Lecture 9
Frequency-domain Representation

ECEN5283 Computer Vision

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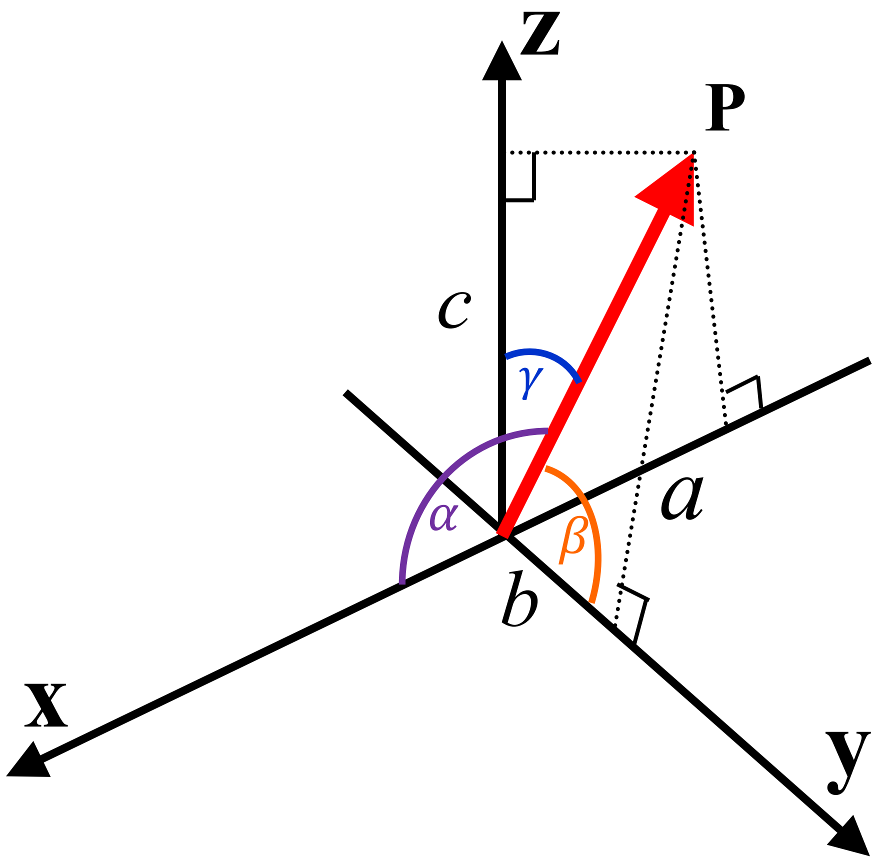
Goals

To represent an image in the frequency domain by the discrete Fourier transform (DFT).

To discuss why it is important to design a linear-phase filter in spatial filter design.

To preview the design of Gabor filters in the frequency domain.

Vector Representation



Vector Analysis

$$a = \langle \mathbf{p}, \mathbf{x} \rangle = \|\mathbf{p}\| \cdot \|\mathbf{x}\| \cos \alpha$$
$$b = \langle \mathbf{p}, \mathbf{y} \rangle = \|\mathbf{p}\| \cdot \|\mathbf{y}\| \cos \beta$$
$$c = \langle \mathbf{p}, \mathbf{z} \rangle = \|\mathbf{p}\| \cdot \|\mathbf{z}\| \cos \gamma$$

$\|\mathbf{p}\|$ the length of vector $\mathbf{p}$

$$\|\mathbf{x}\| = \|\mathbf{y}\| = \|\mathbf{z}\| = \mathbf{1}$$

Vector Synthesis

$$\mathbf{p} = a \cdot \mathbf{x} + b \cdot \mathbf{y} + c \cdot \mathbf{z}$$

Dot Product of Complex Vectors

Let $\mathbf{x} = \{x_1, x_2, \dots., x_N\}$ and $\mathbf{y} = \{y_1, y_2, \dots., y_N\}$ be two vectors in $\mathbf{C}^N$ (N-D complex space), the dot product (inner product) for $\mathbf{C}^N$ is

