# MVC Notes

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## 1 Vectors

$$u \times v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{j}\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{k}\begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \hat{i}(u_2v_3 - u_3v_2) - \hat{j}(u_1v_3 - u_3v_1) + \hat{k}(u_1v_2 - u_2v_1)$$

$$\text{vector projection: } proj_v u = \frac{u \cdot v}{|v|}\hat{v}$$

$$\text{scalar projection: } |proj_v u| = \frac{u \cdot v}{|v|}$$

## 2 3D graphs

The distance d between a point Q and a line $r = r_0 + tv$

$$d = \frac{|v \times \vec{PQ}|}{|v|}$$

where P is any point on the line and v is a vector parallel to the line. This makes sense since $d = |\vec{PQ}| \sin\theta$

For a plane $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$ the normal vector $n = (a, b, c)$

## 2.1 Quadric Surfaces

$$\text{Ellipsoid: } \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\text{Elliptic paraboloid: } z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

$$\text{Hyperboloid of one sheet: } \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

$$\text{Hyperboloid of two sheets: } -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$\text{Elliptic cone } \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z^2}{c^2}$$

$$\text{Hyperbolic paraboloid: } z = \frac{x^2}{a^2} - \frac{y^2}{b^2}$$

# 3 Vector Valued Functions

$$\text{unit tangent vector: } \vec{T} = \frac{\vec{v}}{|\vec{v}|}$$

$$\text{principle unit vector: } \vec{N} = \frac{d\vec{T}/dt}{|d\vec{T}/dt|}$$

The principle unit vector is always perpendicular to the unit tangent vector, which makes sense since it points in the rate of change of the unit tangent vector, which can only change in direction, not length

$$\text{curavture: } \kappa = \left|\frac{d\vec{T}}{ds}\right| = \frac{1}{|\vec{v}|}\left|\frac{d\vec{T}}{dt}\right| = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|^3}$$

$$\text{components of acceleration: } \vec{a} = a_N\vec{N} + a_T\vec{T}$$

$$a_n = \kappa|\vec{v}|^2 = \frac{|\vec{v} \times \vec{a}|}{|\vec{v}|}$$

$$a_T = \frac{d^2s}{dt^2} = \frac{\vec{v} \cdot \vec{a}}{|\vec{v}|}$$

$$\text{unit binormal vector: } \vec{B} = \vec{T} \times \vec{N} = \frac{\vec{v} \times \vec{a}}{|\vec{v} \times \vec{a}|}$$

$$\text{torsion: } \tau = -\frac{d\vec{B}}{ds} \cdot \vec{N} = \frac{(\vec{v} \times \vec{a}) \cdot \vec{a}'}{|\vec{v} \times \vec{a}|^2} = \frac{(\vec{r}' \times \vec{r}'') \cdot \vec{r}'''}{|\vec{r}' \times \vec{r}''|^2}$$

# 4 Derivatives

Chain Rule for one independent variable:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

Chain Rule for two independent variables:

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$$

Implicit Differentiation:

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

Gradient in 3 dimensions:
For a function $f(x, y, z)$,

$$\nabla f(x, y, z) = f_x\hat{\imath} + f_y\hat{\jmath} + f_z\hat{k}$$

$\nabla f(x, y, z)$ will be perpendicular to the level surface (3D surface where the value of f(x,y,z) is constant).

Second Derivative Test:
For a function f(x,y), the discriminant is

$$D(x, y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yx} \end{vmatrix} = f_{xx}f_{yy} - (f_{xy})^2$$

When $D(a, b) > 0$ then the surface has the same general behavior in all directions near $(a, b)$ (use normal partial derivative to determine if it is min or max). There is a saddle point when $D(a, b) < 0$. The test is inconclusive when $D(a, b) = 0$

Lagrange Multipliers:
For an objective function $f(x, y, z)$ and constraint $g(x, y, z) = 0$, at a relative extrema $(a, b, c)$ there exists a Lagrange Multiplier, $\lambda$, where

$$\nabla f(a, b, c) = \lambda \nabla g(a, b, c)$$

Intuitively, the relative extrema is located at a point where objective function surface and the constraint surface are tangent to each other. The gradient for both are orthogonal to the level surface (since it points to the next increasing level surface direction), so the two gradients must be parallel to each other and can be multiplied by a scalar to equal each other.

