The Quaternion Group and Klein Four Group

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November 2023

1 The Quaternion Group

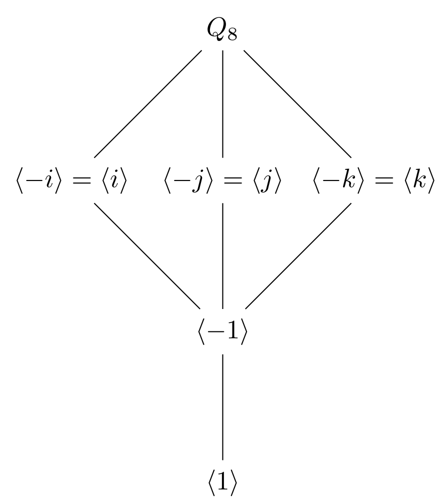
The Quaternion group, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$, is a finite group of order 8. It is not abelian, and its binary operation is as follows: For all $a \in Q_8$,

$$1 \cdot a = a \cdot 1 = a$$

$$(-1) \cdot (-1) = 1$$
$$(-1) \cdot a = a, a \cdot (-1) = -a$$
$$i \cdot i = j \cdot j = k \cdot k = -1$$

$$\begin{aligned} i \cdot j &= k, & j \cdot i &= -k \\ j \cdot k &= i, & k \cdot j &= -i \\ k \cdot i &= j, & i \cdot k &= -j. \end{aligned}$$

The order of 1 is 1, and the order of -1 is 2. The order of the rest of the elements is 4. Q_8 can be generated by any two elements other than 1 and -1, for example, $Q_8 = \langle -i, k \rangle$. All of its subgroups are normal.



Applications

Quaternions in the form $q = w + xi + yj + zk$ where i, j, k multiplication follows the rules above can be used to represent rotations in 3D space. Let

$$\vec{u} = (u_x, u_y, y_z)$$

be a unit vector that represents the axis of rotation. Let

$$q = \sin\left(\frac{\theta}{2}\right) + \cos\left(\frac{\theta}{2}\right)(u_x i + u_y j + u_z k)$$

where θ is the rotation angle, and represent a vector v as

$$v = v_x i + v_y j + v_z k$$

The vector v rotated an angle of θ around axis $\vec{u}$ is

$$v' = q \cdot v \cdot q^{-1}$$

Where q^{-1} is the inverse of q, and can easily be calculated (since it is a unit vector) by

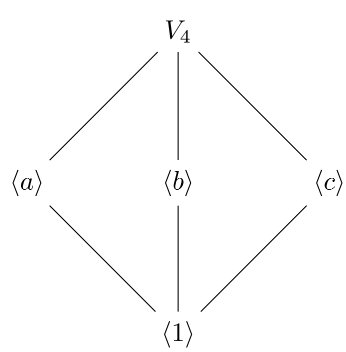
$$q^{-1} = \sin\left(\frac{\theta}{2}\right) - \cos\left(\frac{\theta}{2}\right)(u_x i + u_y j + u_z k)$$

2 The Klein Four Group

The Klein Four Group (Vierergruppe), $V_4 = \{1, a, b, c\}$, is a finite abelian group of order 4. Its binary operation is as follows: For all $x \in V_4$,

$$1x = x1 = x$$
$$x^2 = 1$$
$$ab = c$$
$$bc = a$$
$$ac = b$$

The order of 1 is 1, and the order of the other elements is 2. V_4 can be generated by any two elements from $\{a, b, c\}$. All of its subgroups are normal. Observe that $Q_8/\langle -1 \rangle \cong V_4$.



Applications

The Klein Four Group is used in coding theory. In particular, it is used for error detection/correction, digital data transmission, and data storage.