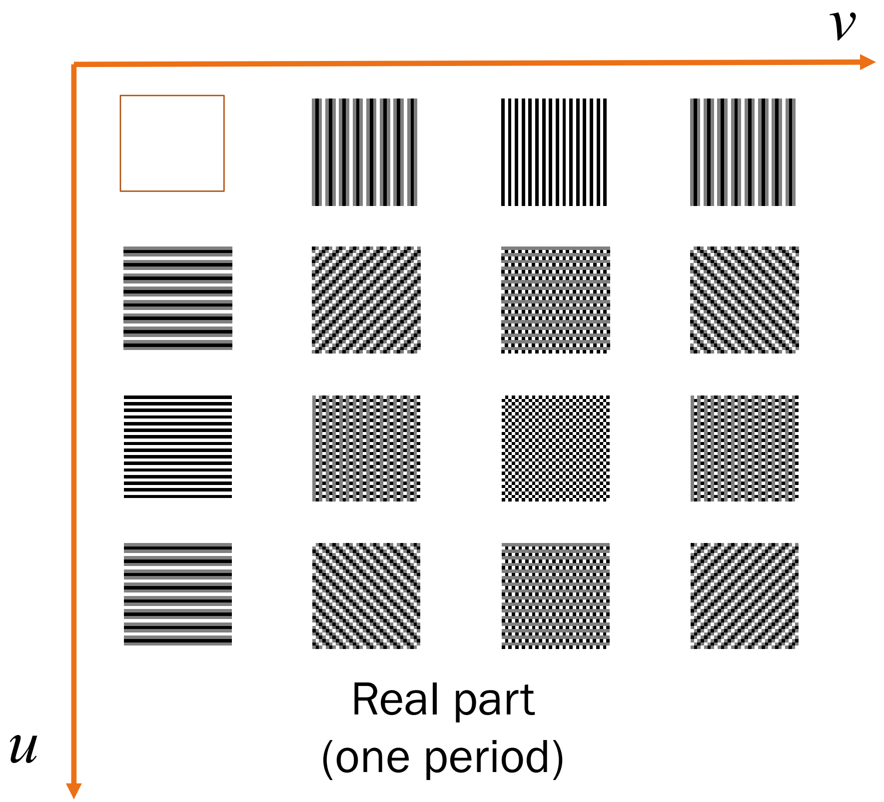
$$\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1^* + x_2 y_2^* + \cdots + x_N y_N^*$$

y_i^* is the complex conjugate of y_i

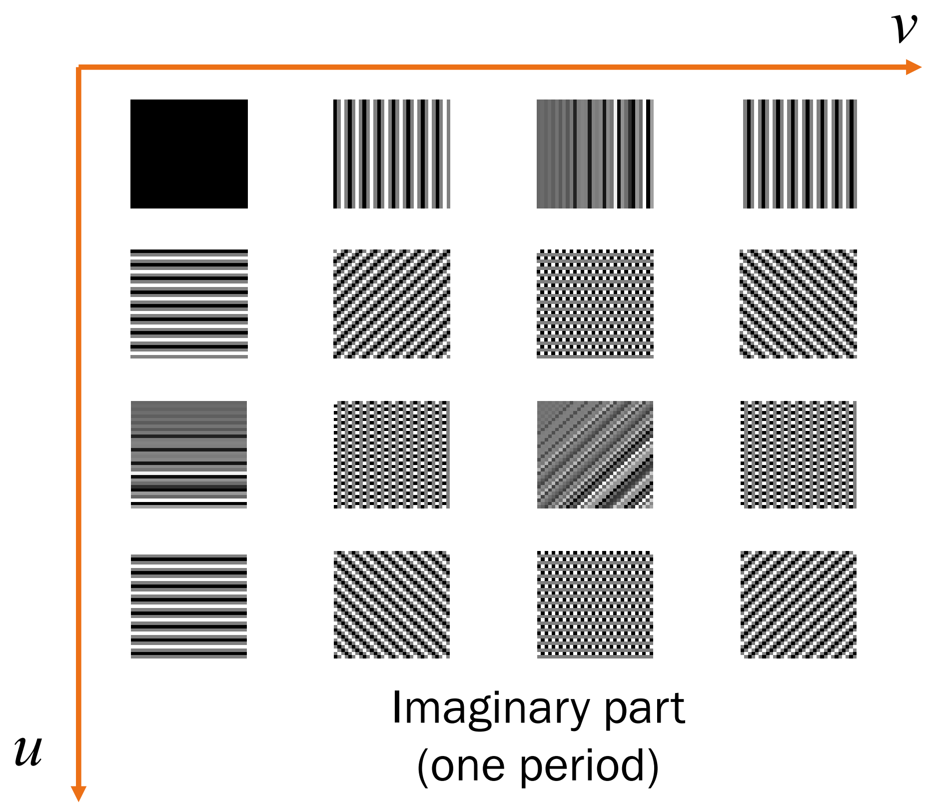
- **Orthogonality:** $\mathbf{x} \perp \mathbf{y}$ if $\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{0}$
- **Vector Length:** $||\mathbf{x}|| = \sqrt{x_1 y_1^* + x_2 y_2^* + \cdots + x_2 y_2^*}$
- **Distance:** $\boldsymbol{d}(\mathbf{x}, \mathbf{y}) = ||\mathbf{x} - \mathbf{y}||$

