ODE Notes

Alex Feng

June 2023

1 First Order Equations

1.1 Linear Equations

An ODE is linear in y if it can be expressed as

$$a_n(x)y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1(x)y - g(x) = 0$$

A linear equation is homogeneous when $g(x) = 0$. A linear first-order equation can be solved using an integrating factor $e^{\int P(x)dx}$. Starting with the standard form,

$$\frac{dy}{dx} + P(x)y = f(x)$$

$$\frac{dy}{dx}e^{\int P(x)dx} + P(x)ye^{\int P(x)dx} = f(x)e^{\int P(x)dx}$$

$$ye^{\int P(x)dx} = c + \int f(x)e^{\int P(x)dx}dx$$

$$y = ce^{-\int P(x)dx} + e^{-\int P(x)dx}\int f(x)e^{\int P(x)dx}dx$$

1.2 Exact Equations

$M(x,y)dx + N(x,y)dy$ is an exact differential if it corresponds to the differential of some function $f(x,y)$.

$$M(x,y)dx + N(x,y)dy = 0$$

is an exact equation since the left hand side is an exact differential, and implies $f(x,y) = c$. Intuitively, $M(x,y)dx + N(x,y)dy$ must be an exact differential if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

After checking that the equation is exact, it can be solved starting with the equation in the form

$$M(x,y)dx + N(x,y)dy = 0$$

$$\frac{\partial f}{\partial x} = M(x,y)$$

$$f(x,y) = g(y) + \int M(x,y)dx$$

(the function $g(y)$ is the "constant" of integration here)

$$\frac{\partial f}{\partial y} = g'(y) + \frac{\partial}{\partial y}\int M(x,y)dx = N(x,y)$$

The last two equations can be used to find a function $f(x,y)$ for the implicit solution $f(x,y) = c$.
A non-exact equation can sometimes be turned into an exact equation using an integrating factor $\mu(x,y)$. To avoid solving a PDE, μ can only be a function of a single variable. For the non-exact equation $M(x,y)dx + N(x,y)dy = 0$, if $(M_y - N_x)/N$ is only a function of x,

$$\mu(x) = e^{\int \frac{M_y - N_x}{N} dx}$$

if $(N_x - M_y)/M$ is only a function of y,

$$\mu(y) = e^{\int \frac{N_x - M_y}{M} dy}$$

1.3 Homogeneous Functions

A function that has the property

$$f(tx,ty) = t^{\alpha} f(x,y)$$

is a homogeneous function of degree α. A first-order DE in differential form

$$M(x,y)dx + N(x,y)dy = 0$$

is homogeneous if M and N are homogeneous equations of the same degree. This type of equation can be solved using the substitution

$$x = uy \text{ or } y = ux$$

For the substitution $x = uy$,

$$y^{\alpha} M(u,1)d(uy) + y^{\alpha} N(u,1)dy = 0$$

$$M(u,1)ydu + M(u,1)udy + N(u,1)dy = 0$$

$$\frac{M(u,1)du}{M(u,1)u + N(u,1)} = -\frac{dy}{y}$$

The other substitution, $y = ux$, is similar.

2 Higher Order Equations

Let $y_1, y_2, \ldots, y_n$ be solutions to a linear nth order differential equation on an interval I. The set of solutions are linearly independent iff the Wronskian,

$$W(y_1, y_2, \ldots, y_n) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix} \neq 0$$

2.1 Reduction of Order

For a homogeneous second order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0$$

if a solution y_1 is already known, the second solution can be written as $y_2 = uy_1$. Substituting y_2 into the equation,

$$a_2(x)(u''y_1 + 2u'y_1' + uy_1'') + a_1(x)(u'y_1 + uy_1') + a_0(x)uy_1 = 0$$

$$a_2(x)(u''y_1 + 2u'y_1') + a_1(x)(u'y_1) + u(a_2(x)y_1'' + a_1(x)y_1' + a_0(x)y_1 = 0$$

$$a_2(x)(u''y_1 + 2u'y_1') + a_1(x)(u'y_1) = 0$$

This equation has now been reduced to a first order equation in terms of u'.

$$u' = e^{\int -(a_1(x)y_1 - 2a_2(x)y_1')/a_2(x)\,dx}$$

$$y_2 = y_1 \int e^{\int -(a_1(x)y_1 - 2a_2(x)y_1')/a_2(x)\,dx} dx$$

2.2 Linear Equations with Constant Coefficients

To find solutions of an auxiliary equation

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

where $\forall i = 0, 1, \ldots n, a_i$ is constant, use the substitution $y = e^{mx}$.

$$e^{mx}(a_n m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0) = 0$$

$$a_n m^n + a_{n-1}m^{n-1} + \cdots + a_1 m + a_0 = 0$$

For any repeated root $m = p$, repeated j times, the general solution will be a linear combination of e^{qx}, for all single roots q, and $e^{px}, xe^{px}, \ldots, x^j e^{px}$ for all repeated roots p. Additionally, for an equation

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$$

the general solution will be the sum of the solution to the corresponding auxiliary equation ($g(x) = 0$) and any particular solution y_p. y_p can be found using the undetermined coefficients method.

