PHY408

Lecture 5: Discrete Fourier Transforms

February 8, 2023

Fourier Transform

Decomposition of $g(t)$ in terms of complex exponentials:

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\omega) e^{i\omega t} \, d\omega \tag{1}$$

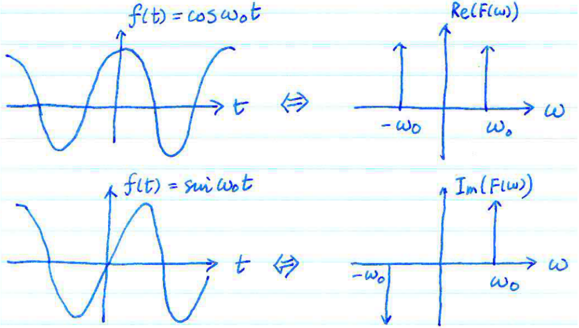
computation of $G(\omega)$

$$G(\omega) = \int_{-\infty}^{\infty} g(t) e^{-i\omega t} dt \tag{2}$$

- $G(\omega)$ is known as the Fourier transform of $g(t)$
- $g(t)$ is the inverse Fourier transform of $G(\omega)$

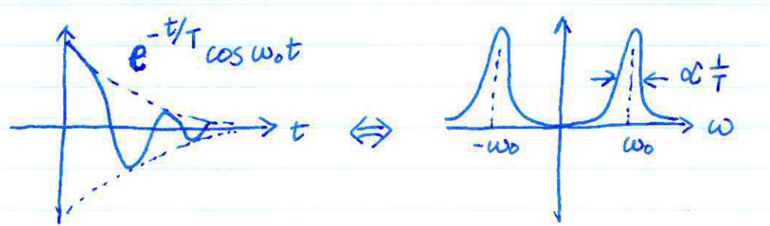
FT examples

- $\cos \omega_0 t = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \Leftrightarrow \pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$
- $\sin \omega_0 t = \frac{e^{i\omega_0 t} - e^{-i\omega_0 t}}{2i} \Leftrightarrow \frac{\pi}{i}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$



FT example

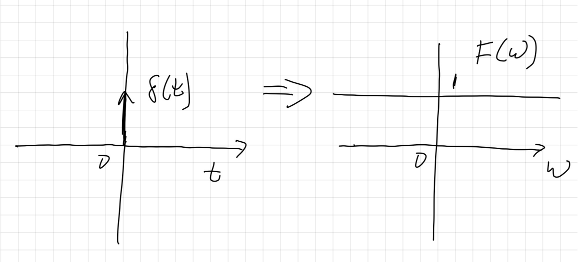
- modulated cosine function with exponential amplitude decay
  $f(t) = e^{-t/T} \cos \omega_0 t \Leftrightarrow F(\omega) = \frac{1}{2} \left[\frac{1}{1/T - i(\omega_0 - \omega)} + \frac{1}{1/T + i(\omega_0 + \omega)} \right]$



FT of a unit impulse

$$F(\omega) = \int_{-\infty}^{\infty} \delta(t) e^{-i\omega t}\, dt \quad (3)$$

$$= 1 \quad (4)$$



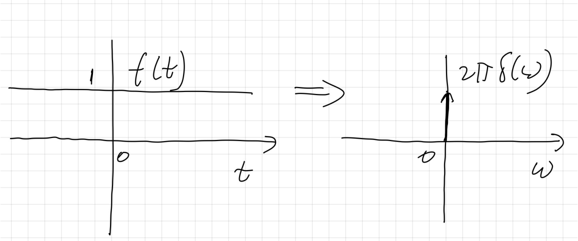
An impulse is composed of all frequencies.

IFT of unit impulse

What is IFT of $\delta(\omega)$?

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{i\omega t}\,d\omega \tag{5}$$

$$= \frac{1}{2\pi} \tag{6}$$



A DC signal is composed of an impulse with zero frequency.

Convolution theorem

Fourier transform of a convolution is simply the product of the Fourier transforms of the two functions being convolved.