2D DFT Basis Functions

$$\varphi_{u,v}(x,y) = e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)} \; (M \times N)$$



Real part
(one period)



Imaginary part
(one period)

2D Discrete Fourier Transform

Given an image $f(x,y)$ of $M \times N$, and the 2D DFT basis functions $\varphi_{u,v}(x,y) = e^{j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$, its 2-D DFT $F(u,v)$ is defined below:

$$F(u,v) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\varphi_{u,v}^{*}(x,y) \quad (u = 0, \dots, M-1; v = 0, \dots N-1)$$

2D frequency coordinates

The inverse DFT (IDFT) is defined as

$$f(x,y) = \sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\varphi_{u,v}(x,y) \quad (x = 0, \dots, M-1; y = 0, \dots N-1)$$

2D spatial coordinates

DFT Spectra

Magnitude (power) spectrum

- (Matlab function: “abs ”)

$$|F(u,v)| = \sqrt{\mathrm{Re}(F(u,v))^2 + \mathrm{Im}(F(u,v))^2}$$

Phase spectrum

- (Matlab function: “angle”)

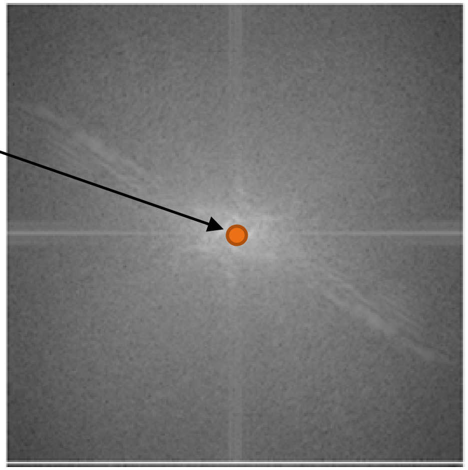
$$\psi(u,v) = \tan^{-1}\left(\frac{\mathrm{Im}(F(u,v))}{\mathrm{Re}(F(u,v))}\right)$$

Real $f(x,y) \rightarrow F(u,v) = F^*(M-u, N-v)$

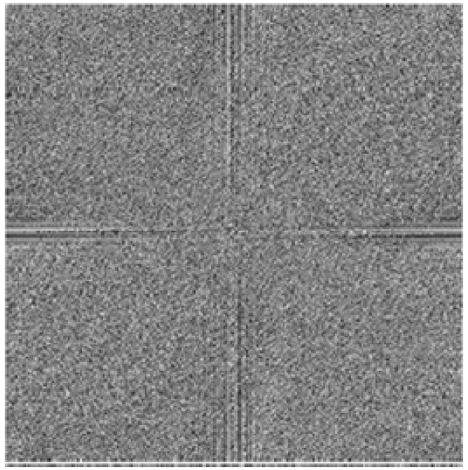
(Conjugate symmetric)

