Linear Algebra Notes

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1 Linear Equations

Definition: Echelon Form

A rectangular matrix is in echelon form (or row echelon form/REF) if it has the following properties:

1. All nonzero rows are above any rows of all zeros
2. Each leading entry of a row is in a column to the right of a leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

A matrix is in reduced echelon form (or reduced row echelon form/RREF) if it also satisfies the additional conditions:

4. The leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

$$\begin{bmatrix} 0 & 1 & 3 & 0 & 0 & 7 \\ 0 & 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is an example of a matrix in RREF form

Theorem 1: Uniqueness of the RREF

Each matrix is row equivalent to one and only one reduced echelon matrix.

Definition: Pivot Position

A pivot position in a matrix A is a location in A that corresponds to a leading 1 in the reduced echelon form of A. A pivot column is a column of A that contains a pivot position.

Convention: Free variables

Parametric descriptions of solutions should be written in terms of free variables (variables corresponding to non-pivot columns). Basic variables correspond to pivot columns.

Theorem 2: Existence and Uniqueness Theorem

A linear system is consistent iff the rightmost column of the augmented matrix is not a pivot column. A consistent system will have either a unique solution (no free variables) or infinitely many solutions (at least one free variable).

Definition: Span

If $\vec{v}_1, \ldots, \vec{v}_p$ are in $\mathbb{R}^n$, then the set of all linear combinations of $\vec{v}_1, \ldots, \vec{v}_p$ is denoted by $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$ and is called the subset of $\mathbb{R}^n$ spanned (or generated) by $\vec{v}_1, \ldots, \vec{v}_p$.

Definition: Matrix Multiplication

If A is an $m \times n$ matrix, with columns $\vec{a}_1, \ldots, \vec{a}_n$, and if $\vec{x}$ is in $\mathbb{R}^n$, then the product of A and $\vec{x}$, denoted by $A\vec{x}$, is the linear combination of the columns of A using the corresponding entries in $\vec{x}$ as weights.

$$A\vec{x} = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n$$

Theorem 3: Matrix Equation Solutions

If A is an $m \times n$ matrix, with columns $\vec{a}_1, \ldots, \vec{a}_n$, and if $\vec{b}$ is in $\mathbb{R}^n$, the matrix equation

$$A\vec{x} = \vec{b}$$

has the same solution set as the vector equation

$$x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}$$

which also has the same solutions set as the system of linear equations whose augmented matrix is

$$\begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \cdots & \vec{a}_n & \vec{b} \end{bmatrix}$$

Theorem 4: Logically Equivalent Matrix Statements

Let A be an $m \times n$ coefficient matrix. Then the following statements are either all true or all false for a particular A.

a. For each $\vec{b}$ in $\mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a solution.
b. Each $\vec{b}$ in $\mathbb{R}^m$ is a linear combination of the columns of A
c. The columns of A span $\mathbb{R}^m$
d. A has a pivot position in every row.

Theorem 5: Matrix Product Properties

If A is an $m \times n$ matrix, $\vec{u}$ and $\vec{v}$ are vectors in $\mathbb{R}^n$, and c is a scalar, then:

$$A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v}$$

$$A(c\vec{u}) = c(A\vec{u})$$

Theorem 6: Homogeneous Equation Solutions

For a consistent equation $A\vec{x} = \vec{b}$, let $\vec{p}$ be a solution. The solution set of $A\vec{x} = \vec{b}$ is the set of all vectors of the form $\vec{w} = \vec{p} + \vec{v}_h$, where $\vec{v}_h$ is any solution of the homogeneous equation $A\vec{x} = 0$

Definition: Linear Dependence

An indexed set of vectors $\{\vec{v}_1, \ldots, \vec{v}_p\}$ in $\mathbb{R}^n$ is said to be linearly independent if the vector equation

$$x_1\vec{v}_1 + x_2\vec{v}_2 + \cdots + x_p\vec{v}_p = 0$$

has only the trivial solution. The set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ is said to be linearly dependent if there exist weights $c_1, \ldots, c_p$, not all zero, such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = 0$$

Theorem 7: Characterization of Linearly Dependent Sets

An indexed set $S = \{\vec{v}_1, \ldots, \vec{v}_p\}$ of two or more vectors is linearly dependent iff at least one of the vectors in S is a linear combination of the others.

