Coordinate Geometry

A Matrix approach

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Problem

Tangents drawn from the point $\begin{bmatrix} -8 \\ 0 \end{bmatrix}$ to the parabola

$$\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -8 & 0 \end{bmatrix} \mathbf{x} = 0$$

touch the parabola at $\mathbf{A}$ and $\mathbf{B}$.

If $\mathbf{F}$ is the focus of this parabola, find the area of $\Delta\mathbf{ABF}$

Introduction

For a curve **S**, if variables are replaced in accordance to $\mathbf{T} = \mathbf{0}$, in coordinate representation, we replace x with $(x + x1)/2$ and x^2 with $x * x1$, and similarly for variable y, to obtain equation of Tangent at point $\begin{bmatrix} x1 & y1 \end{bmatrix}$ on **S**

In this problem, we will take the tangent solution as :

$$\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x1} + \begin{bmatrix} -8 & 0 \end{bmatrix} (\mathbf{x} + \mathbf{x1})/\mathbf{2} = 0$$

where **x1** is the point at which the tangent is drawn.

Approach

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- Using this constraint, apply it in the given parabola equation.
- Finally find the required points and calculate the required area.

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- Also, it is given that this tangent passes through $\begin{bmatrix} -8 \\ 0 \end{bmatrix}$.
- $\begin{bmatrix} -8 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x1} + \begin{bmatrix} -4 & 0 \end{bmatrix} (\mathbf{x1} + \begin{bmatrix} -8 \\ 0 \end{bmatrix}) = 0$

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- Hence, the required points can be derived by putting $\mathbf{x} = \begin{bmatrix} 8 \\ y \end{bmatrix}$ in $\mathbf{x}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} -8 & 0 \end{bmatrix} \mathbf{x} = 0$

  and hence solving for the y value.

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- The solution that we have used has much less time complexity($O(n^2)$)than that of SymPy's Equation Solver($O(n^3)$), but only an approximate solution is obtained(0.01% error), which is acceptable in most cases(Accuracy can be increased at the cost of runtime)

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- If the calculated matrix product and sum from the previously stated equation falls below this threshold, then that value of y is returned
- The linear space is flipped and the procedure is repeated, since we expect at most 2 values for y
- The two values of y are subsequently found, and now we can easily calculate area of the triangle

Results

- Using the above method, the points come out to be

$$\mathbf{A} = \begin{bmatrix} 8 \\ +7.99979997999 \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 8 \\ -7.99979997999 \end{bmatrix}$$

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- The area of $\Delta\mathbf{ABF}$ is hence :

$$1/2 * \begin{vmatrix} 2 & 0 & 1 \\ 8 & 7.99979997999 & 1 \\ 8 & -7.99979997999 & 1 \end{vmatrix} = 47.99879987998 \text{ sq. units.}$$

Solution(Approach B)

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- $\begin{bmatrix} -4 & 0 \end{bmatrix} \mathbf{x1} = \begin{bmatrix} -4 & 0 \end{bmatrix} \begin{bmatrix} -8 \\ 0 \end{bmatrix}$ or simply

$$\begin{bmatrix} 4 & 0 \end{bmatrix} \mathbf{x1} = 32$$

or $\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x1} = 8$

Solution(Approach B)(Cont.d)

- From the question, we have
  $$\mathbf{x1}^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x1} + \begin{bmatrix} -8 & 0 \end{bmatrix} \mathbf{x1} = 0$$

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- And from the previous result, $\begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x1} = 8$
  or, $\begin{bmatrix} 8 & 0 \end{bmatrix} \mathbf{x1} = 64$

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- We see that $LL^T = 64$ or $L^2 = 64$ or $L^T = \pm 8$
- Or, $\begin{bmatrix} 0 & 1 \end{bmatrix} \mathbf{x1} = \pm 8$
- From the previous equations, we have

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- Or,

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \mathbf{x1} = \begin{bmatrix} +8 \\ \pm 8 \end{bmatrix}$$

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- $\mathbf{x1} = \begin{bmatrix} +8 \\ \pm 8 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} => \mathbf{x1} = \begin{bmatrix} +8 \\ \pm 8 \end{bmatrix}$

Results

- Using the above method, the points come out to be

$$\mathbf{A} = \begin{bmatrix} +8 \\ +8 \end{bmatrix}$$
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$$\mathbf{A} = \begin{bmatrix} +8 \\ +8 \end{bmatrix}$$
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- The area of $\Delta\mathbf{ABF}$ is hence :

$$1/2* \begin{vmatrix} 2 & 0 & 1 \\ 8 & 8 & 1 \\ 8 & -8 & 1 \end{vmatrix} = 48 \text{ sq. units.}$$

Plot

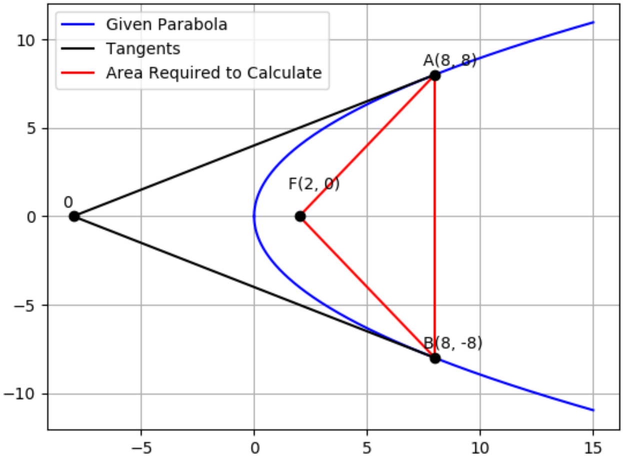

Figure: The Tangents drawn to the parabola, and the required area to be calculated