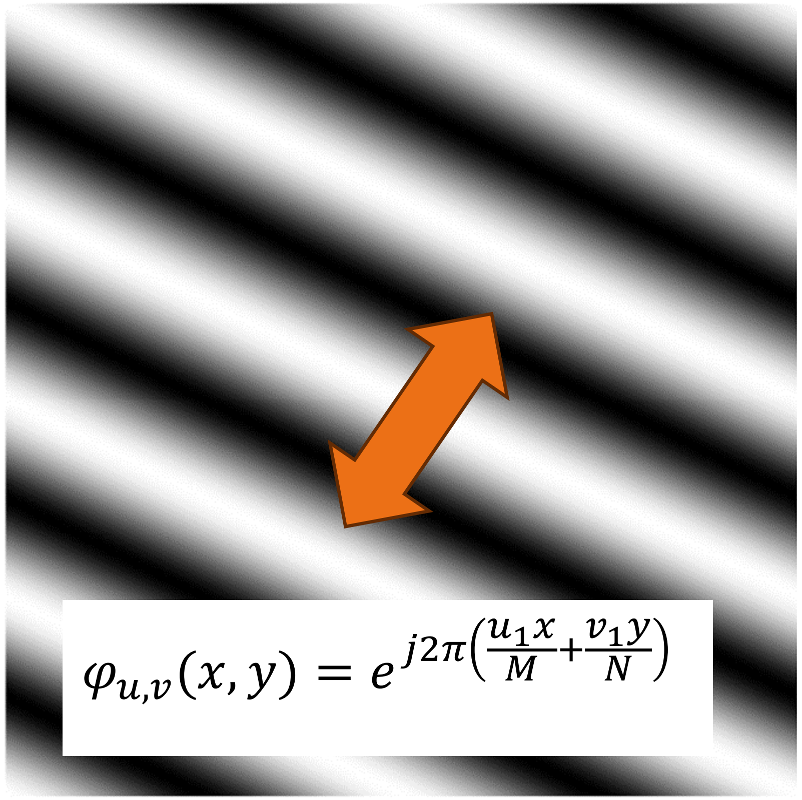
$|F(u,v)|$

$\{\frac{M}{2}, \frac{N}{2}\}$

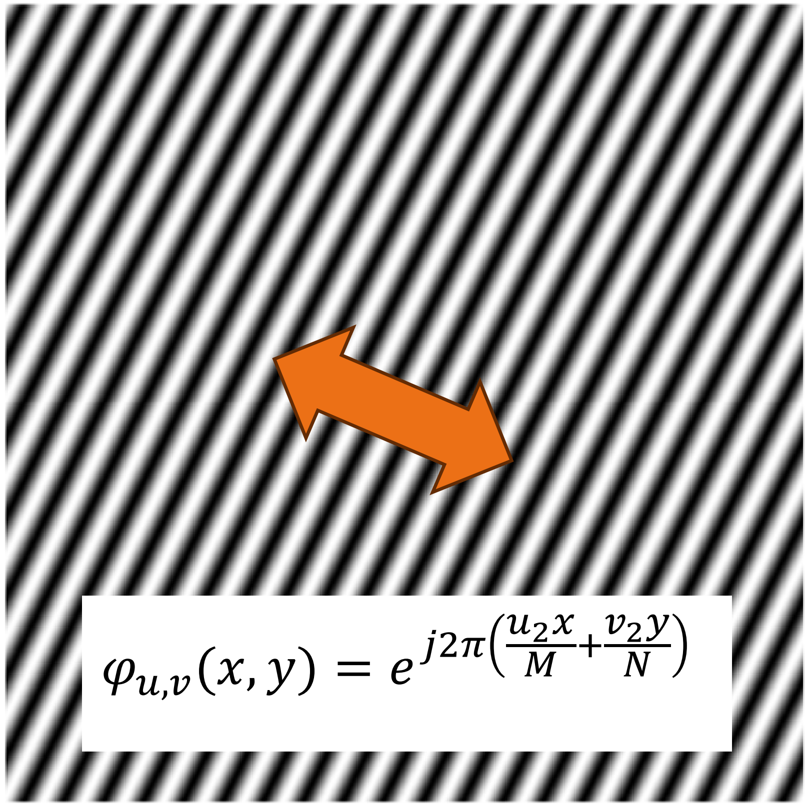


$\psi(u,v)$