Theorem 8: Number of Entries and Vectors for Linear Dependence

If a set contains more vectors than there are entries in each vector, then the set is linearly dependent. That is, any set $\vec{v}_1, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ is linearly dependent if $p > n$.

Theorem 9: Zero Vector and Linear Dependence

If a set $S = \vec{v}_1, \ldots, \vec{v}_p$ in $\mathbb{R}^n$ contains the zero vector, then the set is linearly dependent.

Definition: Linear Transform

A transformation (or mapping) T is linear if
(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}, \vec{v}$ in the domain of T;
(ii) $T(c\vec{u}) = cT(\vec{u})$ for all scalars c and all $\vec{u}$ in the domain of T.

Theorem 10: Unique Matrix for Linear Transformation

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. then there exists a unique matrix A such that

$$T(\vec{x}) = A\vec{x} \text{ for all } \vec{x} \text{ in } \mathbb{R}^n$$

In fact, A is the $m \times n$ matrix whose jth column is the vector $T(\vec{e}_j)$, where $\vec{e}_j$ is the jth column of the identity matrix in $\mathbb{R}^n$:

$$A = \begin{bmatrix} T(\vec{e}_1) & \cdots & T(\vec{e}_n) \end{bmatrix}$$

Definition: Onto

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be onto $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at least one $\vec{x}$ in $\mathbb{R}^n$.

Definition: One-to-one

A mapping $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be one-to-one $\mathbb{R}^m$ if each $\vec{b}$ in $\mathbb{R}^m$ is the image of at most one $\vec{x}$ in $\mathbb{R}^n$.

Theorem 11: Linear Transformation Trivial Solution

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then T is one-to-one iff the equation $T(\vec{x}) = 0$ has only the trival solution.

Theorem 12: Conditions for One-to-one Transformation

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, and let A be the standard matrix for T. then:
a. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$:
b. T is one-to-one iff the columns of A are linearly independent.

2 Matrix Algebra

Theorem 1: Matrix Arithmetic

Let $A, B,$ and C be matrices of the same size, and let r and s be scalars.

a. $A + B = B + A$
b. $(A + B) + C = A + (B + C)$
c. $A + 0 = A$
d. $r(A + B) = rA + rB$
e. $(r + s)A = rA + sA$
f. $r(sA) = (rs)A$

Definition: Matrix Composition

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\vec{b}_1, \ldots, \vec{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\vec{b}_1, \ldots, A\vec{b}_p$. That is,

$$AB = A \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix} = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \end{bmatrix}$$

Theorem 2: Properties of Matrix Multiplication

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined.

a. $A(BC) = (AB)C$ (associative law of multiplication)
b. $A(B + C) = AB + AC$ (left distributive law)
c. $b + C)A = BA + CA$ (right distributive law)
d. $r(AB) = (rA)B = A(rB)$ for any scalar r
e. $I_m A = A = AI_n$ (identity for matrix multiplication)

Theorem 3: Transpose Properties

Let A and B denote matrices whose sizes are appropriate for the following sums and products.

a. $(A^T)^T = A$
b. $(A + B)^T = A^T + B^T$
c. For any scalar r, $(rA)^T = rA^T$
d. $(AB)^T = B^T A^T$

Theorem 4: Two by Two Invertible Matrix

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

If $ad - bc = 0$, then A is not invertible.

Theorem 5: Invertible Matrix Solutions

If A is an invertible $n \times m$ matrix, then for each $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has the unique solution $\vec{x} = A^{-1}\vec{b}$

Theorem 6: Properties of Invertible Matrix

a. If A is an invertible matrix, then A^{-1} is invertible and

$$(A^{-1})^{-1} = A$$

b. If A and B are $n \times n$ invertible matrices, then so is AB, and the inverse of AB is the product of the inverses of A and B in the reverse order. That is,

$$(AB)^{-1} = B^{-1}A^{-1}$$

c. If A is an invertible matrix, then so is A^T, and the inverse of A^T is the transpose of A^{-1}. That is,

$$(A^T)^{-1} = (A^{-1})^T$$

Theorem 7: Conditions for Matrix Invertibility

An $n \times n$ matrix A is invertible iff A is row equivalent to I_n, and in this case, any sequence of elementary row operations that reduced A to I_n also transforms I_n into A^{-1}.

