# Problem Set #1

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## 1

**Solution 1.** (1.3)

1. Suppose $A \in \mathbb{R}$ open. So $A \in \mathbb{G}_1$ by definition. Take $A^c \in \mathbb{R}$ which is closed by properties of open/closed sets.
   $\Rightarrow A^c \notin \mathcal{G}_1$ by definition of $\mathcal{G}_1$
   $\Rightarrow$ G1 not closed under complements. So G1 is not an algebra. $\square$

2. **WTS:** G2 algebra

   - $\emptyset \in \mathcal{G}_2$
     $\Rightarrow \emptyset \in \mathcal{G}_2$
   - Suppose $A_j = \bigcup_{i=1}^{N_j}(a_i, b_i]$
     So then $\bigcup_j^M A_j = \bigcup_j^M (\bigcup_{i=1}^{N_j}((a_i, b_i]) \in \mathcal{G}_2$
   - Suppose $A_j = \bigcup_{i=1}^{N_j}(a_i, b_i]$
     So then complement $(-\infty, a_1] \bigcup (b_n, \infty) \in \mathcal{G}_2$ $\square$

   So G2 is an algebra.

3. **WTS:** G3 sig-alg

   - $\emptyset \in \mathcal{G}_3$
   - Suppose $A_j = \cup_{i=1}^{N_j}((a_i, b_i] \cup (-\infty, b] \cup (a, \infty))$
     So then $\cup_j^\infty A_j = \cup_j^\infty(\cup_{i=1}^\infty((a_i, b_i] \cup (-\infty, b] \cup (a, \infty))) \in \mathbb{G}_3$ $\square$

   So, G3 is a sigma algebra

**Solution 2.** (1.7) Suppose $\mathcal{A}$ is an sigma algebra.
**WTS:** $\{\emptyset, X\} \subset \mathcal{A} \subset \mathcal{P}(X)$
**Pf:** $\emptyset \in S$ for every S sigma algebra. Also, S must be closed under complements therefore $\emptyset^c = X \in S$. So the smallest possible sigma algebra is $\{\emptyset, X\}$. Also, suppose $A \in \mathcal{A}$ so $A \in \mathcal{P}(X)$ because $\mathcal{A} \subset X$
$\therefore \mathcal{A} \subset \mathcal{P}(X)$

**Solution 3.** (1.10)
i) $\emptyset \in S_\alpha \quad \forall \alpha$ by definition of sig-alg. $\Rightarrow \emptyset \in \cap^\alpha S_\alpha$
ii) suppose $A_1, .... \in \cap^\alpha S_\alpha$ this implies $A_1, .... \in S_\alpha \forall \alpha$
So the union of $A_i \in S_\alpha$ for every alpha.

So $\cup A_1, ... \in \cap^{\infty} S_\alpha$
Also, suppose $A \in \bigcap S_\alpha$ now we know that $A \in S_\alpha \quad \forall \alpha$ by definition of sigma-algebra. $A^c \in S_\alpha \quad \forall \alpha$ so, $A^c \in \bigcap S_\alpha$ Therefore intersection is a sigma algebra.

**Solution 4.** (1.17)
i) We know

$$\mu(A \cup B)) = \mu(A) + \mu(B)$$

if

$$A \cap B = \emptyset$$

. Now suppose $A \subset B$ and $B = A \cup U$.
So,

$$\mu(A \cup U) = \mu(A) + \mu(U) \geq \mu(A)$$

because measure is valued on positive reals.

ii) We know,

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) - \mu(\cap_i A_i)$$

So,

$$\mu(\cup_{i=1}^{n} A_i) + \underbrace{\mu(\cap_i^{\infty} A_i)}_{\geq 0} = \sum_{i=1}^{\infty} \mu(A_i)$$

**Solution 5.** (1.18)
**WTS:**

$$\lambda(A) = \mu(A \cap B)$$

**Pf:**

$$A, B \subset S \Rightarrow (A \cap B) \subset S. \Rightarrow \emptyset \cap B = \emptyset$$

