Stochastic Non-Linear Optimization Assignment

Optimization Methods in Algorithms

1. MODEL DESCRIPTION

An important feature of traffic flow density is that the traffic volume predicted for the next time period has a strong but not massively deterministic relationship with the current and recent state. The Markov chain model can express this feature well and it is very suitable for describing the traffic flow density. In the Markov chain model, the state to be predicted obeys the probability distribution, and the probability of the next state depends on the current and previous states. In this section, we adopt this idea and use a Markov Chain model to describe the traffic flow density with demand uncertainty. Firstly, we briefly introduce the Markov Chain model and divide the link state into 4 different modes based on the traffic density of the upstream and downstream of the link, and introduce the dynamic process of the link in 4 state modes. Then, we propose a method which is based on the distribution of the traffic flow density to calculate the transition probability. Finally, we explain some traffic volumes (leaving and receiving flow, entering and accepted flow, etc) in the dynamic process of the traffic network and introduce the dynamic process of state transferring.

1.1 Markov Chain Model for Traffic Flow Density

A Markov chain is a general model that can explain the natural change with a mathematical method. It was proposed by the famous Russian mathematician Markov around 1910. Markov processes are an important aspect of stochastic process theory in probability theory. After a hundred years of development, Markov processes have penetrated into various fields and played an important role. People will find many phenomena with the continuous development of time in the research of practical problems. There are also some phenomena or processes that can be expressed as follows: when the present is known, the future and the past of this process of change are irrelevant. In other words, the future situation of this process does not depend on the past development and change. We call the process with the above properties a Markov process. When the time and state of Markov process are discrete, such Markov process is called Markov chain. The mathematical expression of Markov chain is as follows: Define a random sequence $\{X(t), t \in T\}$, where $T = \{0, 1, 2 \cdots\}$, and the state space is $S = \{s_0, s_1, s_2, \cdots\}$. If at any time t and any state $s_0, s_1, \cdots, s_{t-1}, s_i, s_j$, the random sequence always satisfies with

$$\begin{aligned} &P\{X_{t+1}=s_j|X_t=s_i, X_{t-1}=s_{n-1}, \cdots, X_1=s_1, X_0=s_0\} \\ &= P\{X_{t+1}=s_j|X_t=s_i\}. \end{aligned} \tag{1}$$

then we call this random sequence Markov chain. In Eq. (1), $P\{X_{t+1} = s_j|X_t = s_i\}$ is the transition probability from time step t to time step $t + 1$. The above formula defines the Markov property at the same time as the definition of Markov chain, which is also called "Memorylessness", i.e, the random variable of step $t+1$ is conditionally independent of the rest of the random variables after the random variable of step t is given.

1.2 Link State Modes and Transition Probabilities

In an optimal situation, the shape of the FD (Fundamental Diagram) on each link is supposed to be triangular. Considering the congestion of the link in the real-life signalized traffic networks, the traffic capacity of each link will be reduced to a certain extent, as shown in Fig. 1. Due to the presence of a traffic signal, the shape of the FD at the upstream entrance of the link is triangular, and the shape of the FD at the downstream exit of the link is trapezoidal.

As it is shown in Fig. 1, ρ_c is the critical density, ρ_{cl} and ρ_{cu} are the lower and upper critical traffic flow density when the traffic capacity is limited; ρ_J is the junction traffic flow density, v_f is the free-flow speed, and w_c is the spillback speed. The traffic capacity is limited by the fraction of the green time over the cycle time on the link:

$$Q'_M = \frac{Q_M \cdot g}{c} \tag{2}$$

where Q_M and Q'_M are the capacity and the limited capacity of the FD, g is the green time on the link, and c is the cycle time of the traffic lights.

According to the different degree of congestion at the upper and lower boundary. the link state mode of each link can be classified into four different modes **?**. As shown in Fig. 1, the four state modes can be expressed as:

(1) Free flow-Free flow(FF): the upstream boundary of the link is in free-flow state and the downstream boundary of the link is also in free-flow state.

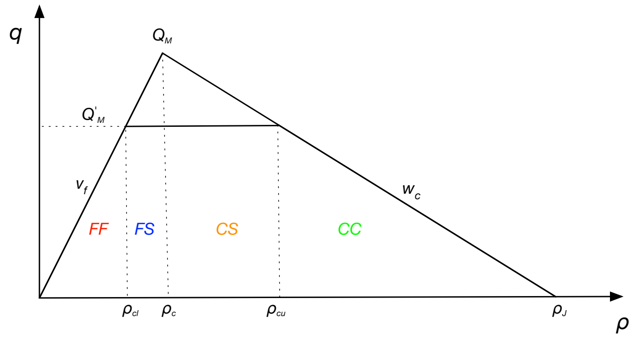

Fig. 1. Illustration for the link state modes: Free flow-Free flow(FF), Free flow-Saturation(FS), Congestion-Congestion(CC), Congestion-Saturation(CS)