Theorem 8: The Invertible Matrix Theorem

Let A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true or all false.

a. A is an invertible matrix.
b. A is row equivalent to the $n \times n$ identity matrix.
c. A has n pivot positions.
d. The equation $A\vec{x} = 0$ only has the trivial solution.
e. The columns of A form a linearly independent set.
f. The linear transformation $\vec{x} \mapsto A\vec{x}$ is one-to-one.
g. The equation $A\vec{x} = \vec{b}$ has at least one solution for each $\vec{b}$ in $\mathbb{R}^n$.
h. the columns of A span $\mathbb{R}^n$.
i. The linear transformation $\vec{x} \mapsto A\vec{x}$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$.
j. There is an $n \times n$ matrix C such that $CA = I$.
k. There is an $n \times n$ matrix D such that $AD = I$.
l. A^T is an invertible matrix.

Theorem 9: Invertible Transformation

Let T: $\mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation and let A be the standard matrix for T. Then T is invertible iff A is an invertible matrix. In that case, the linear transformation S given by $S(\vec{x}) = A^{-1}\vec{x}$ is the unique function satisfying $S(T(\vec{x})) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$ and $T(S(\vec{x})) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^n$

Theorem 10: Column-Row Expansion of AB

If A is $m \times n$ and B is $n \times p$, then

$$AB = \begin{bmatrix} \mathrm{col}_1(A) & \mathrm{col}_2(A) & \cdots & \mathrm{col}_n(A) \end{bmatrix} \begin{bmatrix} \mathrm{row}_1(B) \\ \mathrm{row}_2(B) \\ \vdots \\ \mathrm{row}_n(B) \end{bmatrix}$$

$$= \mathrm{col}_1(A)\mathrm{row}_1(B) + \cdots + \mathrm{col}_n(A)\mathrm{row}_n(B)$$

3 Determinants

Definition: Determinant

For $n \geq 2$, the determinant of an $n \times n$ matrix $A = \begin{bmatrix} a_{ij} \end{bmatrix}$ is the sum of n terms of the form $\pm a_{1j} \det A_{1j}$, with plus and minus signs alternating, where the entries $a_{11}, a_{12}, \ldots, a_{1n}$ are from the first row of A. In symbols,

$$\begin{aligned} \det A &= a_{11} \det A_{11} - a_{12} \det A_{12} + \cdots + (-1)^{1+n} a_{1n} \det A_{1n} \\ &= \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det A_{1j} \end{aligned}$$

Theorem 1: Cofactor

The determinant of an $n \times n$ matrix A can be computed by a cofactor expansion across any row or down any column. The expansion across the ith row using the cofactors in $C_{ij} = (-1)^{i+j} \det A_{ij}$ is

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$

The cofactor expansion down the jth column is

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$

Theorem 2: Determinant of Triangular Matrix

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

Theorem 3: Row Operations for Determinants

Let A be a square matrix

a. If a multiple of one row of A is added to another row to produce a matrix B, then $\det B = \det A$.
b. If two rows of A are interchanged to produce B, then $\det B = -\det A$
c. If one row of A is multiplied by k to produce B, then $\det B = k \det A$

Theorem 4: Determinants and Invertibility

A square matrix A is invertible iff $\det A \neq 0$

Theorem 5: Determinant of a Transpose

If A is an $n \times n$ matrix, then $\det A^T = \det A$

Theorem 6: Multiplicative Property

If A and B are $n \times n$ matrices, then $\det AB = (\det A)(\det B)$

4 Vector Spaces and Subspaces

Definition: Vector Space

A vector space is a nonempty set V of objects, called vectors, on which are defined two operations, called addition and multiplication by scalars (real numbers), subject to the ten axioms listed below. The axioms must hold for all vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in V and for all scalars c and d.