So, i)

$$\lambda(\emptyset) = \mu(\emptyset) = 0$$

And because intersection is in S and ,

$$\lambda(A) = \mu(A \cap B) \Rightarrow \lambda(\cup^{\infty} A_i) = \mu(\cup^{\infty}(A_i \cap B)) = \sum^{\infty} \underbrace{\mu(A_i \cap B)}_{\lambda(A_i)} = \sum^{\infty} \lambda(A_i)$$

**Solution 6.** (1.20)

$$\mu(A_1) - \mu(\lim_{n\to\infty}(A_n))$$
$$= \mu(A_1 \bigcap \lim_{n\to\infty} A_n)$$
$$= \mu(\lim_{n\to\infty}(A_1 \bigcap A_n))$$
$$= \lim_{n\to\infty} \mu(A_1 \bigcap A_n)$$
$$= \lim_{n\to\infty} \mu(A_1) - \mu(A_n))$$
$$= \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$$
$$\therefore \mu(\lim_{n\to\infty}(A_n)) = \lim_{n\to\infty} \mu(A_n)$$

# 2

**Solution 7.** (2.10) We know from the countable subadditivity of the outer measure that

$$\mu^*(B) \leq \mu^*(B \cap E) + \mu^*(B \cap E^c)$$

So the if $\geq$ from the theorum holds, and $\leq$ holds from definition of outer measure and it is equivalent to replace it with equality.

**Solution 8.** (2.14) **WTS:** $\sigma(\mathcal{O}) \subset \mathcal{M}$ From Caratheodory and construction of lebasque measure as a infinite ocllection of the form $(a, b]$ and $(-\infty, a]$ we know

$$\sigma(\mathcal{A}) \subset \mathcal{M} \tag{1}$$
$$if \quad o \in \sigma(\mathcal{O}) \quad then \quad o \in \sigma(\mathcal{A}) \tag{2}$$
$$so, \quad o \in \mathcal{M} \tag{3}$$

# 3

**Solution 9.** (3.1) Suppose $r = \bigcup_{n\in\mathbb{N}} r_i$
So $\mu(\bigcup_{n\in\mathbb{N}} r_i) = \sum_{n\in\mathbb{N}} \mu(r_i)$
but by construction of lebesque measure, $\mu(r_i) = 0, \forall i$
So, $\mu(r) = \mu(\bigcup_{n\in\mathbb{N}} r_i) = 0$

**Solution 10.** (3.4)
Because the set of all measureable sets is a sigma-algebra, it is closed under complements. So those conditions provided each are complements of each other.
I'll show that the sets being measurable are all equivalent statements.
$f^{-1}((-\infty, a))$ is measurable $\iff$ $f^{-1}([a, \infty))$ is measurable (they are complements).
**WTS:**

$$f^{-1}((-\infty, a)) \in \mathcal{M} \iff f^{-1}((-\infty, a]) \in \mathcal{M}$$

Left to Right: Suppose sets of the form $f^{-1}((-\infty, a]) \in \mathcal{M}$.
Now, we construct a sequence of sets $E_{i,n} = f^{-1}((-\infty, a - \frac{1}{n}]) \in \mathcal{M}$. This countable union $\cup_{n=1}^{\infty} = f^{-1}((-\infty, a])$ is in $\mathcal{M}$

Right to Left: Suppose sets of the form $f^{-1}((-\infty, a)) \in \mathcal{M}$. Then their complements, sets of the form $f^{-1}([a, \infty))$ are also $in \mathcal{M}$. We can use a similar argument, employing sets of the form $f^{-1}([a + \frac{1}{n}, \infty))$ to show that sets of the form $f^{-1}((a, \infty)) \in \mathcal{M}$. This shows that the complements of these sets, $f^{-1}((-\infty, a])$, are also elements of $\mathcal{M}$