2.3 Variation of Parameters

Variation of parameters can be used to solve an equation in the form

$$y'' + P(x)y' + Q(x)y = f(x)$$

Let y_1 and y_2 be in the fundamental set of solutions for the associated homogeneous equation. The solution must be in the form

$$y_p = u_1(x)y_1(x) + u_2(x) + y_2(x)$$

$$y_p' = u_1'y_1 + y_1'u_1 + u_2'y_2 + y_2'u_2$$

$$y_p'' = u_1''y_1 + u_1'y_1' + u_1'y_1' + u_1y_1'' + u_2''y_2 + u_2'y_2' + u_2'y_2' + u_2y_2''$$

$$y'' + P(x)y' + Q(x)y$$

$$= u_1''y_1+u_1'y_1'+u_1'y_1'+u_1y_1''+u_2''y_2+u_2'y_2'+u_2'y_2'+u_2y_2''+P(u_1'y_1+y_1'u_1+u_2'y_2+y_2'u_2)+Q(u_1y_1+u_2+y_2)$$

$$= u_1(y_1''+Py_1'+Qy_1)+u_2(y_2''+Py_2'+Qy_2)+u_1''y_1+u_1'y_1'+u_2''y_2+u_2'y_2'+P(y_1u_1'+y_2u_2')+y_1'u_1'+y_2'u_2'$$

$$= \frac{d}{dx}(u_1'y_1 + u_2'y_2) + P(y_1u_1' + y_2u_2') + y_1'u_1' + y_2'u_2' = f(x)$$

Assuming

$$u_1'y_1 + u_2'y_2 = 0$$

$$y_1'u_1' + y_2'u_2' = f(x)$$

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}}{W}$$

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}}{W}$$

For the Wronskian of y_1 and y_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$y_p = -y_1 \int \frac{y_2 f(x)}{W} dx + y_2 \int \frac{y_1 f(x)}{W} dx$$

For an nth order equation,

$$W_m = \begin{vmatrix} y_1 & y_2 & \cdots & y_{m-1} & 0 & y_{m+1} & \cdots & y_n \\ y_1' & y_2' & \cdots & y_{m-1}' & 0 & y_{m+1}' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_{m-1}^{(n-2)} & 0 & y_{m+1}^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_{m-1}^{(n-1)} & f(x) & y_{m+1}^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

$$u_m' = \frac{W_m}{W(y_1, y_2, \ldots, y_n)}$$

$$y = \sum_{j=1}^{n} y_j u_m = \sum_{j=1}^{n} y_j \int \frac{W_j}{W(y_1, y_2, \ldots, y_n)} dx$$

2.4 Cauchy-Euler Equation

An equation in the form

$$a_n x^n \frac{d^n y}{dx^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1 x \frac{dy}{dx} + a_0 y = g(x)$$

with constant coefficients $a_0, a_1, \ldots, a_n$ can be solved using the substitution $y = x^m$. The auxiliary equation becomes

$$\sum_{k=0}^{n} a_k \frac{m!}{(m-k)!} x^m = 0$$

solving for m leads to the general solution. When m is a repeated root, reduction of order can be used to find that the solution would be in the form

$$y = c_1 x^{m_1} + c_2 x^{m_1} \ln x$$

3 Series Solutions

3.1 Ordinary Point

Equations can be solved using the series

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

where x_0 is an ordinary point ($P(x)$ and $Q(x)$ in the standard form are both analytic). The radius of convergence is the distance from x_0 to the nearest singular point ($P(x)$ and $Q(x)$ are not analytic).

3.2 Regular Singular Point

x_0 is a regular singular point if $p(x) = (x - x_0)P(x)$ and $q(x) = (x - x_0)^2 Q(x)$ are both analytic at x_0. A solution can be found using Frobenius' Theorem,

$$y = (x - x_0)^r \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

where r is a constant to be determined.