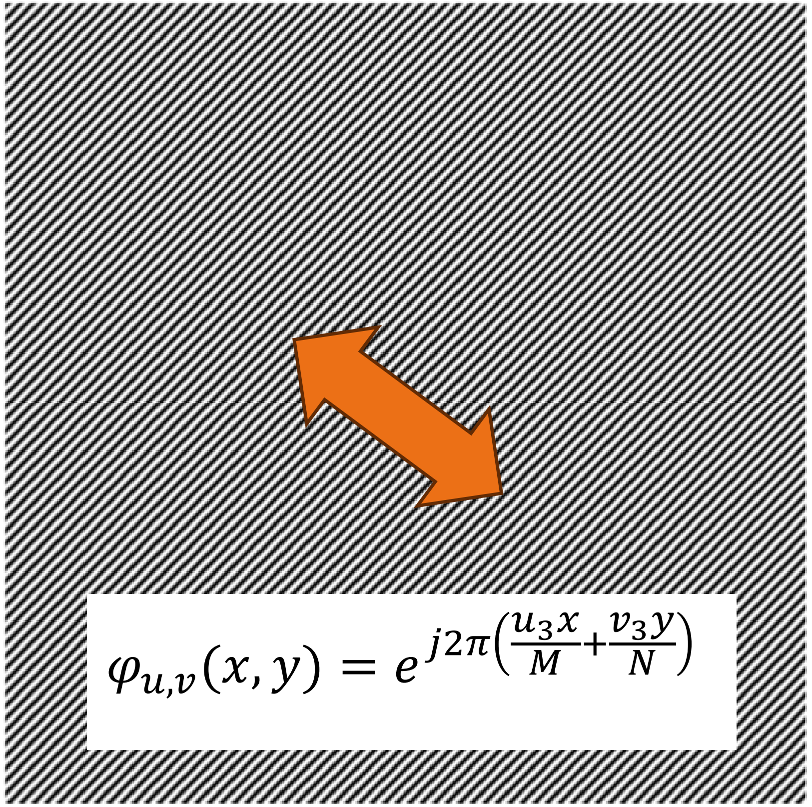




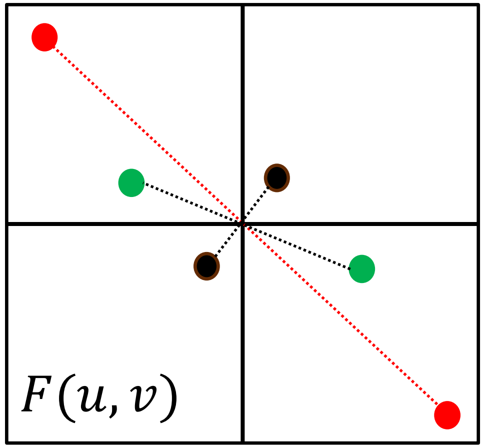
$$\varphi_{u,v}(x,y) = e^{j2\pi\left(\frac{u_1 x}{M}+\frac{v_1 y}{N}\right)}$$



