $f(t)$ is of finite duration ($0 \leq t \leq t_4$). However, $w(t)$ may be of infinite duration with decreasing values at large time. For discrete implementation, $w(t)$ has to be truncated ($0 \leq t \leq t_2$), and according to the previous slide, the convolved time series $g(t)$ is non-zero between 0 and $t_2 + t_4$.

1. What is the effect of truncating $w(t)$? Are all the values of $g(t)$ between 0 and $t_2 + t_4$ accurate?

From Lecture 3: Convolution with a boxcar



Given

1. $f(t) \neq 0,\ 0 \leq t \leq t_4$
2. $w(t) \neq 0,\ 0 \leq t < \infty$ but truncated between $[0, t_2]$

only $g(t) \neq 0,\ 0 \leq t \leq t_2$ may be accurately computed by convolution.

Fourier Transform

Decomposition of $g(t)$ in terms of complex exponentials:

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} \, d\omega \tag{4}$$

computation of $G(\omega)$

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt \tag{5}$$

- $G(\omega)$ is known as the Fourier transform of $g(t)$
- $g(t)$ is the inverse Fourier transform of $G(\omega)$

Properties of Fourier Transform (FT)

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt \tag{6}$$

- FT exists for a variety of functions (integrable, stable, finite-energy); in comparison, Laplace transform exists for all functions because of e^{-st}.
- interchangeability: completely symmetric except $\frac{1}{2\pi} \Rightarrow \frac{1}{\sqrt{2\pi}}$, $e^{i\omega t} \Rightarrow e^{-i\omega t}$
- $g(t) = \int_{-\infty}^{\infty} G(f) e^{i2\pi ft}\, df$
- $t \leftrightarrow \omega$, $x \leftrightarrow k$ (space-wavenumber)

Properties of FT

- self-consistent: then

$$
\begin{aligned}
f(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(t')e^{-i\omega t'}dt'\right]e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\iint_{-\infty}^{\infty} f(t')e^{i\omega(t-t')}dt'd\omega \\
&= \int_{-\infty}^{\infty} f(t')\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t')}d\omega\right]dt' \\
&= \int_{-\infty}^{\infty} f(t')\delta(t-t')dt' = f(t)
\end{aligned}
$$

using the fact that

$$\delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}d\omega$$

More on $\delta(t)$ function

$$\delta(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega t} d\omega \tag{7}$$

The RHS can be approximated by

$$\delta(t) = \lim_{\Omega \to \infty} \frac{1}{2\pi} \int_{-\Omega}^{\Omega} e^{i\omega t} d\omega \tag{8}$$

define the boxcar function in the frequency domain

$$B_\Omega(\omega) = \begin{cases} 1 & -\Omega \leq \omega \leq \Omega \\ 0 & \text{else} \end{cases} \tag{9}$$

then

$$\delta(t) = \lim_{\Omega \to \infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} B_\Omega(\omega) e^{i\omega t} d\omega \equiv \lim_{\Omega \to \infty} b_\Omega(t) \tag{10}$$

What is the IFT of the boxcar function $B(\omega)$?

$$
\begin{aligned}
b_\Omega(t) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} B_\Omega(\omega)e^{i\omega t}d\omega = \frac{1}{2\pi}\int_{-\Omega}^{\Omega} e^{i\omega t}d\omega \\
&= \frac{1}{2\pi}\frac{1}{it}e^{i\omega t}\Big|_{-\Omega}^{\Omega} = \frac{1}{\pi t}\left[\frac{e^{i\Omega t}-e^{-i\Omega t}}{2i}\right] = \frac{\sin\Omega t}{\pi t} = \frac{\Omega}{\pi}\frac{\sin\Omega t}{\Omega t} \\
&= \frac{\Omega}{\pi}\,\mathrm{sinc}\left(\frac{\Omega t}{\pi}\right)
\end{aligned}
$$

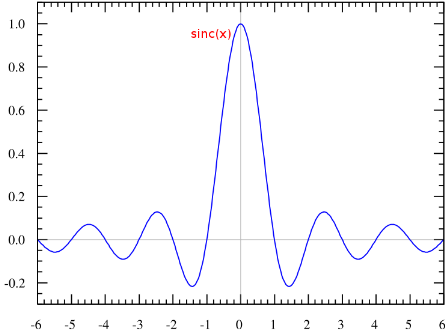
where we define

$$
\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \tag{11}
$$

sinc function

where we define

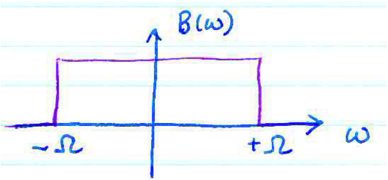
$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x} \tag{12}$$

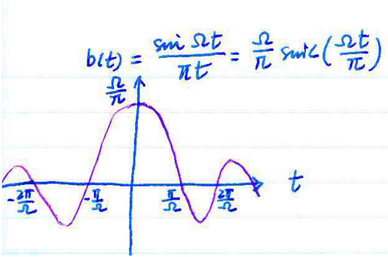


1. $\mathrm{sinc}(x = 0) = 1$
2. $\mathrm{sinc}(x = n) = 0$, half-width of the central peak is 1.