1. The sum of $\vec{u}$ and $\vec{v}$, denoted by $\vec{u} + \vec{v}$, is in V.
2. $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
3. $(\vec{u} + \vec{v}) + w = \vec{u} + (\vec{v} + \vec{w})$.
4. There is a zero vector 0 in V such that $\vec{u} + 0 = \vec{u}$.
5. For each u in V, there is a vector $-u$ in V such that $\vec{u} + (-\vec{u}) = 0$.
6. $c\vec{u}$ is in V
7. $c(\vec{u} + \vec{v}) = c\vec{u} + d\vec{u}$
8. $(c + d)\vec{u} = c\vec{u} + d\vec{u}$
9. $c(d\vec{u}) = (cd)\vec{u}$
10. $1\vec{u} = \vec{u}$

Definition: Subspace

A subspace of a vector space V is a subset H of V that has three properties:

a. The zero vector of V is in H.
b. H is closed under vector addition. That is, for each $\vec{u}$ and $\vec{v}$ in H, the sum $\vec{u} + \vec{v}$ is in H.
c. H is closed under multiplication by scalars. That is, for each $\vec{u}$ in H and each scalar c, the vector $c\vec{u}$ is in H.

Theorem 1: Subspace and Span

If $\vec{v}_1, \ldots, \vec{v}$ are in vector space V, then $\text{Span}\{\vec{v}_1, \ldots, \vec{v}_P\}$ is a subspace of V

Definition: Null Space

The null space of an $m \times n$ matrix A, written as $\text{Nul}A$ or $\mathcal{N}(A)$, is the set of all solutions of the homogeneous equation $A\vec{x} = 0$. In set notation,

$$\mathcal{N}(A) = \{\vec{x} : \vec{x} \text{ is in } \mathbb{R}^n \text{ and } A\vec{x} = 0\}$$

Theorem 2: Null Space Subspace

The null space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^n$. Equivalently, the set of all solutions to a system $A\vec{x} = 0$ of m homogeneous linear equations in n unknowns is a subspace of $\mathbb{R}^n$.

Definition: Column Space

The column space of an $m \times n$ matrix A, written as Col$A$ or $\mathcal{C}(A)$, is the set of all linear combinations of the columns of A. If $A = \begin{bmatrix} \vec{a}_1 & \cdots & \vec{a}_n \end{bmatrix}$, then

$$\mathcal{C}(A) = \text{Span}\{\vec{a}_1, \ldots, \vec{a}_n\}$$

Theorem 3: Column Space Subspace

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$

Definition: Linear Transformation

A linear transformation T from a vector space V into a vector space W is a rule that assigns to each vector $\vec{x}$ in V a unique vector $T(\vec{x})$ in W, such that

(i) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$ for all $\vec{u}$, $\vec{v}$ in V, and

(ii) $T(c\vec{u}) = cT(\vec{u})$ for all $\vec{u}$ in V and all scalars c.

Theorem 4: Linear Dependence

An indexed set $\{\vec{v}_1, \ldots, \vec{v}_p\}$ of two or more vectors, with $\vec{v}_1 \neq 0$, is linearly dependent iff some $\vec{v}_j$ (with $j > 1$) is a linear combination of the preceding vectors, $\vec{v}_1, \ldots, \vec{v}_{j-1}$.

Definition: Basis

Let H be a subspace of vector space V. A set of vectors β in V is a basis for H if

(i) β is a linearly independent set, and

(ii) the subspace spanned by β coincides with H; that is,

$$H = \text{Span}\beta$$

Theorem 5: The Spanning Set Theorem

Let $S = \{\vec{v}_1, \ldots \vec{v}_p\}$ be a set in a vector space V, and let $H = \text{Span}\{\vec{v}_1, \ldots, \vec{v}_p\}$.

a. If a vector $\vec{v}_k \in S$ is a linear combination of the remaining vectors in S,

$$\text{Span}(S - \{\vec{v}_k\}) = \text{Span}H$$

b. If $H \neq \{0\}$, some subset of S is a basis for H.

Theorem 6: Basis of Column Space

The pivot columns of a matrix A form a basis for $\mathcal{C}(A)$.

Theorem 7: Basis of Row Space

If two matrices A and B are row equivalent, then their row spaces are the same. If B is in echelon form, the nonzero rows of B form a basis for the row space of A as well as for that of B.