$$\varphi_{u,v}(x,y) = e^{j2\pi\left(\frac{u_2 x}{M}+\frac{v_2 y}{N}\right)}$$



$$\varphi_{u,v}(x,y) = e^{j2\pi\left(\frac{u_3 x}{M}+\frac{v_3 y}{N}\right)}$$



Magnitude and Phase Spectra

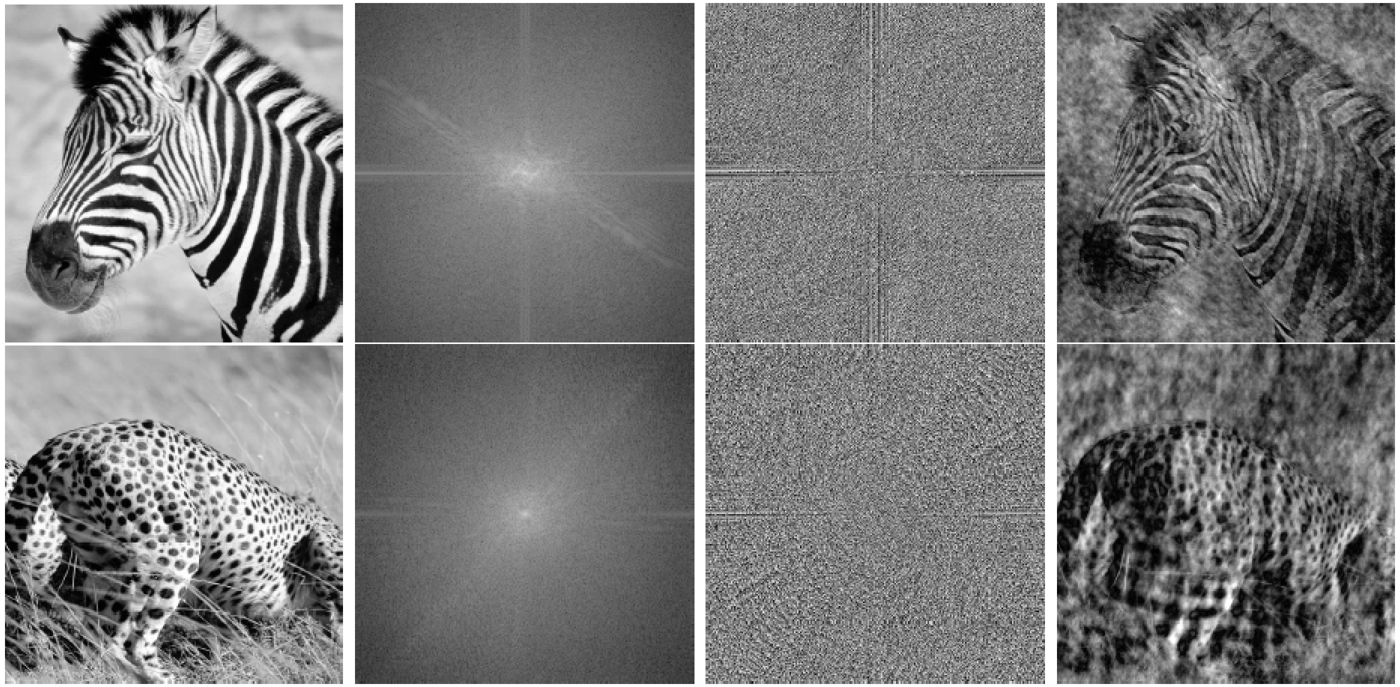


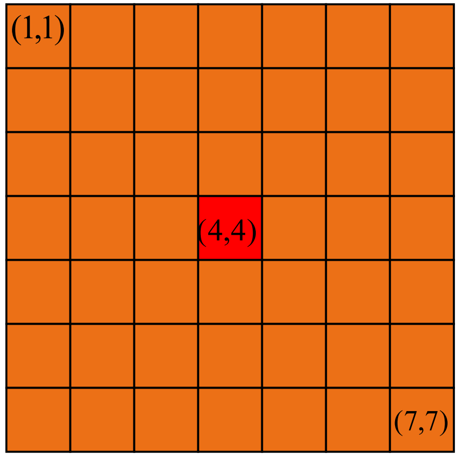
Figure 8.5. The second image in each row shows the log of the magnitude spectrum for the first image in the row; the third image shows the phase spectrum, scaled so that $-\pi$ is dark and π is light. The final images are obtained by swapping the magnitude spectra. While this swap leads to substantial image noise, it doesn't substantially affect the interpretation of the image, suggesting that the phase spectrum is more important for perception than the magnitude spectrum.

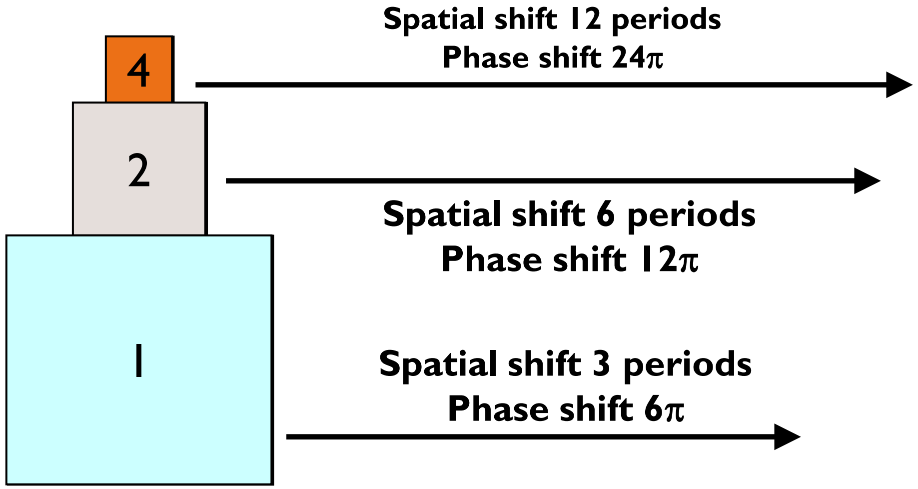
Linear Phase Filtering

The *relative location* of different frequency components are shifted linearly proportional to their frequency to keep the frequency integrity of a signal.

- The filter is designed to odd-sized $(2k+1, 2k+1)$ be symmetric about the center $(k+1, k+1)$.
- Then the filtered image will be shifted by k pixels (to the left and downward).
- We can perform zero-phase filtering (no shift) by shifting back the filtered image k pixels.

