Mixed Integer Linear Programming Formulation for Optimal Reactive Compensation and Voltage Control of Distribution Power Systems

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***Abstract*—This work proposes a mixed integer linear programming (MILP) formulation, based on a linear power flow that has recently been proposed in the literature, for the reactive power compensation of distribution networks through the optimal selection of capacitor banks (CB) as well as the optimal tap selection of voltage regulators (VRs) and on load tap changers (OLTCs) in order to maintain the system voltages within bounds. In the proposed formulation the voltage dependent load model is taken into account and the objective is to minimize the supply energy cost on a day-long time period. In order to evaluate the proposed model, several simulations are shown for a 34-bus radial distribution system. The results are duly discussed.**

***Index Terms*—Linear power flow, load model, mixed integer linear programming, optimal reactive compensation, voltage control, distribution systems.**

I. Introduction

The demand increase combined with the continuous expansion of the power distribution networks require the development of new operation techniques in order to improve their efficiencies, mainly regarding loss minimization and consequently demand reduction [1]–[3]. Besides improving the operation cost of the networks, the possibility of operating the grid in less stressing conditions is of great interest from the point of view of security and reliability, since the amount of power through the feeders can be relieved.

There are several works addressing the reactive power compensation and voltage control problem. For instance, in [4] a framework for optimally controlling CB and VRs using a quadratically constrained programming problem is proposed. Reference [5] presents an experience with volt-var optimization integrating smart metering. References [6] and [7] present a fuzzy optimization approach and a strategy for placing capacitors at multiple locations for demand reduction, respectively.

Among the methodologies used for volt-var optimization, an approach based on the insertion of devices in the distribution network can be found in [8]. A mixed-integer second-order cone programming model is presented in [9] for obtaining the optimal network operation considering CBs, VRs, distributed generation and energy storage devices. In the same way, a volt-var control via multiobjective optimization is addressed in [10]. This work proposes a strategy based on the next day/week forecast, taking into account the active power intake reduction, and the voltage deviation.

On the other hand, a method for the optimal allocation of CB and VRs is proposed in [11], where some approximations are made in order to linearize the constraints. These works are based on the power network balance and take into account some considerations, such as, the load is represented as constant real and reactive power and the power losses on branches are concentrated in one of the branch terminal nodes.

This paper proposes a formulation for the inclusion of CBs and VRs in a linear power flow model proposed in [12] in order to obtain the optimal tap positions for the regulators and the number of units of capacitor banks for loss minimization. Also, by considering voltage dependent load models and the linear power flow model, the trade-off between computational burden and accuracy is improved. A 34-bus radial distribution system is used to simulate different operating scenarios, such as, considering reactive compensation, considering voltage control and different load models, and so on. All simulations were carried out for a day-long time period.

II. Linear Power Flow Model

One of the main characteristics of distribution power systems is the unbalance of phase currents. This fact requires the use of three-phase power flow models for several case studies [13], [14]. However, for some specific applications (especially for primary distribution networks), the balanced network model can also provide the needed accuracy. The conventional power flow is usually formulated with constant real and reactive power loads, which makes the problem nonlinear and iterative methods are required to find the solution. However, several works available in the literature formulate the load flow problem as a linear problem [15], [12]. Since the nodal voltages and currents are related by the admittance matrix, we have

$$\begin{bmatrix} I_S \\ I_N \end{bmatrix} = \begin{bmatrix} Y_{SS} & Y_{SN} \\ Y_{NS} & Y_{NN} \end{bmatrix} \begin{bmatrix} V_S \\ V_N \end{bmatrix} \tag{1}$$

where S represents the slack node and N is the set of remaining nodes. The net currents (per unit representation) in

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each bus can be related to the nodal voltages through constant impedance, current and power models as follows.

$$I_K = \frac{S_{PK}^*}{V_K^*} + S_{IK}^* + S_{ZK}^* V_K \tag{2}$$

where the PK, IK and ZK subscripts represent the constant power, current and impedance load models at bus K, respectively. One can see that (2) is linear except for the first term, which represents the constant power loads. For that reason, in [12] linearization is proposed, based on the Taylor series around zero, thus, (2) is approximated by (3).

