Homework 1. Approximation theory.

Submit a single PDF file with all your derivations and reports of numerical problems. Include figures in your PDF file. Include links to your codes (to GitHub, Google Drive, or DropBox).

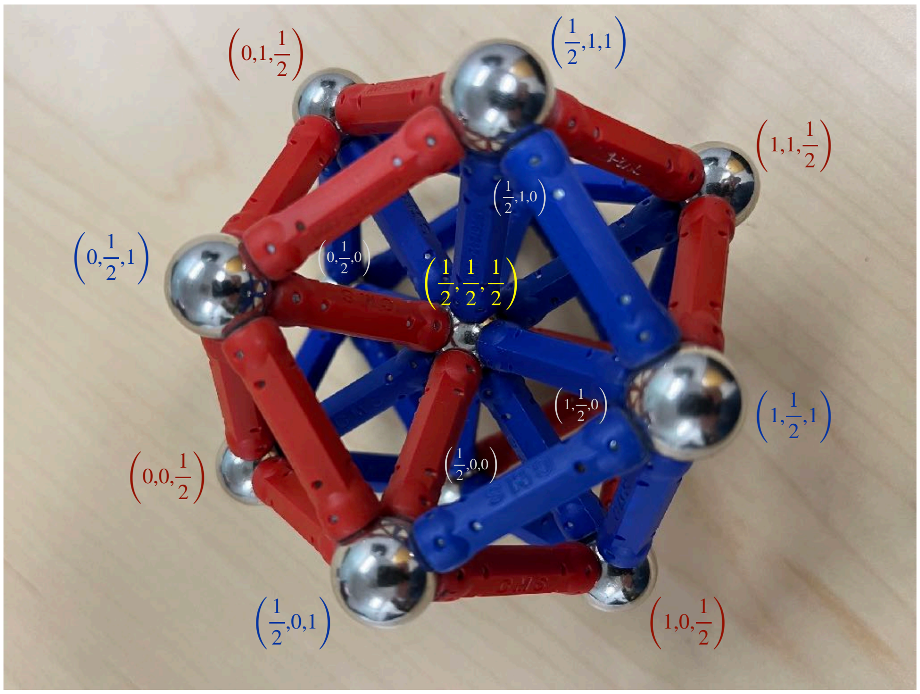

Figure 1: A model of the support of the function $\phi(x,y,z)$ in 3D assembled out of a magnetic constructor. This structure is a piece of the face-centered cubic lattice. The metal balls correspond to mesh nodes. Imagine that you a looking at this structure from above, i.e. the z-axis is directed upwards, and that this structure is embedded into a unit cube so that the nodes have the coordinate as marked in the figure. The points with $z = 0$, $z = \frac{1}{2}$, and $z = 1$ are marked with silver, dark red, and dark blue coordinate labels, respectively, except for the center of the cube, $\left(\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)$, has a yellow label. The edges (the magnetic sticks) connect the all pairs of nodes at distance $\frac{1}{\sqrt{2}}$.

1. (a) Calculate the maximal error of the approximation x^2 by its linear interpolant on the interval $[a,b]$ and find the point where this maximal error is achieved.

   (b) Fill the gaps left in the proof of Proposition 3 in `ApproximationTheory.pdf`. Let $f(x) = x^2$, $f_0(x) = x$. Use the result in the previous item to justify the recursion

$$f_m(x) = f_{m-1}(x) - \frac{g_m(x)}{2^{2m}}, \quad m = 1,2,\ldots, \tag{1}$$

where $g_m(x)$ is the sawtooth function on $[0,1]$ with 2^m teeth defined in `ApproximationTheory.pdf`, and prove that the error at the mth iteration

$$e_m(x) = f_m(x) - x^2, \quad m = 1, 2, \ldots, \tag{2}$$

safisfies

$$\max_{x\in[0,1]} e_m(x) = 2^{-2m-2} \tag{3}$$

2. Let $f(x,y,z)$ be a twice continuously differentiable function defined on the unit cube $[0,1]^3 \subset \mathbb{R}^3$. We partition the unit cube into N^3 subcubes with side $h = \frac{1}{N}$. The we define a mesh so that its nodes, $\{(x_j, y_j, z_j)\}_{j=1}^{N_{\mathsf{nodes}}}$, are the face centers of these cubes and the edges connect all pairs of the face centers at distance $\frac{h}{\sqrt{2}}$ – see Fig. 1 for the case $h = 1$. The mesh defined this way is exactly the face-centered cubic (FCC) lattice, a crystal structure common in nature. It has tetrahedral cells and straight square pyramidal (half-octahedral) cells.

   Consider the linear interpolant of f, denoted by $I_h(f)$, a piecewise linear function that is linear within each mesh tetrahedron and equal to f at each mesh node. To construct it, we define the function $\phi(x,y,z)$ so that

   - it is continuous and piecewise linear,
   - linear within each tetrahedral and half-octahedral cell,
   - equal to one at $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$,
   - equal to zero at the twelve nodes labeled in Fig. 1,
   - and equal to zero outside the convex envelope of these twelve neighbors.

   Then the linear interpolant can be written as

$$I_h f(x,y,z) = \sum_{j=1}^{N_{\mathsf{nodes}}} f(x_j, y_j, z_j)\phi\left(\frac{x - x_j}{h} + \frac{1}{2}, \frac{y - y_j}{h} + \frac{1}{2}, \frac{z - z_j}{h} + \frac{1}{2}\right). \tag{4}$$

   (a) Express the function ϕ as a `ReLU` neural network by extending the construction from Ref. [1] presented in Section 3.10 in `ApproximationTheory.pdf`. Justify your result.

   (b) What is the architecture of the resulting `ReLU` neural network that represents the linear interpolant $I_h f$?

References

[1] J. He, L. Li, and J. Xu, "Relu deep neural networks from the hierarchical basis perspectiveimage 1," *Computers Mathematics with Applications*, vol. 120, pp. 105–114, 2022.