**Problem 1.** For this case find an algebraic solution for the policy function, $k_{t+1} = \phi(k_t, z_t)$.

*Proof.* We first guess, that there exists $A$ defining the policy function $K_{t+1} = Ae^{zt}K_t^{\alpha}$ and plugging it into the Euler equation. This gives us:

$$\frac{1}{e^{z_t}K_t^{\alpha} - K_{t+1}} = \beta E_t\{\frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{e^{z_{t+1}}K_{t+1}^{\alpha} - K_{t+2}}\}$$
$$\frac{1}{(1-A)e^{z_t}K_t^{\alpha}} = \beta E_t\{\frac{\alpha e^{z_{t+1}}K_{t+1}^{\alpha-1}}{(1-A)e^{z_{t+1}}K_{t+1}^{\alpha}}\}$$
$$\frac{1}{e^{z_t}K_t^{\alpha}} = \beta\frac{\alpha}{K_{t+1}}$$
$$\frac{1}{e^{z_t}K_t^{\alpha}} = \beta\frac{\alpha}{Ae^{zt}K_t^{\alpha}}$$
$$1 = \frac{\alpha\beta}{A}$$
$$A = \alpha\beta$$

□

**Problem 2.** Given following functional forms,

$$u(c_t, \ell_t) = \ln\, c_t + a\, \ln\, (1 - \ell_t)$$
$$F(K_t, L_t, z_t) = e^{z_t}K_t^{\alpha}L_t^{1-\alpha}$$

find the characterizing equations.

*Proof.*

$$c_t = (1-\tau)(w_t l_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1}$$
$$\frac{1}{c_t} = \beta E_t\{\frac{1}{c_{t+1}}((r_{t+1} - \delta)(1-\tau) + 1)\}$$
$$\frac{a}{1 - l_t} = \frac{w_t}{c_t}(1-\tau)$$
$$r_t = \alpha e^{zt}K_t^{\alpha-1}L_t^{1-\alpha}$$
$$w_t = (1-\alpha)e^{zt}K_t^{\alpha}L_t^{-\alpha}$$
$$\tau(w_t l_t + (r_t - \delta)k_t) = T_t$$
$$z_t = (1-p_z)\bar{z} + p_z z_{t-1} + \epsilon_t^z, \epsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)$$

□

**Problem 3.** Using CRRA utility for consumption,

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\, \ln\, (1 - \ell_t)$$
$$F(K_t, L_t, z_t) = e^{z_t}K_t^{\alpha}L_t^{1-\alpha}$$

*Proof.* New characterizing equations are similar

$$c_t = (1-\tau)(w_t l_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1}$$
$$\frac{1}{c_t^{-\gamma}} = \beta E_t\Big\{\frac{1}{c_{t+1}^{-\gamma}}((r_{t+1} - \delta)(1-\tau) + 1)\Big\}$$
$$\frac{a}{1-l_t} = \frac{w_t}{c_t^{-\gamma}}(1-\tau)$$
$$r_t = \alpha e^{zt} K_t^{\alpha-1} L_t^{1-\alpha}$$
$$w_t = (1-\alpha)e^{zt} K_t^{\alpha} L_t^{-\alpha}$$
$$\tau(w_t l_t + (r_t - \delta)k_t) = T_t$$
$$z_t = (1-p_z)\bar{z} + p_z z_{t-1} + \epsilon_t^z, \epsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)$$

□

**Problem 4.** Using CRRA utility for consumption and leisure and a new production function.

$$u(c_t, \ell_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma} + a\frac{(1-\ell_t)^{1-\xi} - 1}{1-\xi}$$
$$F(K_t, L_t, z_t) = e^{z_t}\left[\alpha K_t^{\eta} + (1-\alpha)L_t^{\eta}\right]^{\frac{1}{\eta}}$$

*Proof.* New characterizing equations are similar

$$c_t = (1-\tau)(w_t l_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1}$$
$$\frac{1}{c_t^{-\gamma}} = \beta E_t\Big\{\frac{1}{c_{t+1}^{-\gamma}}((r_{t+1} - \delta)(1-\tau) + 1)\Big\}$$
$$\frac{a}{(1-l_t)^{-\xi}} = \frac{w_t}{c_t^{-\gamma}}(1-\tau)$$
$$r_t = \alpha e^{zt} K_t^{\eta-1}(\alpha K_t^{\eta} + (1-\alpha)L_t^{\eta})^{\frac{1-\eta}{\eta}}$$
$$w_t = (1-\alpha)e^{zt} L_t^{\eta-1}(\alpha K_t^{\eta} + (1-\alpha)L_t^{\eta})^{\frac{1-\eta}{\eta}}$$
$$\tau(w_t l_t + (r_t - \delta)k_t) = T_t$$
$$z_t = (1-p_z)\bar{z} + p_z z_{t-1} + \epsilon_t^z, \epsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)$$

□

**Problem 5.** Using the new given functional forms. Write out characterizing equations.

$$u(c_t) = \frac{c_t^{1-\gamma} - 1}{1-\gamma}$$
$$F(K_t, L_t, z_t) = K_t^{\alpha}(L_t e^{z_t})^{1-\alpha}$$

We have 6 because intratemporal labor/leisure euler equation drops out.