$$I_K \approx PK^* \left(2 - V_K^*\right) + S_{IK}^* + S_{ZK}^* V_K \tag{3}$$

Equation (3) is complex and represents the load current, however, in order to model the CB and VRs, it is necessary to separate its real and imaginary terms, as follows.

$$\begin{aligned} I_K^r = &\rho_K (P_K (2 - V_K^r) + Q_K V_K^i) + \\ &\gamma_K P_K + \alpha_K (P_K V_K^r + Q_K V_K^i) \end{aligned} \tag{4}$$

$$\begin{aligned} I_K^i = &\rho_K (-Q_K (2 - V_K^r) + P_K V_K^i) + \\ &\gamma_K (-Q_K) + \alpha_K (P_K V_K^i - Q_K V_K^r) \end{aligned} \tag{5}$$

Eqs. (4) and (5) correspond to the real and imaginary components of (3) respectively. ρ_K, γ_K and α_K represent the percentage of load model (constant power, current and impedance). Finally, V_K^r and V_K^i are the real and imaginary components of the nodal voltage. Through (1), (4) and (5), the current balance current in the power flow representation can be obtained, as it will be shown in the next section.

III. Mathematical Model for Volt-Var Control

This work proposes the allocation of capacitors banks and voltage regulators to compensate the system's reactive power and to maintain the voltage magnitudes within their limits along the network, respectively [16]. Keeping this idea in mind, the model for CB and VRs is presented next.

A. Capacitor Bank (CB)

In Fig. 1 a capacitor bank with n units is connected at bus i, where each unit injects a certain amount of reactive power Q_n. Subscript d corresponds to the load level. Since the power flow formulation is based on the current balance (equations (4) and (5)), the current injection from the CB is given by

$$\hat{I}_{i,d} = \frac{-jQ_n n_{i,d}}{(\hat{V}_{i,d})^*}, \tag{6}$$

where $\hat{V}_{i,d}$ is the voltage at bus i for load level d. $n_{i,d}$ corresponds to the number of CB units in operation. Equation (6) is complex, therefore, it must be separated into real and imaginary components as follows.

$$I_{i,d}^r = \frac{n_{i,d} Q_n V_{i,d}^i}{\left(V_{i,d}^r\right)^2 + \left(V_{i,d}^i\right)^2} \tag{7}$$

$$I_{i,d}^i = \frac{-n_{i,d} Q_n V_{i,d}^r}{\left(V_{i,d}^r\right)^2 + \left(V_{i,d}^i\right)^2} \tag{8}$$

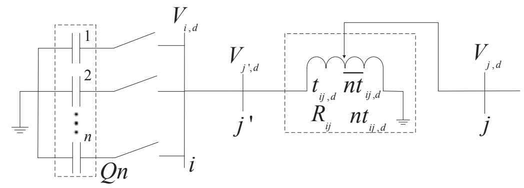

Fig. 1. Capacitor Bank and Voltage regulator

Note that the expression for the real component of the current contains the imaginary component of the voltage, and the expression for the imaginary component of the current contains the real component of the voltage. Also, both equations contain the squared voltage magnitudes. According to [8], Eqs. (9)-(13) limit the number of operation changes in CBs.

$$0 \leq n_{i,d} \leq N_i \tag{9}$$

$$\sum_{d \epsilon D} (\overset{+}{n}_{i,d} + \overset{-}{n}_{i,d}) \leq \Delta_i \tag{10}$$

$$n_{i,d} - n_{i,d-1} = \overset{+}{n}_{i,d} - \overset{-}{n}_{i,d} \tag{11}$$

$$\overset{+}{n}_{i,d} \geq 0 \tag{12}$$

$$\overset{-}{n}_{i,d} \geq 0 \tag{13}$$

where Δ_i represents the maximum number of changes in the operation of switchable units during the period of analysis, N_i is the maximum number of units connected at a node, D is the set of the load levels and $\overset{+}{n}_{i,d}$, $\overset{-}{n}_{i,d}$ are auxiliary variables that indicate a positive and negative change in the number of the CB units.