- Proof by direct substitution:

$$g(t) = \int_{-\infty}^{\infty} w(\tau) f(t-\tau) d\tau \tag{7}$$

using IFT

$$\begin{aligned} w(\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} W(\omega') e^{i\omega'\tau} d\omega' \\ f(t-\tau) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega(t-\tau)} d\omega \end{aligned} \tag{8}$$

substitute into expression of $g(t)$:

$$\begin{aligned} g(t) &= \frac{1}{4\pi^2} \iiint W(\omega') F(\omega) e^{i\omega'\tau} e^{i\omega(t-\tau)} d\omega' d\omega d\tau \\ &= \frac{1}{2\pi} \iint W(\omega') F(\omega) \left[\frac{1}{2\pi} \int e^{i(\omega'-\omega)\tau} d\tau \right] e^{i\omega t} d\omega' d\omega \end{aligned}$$

Convolution theorem

$$
\begin{aligned}
g(t) &= \frac{1}{2\pi}\iint W(\omega')F(\omega)\delta(\omega'-\omega)e^{i\omega t}d\omega' d\omega \\
&= \frac{1}{2\pi}\int W(\omega)F(\omega)e^{i\omega t}d\omega \qquad (10)
\end{aligned}
$$

by inspection,

$$G(\omega) = W(\omega)F(\omega) \qquad (11)$$

According to the interchangeability of $t \leftrightarrow \omega$,

$$g(t) = w(t)f(t) \Leftrightarrow G(\omega) = W(\omega) * F(\omega) \qquad (12)$$

Fourier Series for periodic functions

Rewrite FT in terms of $t \leftrightarrow f$:

$$
\begin{aligned}
G(f) &= \int_{-\infty}^{\infty} g(t)e^{-i2\pi ft}\,dt \\
g(t) &= \int_{-\infty}^{\infty} G(f)e^{i2\pi ft}\,df
\end{aligned} \tag{13}
$$

1. Recall $\cos\omega_0 t \Leftrightarrow \pi[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)]$, $\delta(t)$ is a 'distribution' or 'measure', only defined in an integral sense.
2. If $g(t)$ is periodic with period of T, then it can be exactly represented over that range by all sinusoids with frequencies that are integer multiples of the fundamental frequency $1/T$ (i.e., a discrete spectrum).

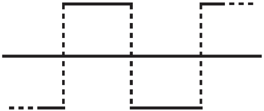
$$
f(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} F_k e^{i\omega_k t}, \quad \text{where } \omega_k = 2\pi k/T \tag{14}
$$

where

$$
F_k = \int_0^T f(t)e^{-i\omega_k t}dt \tag{15}
$$

Fourier Series for periodic functions

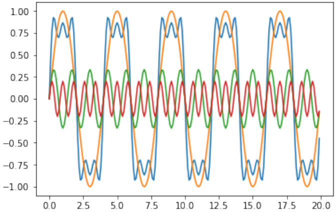
Consider the periodic step function



This can be represented by the series

$$f = \frac{4}{\pi}\sin(\omega_0 t) + \frac{4}{3\pi}\sin(3\omega_0 t) + \frac{4}{5\pi}\sin(5\omega_0 t) + ...$$

For the first three terms in the series we have



Fourier Series for periodic functions

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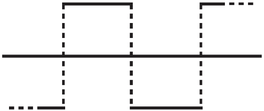

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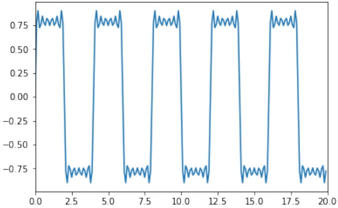
$$f = \frac{4}{\pi}\sin(\omega_0 t) + \frac{4}{3\pi}\sin(3\omega_0 t) + \frac{4}{5\pi}\sin(5\omega_0 t) + ...$$

With the first six terms in the series we obtain:



Question

1. Given a continuous function, if only the segment between $[0, T]$ is know, what is the result of summing FS over $[0, T]$?

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2. reconstruction is only accurate over $[0, T]$, and periodic outside.
3. discontinuity jumps: converge to the middle of discontinuities (Gibbs phenomenon)

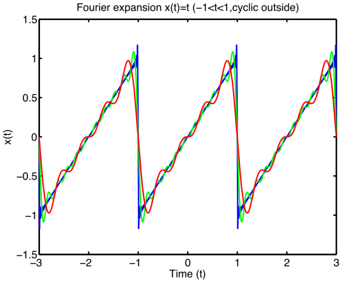
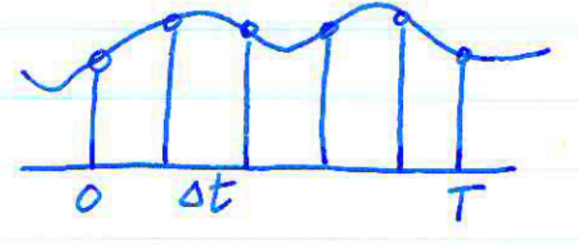


Figure 1: Fourier series expansion of x(t)=t and Gibb's phenomenon. Red is N=4, Green N=10 and Blue N=100.

Finite time series

For a continuous process $f(t)$, we can only obtain finite number of discrete samples $f_k, k = 0, 1, \cdots N-1$.



