Numerical Analysis — FMN011 — 2016/06/03
SOLUTIONS

The exam lasts 5 hours and has 14 questions. A minimum of 35 points out of the total 70 are required to get a passing grade. These points will be added to those you obtained in your two home assignments, and the final grade is based on your total score.

Justify all your answers and write down all important steps. Unsupported answers will be disregarded.

During the exam you are allowed a pocket calculator, but no textbook, lecture notes or any other electronic or written material.

1. **(5p)** What vector norm should I use for each of the following tasks?

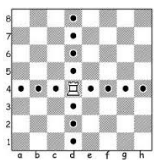
   (a) Calculate the distance I need to walk to go from Lund's cathedral to Lund's market.
   Solution: 1-norm

   (b) Calculate if every person on a list can pass under a 1.9 m door without bending down.
   Solution: ∞-norm

   (c) Calculate the magnitude of the residual after fitting a straight line to given data, using least squares.
   Solution: 2-norm

   (d) Make sure that the values of an error vector are all under 10^{-3}.
   Solution: ∞-norm

   (e) Calculate how many squares a rook needs to move over to get to a specific position on the chessboard (may require more than one move).
   Solution: 1-norm

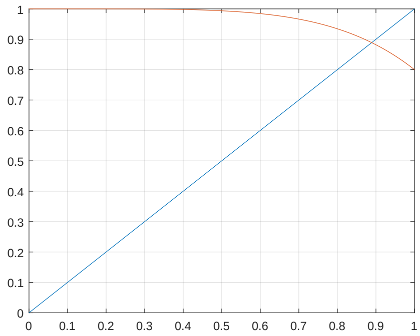


Figur 1: These are the possible positions of a rook after a single move.

2. Consider the polynomial $p(x) = x^5 + 5x - 5$.

   (a) **(2p)** Plot the curves $y = x$ and $y = 1 - \frac{1}{5}x^5$, and show that p has a unique root in $(0, 1)$.
   Solution: See figure.

   (b) **(3p)** Show that if we take $g(x) = 1 - \frac{1}{5}x^5$, then we can find the root using a fixed point iteration, $x_n = g(x_{n-1})$. (Do not find the root)
   Solution: $g(x) = 1 - x^5/5 \Rightarrow |g'(x)| = x^4 < 1$ if $x \in (0, 1)$.



Figur 2: One is a decreasing function and the other an increasing function, so they intersect only once.

3. **(5p)** The function $f(x) = x^3 - 4x$ has a root at $r = 2$. If the error $e_i = x_i - r$ after four steps of Newton-Raphson's method is $e_4 = 10^{-4}$, estimate e_5.
Solution: Quadratic convergence:
$$\begin{aligned} \frac{e_5}{{e_4}^2} &\approx k \\ e_5 &\approx k{e_4}^2 \\ e_5 &\approx 10^{-8} \end{aligned}$$

4. **(5p)** Suppose you want to solve the system $Ax = b$, where A is an $n \times n$ matrix, and you have the LU factorization, $A = LU$. What are the steps required for solving this problem using this factorization, and what is the (approximate) number of computations needed for each step?
Solution:
$$\begin{aligned} Ly &= b \text{ (forward substitution) } O(n^2) \\ Ux &= y \text{ (back substitution) } O(n^2) \end{aligned}$$

5. **(5p)** Given data points $(x, y, z) = (0,0,3), (0,1,2), (1,0,3), (1,1,5)$, you need to find the plane $z = c_0 + c_1 x + c_2 y$ that best fits the data.

   (a) Write down the overdetermined system for this problem.
   Solution:
$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 3 \\ 5 \end{pmatrix}$$

   (b) Write down the square system you need to solve to get the least squares solution to this problem. (Do not solve)
   Solution: $A^T A c = A^T z$
$$\begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \\ 7 \end{pmatrix}$$

6. **(5p)** Use Lagrange interpolation to find a polynomial that passes through the points (0,1), (2,3), and (3,0).
Solution:

$$
\begin{aligned}
l_0(x) &= \frac{(x-2)(x-3)}{(0-2)(0-3)} = \frac{1}{6}(x-2)(x-3) \\
l_1(x) &= \frac{x(x-3)}{2(2-3)} = -\frac{1}{2}x(x-3) \\
P(x) &= \frac{1}{6}(x-2)(x-3) - \frac{3}{2}x(x-3) = -\frac{4}{3}x^2 + \frac{11}{3}x + 1
\end{aligned}
$$

7. (a) **(3p)** Name at least three properties of the matrix involved in the construction of a cubic spline.
Solution: strictly diagonally dominant, tridiagonal, sparse