B. Voltage Regulator (VR) and OLTC

Fig. 1 shows a voltage regulator connected between nodes i and j, where $V_{j',d}$ corresponds to the magnitude of the non-regulated voltage, R_{ij} represents the regulation percentage of the voltage regulator and $t_{ij,d}$ is the tap of the VR at load level d. The relationship between R_{ij} and $t_{ij,d}$ is

$$t_{ij,d} = 1 + R_{ij} \frac{nt_{ij,d}}{\overline{nt}_{ij}}. \tag{14}$$

In [8], the following model for tap voltage regulators and OLTCs is presented.

$$-\overline{nt}_{ij} \leq nt_{ij,d} \leq \overline{nt}_{ij} \tag{15}$$

$$\sum_{d \epsilon D} (\overset{+}{nt}_{ij,d} + \overset{-}{nt}_{ij,d}) \leq Nt_{ij} \tag{16}$$

$$nt_{ij,d} - nt_{ij,d-1} = \overset{+}{nt}_{ij,d} - \overset{-}{nt}_{ij,d} \tag{17}$$

$$\overset{+}{nt}_{ij,d} \geq 0 \tag{18}$$

$$\overset{-}{nt}_{ij,d} \geq 0 \tag{19}$$

Equations (15)-(19) represent the tap operation of the VR for each load level. $nt_{ij,d}$ corresponds to the number of tap steps, $\overline{nt}_{ij}$ is the maximum number of steps, Nt_{ij} is the maximum variation of steps in the time period and $\overset{+}{nt}_{ij,d}$, $\overset{-}{nt}_{ij,d}$ are auxiliary variables that indicate a positive and negative change in the tap steps of the VRs. Since the tap relates the regulated and non-regulated voltages, it is possible to represent its real and imaginary parts as follows

$$V^r_{j,d} = (1 + R_{ij}\frac{nt_{ij,d}}{\overline{nt}_{ij}})V^r_{i,d} \tag{20}$$

$$V^i_{j,d} = (1 + R_{ij}\frac{nt_{ij,d}}{\overline{nt}_{ij}})V^i_{i,d}. \tag{21}$$

C. Linearization of CBs and VRs Expressions

From the set of equations showed earlier corresponding to the CB and VRs models, (7), (8), (20) and (21) are non-linear. Therefore, a linearization must be made to get a linear approach. Since an integer variable can be represented as a set of binary variables and there are products between binary and continuous variables, the disjunctive formulation can be used in order to obtain linear expressions, resulting in

$$Int = \sum_{e=1}^{\beta} e\mu_{C,d,e} \tag{22}$$

$$\sum_{e=1}^{\beta} \mu_{C,d,e} = 1 \tag{23}$$

$$\mu_{C,d,e} = \{0,1\} \tag{24}$$

where $Int = n_{i,d}$ and $\beta = N_i$ for the CB case and $Int = nt_{ij,d}$ and $\beta = 2\overline{nt}_{ij}$ for the VRs case.

Equations (22)-(24) represent the integer variable as a set of binary variables. The product of binary and continuous variables is linearized by defining $A_{i,d,e} = V^{comp}_{i,d}\mu_{i,d,e}$ which is represented by the next three equations.

$$V_{min}\mu_{i,d,e} \le A_{i,d,e} \le V_{max}\mu_{i,d,e} \tag{25}$$

$$V^{comp}_{i,d} - (1-\mu_{i,d,e})\,V_{max} \le A_{i,d,e} \tag{26}$$

$$A_{i,d,e} \le V^{comp}_{i,d} - (1-\mu_{i,d,e})\,V_{min} \tag{27}$$

where $V^{comp}_{i,d}$ represents the continuous variable. It can be a real or imaginary voltage component, in both the CB or VR equations. V_{min} and V_{max} are the minimum and maximum voltage magnitudes, which are parameters of the problem. The linearization procedure is summarized below.

1) Represent the integer variable Int as a set of binary variables (Eqs. (22)-(24)).
2) Replace $n_{i,d}$ and $nt_{ij,d}$ in (7), (8), (20) and (21) by the representation made in 1).
3) Perform the substitution $A_{i,d,e} = V^{comp}_{i,d}\mu_{i,d,e}$ and represent it by (25), (26) and (27).