FT of boxcar function

$$b_\Omega(t) = \frac{\Omega}{\pi} \operatorname{sinc}\left(\frac{\Omega t}{\pi}\right) \tag{13}$$





1. A boxcar with a half width Ω in one domain (t or ω) transforms into a sinc function with a central peak of half width π/Ω and maximum value Ω/π in the other domain (ω or t), i.e. the product of half widths is constant $\sim \pi$
2. Area under $b_\Omega(t) \sim 1$.

Approximation of $\delta(t)$ by sinc function

$$
\begin{aligned}
\delta(t) &= \lim_{\Omega\to\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} B_\Omega(\omega) e^{i\omega t} d\omega \\
&= \lim_{\Omega\to\infty} b_\Omega(t) = \lim_{\Omega\to\infty} \frac{\Omega}{\pi} \text{sinc}\left(\frac{\Omega t}{\pi}\right)
\end{aligned}
\tag{14}
$$

1. half-width of central peak is $\frac{\pi}{\Omega} \to 0$
2. $\delta(t = 0) \sim \frac{\Omega}{\pi} \to \infty$
3. $\delta(t) \Leftrightarrow 1$ confirmed

Properties of FT

For a general complex function $f(t)$ and its Fourier transform $F(\omega)$,

$$f(t) = f_R(t) + if_I(t), \quad F(\omega) = F_R(\omega) + iF_I(\omega) \tag{15}$$

What is the relationship between $f_R(t)$, $f_I(t)$ and $F_R(\omega)$, $F_I(\omega)$?

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What is the relationship between $f_R(t)$, $f_I(t)$ and $F_R(\omega)$, $F_I(\omega)$?

$$F_R(\omega) + iF_I(\omega) = \int_{-\infty}^{\infty} [f_R(t) + if_I(t)]e^{-i\omega t} dt \tag{16}$$

which gives

$$\begin{aligned} F_R(\omega) &= \int_{-\infty}^{\infty} [f_R(t)\cos\omega t + f_I(t)\sin\omega t] dt \\ F_I(\omega) &= \int_{-\infty}^{\infty} [f_I(t)\cos\omega t - f_R(t)\sin\omega t] dt \end{aligned} \tag{17}$$

Consider

- $f(t)$ real, i.e. $f_I(t) = 0$, then

FT of a real signal

$f(t)$ real,

$$\begin{aligned} F_R(\omega) &= \int_{-\infty}^{\infty} [f(t)\cos\omega t] dt \\ F_I(\omega) &= -\int_{-\infty}^{\infty} [f(t)\sin\omega t] dt \end{aligned} \tag{18}$$

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1. $F_R(\omega)$ even, $F_I(\omega)$ odd

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1. $F_R(\omega)$ even, $F_I(\omega)$ odd
2. Amplitude spectrum $F_A(\omega)$ even, phase spectrum $F_\phi(\omega)$ odd.

FT of a real signal

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1. $F_R(\omega)$ even, $F_I(\omega)$ odd
2. Amplitude spectrum $F_A(\omega)$ even, phase spectrum $F_\phi(\omega)$ odd.
3. $F(-\omega) = F^*(\omega)$: spectrum for neg. freq. is the complex conjugate of the spectrum for pos. freq.

FT of a real and even signal

$f(t)$ real and even: $f_I(t) = 0$ and $f_R(t) = f_R(-t)$. From

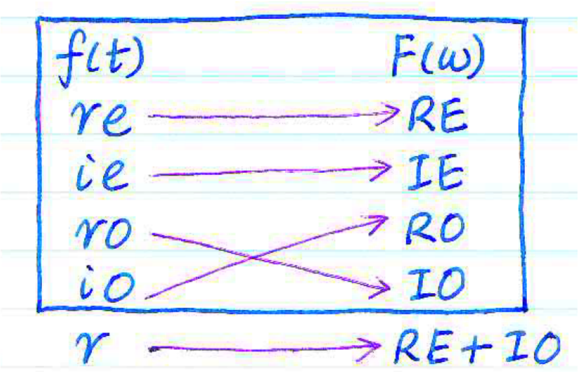
$$\begin{aligned} F_R(\omega) &= \int_{-\infty}^{\infty} [f(t)\cos\omega t]dt \\ F_I(\omega) &= -\int_{-\infty}^{\infty} [f(t)\sin\omega t]dt \end{aligned} \tag{19}$$

we obtain:

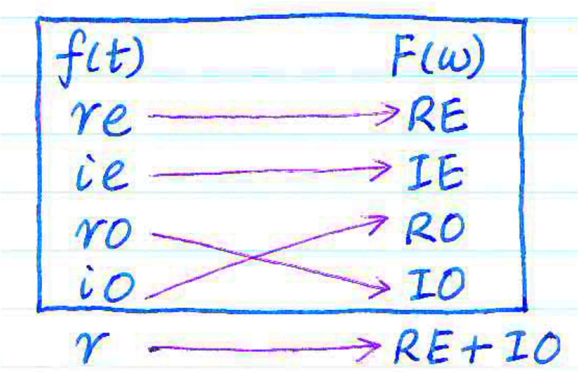
$$F_I(\omega) = 0, \quad F(\omega) = F_R(\omega) \text{ real even} \tag{20}$$

$f(t)$		$F(\omega)$
re	$\longrightarrow$	RE
ie	$\longrightarrow$	IE
ro	$\longrightarrow$	IO
io	$\longrightarrow$	RO

$r \longrightarrow RE + IO$



- $r = re + ro \Leftrightarrow RE + IO$
- what signal gives a real FT?:



- $r = re + ro \Leftrightarrow RE + IO$
- what signal gives a real FT?: $R = RE + RO \Leftrightarrow re + io$

FT: shifting and scaling

if $f(t) \Leftrightarrow F(\omega)$, then

$$g(t) = f(t+a) \Leftrightarrow G(\omega)? \tag{21}$$

$$\begin{aligned} G(\omega) &= \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt = \int_{-\infty}^{\infty} f(t+a) e^{-i\omega t} dt \\ &= \int_{-\infty}^{\infty} f(t') e^{-i\omega(t'-a)} dt' = \left[\int_{-\infty}^{\infty} f(t') e^{-i\omega t'} dt' \right] e^{i\omega a} \\ &= F(\omega) e^{i\omega a} \end{aligned} \tag{22}$$

i.e.

$$f(t+a) \Leftrightarrow F(\omega) e^{i\omega a} \tag{23}$$

i.e time shift corresponds to an additional linear phase shift.

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Scaling in time:

$$f(at) \Leftrightarrow \frac{1}{|a|}F(\omega/a). \tag{24}$$

FT Properties

1. $$f(t) = \int F(\omega)e^{i\omega t}d\omega \Rightarrow \dot{f}(t) \Leftrightarrow (i\omega)F(\omega) \tag{25}$$

2. $$\frac{d^n f}{dt^n} \Leftrightarrow (i\omega)^n F(\omega) \tag{26}$$

3. $$\int f(t)dt \Leftrightarrow \frac{1}{i\omega}F(\omega) \tag{27}$$

4. $$\int [f(t)e^{iat}]e^{-i\omega t}dt = \int f(t)e^{-i(\omega-a)t}dt \Leftrightarrow F(\omega - a) \tag{28}$$

5. $$f(t/a) \Leftrightarrow |a|F(a\omega) \tag{29}$$

6. $$f(-t) \Leftrightarrow F(-\omega) = F^*(\omega) \tag{30}$$

FT examples

Consider the IFT of $\delta(\omega - \omega_0)$

$$\begin{aligned} f(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \delta(\omega - \omega_0) e^{i\omega t} \, d\omega \\ f(t) &= \frac{1}{2\pi} e^{i\omega_0 t} \end{aligned} \tag{31}$$

So we have

$$e^{i\omega_0 t} \Leftrightarrow 2\pi\delta(\omega - \omega_0) \tag{32}$$

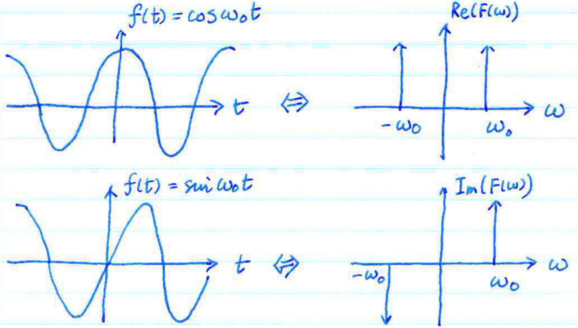
and

$$e^{-i\omega_0 t} \Leftrightarrow 2\pi\delta(\omega + \omega_0) \tag{33}$$

The exponential $e^{i\omega_0 t}$ is associated with an impulse at a non-zero frequency.

