Project 1. Chebyshev Spectral Methods.

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1. Prove the discrete orthogonality relations for Chebyshev polynomials on the extrema of $T_n(x)$ grid $x_j = \cos\frac{\pi j}{n}$, $0 \le j \le n$ ([1], Section 3.3.1, property (d)):

$$I_n := \frac{1}{2}T_r(-1)T_s(-1)+\frac{1}{2}T_r(1)T_s(1)+\sum_{j=1}^{n-1}T_r(x_j)T_s(x_j) = \begin{cases}0, & r \ne s\\ \frac{1}{2}n, & 1 \le r = s \le n-1\\ n, & r = s = 0,\ n\end{cases}. \quad (1)$$

2. Consider the Chebyshev interpolant of a function $f(x)$ on the extrema grid $x_j = \cos\frac{\pi j}{n}$, $0 \le j \le n$,

$$p_n(x) = \frac{1}{2}c_0T_0(x) + \frac{1}{2}c_nT_n(x) + \sum_{k=1}^{n-1}c_kT_k(x). \quad (2)$$

The interpolant means $p_n(x_j) = f(x_j)$ for all $0 \le j \le n$.

Prove that the coefficients c_k are given by ([1], Section 3.4.1, Chebyshev interpolation of the second kind)

$$c_k = \frac{2}{n}\left(\frac{1}{2}T_k(x_0)f(x_0) + \frac{1}{2}T_k(x_n)f(x_n) + \sum_{j=1}^{n-1}f(x_j)T_k(x_j)\right) \quad (3)$$

or, equivalently, as $T_k(\cos t)\cos kt$,

$$c_k = \frac{2}{n}\left(\frac{1}{2}(-1)^kf(-1) + \frac{1}{2}f(1) + \sum_{j=1}^{n-1}f\left(\cos\left(\frac{\pi j}{n}\right)\right)\cos\left(\frac{\pi kj}{n}\right)\right). \quad (4)$$

Hint: use the orthogonality relation from the previous exercise.

3. Adapt Clenshaw's method (see Section 3.7.1 in Ref. [1]) for the extrema Chebyshev grid. Show that the Chebyshev sum at a query point x,

$$p_n(x) = \frac{1}{2}c_0T_0(x) + \frac{1}{2}c_nT_n(x) + \sum_{k=1}^{n-1}c_jT_k(x) \equiv \frac{1}{2}\left(b_0 - b_2 - b_n\cos(n\arccos x)\right), \quad (5)$$

where b is defined in Eq. (3.102) in [1].

4. The goal of this exercise is to understand how to derive the Chebyshev differentiation matrix in ([2], Chapter 6, Theorem 7, page 53 of the book or page 73 of the PDF file of the book). It is composed of Exercise 6.1 from [2].

   (a) Let $x_0 < x_1 < \ldots < x_N$ be finite distinct points. The Lagrange interpolation polynomial p_N of a function u with data at these points is given by

   $$p_N(x) = \sum_{j=0}^{N} f(x_j) L_j(x) \tag{6}$$

   where L_js are the Lagrangian or cardinal functions defined as

   $$L_j(x) := a_j^{-1} \prod_{\substack{k=0\\k\neq j}}^{N} (x - x_k), \quad a_j := \prod_{\substack{k=0\\k\neq j}}^{N} (x_j - x_k). \tag{7}$$

   Show that

   $$L_j'(x) = L_j(x) \sum_{\substack{k=0\\k\neq j}}^{N} \frac{1}{x - x_k}. \tag{8}$$

   (b) The vector of derivatives of $p_N(x)$ at nodes x_j is given by

   $$\begin{aligned}
   \begin{bmatrix} p_N'(x_0) \\ p_N'(x_1) \\ \cdots \\ p_N'(x_N) \end{bmatrix} &= \begin{bmatrix} \sum_{j=0}^{N} f(x_j) L_j'(x_0) \\ \sum_{j=0}^{N} f(x_j) L_j'(x_1) \\ \vdots \\ \sum_{j=0}^{N} f(x_j) L_j'(x_N) \end{bmatrix} \\
   &= \begin{bmatrix} L_0'(x_0) & L_1'(x_0) & \cdots & L_N'(x_0) \\ L_0'(x_1) & L_1'(x_1) & \cdots & L_N'(x_1) \\ \vdots & \vdots & & \vdots \\ L_0'(x_N) & L_1'(x_N) & \cdots & L_N'(x_N) \end{bmatrix} \begin{bmatrix} f(x_0) \\ f(x_1) \\ \cdots \\ f(x_N) \end{bmatrix} && (9) \\
   &=: D_N \begin{bmatrix} f(x_0) \\ f(x_1) \\ \cdots \\ f(x_N) \end{bmatrix}. && (10)
   \end{aligned}$$