The step 3) shown above must be performed for both real and imaginary voltage components presented in the formulation. In the VRs case the linearized expressions are

$$V^r_{j,d} = V^r_{i,d} + \frac{R_{ij}}{\overline{nt}_{ij}}\sum_{e=1}^{\beta}(eA_{i,d,e}) \tag{28}$$

$$V^i_{j,d} = V^i_{i,d} + \frac{R_{ij}}{\overline{nt}_{ij}}\sum_{e=1}^{\beta}(eA_{i,d,e}). \tag{29}$$

However, (7) and (8) still remain non-linear, because there are quadratic terms in the denominator. That is why, squared voltage terms are approximated by previously obtained values (V^r_0 and V^i_0) as shown in (30) and (31)

$$I^r_{i,d} = \sum_{e=1}^{N_i}\left[\frac{eQ_nA_{i,d,e}}{(V^r_0)^2 + (V^i_0)^2}\right] \tag{30}$$

$$I^i_{i,d} = \sum_{e=1}^{N_i}\left[\frac{-eQ_nA_{i,d,e}}{(V^r_0)^2 + (V^i_0)^2}\right]. \tag{31}$$

Since each constraint with binary variables can be replaced by a relaxed constraint, where each variable take a different value within the interval $[0,1]$ [17], the approximation above can be used through solving consecutively the non-linear and linear models, as shown below.

1) Relax integrality and solve the non-linear model to find an operation point (V^r_i and V^i_i).
2) Use the operation point found in 1) to set V^r_0 and V^i_0.
3) Solve the linear model.
4) Fix the binary and integer variables to the values found in 3).
5) Solve the non-linear model and the optimal solution is found.

The procedure above allows the use of approximated values to linearize the problem and to avoid infeasible operation points.

D. Mixed integer linear programming

Since this work is aimed to minimizing the supply energy cost on a day-long time period, the objective function is

$$\min\left\{cp\Delta h(V^r_{s,d}I^r_{i,d} + V^i_{s,d}I^i_{i,d})\right\} \tag{32}$$

where cp corresponds to the purchase price of the supplied energy from the substation, Δh is the time duration of each load level h and $V^r_{s,d} = 1$ and $V^i_{s,d} = 0$, which correspond to the substation's voltage fixed in 1 pu. The constraints of the problem are (1), (4) and (5)to represent the network's model, (9)–(13), (30) and (31) to include CBs. Eqs. (15)–(19), (28) and (29) for adding VRs. Finally, (22)–(27) for the respective linearizations of both CBs and VRs equations.

IV. Simulations and Results

The proposed formulation was tested for the 34-bus radial distribution network shown in Fig. 2. The network data is available in [18]. Different energy prices were assumed for each time period. The base case corresponds to the power flow without CBs and VRs for 24 load levels. All the results were modeled in AMPL (A Mathematical Programming Language) and commercial solvers were used to solve the problem.