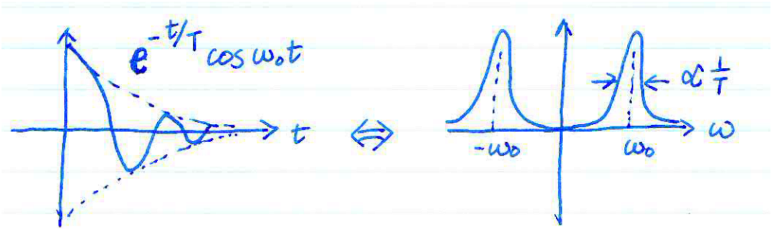
FT examples

- $\cos\omega_0 t = \frac{e^{i\omega_0 t}+e^{-i\omega_0 t}}{2} \Leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$
- $\sin\omega_0 t = \frac{e^{i\omega_0 t}-e^{-i\omega_0 t}}{2i} \Leftrightarrow \frac{\pi}{i}[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)]$



FT example

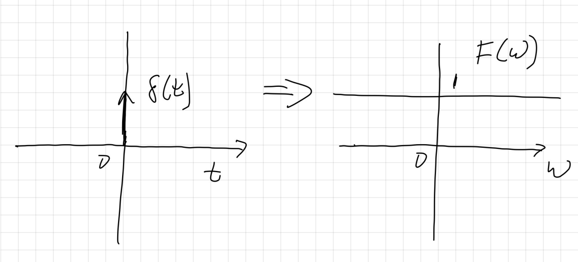
- modulated cosine function with exponential amplitude decay
  $f(t) = e^{-t/T} \cos \omega_0 t \Leftrightarrow F(\omega) = \frac{1}{2}\left[\frac{1}{1/T - i(\omega_0 - \omega)} + \frac{1}{1/T + i(\omega_0 + \omega)}\right]$



FT of a unit impulse

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t}\, dt \quad (34)$$

$$= 1 \quad (35)$$



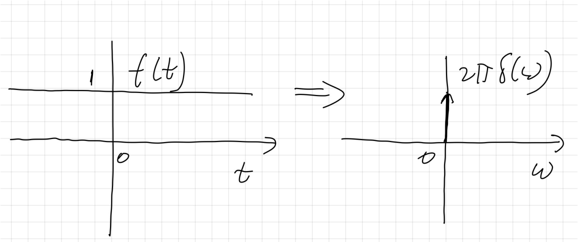
An impulse is composed of all frequencies.

IFT of unit impulse

What is IFT of $\delta(\omega)$?

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{i\omega t}\, d\omega \quad (36)$$

$$= \frac{1}{2\pi} \quad (37)$$



A DC signal is composed of an impulse with zero frequency.

Convolution theorem

Fourier transform of a convolution is simply the product of the Fourier transforms of the two functions being convolved.

- Proof by direct substitution:

$$g(t) = \int_{-\infty}^{\infty} w(\tau) f(t-\tau) d\tau \tag{38}$$

using IFT

$$\begin{aligned} w(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} W(\omega') e^{i\omega'\tau} d\omega' \\ f(t-\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega(t-\tau)} d\omega \end{aligned} \tag{39}$$

substitute into expression of $g(t)$:

$$\begin{aligned} g(t) &= \frac{1}{4\pi^2} \iiint W(\omega') F(\omega) e^{i\omega'\tau} e^{i\omega(t-\tau)} d\omega' d\omega d\tau \\ &= \frac{1}{2\pi} \iint W(\omega') F(\omega) \left[\frac{1}{2\pi} \int e^{i(\omega'-\omega)\tau} d\tau \right] e^{i\omega t} d\omega' d\omega \end{aligned}$$

Convolution theorem

$$
\begin{aligned}
g(t) &= \frac{1}{2\pi}\iint W(\omega')F(\omega)\delta(\omega'-\omega)e^{i\omega t}d\omega' d\omega \\
&= \frac{1}{2\pi}\int W(\omega)F(\omega)e^{i\omega t}d\omega
\end{aligned} \tag{41}
$$

by inspection,

$$
G(\omega) = W(\omega)F(\omega) \tag{42}
$$

According to the interchangeability of $t \leftrightarrow \omega$,

$$
g(t) = w(t)f(t) \Leftrightarrow G(\omega) = \frac{1}{2\pi}W(\omega) * F(\omega) \tag{43}
$$