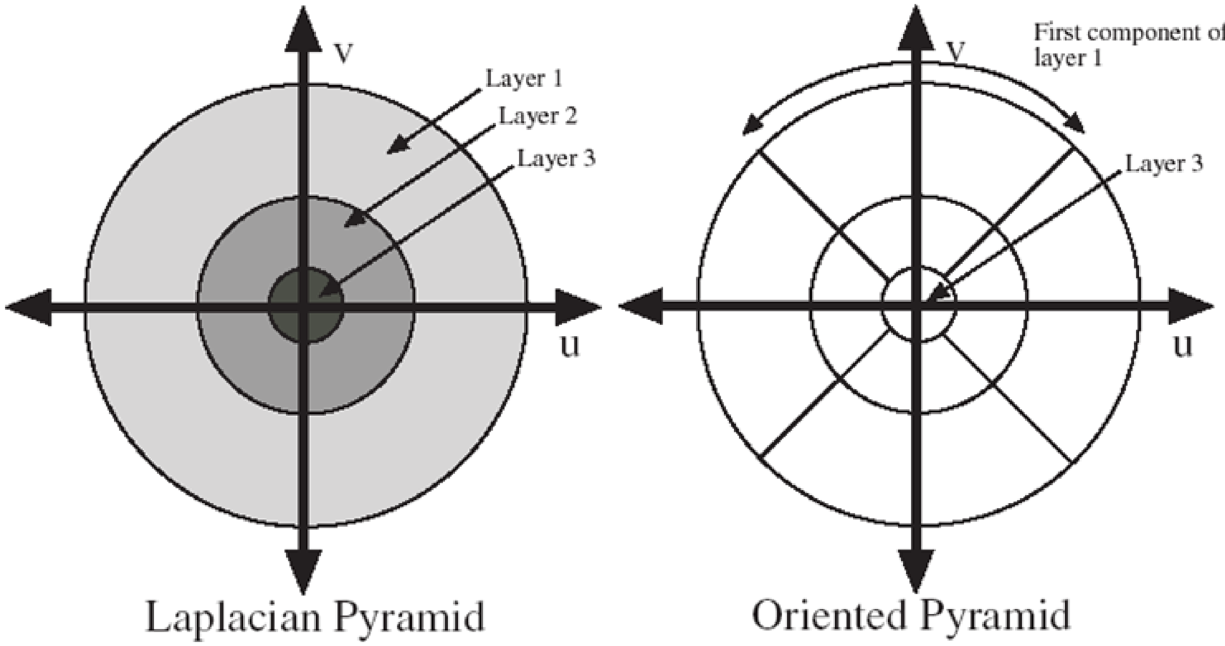
$k = 3$





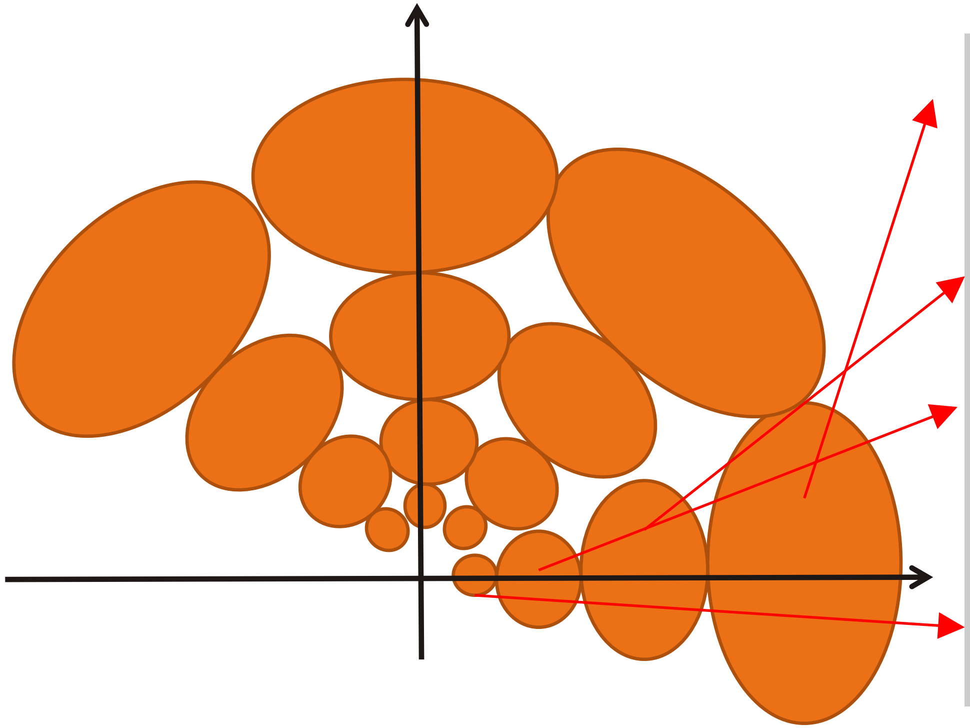
Frequency-domain Filter Design

The frequency-domain representation can give us some insights to a systematic design of a filter bank for texture analysis.

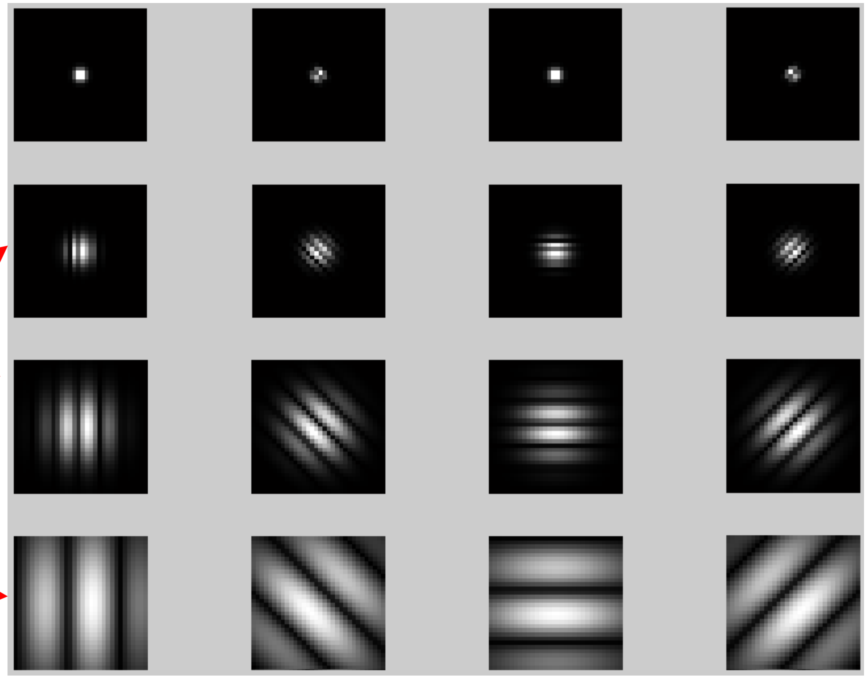


Oriented filters can reveal more distinct frequency characteristics between different textures.