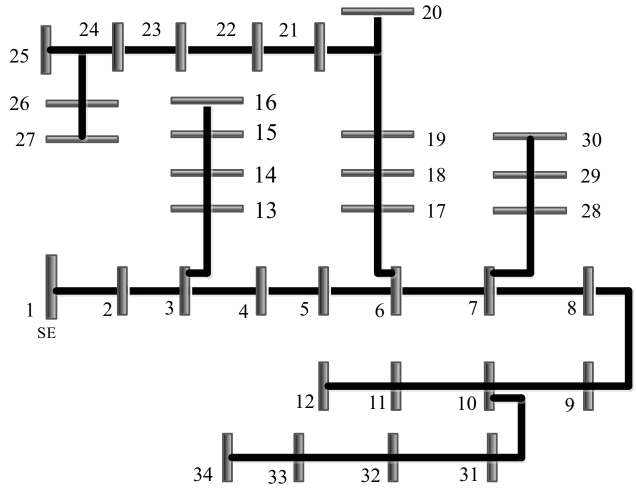

Fig. 2. Distribution test network

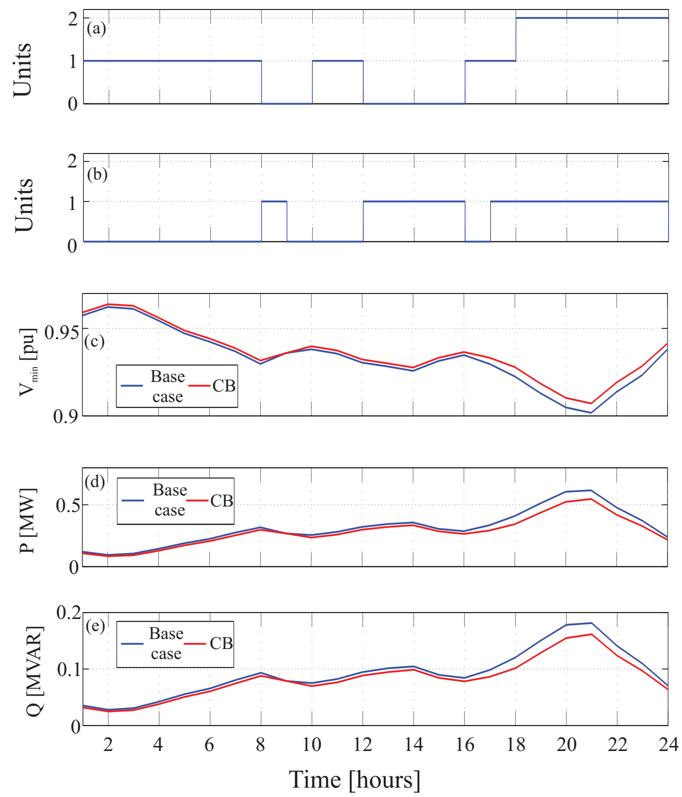

Fig. 3. CBs at buses 26 and 27. (a) CB switching at bus 26. (b) CB switching at bus 27. (c) Lowest voltage for each load level. (d) Total active power losses. (e) Total reactive power losses.

A. Reactive compensation

After running a power flow for the base case with $\rho_K = 0.4$, $\gamma_K = 0.3$ and $\alpha_K = 0.3$ for the polynomial load model and identifying the critical buses (the ones with the lowest bus voltage magnitudes), capacitor banks are then allocated (hence, their locations are assumed to be known). Each CB has 3 units with 300 kVAr capacity. Fig. 3 illustrates the operation with CBs at buses 26 and 27. One can see that the losses are reduced, mainly at the heavy load period (20-22h). The voltage profile is also improved when the system is compensated. With this losses reduction, lower power flows are observed in the feeders, thus, the power substation is required to supply less power, resulting in a lower operation cost of the network.

In order to minimize the operation cost of the distribution grid and to perform the volt-var control, the optimal operation of the CB must also be achieved. For that reason, Figs. 3 (a) and (b) show the operation of the CB units in a way the objective function is met, always satisfying all the constrains. For this case, one unit only (from the three available) for each CB operates in each different period. However, for the critical period both CB modules are operating at the same time. Even for the CB at bus 26 a second unit is connected.

A better compensation is obtained when one more CB is connected at bus 25. In this case the power losses and voltage profile are shown in Fig. 4. One can see the power losses reduced as increases the number of CB in the system and how the voltage drops are also reduced.

B. Reactive compensation and voltage control

A voltage regulator is connected at branch 26-27, since this region of the feeder presents the lowest voltage magnitudes, and an OLTC is connected at branch 1-2 as a way to regulate the substation voltage. Both equipment have 32 tap steps and $\pm 10\%$ regulation. The proposed model determines the tap position for an optimal voltage control. The results are