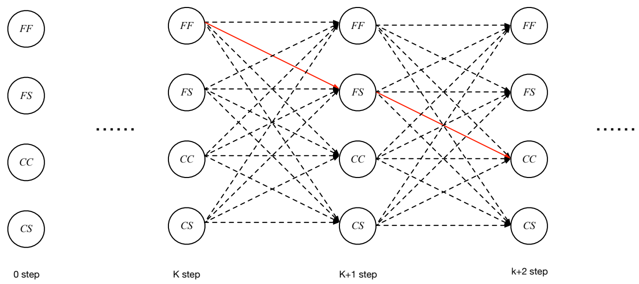

Fig. 2. Iterative Process of Markov Traffic Model

Table 1. State Transition Probability

Transition Probability	FF	FS	CS	CC
FF	0.7	0.2	0.09	0.01
FS	0.19	0.6	0.2	0.01
CS	0.01	0.19	0.6	0.2
CC	0.01	0.09	0.2	0.7

(2) Free flow-Saturation(FS): the upstream boundary of the link is in free-flow state and the downstream boundary of the link is in saturation state.
(3) Congestion-Congestion(CC): the upstream boundary of the link is in congestion state and the downstream boundary of the link is also in congestion state.
(4) Congestion-Saturation(CS): the upstream boundary of the link is in congestion state and the downstream boundary of the link is also in saturation state.

As Fig. 2 shows, the state mode set of the Markov traffic model is $M = \{FF, FS, CC, CS\}$, and it is possible for each state mode to switch to any other state mode in the state mode set. The transition probability transferring from mode n to m at time step k is defined as $P_{n,m}(k)(n, m \in M)$. The state transition probabilities can be statistically obtained through history data. In this problem, the state transition probabilities are given in Table 1.

1.3 Link Models for Different Link State Modes

Considering the different modes, entering and leaving flows and traffic signals, the dynamic evolution of the traffic flow density at time step k on a link can be formulated as

$$\rho(k+1) = A\rho(k) + B_0\rho(k)U(k) + B_1U(k) + Dd(k) + C, \tag{3}$$

$$U(k) = \beta(k)\gamma(k), \tag{4}$$

where A, B_0, B_1, D and C are constant parameters, $U(k)$ is the scalar control input at time step k, which is assumed to be a deterministic value; it is composed of the turning rate vector $\beta(k)$ and the vector of green time splits $\gamma(k)$ of the traffic signal at the intersection at time step k. More over $d(k)$ includes the stochastic disturbances from outside the link:

$$d(k) = [q_E(k)d_i(k)d_0(k)q_A(k)], \tag{5}$$

where q_E is the input flow of the link, which composed of all the leaving flows of the upstream links, q_A is the output flow of the link, which is determined by all available receive flow of the downstream links, and d_i and d_o are the disturbance flows that get in and out of the link.

The control input $U(k)$ can be defined as

$$U(k) = \beta_{th}(k)\gamma_{th}(k) + \beta_l(k)\gamma_l(k), \tag{6}$$

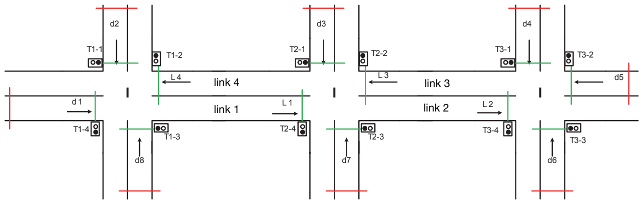

Fig. 3. A small test network

where $\beta_{th}(k)$ and $\beta_l(k)$ are the ratios of going straight and turning left at the intersection at time step k, and $\gamma_{th}(k)$ and $\gamma_l(k)$ are the green signal splits of going straight and turning left at intersections at time step k.

In the stochastic link flow model, we can further write four different dynamic models according to the different probability of the link mode. For the link state mode FF:

$$\rho(k+1) = A\rho(k) + B_0\rho(k)U(k) + Dd(k), \tag{7}$$

where $A = 1 + \beta_r B_0$, $B_0 = -\frac{T_s}{l}v_f$, $D = [\frac{T_s}{l}000]$, and l is the length of the link, T_s is the simulation time step.

For the link state mode FS:

$$\rho(k+1) = A\rho(k) + B_1U(k) + Dd(k) + C, \tag{8}$$

where $A = 1$, $B_1 = -\frac{T_s}{l}Q_M$, $D = [\frac{T_s}{l}000]$, and $C = \beta_r B_1$.

For the link state mode CC:

$$\rho(k+1) = A\rho(k) + Dd(k) + C, \tag{9}$$

where $A = 1 + \frac{T_s}{l}w_c$, $D = [000-\frac{T_s}{l}]$, and $C = -\frac{T_s}{l}w_c\rho_J$.

For the link state mode CS:

$$\rho(k+1) = A\rho(k) + B_1U(k) + Dd(k) + C, \tag{10}$$

where $A = 1 + \frac{T_s}{l}w_c$, $B_1 = -\frac{T_s}{l}Q_M$, $C = -\frac{T_s}{l}w_c + \beta_r B_1$, $D = [\frac{T_s}{l}000]$.