*Proof.*

$$
\begin{aligned}
c_t &= (1-\tau)(w_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1} \\
\frac{1}{c_t^{-\gamma}} &= \beta E_t\Big\{\frac{1}{c_{t+1}^{-\gamma}}((r_{t+1}-\delta)(1-\tau)+1)\Big\} \\
r_t &= \alpha(e^{zt})^{(1-\alpha)}K_t^{\alpha-1} \\
w_t &= (1-\alpha)(e^{zt})^{(1-\alpha)}K_t^{\alpha} \\
\tau(w_t + (r_t-\delta)k_t) &= T_t \\
z_t &= (1-p_z)\bar{z} + p_z z_{t-1} + \epsilon_t^z, \epsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)
\end{aligned}
$$

In the non-stochastic steady state this becomes:

$$
\begin{gathered}
\bar{c} = (1-\tau)(\bar{w} + (\bar{r}-\delta)\bar{k}) + \bar{T} \\
\frac{1}{\bar{c}^{-\gamma}} = \beta\frac{1}{\bar{c}^{-\gamma}}((\bar{r}-\delta)(1-\tau)+1) \\
\bar{r} = \alpha(e^{\bar{z}})^{1-\alpha}\bar{k}^{\alpha-1} \\
\bar{w} = (1-\alpha)(e^{\bar{z}})^{1-\alpha}\bar{k}^{\alpha} \\
\tau(\bar{w} + (\bar{r}-\delta)\bar{k}) = \bar{T}
\end{gathered}
$$

With $\bar{z} = 0$. Solving algebraically for the steady state values of capital, output, and investment, giving us:

$$
\begin{gathered}
\bar{r} = \frac{\frac{1}{\beta}-1}{1-\tau} + \delta = 0.12 \\
\bar{k} = \Big(\frac{\bar{r}}{\alpha}\Big)^{\frac{1}{\alpha-1}} = 7.28 \\
\bar{I} = \delta\bar{k} = 0.73 \\
\bar{Y} = \bar{k}^{\alpha} = 2.21
\end{gathered}
$$

Numerical solution is similar. See jupyter notebook attached below. □

**Problem 6.** Using the new given functional forms,

$$
\begin{aligned}
u(c_t, \ell_t) &= \frac{c_t^{1-\gamma}-1}{1-\gamma} + a\frac{(1-\ell_t)^{1-\xi}-1}{1-\xi} \\
F(K_t, L_t, z_t) &= K_t^{\alpha}(L_t e^{z_t})^{1-\alpha}
\end{aligned}
$$

*Proof.* We have similar characterizing equations:

$$
\begin{gathered}
c_t = (1-\tau)(w_t + (r_t - \delta)k_t) + k_t + T_t - k_{t+1} \\
\frac{1}{c_t^{-\gamma}} = \beta E_t\Big\{\frac{1}{c_{t+1}^{-\gamma}}((r_{t+1}-\delta)(1-\tau)+1)\Big\} \\
\frac{a}{(1-l_t)^{-\xi}} = \frac{w_t}{c_t^{-\gamma}}(1-\tau)
\end{gathered}
$$

$$r_t = \alpha(e^{zt})^{1-\alpha} K_t^{\alpha-1} L_t^{1-\alpha}$$
$$w_t = (1-\alpha)(e^{zt})^{1-\alpha} K_t^{\alpha} L_t^{-\alpha}$$
$$\tau(w_t + (r_t - \delta)k_t) = T_t$$
$$z_t = (1-p_z)\bar{z} + p_z z_{t-1} + \epsilon_t^z, \epsilon_t^z \sim \text{i.i.d}(0, \sigma_z^2)$$

which in the steady state become:

$$\bar{c} = (1-\tau)(\bar{w} + (\bar{r} - \delta)\bar{k}) + \bar{T}$$
$$\frac{1}{\bar{c}^{-\gamma}} = \beta \frac{1}{\bar{c}^{-\gamma}}((\bar{r} - \delta)(1-\tau) + 1)$$
$$\frac{a}{(1-\bar{l})^{-\xi}} = \frac{\bar{w}}{\bar{c}^{-\gamma}}(1-\tau)$$
$$\bar{r} = \alpha(e^{\bar{z}})^{1-\alpha} \bar{k}^{\alpha-1} \bar{l}^{1-\alpha}$$
$$\bar{w} = (1-\alpha)(e^{\bar{z}})^{1-\alpha} \bar{k}^{\alpha} \bar{l}^{-\alpha}$$
$$\tau(\bar{w} + (\bar{r} - \delta)\bar{k}) = \bar{T}$$

where $\bar{z} = 0$. Numerical solution is similar. See jupyter notebook attached below. $\square$