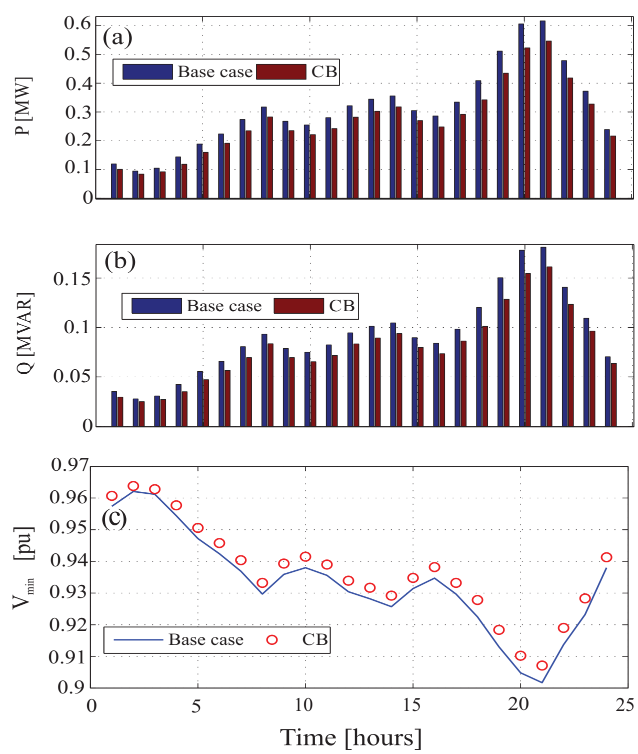

Fig. 4. CBs at buses 25, 26 and 27. (a) Total active power losses. (b) Total reactive power losses. (c) Lowest voltage for each load level.

summarized in Table I, considering different load models. A comparison with the base case is also shown.

TABLE I
CB AND VRs RESULTS

Load Model	Test	Total P [kW]	V_{min} [pu]	O.F [U$]
P	Base case	717.11	0.8923	107938.14
	CB and VR	634.12	0.8998	107131.08
I	Base case	580.69	0.9016	102421.69
	CB and VR	487.06	0.9047	95328.86
Z	Base case	523.26	0.9012	97777.22
	CB and VR	432.98	0.9099	89989.17

The third column of Table I corresponds to the active power losses. The fourth column shows the minimum voltage magnitude, both results for the peak load period (21h). One can see that the constant power model is the most critical one, since the objective functions for this case present the highest values.

Finally, Fig. 5 shows the voltage profile for all buses in each load period after the reactive compensation and the voltage control, with the load model proposed in *A*. Since an appropriate percentage of regulation is fixed and the capacity of the CB is enough to supply a considerable reactive power, the system voltages are within the predefined range ($V_{min} = 0.9$ pu and $V_{max} = 1$ pu), as can be seen in Fig. 5.

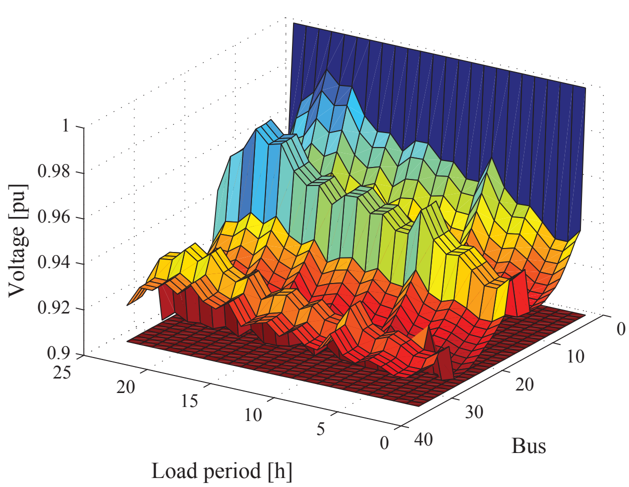

Fig. 5. Voltage profile after reactive compensation and voltage control

V. CONCLUSION

The mixed integer linear programming model proposed in this paper allows the reactive compensation and the voltage control of distribution systems through the optimal operation of CBs and VRs. The model guarantees lower power supply from the substation, and therefore, minimum power losses. Since the voltage dependent load model is appropriately considered and the load condition within a day-long time period is taken into account, the proposed formulation is closer to reality. The inclusion of distributed generation (dispatchable and non-dispatchable), energy storage devices and the stochastic nature of the load and generation is proposed as a further development of this work, as well as, the expansion of the model to a three-phase power distribution network for unbalanced operation, where a possible idea would be to use a three-phase model for obtaining the phase voltages as balanced as possible after volt-var control.

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