Numerical Analysis — FMN011 — 2015/06/03
Solutions

The exam lasts 4 hours and has 14 questions. A minimum of 35 points out of the total 70 are required to get a passing grade. These points will be added to those you obtained in your two home assignments, and the final grade is based on your total score.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

1. **(5p)** What vector norm should I use in each of the following cases?

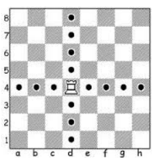
   (a) Calculate the distance I need to walk to go from Lund's cathedral to Lund's library.
   Solution: 1-norm

   (b) Calculate the height of a door so that every person under 1.4 meters can pass under it without bending down.
   Solution: ∞-norm

   (c) Calculate the magnitude of the residual after fitting a straight line to given data, using least squares.
   Solution: 2-norm

   (d) Make sure that the values of an error vector are all under 10^{-3}.
   Solution: ∞-norm

   (e) Calculate how many squares a rook needs to move over to get to a specific position on the chessboard (may require more than one move).
   Solution: 1-norm



Figur 1: These are the possible positions of a rook after a single move.

2. **(5p)** I need to solve the following equation for x:

$$x - \frac{1}{2}y = \frac{h(x) + h(y)}{20}$$

   where $y = 1$ and $h(x) = x^5$. Choose one of the following methods and write out the iteration I must perform for this particular problem (in detail). Do not solve.

(a) $x_{n+1} = x_n - f(x_n)/f'(x_n)$

(b) $x_{n+1} = Ax_n$

(c) $x_{n+1} = g(x_n)$

Solution: (a) Newton-Raphson

$$\begin{aligned} f(x) &= x - \frac{1}{20}x^5 - \frac{11}{20} = 0 \\ f'(x) &= 1 - \frac{1}{4}x^4 \\ x_{n+1} &= x_n - \frac{x_n - \frac{1}{20}x_n^5 - \frac{11}{20}}{1 - \frac{1}{4}x^4} \end{aligned}$$

3. **(4p)** After solving a problem with 7 iterations of a certain iterative method, I get the following errors:
0.6 0.54 0.45 0.3 0.14 0.028 0.0012
Calculate what is the rate of convergence of the method, with the help of

$$\|e_{k+1}\| \leq c \cdot \|e_k\|^r$$

Solution: We get $\|e_{k+1}\| \approx 1.5 \cdot \|e_k\|^2$, so the rate of convergence is 2.

k	1	2	3	4	5	6
e_{k+1}/e_k	0.9	0.83	0.67	0.47	0.20	0.43
e_{k+1}/e_k^2	1.5	1.54	1.48	1.56	1.43	1.53

4. **(6p)** Explain what the backslash operator in Matlab does (what method does it use, what particular features) when we write the command $x = A\backslash b$, and

(a) A is a full 15×15 matrix
Solution: Gauss elimination with pivoting

(b) A is a 150×5 matrix
Solution: QR for least squares

(c) A is an upper triangular matrix of dimension 25 by 25.
Solution: Backward substitution

5. The projection of a vector v onto a vector w is

$$\text{proj}_w v = \frac{v^t w}{w^t w} w.$$

(a) **(2p)** Construct an orthogonal basis for the subspace spanned by vectors $(1, 1, 0)$ and $(2, 0, 3)$.
Solution:
$x_1 = 1/\sqrt{2}(1, 1, 0)$
$\text{proj}_{(1,1,0)}(2, 0, 3) = (1, 1, 0)$
$\bar{x_2} = (2, 0, 3) - (1, 1, 0) = (1, -1, 3)$
$x_2 = 1/\sqrt{11}(1, -1, 3)$

(b) **(2p)** What is the relation between the basis you constructed and the QR factorization of A?
Solution: x_1 and x_2 are the first two columns of Q.

6. **(6p)** Let A be an invertible $n \times n$ real matrix.

   (a) Prove that the eigenvalues of A and A^T are the same. (Hint: we know that $\det(A) = \det(A^T)$.)
   Solution:

$$\begin{aligned} \lambda = \mathrm{eig}(A) &\Rightarrow \det(\lambda I - A) = 0 \\ &\Rightarrow \det((\lambda I - A)^t) = 0 \\ &\Rightarrow \det(\lambda I - A^t) = 0 \\ &\Rightarrow \lambda = \mathrm{eig}(A^t) \end{aligned}$$

   (b) Prove that if λ is a non-zero eigenvalue of A then $1/\lambda$ is an eigenvalue of A^{-1}.
   Solution: $Ax = \lambda x \Rightarrow \lambda^{-1}x = A^{-1}x$

   (c) Prove that if A is an orthogonal matrix, its real eigenvalues are ± 1.
   Solution: Uses $A^tA = I$

$$\begin{aligned} Ax &= \lambda x \\ x^tA^t &= \lambda x^t \\ x^tA^tAx &= \lambda^2 x^t x \\ x^t x &= \lambda^2 x^t x \\ \lambda^2 &= 1 \quad \Rightarrow \quad |\lambda| = \pm 1 \quad \text{if} \quad \lambda \in \mathbb{R} \end{aligned}$$

7. **(5p)** Comment on the following statements:

   (a) There is one unique polynomial of degree n that interpolates n data points.
   Solution: There is one unique polynomial of degree n that interpolates $n+1$ data points.