Theorem 8: The Unique Representation Theorem

Let $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a vector space V. Then for each $\vec{x}$ in V, there exists a unique set of scalars $c_1, \ldots, c_n$ such that

$$\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$$

Definition: Basis Coordinates

Supposed $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$ is a basis for a vector space V and $\vec{x} \in V$. The coordinates of $\vec{x}$ relative to the basis (or the β-coordinates of $\vec{x}$) are the weights of $c_1, \ldots, c_n$ such that $\vec{x} = c_1\vec{b}_1 + \cdots + c_n\vec{b}_n$

Theorem 9: Coordinate Mapping One-to-One

Let $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$ be a basis for a vector space V. Then the coordinate mapping $\vec{x} \mapsto [\vec{x}]_\beta$ is a one-to-one linear transformation from V onto $\mathbb{R}^n$.

Theorem 10: Basis and Linear Dependence

If a vector space V has a basis $\beta = \{\vec{b}_1, \ldots, \vec{b}_n\}$, then any set in V containing more than n vectors must be linearly dependent.

Theorem 11: Entries in Basis

If a vector space V has a basis of n vectors, then every basis of V must consist of exactly n vectors.

Definition: Dimension

Let V be a vector space. V is finite-dimensional if it is spanned by a finite set. The dimension of V (written as $\dim V$) is the number of vectors in a basis for V. $\dim(\{0\}) \equiv 0$. V is infinite-dimensional if it is not spanned by a finite set.

Theorem 12: Counterpart to Spanning Set Theorem

Let H be a subspace of a finite-dimensional vector space V. Any linearly independent set in H can be expanded, if necessary, to a basis for H. Also, H is finite-dimensional and

$$\dim H \leq \dim V$$

Definition: Rank and Nullity

The rank of an $m \times n$ matrix A is $\dim \mathcal{C}(A)$ and the nullity of A is $\dim \mathcal{N}(A)$.

Theorem 14: The Rank Theorem

Let A be an $m \times n$ matrix.

$$\text{rank } A + \text{nullity } A = \text{number of columns in } A$$

Theorem: The Invertible Matrix Theorem (cont.)

Let A be an $n \times n$ matrix. The following statements are equivalent the the statement that A is an invertible matrix.

m. The columns of A form a basis of $\mathbb{R}^n$
n. $\mathcal{C}(A) = \mathbb{R}^n$
o. rank $A = n$
p. nullity $A = 0$
q. $\mathcal{N}(A) = \{0\}$

Theorem 15: Change of Coordinates Matrix

Let $\beta = \{\vec{b}_1, \dots, \vec{b}_n\}$ and $\gamma = \{\vec{g}_1, \dots, \vec{g}_n\}$ be bases of a vector space V. Then there is a unique $n \times n$ matrix $P_{\gamma \leftarrow \beta}$ such that

$$[\vec{x}]_\gamma = \underset{\gamma \leftarrow \beta}{P} [\vec{x}]_\beta$$

The columns of $P_{\gamma \leftarrow \beta}$ are the γ-coordinate vectors of the vectors in the basis β. That is

$$\underset{\gamma \leftarrow \beta}{P} = \begin{bmatrix} [\vec{b}_1]_\gamma & [\vec{b}_2]_\gamma & \cdots & [\vec{b}_n]_\gamma \end{bmatrix}$$

5 Eigenvalues and Eigenvectors

Definition: Eigenvalues and Eigenvectors

An eigenvector of an $n \times n$ matrix A is a nonzero vector $\vec{x}$ such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an eigenvalue of A if there is a nontrivial solution $\vec{x}$ of $A\vec{x} = \lambda\vec{x}$; such an $\vec{x}$ is called an eigenvector corresponding to λ.

Theorem 1: Eigenvalues of Triangular Matrix

The eigenvalues of a triangular matrix are the entries on its main diagonal.

Theorem 2: Eigenvalues and Linear Independence

If $\vec{v}_1, \dots, \vec{v}_r$ are eigenvectors that correspond to distinct eigenvalues $\lambda_1, \dots, \lambda_r$ of an $n \times n$ matrix A, then the set $\{\vec{v}_1, \dots, \vec{v}_r\}$ is linearly independent.