(b) **(2p)** In what way is the structure of the matrix affected by the end conditions of the cubic spline?
Solution: only first and last rows are affected

8. **(5p)** Name the method illustrated here, and describe what it calculates. Show that the result given by the algorithm is correct.

$$
A = \begin{bmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{bmatrix} \quad \text{and} \quad x_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}
$$

i	1	2	3	4	...	14	15
$\frac{Ay_{i-1}}{\Vert Ay_{i-1}\Vert}$	0.3162 0.9487	0.5145 0.8575	0.6139 0.7894	0.6616 0.7498	... ...	0.7070 0.7072	0.7071 0.7071
$y_i^T Ay_i$	1.50	1.80	1.941	1.985	...	1.999	2.000

Solution: Normalized power method, calculates the dominant eigenvalue, $\lambda = 2$, and corresponding eigenvector, $u = (0.7071, 0.7071)^T$.

$$
\begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix} \begin{pmatrix} 0.7071 \\ 0.7071 \end{pmatrix} = 2 \begin{pmatrix} 0.7071 \\ 0.7071 \end{pmatrix}
$$

9. **(5p)** Let A be an $n \times n$ orthogonal matrix. Suppose you want to solve the overdetermined system $A_m x = b$, where A_m is the $n \times m$ matrix made up of m columns of A.

(a) Explain why orthogonality is important for finding the least squares solution to $A_m x = b$.
Solution: Least squares solution is the solution to the normal equations

$$
A_m^T A_m x = A_m^T b \Rightarrow x = A_m^T b
$$

as $A_m^T A_m = I$ because of orthogonality.

(b) Do an operations count for this problem.
Solution: Matrix-vector multiplication with $m \times n$ matrix, $n \times 1$ vector needs $m(2n-1)$ operations.

10. **(5p)** True or false:

(a) The PageRank algorithm is based on the singular value decomposition of the Google matrix.
Solution: F (power method)

(b) The eigenvalues of a real matrix may be complex.
Solution: T (zeros of polynomials)

(c) The singular values of a real matrix may be complex.
Solution: F ($A^T A$ has real eigenvalues)

(d) If x is an eigenvector of A, then so is $-x$.
Solution: T (cx is also an eigenvalue for any $c \neq 0$)

(e) If u is a singular vector of A then so is $-u$.
Solution: T (u is eigenvector of AA^T)

11. **(5p)** From the DFT interpolation theorem we know that the interpolating trigonometric polynomial related to the DFT is

$$P_n(t) = \frac{a_0}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=1}^{n/2-1} \left(a_k \cos \frac{2k\pi(t-c)}{d-c} - b_k \sin \frac{2k\pi(t-c)}{d-c} \right) + \frac{a_{n/2}}{\sqrt{n}} \cos \frac{n\pi(t-c)}{d-c}$$

What is the discrete Fourier transform of a pure cosine wave $f(t) = \cos 2\pi t$ sampled at 4 equally spaced points on [0,1)?
Solution: $t_1 = 0, 0.25, 0.5, 0.75$,

$$P(t) = \frac{a_0}{2} + a_1 \cos 2\pi t + \frac{a_2}{2} \cos 4\pi t$$

$f(t) = \cos 2\pi t \Rightarrow a_0 = 0, a_1 = 1, b_1 = 0, a_2 = 0,\ y = (0, 1, 0, 1)^T$

12. **(5p)** Explain why the computational complexity of the inverse Fourier transform is the same as that of the DFT.
Solution: Inverse DFT is a multiplication by an $n \times n$ matrix, just as DFT is ($F^{-1}y = \overline{F}y$).

13. **(5p)** Describe the steps taken to compress an image in a jpeg file.
Solution:

(a) DFT on 8×8 submatrices

(b) Quantization

(c) Huffman coding for DC component

(d) Huffman coding for AC components

14. **(5p)** Draw a Huffman tree and convert the message

FUZZY WUZZY

including spaces, to a bit stream by using Huffman coding. Compare the Shannon information

$$I = -\sum p_i \log_2 p_i$$

with the average number of bits needed per symbol. Could we hope for a better compression? Explain your answer.
Solution: A possible coding is

F	1	0000	4
-	1	0001	4
W	1	001	3
U	2	010	6
Y	2	011	6
Z	4	1	4
	11		27/11≈ 2.45bits/symbol

$$I = 3(\frac{1}{11}\log_2 11) + 2(\frac{2}{11}\log_2\frac{11}{2}) + \frac{4}{11}\log_2\frac{11}{4} approx 2.37\text{bits/symbol}$$

I gives the minimum possible number of bits, which for 11 symbols would be 26.05, so 27 bits/symbol is the best one can hope for. This is the same we got with Huffman coding.

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