   where D_N is the differentiation matrix. Using item (a), derive the following expressions for the entries of D_N:

   $$[D_N]_{i,j} = \frac{a_i}{a_j(x_i - x_j)}, \quad i \neq j, \tag{11}$$

   $$[D_N]_{j,j} = \sum_{\substack{k=0\\k\neq j}}^{N} \frac{1}{x_j - x_k}. \tag{12}$$

(c) Theorem 7 in Section 6 in Ref. [2] holds for Chebyshev extrema grid, i.e., if $x_j = \cos\left(\frac{\pi j}{N}\right)$. Look up the property of Chebyshev polynomials with derivatives in [1] and Gottlieb's work of (1984) (pages 9–11). This is just a reading task.

5. Solve a homogeneous fourth-order boundary-value problem on $[-1,1]$ using Chebyshev spectral method:

$$u^{(4)} - 4u'' + 3u = g(x), \quad u(-1) = u'(-1) = u(1) = u'(1) = 0, \tag{13}$$

where $g(x)$ is chosen so that the exact solution is

$$u_{\mathsf{exact}} = \cos(\pi x) + 1. \tag{14}$$

Note that a fourth-order equation requires four boundary conditions and that the exact solution satisfies the boundary conditions.

Plot the exact solution and numerical solutions at different numbers of Chebyshev nodes N. Also plot numerical error versus x at different N. Determine what N do you need to take to achieve a machine precision.

6. Solve an nonhomogeneous fourth-order boundary-value problem on $[0,5]$ using Chebyshev spectral method:

$$u^{(4)} + u' + u = g(x), \tag{15}$$

where $g(x)$ and boundary conditions are chosen so that the exact solution is

$$u_{\mathsf{exact}} = \frac{1}{1+x^2}: \tag{16}$$

$$u(0) = 1, \quad u'(0) = 0, \quad u(5) = \frac{1}{26}, \quad u'(5) = -\frac{10}{676}, \tag{17}$$

$$g(x) = \frac{24 - 240x^2 + 120x^4}{(1+x^2)^5} + \frac{1}{1+x^2} - \frac{2x}{(1+x^2)^2}. \tag{18}$$

Plot the exact solution and numerical solutions at different numbers of Chebyshev nodes N. Also plot numerical error versus x at different N. Determine what N do you need to take to achieve a machine precision.

Hint: Start as follows. First rescale g and u on $[-1,1]$. The decompose $u = v + b$ where v satisfies homogeneous boundary conditions (all boundary data are zero), and

b is any function that satisfies the boundary conditions (17). You can construct $b(x)$ as follows:

$$b(x) = p_0(x)u(0) + q_1(x)u'(0) + p_1(x)u(5) + q_1(x)u'(5), \tag{19}$$

where p_0, q_0, p_1, q_1 are cubic polynomials such that

$$\begin{aligned}
p_0(-1) = 1, \quad p_0'(-1) = 0, \quad p_0(1) = 0, \quad p_0'(1) = 0,\\
q_0(-1) = 0, \quad q_0'(-1) = 1, \quad q_0(1) = 0, \quad q_0'(1) = 0,\\
p_1(-1) = 0, \quad p_1'(-1) = 0, \quad p_1(1) = 1, \quad p_1'(1) = 0,\\
q_1(-1) = 0, \quad q_1'(-1) = 0, \quad q_1(1) = 0, \quad q_1'(1) = 1.
\end{aligned}$$

Thus, we find

$$\begin{aligned}
p_0(x) &= \frac{(x-1)^2(x+2)}{4}, \quad q_0(x) = \frac{1}{4}(x-1)^2(x+1),\\
p_1(x) &= \frac{(x+1)^2(2-x)}{4}, \quad q_1(x) = \frac{1}{4}(x+1)^2(x-1).
\end{aligned}$$

References

[1] J. Gil, P. Segura, and N. Temme, *Numerical Methods for Special Functions*. SIAM, 2007.

[2] L. N. Trefethen, *Spectral Methods in MATLAB*. Philadelphia: Society for Industrial and Applied Mathematics (SIAM), 2000.