Theorem 3: Properties of Determinants

$A, B \in M_{n\times n}$

a. A is invertible iff $\det A \neq 0$

b. $\det AB = \det(A)\det(B)$

c. $\det A^T = \det A$

d. If A is triangular, $\det A = \prod_{k=1}^{n} a_{kk}$

e. A row replacement operation on A does not change $\det A$. A row interchange changes the sign of the determinant. A row scaling also scales the determinant by the same scalar factor.

Theorem: The Invertible Matrix Theorem (cont.)

$A \in M_{n\times n}$. A is invertible iff

r. 0 is not en eigenvalue of A.

Theorem 4: Eigenvalues and Similarity

$A, B \in M_{n\times n}$. If A and B are similar then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).

Theorem 5: The Diagonalization Theorem

$A \in M_{n\times n}$. A is diagonalizable iff it has n linearly independent eigenvectors. In fact, $A = PDP^{-1}$, with D a diagonal matrix, iff the columns of P are n linearly independent eigenvectors of A. In this case, the diagonal entries of D are eigenvalues of A that correspond, respectively, to the eigenvectors in P.

Theorem 6: Eigenvalues and Diagonalizability

If $A \in M_{n\times n}$ has n distinct eigenvalues then it is diagonalizable.

Theorem 7: Eigenvectors and Dimension

Let $A \in M_{n\times n}$ have distinct eigenvalues $\lambda_1, \ldots, \lambda_p$

a. For $1 \leq k \leq p$, the dimension of the eigenspace for λ_k is less than or equal to the mulitiplicity of eigenvalue λ_k

b. A is diagonazliable iff the sum of the dimensions of the eigenspaces equals n.

c. If A is diagonalizable and β_k is a basis for the eigenspace corresponding to λ_k for each k, then the total collection of vectors in the sets $\beta_1, \ldots, \beta_p$ forms an eigenvector basis for $\mathbb{R}^n$.

Definition: Eigenvector Transformation

Let V be a vector space. An eigenvector of a linear transformation $T : V \to V$ is $\vec{x} \in V, \vec{x} \neq 0$ such that $T(\vec{x}) = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an eigenvalue of T if there is a nontrivial solution $\vec{x}$ of $T(\vec{x}) = \lambda\vec{x}$; such an $\vec{x}$ is called an eigenvector corresponding to λ.

Theorem 8: Diagonal Matrix Representation

Suppose $A = PDP^{-1}$, where $D \in M_{n\times n}$ is diagonal. If β is the basis for $\mathbb{R}^n$ formed from the columns of P, then D is the β-matrix for the transformation $\vec{x} \mapsto A\vec{x}$.

6 Orthogonality

Theorem 1: Inner Product Properties

Let $\vec{u}, \vec{v},$ and $\vec{w} \in \mathbb{R}^n, c \in \mathbb{R}$.
a. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
b. $(\vec{u} + \vec{v}) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$
c. $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$
d. $\vec{u} \cdot \vec{u} \geq 0$, and $\vec{u} \cdot \vec{u} = 0 \iff \vec{u} = 0$

Definition: Norm

The norm (length) of $\vec{v}$ is $\|\vec{v}\| \in \mathbb{R}_{\geq 0}$ defined by

$$\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$$

Definition: Distance

For $\vec{u}, \vec{v} \in \mathbb{R}^n$, the distance between $\vec{u}$ and $\vec{v}$, written as $\text{dist}(\vec{u}, \vec{v})$, is the norm of vector $\vec{u} - \vec{v}$, or

$$\text{dist}(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$

Definition: Orthogonal

$\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal to each other if $\vec{u} \cdot \vec{u} = 0$.

Theorem 2: The Pythagorean Theorem

$\vec{v}, \vec{u} \in \mathbb{R}^n$ are orthogonal to each other iff $\|\vec{u} + \vec{u}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$.

Theorem 3: Null Space Orthogonal Complement

Let $A \in M_{m\times n}$

$$(\text{Row}(A))^\perp = \mathcal{N}(A) \text{ and } (\mathcal{C}(A))^\perp = \mathcal{N}(A^T)$$

Theorem 4: Orthogonal Set Subspace

If $S = \{\vec{u}_1, \ldots, \vec{u}_p\}$ is an orthogonal set of nonzero vectors in $\mathbb{R}^n$, then S is linearly independent and hence is a basis for the subspace spanned by S.