1.4 Leaving Flows of the Link

In the previous section, we defined four different state modes for the link based on the different congestion levels of the link. The leaving flow and the receiving flow of the link are exactly related to the congestion levels of the link. So when the link is in different traffic modes, the leaving flow and the receiving flow are also different. According to the Fundamental Diagram in Fig. 1, the leaving flow of the link can be written as

$$q_{L,FF}(k) = \rho(k)V_fU(k), \tag{11}$$

$$q_{L,FS}(k) = q_{L,CS}(k) = Q_MU(k), \tag{12}$$

$$q_{L,CC}(k) = Q_MU(k). \tag{13}$$

1.5 Entering Link Flows

The entering flow of link i can be expressed as the sum of all the flows from upstream links as

$$q_{E,i}(k) = \sum_{u\in I_i} \beta_{u,i}q_u(k), \tag{14}$$

where $\beta_{u,i}$ is the turning ratio of the flow turning from link u to link i, $q_u(k)$ is the leaving flow of link u at time step k, and $q_{E,i}(k)$ is the entering flow of link i at time step k.

1.6 State Transferring

Given the transition probability of the traffic flow density of Markov traffic model, during the iteration of the model, we suppose that the link state will jump to the state mode with the highest transition probability at each iteration. As is shown in Fig. 2, we assume the link is in the FF mode at time step k. With the increase of traffic demand, the traffic density in the road network will gradually accumulate and the probability of transferring to the FS mode will become the highest. Then the link state mode will switch from the FF mode to the FS mode at time step $k+1$. If the demand keeps on increasing, the probability of transferring will be changed and the probability of transferring to the CC mode will become the highest. The link state mode will switch from the FS mode to the CC mode at time step $k+2$, in which the traffic congestion will be propagated to the upstream link.

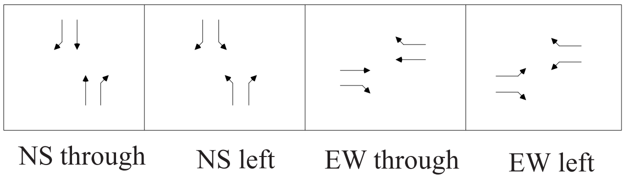

Fig. 4. The phases of traffic signal

Table 2. State Transition Probability

$v_{\rm f}$	$T = T_{\rm s}$	$Q_{\rm M}$	$w_{\rm C}$
60 km/h	100 s	3000 veh/h	-30 km/h
$\rho_{\rm c}$	$\rho_{\rm J}$	l	$d_i = d_o$
50 veh/km	150 veh/km	800 m	0
$\beta_{\rm th}$	$\beta_{\rm l}$	$\beta_{\rm r}$	
0.4	0.3	0.3	

2. OPTIMIZATION

As shown in Fig. 3, the considered traffic network is quite simple, which has 3 signalized intersections, 12 links in the network, and 12 sets of green time splits for the 3 intersections. The stochastic traffic demand is provided to the network through 8 external access links, and

$$q_0(k) = \begin{cases} 600 + E_1 * w_0 veh/h \quad k < 10 \\ 2000 + E_2 * w_0 veh/h \quad 10 \leq k < 20 \\ 500 + E_1 * w_0 veh/h \; k \geq 20 \end{cases} \tag{15}$$

where w_0 is a random value generated within [0,1], $E_1 = 50$ and $E_2 = 100$ is a given variable. When $k = 0$, the densities on all the links are 0 veh/km per lane. Each link has 3 lanes. Parameters are given in Table 2. As shown in Fig. 4, there are four traffic signal stages: North South through, North South left, East West through, and East West left. For each intersection, the sum of the traffic signal durations of the four phases equals to the cycle time at time step k, that is the sum of the traffic signal splits of the four phases equals to 1 at time step k:

$$\sum_{p \in P} \gamma_p(k) = 1, \tag{16}$$

and the traffic signal splits, i.e. the duration of the green time dividing the cycle time, are the variables that need to be optimized.

Total Time Spent (TTS) of all the links in time interval $[kT,\ (k+1)T]$ is defined as the output of the system, as

$$y(k) = \sum_{l \in L} \rho_l(k), \tag{17}$$

which is optimization objective function, and T is the time interval and it is equal to the cycle time of the traffic lights.

In this optimization, we suppose that all the spaces of the downstream links are infinite, that is we suppose the downstream links are all unlimited. The optimized variables are the traffic signal splits $\gamma(k)$, which is $0 \leq \gamma(k) \leq 1$. Traffic demands entering the traffic network are the system inputs. Since this is a practical problem, all the variables are positive.

3. PROBLEMS

(1) Write the model of the system to show the relationship between system inputs and outputs.
(2) Formulate the optimization problem for a time period of 50 minutes ($[kT,\ (k+30)T]$), to minimize the objective in the time period, by adjusting the optimized variable, i.e. the green time splits.
(3) Select a proper optimization algorithm (explain the reasons) write the code with MATLAB or other software.
(4) Draw the dynamic figures for the inputs and the outputs of the optimized results, and discuss the effect of the optimization algorithm (including convergence speed and effect, computation time, local or global, etc.).
(5) Select two other optimization algorithms to solve the problem, compare the results, and analyze the disadvantages and advantages of the three algorithms, then draw some conclusions.