# 5 Integration

## 5.1 Alternate Coordinates

Polar Coordinates:

$$x = r\cos\theta,\ y = r\sin\theta,\ r^2 = x^2 + y^2$$

$$\iint\limits_R f(x,y)dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta$$

Cylindrical Coordinates:

$$x = r\cos\theta,\ y = r\sin\theta,\ r^2 = x^2 + y^2 + z^2,\ z = z$$

$$\iiint\limits_D f(x,y,z)dV = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} \int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)rdzdrd\theta$$

Spherical Coordinates:

$$x = \rho\sin\phi\cos\theta,\ y = \rho\sin\phi\sin\theta,\ z = \rho\cos\phi,\ \rho^2 = x^2 + y^2 + z^2$$

$$\iiint\limits_D f(x,y,z)dV = \int_\alpha^\beta \int_a^b \int_{G(\phi,\theta)}^{H(\phi,\theta)} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\rho^2\sin\phi d\rho d\phi d\theta$$

Jacobian:
The Jacobian is the "u-sub" for integrating functions of multiple variables. It is convenient for simplifying the integrand (similar to single variable calculus), but it is also used to simplify the bounds of multivariable integrals.
For two independent variables,

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$

$$\iint\limits_R f(x,y)dA = \iint\limits_S f(g(u,v), h(u,v))|J(u,v)|dA$$

Where S is the region R transformed from the $xy$-plane to the $uv$-plane, and $J(u,v)$ is the Jacobian for the transformation $x = g(u,v)$,$y = h(u,v)$.
For three independent variables,

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

$$\iiint\limits_D f(x,y,z)dV = \iiint\limits_S f(g(u,v,w), h(u,v,w), p(u,v,w))|J(u,v,w)|dV$$

Where S is the region D transformed from $xyz$-space to the $uvw$-space, and $J(u,v,w)$ is the Jacobian for the transformation $x = g(u,v,w)$, $y = h(u,v,w)$, $z = p(u,v,w)$.

## 5.2 Line and Surface integrals

Line integrals:
For a scalar function $f(x(t), y(t), z(t))$,

$$\int\limits_C f(x(t), y(t), z(t))ds = \int\limits_C f(x(t), y(t), z(t))|r'(t)|dt$$

$$= \int_a^b f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt$$

A conservative vector field $\vec{F}$ always has a potential function $\phi$ where $\vec{F} = \nabla\phi$.

$$\int\limits_C \vec{F} \cdot d\vec{r} = \int\limits_C \nabla\phi \cdot d\vec{r} = \phi(B) - \phi(A)$$

for C in R from A to B. Intuitively,

$$\oint\limits_C \vec{F} \cdot d\vec{r} = 0$$

for a closed curve C. Alternatively, a method for vector field line integrals that works for non-conservative fields is

$$\int\limits_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)dt$$

Surface integrals:
For a function $f$ on a smooth surface S given parametrically by $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$, the surface integral of f over S is

$$\iint\limits_S f(x, y, z)dS = \iint\limits_R f(x(u, v), y(u, v), z(u, v))\left|\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial v}\right|dA$$

For a surface S explicitly defined by $z = g(x, y)$, the surface integral of a function $f$ over S is

$$\iint\limits_S f(x, y, z)dS = \iint\limits_R f(x, y, g(x, y))\sqrt{z_x^2 + z_y^2 + 1}dA$$

The surface integral of vector field $\vec{F} = \langle f, g, h\rangle$ over surface S defined parametrically as $\vec{r}(u, v) = \langle x(u, v), y(u, v), z(u, v)\rangle$ is

$$\iint\limits_S \vec{F} \cdot \vec{n}dS = \iint\limits_R \vec{F} \cdot \left(\frac{\partial\vec{r}}{\partial u} \times \frac{\partial\vec{r}}{\partial u}\right)dA$$

For S defined in the form $z = s(x, y)$

$$\iint\limits_S \vec{F} \cdot \vec{n}dS = \iint\limits_R (-fz_x - gz_y + h)dA$$

# 6 Theorems

Divergence:
For a differentiable vector field $\vec{F} = \langle f, g, h \rangle$

$$\nabla \cdot \vec{F} = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$$

A "source free" vector field has a divergence of 0. For a differentiable vector field

$$\vec{F} = \frac{\vec{r}}{|\vec{r}|^p} = \frac{\langle x, y, z \rangle}{(x^2 + y^2 + z^2)^{p/2}}$$

$$\nabla \cdot \vec{F} = \frac{3 - p}{|\vec{r}|^p}$$

Curl:
For a differentiable vector field $\vec{F} = \langle f, g, h \rangle$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = \left( \frac{\partial h}{\partial y} - \frac{\partial g}{\partial z} \right) \hat{i} + \left( \frac{\partial f}{\partial z} - \frac{\partial h}{\partial x} \right) \hat{j} + \left( \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) \hat{k}$$

A vector field with 0 curl is irrotational. The general rotation vector field is $\vec{F} = \vec{a} \times \vec{r}$ where $\vec{a} = \langle a_1, a_2, a_3 \rangle$ is the axis of rotation and $\vec{r} = \langle x, y, z \rangle$. $|\nabla \times \vec{F}| = 2|\vec{a}|$ and $\nabla \cdot \vec{F} = 0$. The constant angular speed of the field is

$$\omega = |\vec{a}| = \frac{1}{2} |\nabla \times \vec{F}|$$

Stokes' Theorem:

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\nabla \times \vec{F}) \cdot \vec{n} dS$$

Divergence Theorem:

$$\iint_S \vec{F} \cdot \vec{n} dS = \iiint_D \nabla \cdot \vec{F} dV$$