Definition: Orthogonal Basis

An orthogonal basis for a subspace W of $\mathbb{R}^n$ is a basis for W that is also an orthogonal set.

Theorem 5: Linear Combination of Orthogonal Basis

Let $\{\vec{u}_1, \ldots, \vec{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. For each $\vec{y} \in W$,

$$\vec{y} = c_1\vec{u}_1 + \cdots + c_p\vec{u}_p$$

$$c_j = \frac{\vec{y} \cdot \vec{u}_j}{\vec{u}_j \cdot \vec{u}_j} \quad (j = 1, \ldots, p)$$

Theorem 6: Orthonormal Columns

$U \in M_{m\times n}$ has orthonormal columns iff $U^T U = I$

Theorem 7: Orthonormal Column Properties

$U \in M_{m\times n}$ with orthonormal columns, $\vec{x}, \vec{y} \in \mathbb{R}^n$.

a. $\|U\vec{x}\| = \|\vec{x}\|$
b. $(U\vec{x}) \cdot (U\vec{y}) = \vec{x} \cdot \vec{y}$
c. $(U\vec{x}) \cdot (U\vec{y}) = 0 \iff \vec{x} \cdot \vec{y} = 0$

Theorem 8: The Orthogonal Decomposition Theorem

Let W be a subspace of $\mathbb{R}^n$. Each $\vec{y} \in \mathbb{R}^n$ can be written uniquely in the form

$$\vec{y} = \hat{y} + \vec{z}$$

where $\hat{y} \in W, \vec{z} \in W^{\perp}$. In fact, if $\{\vec{u}_1, \ldots, \vec{u}_p\}$ is any orthogonal basis of W, then

$$\hat{y} = \frac{\vec{y} \cdot \vec{u}_1}{\vec{u}_1 \cdot \vec{u}_1}\vec{u}_1 + \cdots + \frac{\vec{y} \cdot \vec{u}_p}{\vec{u}_p \cdot \vec{u}_p}\vec{u}_p$$

and $\vec{z} = \vec{y} - \hat{y}$.

Theorem 9: The Best Approximation Theorem

Let W be a subspace of $\mathbb{R}^n$, $\vec{y} \in \mathbb{R}^n$, $\hat{y} = \text{proj}_W\vec{y}$

$$\forall \vec{v} \in W, \vec{v} \neq \hat{y} \implies \|\vec{y} - \hat{y}\| < \|\vec{y} - \vec{v}\|$$

Theorem 10: Projection onto a Subspace

If $\{\vec{u}_1, \ldots, \vec{u}_p\}$ is an orthonormal basis for the subspace W of $\mathbb{R}^n$, then

$$\text{proj}_W\vec{y} = (\vec{y} \cdot \vec{u}_1)\vec{u}_1 + (\vec{y} \cdot \vec{u}_2)\vec{u}_2 + \cdots + (\vec{y} \cdot \vec{u}_p)\vec{u}_p$$

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_p \end{bmatrix} \implies \forall \vec{y} \in \mathbb{R}^n, \text{proj}_W\vec{y} = UU^T\vec{y}$$

Theorem 11: The Gram-Schmidt Process

Given a basis $\{\vec{x}_1, \ldots, \vec{x}_p\}$ for a nonzero subspace W of $\mathbb{R}^n$, define

$$\vec{v}_1 = \vec{x}_1$$

$$\vec{v}_2 = \vec{x}_2 - \frac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$$

$$\vec{v}_3 = \vec{x}_3 - \frac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$\vdots$$

$$\vec{v}_p = \vec{x}_p - \frac{\vec{x}_p \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{x}_p \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 - \cdots - \frac{\vec{x}_p \cdot \vec{x}_{p-1}}{\vec{v}_{p-1} \cdot \vec{v}_{p-1}}\vec{v}_{p-1}$$

Then $\{\vec{v}_1, \ldots, \vec{v}_p$ is an orthogonal basis for W. In addition

$$\text{Span}\{\vec{v}_1, \ldots, \vec{v}_k\} = \text{Span}\{\vec{x}_1, \ldots, \vec{x}_k\} \quad \text{for } 1 \geq k \geq p$$

Theorem 12: The QR Factorization

If $A \in M_{m \times n}$ with linearly independent columns, then A can be factored as $A = QR$, where $Q \in M_{m \times n}$ matrix whose columns form an orthonormal basis for $\mathcal{C}(A)$ and $R \in M_{n \times n}$ is an upper triangluar invertible matrix with positive entries on its diagonal.