Gabor Filter Kernels (Magnitude): 4 Scales and 4 Orientations

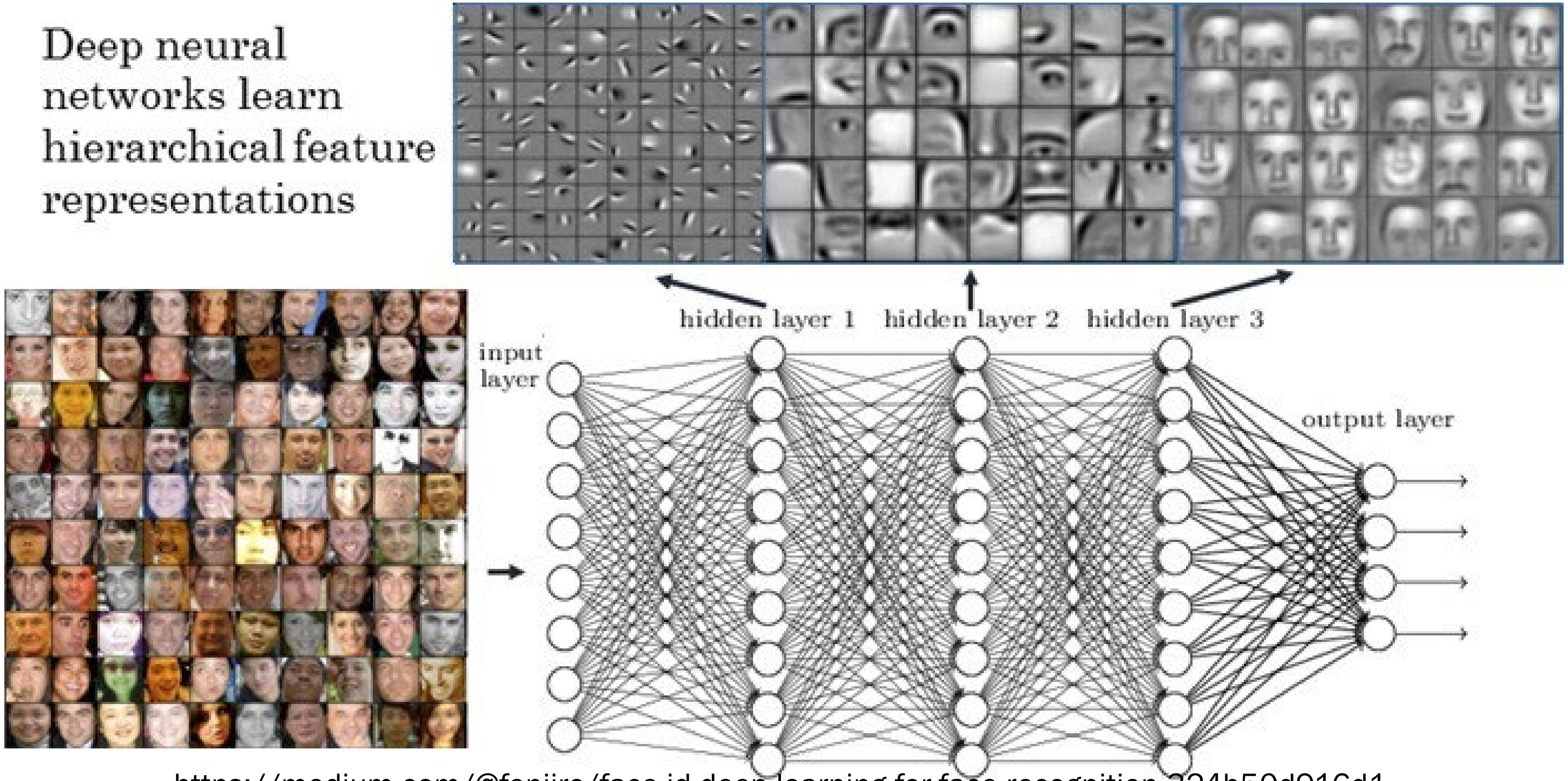


Frequency representation
of the Gabor filter design



Spatial representation of
of Gabor filter kernels

Face Features from Deep Learning



https://medium.com/@fenjiro/face-id-deep-learning-for-face-recognition-324b50d916d1