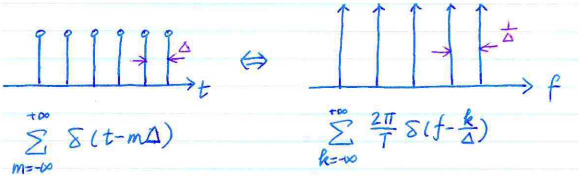
1. effect of discretization by Δt?
2. effect of truncation over $[0, T]$?

Dirac comb

Define the Dirac comb function (sum of shifted Dirac functions)

$$\sum_{m=-\infty}^{\infty} \delta(t - m\Delta) \Leftrightarrow \sum_{k=-\infty}^{\infty} \frac{2\pi}{\Delta}\delta(f - \frac{k}{\Delta}) \quad (16)$$

i.e. the FT of a Dirac comb is still a Dirac comb, and the product of Dirac function spacings in t and f domain $= \Delta \times \frac{1}{\Delta} = 1$.



a discrete time series (Δ) has a periodic frequency spectrum (period of $1/\Delta$)

FT of the Dirac comb

$$c(t) = \sum_{m=-\infty}^{\infty} \delta(t - m\Delta) \tag{17}$$

The FT is given by

$$\begin{aligned} C(\omega) &= \int_{-\infty}^{\infty} c(t) e^{-i\omega t} dt \quad (18) \\ &= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(t - m\Delta) e^{-i\omega t} dt \quad (19) \\ &= \sum_{m=-\infty}^{\infty} e^{-i\omega m\Delta} \quad (20) \end{aligned}$$

FT of the Dirac comb

$$C(\omega) = \sum_{m=-\infty}^{\infty} e^{-i\omega m\Delta} \tag{21}$$

can be approximated by

$$C(\omega) = \lim_{M\to\infty} \sum_{m=-M}^{M} e^{-i\omega m\Delta} \tag{22}$$

This is a geometric series which can be expressed as

$$C(\omega) = \lim_{M\to\infty} \frac{e^{iM\omega\Delta} - e^{-i(M+1)\omega\Delta}}{1 - e^{-i\omega\Delta}}, \tag{23}$$

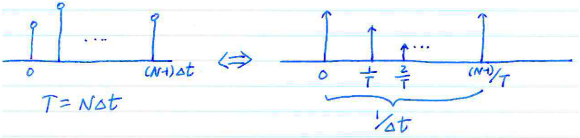
which yields

$$C(\omega) = \sum_{k=-\infty}^{\infty} \frac{2\pi}{\Delta}\delta\left(\omega - \frac{2\pi k}{\Delta}\right). \tag{24}$$

Discrete Fourier transform (DFT)

Now sample the continuous signal $g(t)$ at interval Δ as $\{g_0, g_1, \cdots g_{N-1}\}$:

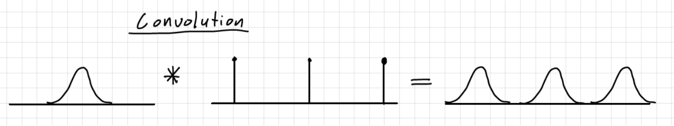
1. Total length of the signal $N * \Delta = T$: According to Fourier Series, we only need discrete harmonics whose freqs are multiples of $1/T$ to represent $\{g_i\}$.
2. Sampling interval Δ: multiplication in the time domain by Dirac comb with Δ spacing $\rightarrow$ convolution in the freq domain by $\sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{\Delta})$ (period of $1/\Delta$)
3. How many independent points are there in the frequency spectrum of g_K? $\frac{1}{\Delta}/\frac{1}{T} = T/\Delta = N$ points $G_0, G_1, \cdots G_{N-1}$!



4. G_k's are called the Discrete Fourier Transform of F_j's.

Discrete Fourier transform (DFT) - Sampling

Convolution with the Dirac comb produces a period signal.



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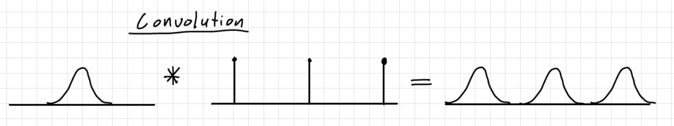
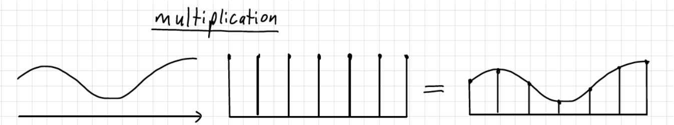


Discrete sampling is equivalent to multiplying a continuous signal by the Dirac comb.