Definition: Least-Squares Solution

If $A \in M_{m \times n}, \vec{b} \in \mathbb{R}^m$, a least-squares solution of $A\vec{x} = \vec{b}$ is an $\hat{x} \in \mathbb{R}^n$ such that

$$\forall \vec{x} \in \mathbb{R}^n, \|\vec{b} - \hat{Ax}\| \leq \|\vec{b} - A\vec{x}\|$$

Theorem 13: Least-Squares Solution Set

The set of least-squares solutions of $A\vec{x} = \vec{b}$ coincides with the nonempty set of solutions of the normal equations $A^T A\vec{x} = A^T\vec{b}$.

Theorem 14: Least-Squares Statments

$A \in M_{m \times n}$. The following statements are logically equivalent:

a. The equation $A\vec{x} = \vec{b}$ has a unique least-squares solution for each $\vec{b} \in \mathbb{R}^m$.

b. The columns of A are linearly independent.

c. The matrix $A^T A$ is invertible

When these statements are true, the least-squares solution $\hat{x}$ is given by

$$\hat{x} = (A^T A)^{-1} A^T \vec{b}$$

Theorem 15: QR Factorization and Least-Squares Solution

Given $A \in M_{m\times n}$ with linearly independent columns, let $A = QR$ be a QR factorization of A as in Theorem 12. Then, for each $\vec{b} \in \mathbb{R}^m$, the equation $A\vec{x} = \vec{b}$ has a unique least-squares solution, given by

$$\hat{x} = R^{-1}Q^T\vec{b}$$

Definition: Inner Product Space

An inner product on a vector space V is a function that, to each pair of vectors $\vec{u}, \vec{v} \in V$, associates a real number $\langle \vec{u}, \vec{v} \rangle$ and satisfies the following axioms, $\forall \vec{u}, \vec{v}, \vec{w} \in V, \forall c \in \mathbb{R}$:

1. $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$
2. $\langle \vec{u} + \vec{v}, \vec{w} \rangle = \langle \vec{u}, \vec{w} \rangle + \langle \vec{v}, \vec{w} \rangle$
3. $\langle c\vec{u}, \vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$
4. $\langle \vec{u}, \vec{u} \rangle \geq 0$ and $\langle \vec{u}, \vec{u} \rangle = 0 \iff \vec{u} = 0$

A vector space with an inner product is called an inner product space.

Theorem 16: The Cauchy-Schwarz Inequality

$$\forall \vec{u}, \vec{v} \in V, |\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\|\|\vec{v}\|$$

Theorem 17: The Triangle Inequality

$$\forall \vec{u}, \vec{v} \in V, \|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$

7 Symmetric Matrices and Quadratic Forms

Theorem 1: Orthogonal Eigenvectors

If A is symmetric, then any two eigenvectors form different eigenspaces are orthogonal.

Theorem 2: Orthogonal Diagonalizability

$A \in M_{n\times n}$ is orthogonally diagonalizable iff A is a symmetric matrix.

Theorem 3: The Spectral Theorem for Symmetric Matrices

If $A \in M_{n\times n}$ is symmetric:

a. A has n real eigenvalues, counting multiplicities

b. The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation.

c. The eigenspaces are mutually orthogonal

d. A is orthogonally diagonalizable.

Theorem 4: The Principal Axes Theorem

Let $A \in M_{n\times n}$ be symmetric. Then there is an orthogonal change of variable, $\vec{x} = P\vec{y}$, that transforms the quadratic form $\vec{x}^T A\vec{x}$ into a quadratic form $\vec{y}^T D\vec{y}$ with no cross-product term.