   (b) Interpolation is usually not appropriate if the data points are subjected to experimental errors.
   Solution: In that case it is better to use least squares.

   (c) When doing high-degree polynomial interpolation the curve may wiggle so much that it will not pass through all the data points.
   Solution: In interpolation the curve passes through each and every point.

   (d) Chebyshev interpolation points are bunched near the center of the interval.
   Solution: They are bunched towards the two ends.

   (e) A spline is a piecewise polynomial of degree k that is continuously differentiable k times.
   Solution: A spline is a piecewise polynomial of degree k that is continuously differentiable $k-1$ times.

8. **(5p)** Give two examples of additional conditions that might be imposed to determine the cubic spline interpolant to a set of data points.

Solution: Two possible sets of conditions are

$$\begin{array}{llcl} (1) & s'(t_0) = s_0 & \text{and} & s'(t_f) = s_f \\ (2) & s''(t_0) = 0 & \text{and} & s''(t_f) = 0 \end{array}$$

9. **(6p)** After applying a QR iteration to each of two matrices, A and B, the results are, respectively:

```
ans =
          4.8979  -1.5046e-16  -3.5827e-16  -2.7318e-16
           0        0.60394      2.8797e-16  -2.4716e-17
           0            0        0.14022     -7.0097e-16
           0            0            0         0.00041375
```

and

```
ans =
     -3.0025       0.24446      -3.663        1.2072
           0       0.51801       0.025815     0.32126
           0            0       -0.21637      0.071929
           0            0            0       -0.00050994
```

One of the original matrices is real symmetric. Which one is it, and how can you tell?
Solution: The first one, because it is a diagonal matrix up to round-off errors, and real symmetric matrices converge to diagonal matrices.

10. **(5p)** Count the number of operations involved in carrying out k steps of the unnormalized power method ($x_{i+1} = Ax_i$) for an $n \times n$ matrix.
Solution: For each row of A the matrix-vector multiplication does n multiplications and $n-1$ additions. Therefore, it takes $n(2n-1)$ operations for each iteration. For k iterations it will take $kn(2n-1)$ operations.

11. **(4p)** The DFT of a real vector of length n can be interpreted as interpolation by a set of n trigonometric basis functions.

    (a) Why does computing the DFT *not* require the solution of an $n \times n$ linear system by matrix factorization in order to determine the coefficients of the basis functions?
    Solution: The coefficients are the entries of the DFT vector.

    (b) What is the worst-case computational complexity for computing the DFT? For what values of n is this the case?
    Solution: $O(n^2)$ if n is a prime number.

    (c) What is the best-case computational complexity for computing the DFT? For what values of n is this the case?
    Solution: $O(n \log_2 n)$ if $n = 2^N$.

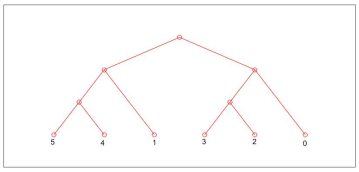
    (d) Explain briefly the reason for the difference, or the lack of it, between the complexities in parts *b* and *c*.
    Solution: FFT is based on repeatedly halving a DFT of size s until only DFTs of size 1 remain.

12. **(5p)** Use Huffman coding to construct a code for the elements of the following matrix.

$$\begin{pmatrix} 5 & 3 & 2 & 2 \\ 3 & 4 & 1 & 1 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Solution:

5	4	3	2	1	0
1/16	2/16	2/16	2/16	3/16	6/16
000	001	100	101	01	11



13. **(5p)** Let A be an $m \times n$ real matrix, and

$$\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \lambda \begin{pmatrix} u \\ v \end{pmatrix}$$

Show that $|\lambda|$ is a singular value of A.
Solution: s is a singular value of A if s^2 is an eigenvalue of A^tA.

$$\begin{aligned} Av &= \lambda u \quad \text{and} \quad A^t u = \lambda v \\ A^tAv &= \lambda A^t u \\ &= \lambda^2 v \\ \lambda^2 &= \text{eig}(A^tA) \end{aligned}$$

14. **(5p)** Mention and briefly describe the different types of compression that are used in JPEG.

Solution:

Lossy compression: With DCT applied to 8 by 8 blocks to filter out less important frequencies, then quantization to compress a range of values into a single one.

Lossless compression: Huffman coding, assigning shorter codes to symbols that appear more frequently.