How to compute G_k's?

According to Fourier Series:

$$G_k = \int_0^T g(t) e^{-i\omega_k t} dt \sim \sum_j g_j e^{-i2\pi \frac{k}{T}(j\Delta t)} \Delta t = \sum_j g_j e^{-i2\pi kj/N} \Delta t \quad (25)$$

based on $T = N\Delta t$. Similarly, reconstruction of the time series

$$g(t_j) = g_j = \frac{1}{T} \sum_{k=0}^{N-1} G_k e^{i2\pi \frac{k}{T}(j\Delta t)} = \frac{1}{\Delta t} \frac{1}{N} \sum_{k=0}^{N-1} G_k e^{i2\pi kj/N} \quad (26)$$

DFT:

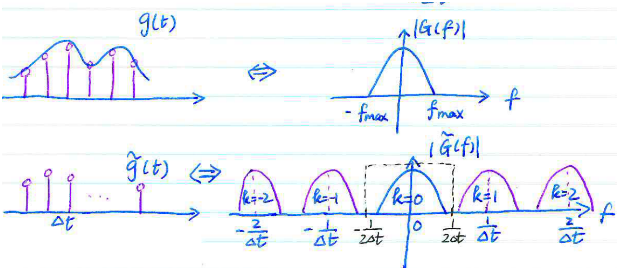
$$G_k = \Delta t \sum_j g_j e^{-i2\pi kj/N}, \qquad g_j = \frac{1}{N\Delta t} \sum_{k=0}^{N-1} G_k e^{i2\pi kj/N} \quad (27)$$

- how good of a representation G_k is to $G(\omega)$?

Sampling theorem

If $G(f)$ is the true spectrum of the continuous time series $g(t)$, then the spectrum of the sampled time series is

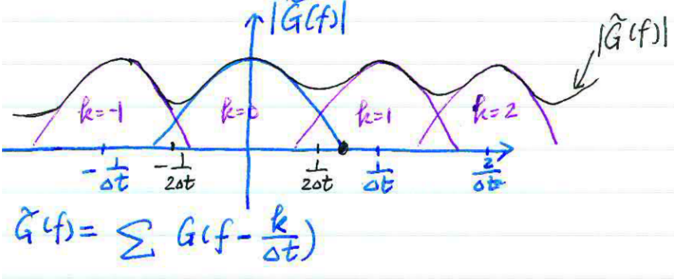
$$\tilde{G}(f) \sim G(f) * \sum_k \delta(f - \frac{k}{\Delta t}) = \sum_k G(f - \frac{k}{\Delta t}) \qquad \text{Periodic!} \qquad (28)$$



Sampling theorem: if $f_{max} < \frac{1}{2\Delta t} = \frac{1}{2} f_s$ (known as Nyquist frequency, a minimum two points to sample one cycle), then $G(f)$ can be fully recovered from $\tilde{G}(f)$, in other words, the discrete time series g_k can fully reconstruct the original continuous signal $g(t)$.

Aliasing

If the sampling criterion is not satisfied, i.e. $f_{max} \geq \frac{1}{2} f_s$



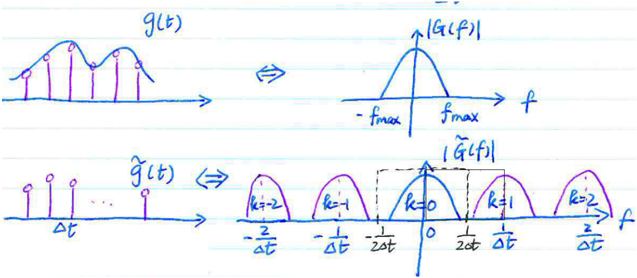
1. high and low frequency content overlap: not possible to extract $G(f)$ accurately from $\tilde{G}(f)$.
2. anti-aliasing filter: given a sampling rate f_s, first low-pass filter the continuous input signal $g(t)$ below $f_{max} = \frac{1}{2} f_s$ before digitization.

Reconstruction of the original continuous signal

1. $\tilde{G}(f)$ periodic:

$$\tilde{G}(f) \sim G(f) * \sum_k \delta(f - \frac{k}{\Delta t}) = \sum_k G(f - \frac{k}{\Delta t}) \qquad (29)$$

therefore we only need to examine $\tilde{G}(f)$ in the freq range of $[0, \frac{1}{\Delta t}]$ (implemented by most numerical algorithms) or more physically $[-\frac{1}{2}\frac{1}{\Delta t}, \frac{1}{2}\frac{1}{\Delta t}]$ (*fftshift*).



The Fourier transform of which should correspond to a continuous signal in the time domain.