4 Laplace Transform

4.1 Definition

Integral transforms are in the form

$$\int_0^{\infty} K(s,t) f(t) dt$$

where $K(s,t)$ is the kernel of a transform. The Laplace Transform uses $k(s,t) = e^{-st}$. The Laplace Transform of a function $f(t)$ defined for $t \geq 0$ is

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt$$

$\mathscr{L}\{f(t)\}$ exists for $s > c$ if f is piecewise continuous on $[0, \infty)$ and of exponential order c.
Transforms of Basic Functions:

$$\mathscr{L}\{1\} = \frac{1}{s}$$

$$\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad n = 1, 2, 3, \ldots$$

$$\mathscr{L}\{e^{at}\} = \frac{1}{s-a}$$

$$\mathscr{L}\{\sin kt\} = \frac{k}{s^2 + k^2}$$

$$\mathscr{L}\{\cos kt\} = \frac{s}{s^2 + k^2}$$

$$\mathscr{L}\{\sinh kt\} = \frac{k}{s^2 - k^2}$$

$$\mathscr{L}\{\cosh kt\} = \frac{s}{s^2 - k^2}$$

4.2 Properties

If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$,

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

where $F(s) = \mathscr{L}\{f(t)\}$.

If $\mathscr{L}\{f(t)\} = F(s)$, then

$$\forall a \in \mathbb{R}, \mathscr{L}\{e^{at} f(t)\} = F(s-a)$$

The unit step function $\mathscr{U}(t-a)$ is defined to be

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

If $F(s) = \mathscr{L}\{f(t)\}$ and $a > 0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\} = e^{-as} F(s)$$

If $F(s) = \mathscr{L}\{f(t)\}$, then

$$\forall n \in \mathbb{N}, \mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

The convolution of two piecewise continuous functions on the inverval $[0, \infty)$, f and g, is defined as

$$f * g = \int_0^t f(\tau) g(t - \tau) d\tau$$

The convolution theorem for Laplace transforms states that

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\mathscr{L}\{g(t)\} = F(s)G(s)$$

The transform of $f(t)$, which is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T is

$$\mathscr{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t) dt$$

5 Useful Functions

5.1 Dirac Delta

The delta function is a piecewise function defined as

$$\delta_a(t - t_0) = \begin{cases} 0, & 0 \leq t < t_0 - a \\ \frac{1}{2a}, & t_0 - a \leq t < t_a + a \\ 0, & t \geq t_0 + a \end{cases}$$

It is called a unit impulse since

$$\int_0^\infty \delta_a(t - t_0) dt = 1$$

The Dirac delta "function" is defined as

$$\delta(t - t_0) = \lim_{a \to 0} \delta_a(t - t_0)$$

It has the following properties:

$$\delta(t - t_0) = \begin{cases} \infty, & t = t_0 \\ 0, t & \neq t_0 \end{cases}$$

$$\int_0^\infty \delta(t - t_0) dt = 1$$

$$\mathscr{L}\{\delta(t - t_0)\} = e^{-st_0}$$

5.2 Gamma

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}$$
$$\forall n \in \mathbb{N}, \Gamma(n+1) = n!$$

5.3 Bessel Functions

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0$$

is called a Bessel's equation of order ν. The solution is

$$y = c_1 J_\nu(x) + c_2 J_{-\nu}(s), \quad \nu \notin \mathbb{Z}$$

where $J_\nu(x)$ is the Bessel function of the first kind,

$$J_\nu = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(1+\nu+n)} \left(\frac{x}{2}\right)^{2n+\nu}$$

Additionally, another solution is

$$y = c_1 J_\nu(s) + c_2 Y_\nu(x), \quad \nu \notin \mathbb{Z}$$

where $Y_\nu(x)$ is the Bessel function of the second kind,

$$Y_\nu(x) = \frac{\cos \nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin \nu\pi}$$

The modified Bessel equation of order ν is

$$x^2y'' + xy' - (x^2 + \nu^2)y = 0$$

The solution, written in terms of the modified Bessel function of the first kind and the modified Bessel function of the second kind is

$$y = c_1 I_\nu(x) + c_2 K_\nu(x)$$

where

$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$
$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin \nu\pi}$$

Another common equation is

$$y'' + \frac{1-2a}{x}y' + \left(b^2c^2x^{2c-2} + \frac{a^2 - p^2c^2}{x^2}\right) y = 0, \quad p \geq 0$$

which has the solution

$$y = x^a \left(c_1 J_p(bx^c) + c_2 Y_p(bx^c)\right)$$