**Solution 11.** (3.7)
**WTS:**

1. $f + g$
2. $f \cdot g$
3. $\max(f, g)$
4. $\min(f, g)$
5. $|f|$

**Pf:**

1. Take $F(f(x) + g(x)) = f(x) + g(x)$. So $F$ is cont. and (part 4) measurable. So, $f + g$ is measurable.

2. Take $F(f(x) + g(x)) = f(x)g(x)$. So $F$ is cont. and (part 4) measurable. So, $f \cdot g$ is measurable.

3. As $f$ and $g$ are measurable functions on $(X, \mathcal{M})$,
   $\forall a \in \mathbb{R}$, $\{x \in X : f(x) < a\} \in \mathcal{M}$ and $\{x \in X : g(x) < a\} \in \mathcal{M}$.
   So,
   $$\{x \in X : \max(f(x), g(x)) < a\} = \{x \in X : f(x) < a\} \cap \{x \in X : g(x) < a\}$$
   . $\mathcal{M}$ is closed under countable intersections, therefore, $\{x \in X : \max(f(x), g(x)) < a\} \in \mathcal{M}$, so that $\max(f(x), g(x))$ is measurable.

4. $\{x \in X : \min(f(x), g(x)) > a\} = \{x \in X : f(x) > a\} \cap \{x \in X : g(x) > a\}$. $\mathcal{M}$ is closed under countable intersections,
   So, $\{x \in X : \min(f(x), g(x)) > a\} \in \mathcal{M}$, so that $\min(f(x), g(x))$ is measurable.Basically the same proof as above.

5. Because $\{x \in X : |f(x)| > a\} = \{x \in X : f(x) < -a\} \cup \{x \in X : f(x) > a\}$. Both of these sets are in $\mathcal{M}$. $\mathcal{M}$ is closed under countable unions, therefore, $\{x \in X : |f(x)| > a\} \in \mathcal{M}$, so that $|f(x)|$ is measurable.

**Solution 12.** (3.14)
**WTS:**

$$\forall \epsilon > 0, \exists N = N(\epsilon) \text{ such that } n \geq N \implies |f(x) - s_n(x)| < \epsilon, \forall x \in X$$

Let $\epsilon > 0$. Suppose $f(x) < M$. Pick $N > M$ in the naturals. Then $f(x) < N \quad \forall x$ and $x \notin E_\infty^N$. We also see that there exists $N_1$ such that

$$N_1 > N \text{ and } \frac{1}{2^{N_1}} < \epsilon$$

Now, it follows that for $n > N_1$,

$$\forall x \in X, x \in E_i^n \text{ for some index } 0 \leq i \leq N_1, i \in \mathbb{N}$$

Then $f(x) \in [\frac{i-1}{2^n}, \frac{i}{2^n})$ and our simple function in this interval is $s_n(x) = \frac{i-1}{2^n}$
But because we chose this $N_1$ so that it works for every single $x \in X$, and so $|f(x) - s_n(x)| < \frac{1}{2^n} < \frac{1}{2^{N^2}} < \epsilon$ we have uniform convergence.

**Solution 13.** (4.13)
**WTS:** $f \in \mathcal{L}^1(\mu, E)$
Which is to say,
**WTS:** $\int_E f^+ d\mu$ finite
**WTS:** $\int_E f^- d\mu$ finite

We are given $||f|| = f^+ + f^-$.
$0 \leq f^+$ and $0 \leq f^-$ from the definition
Because $||f|| < M$ on $E$, then $0 \leq f^+ < M$ and $0 \leq f^- < M$ on $E$.
Because $\mu(E) < \infty$, we have that,

$$\int_E f^+ d\mu < M\mu(E) < \infty$$
$$\int_E f^- d\mu < M\mu(E) < \infty$$

This is what we needed.
$\therefore \quad f \in \mathcal{L}^1(\mu, E)$.

**Solution 14.** (4.14)
Contrapositive is easier to show.

Suppose there exists a measurable set $\hat{E} \subset E$ such that $f$ is infinite on $\hat{E}$.
WLOG: $f = +\infty$

$$\infty = \int_{\hat{E}} f d\mu \leq \int_E f d\mu \leq \int_E ||f|| d\mu$$

But this implies that $f \notin \mathcal{L}^1(\mu, E)$. Contradiction so our statement is true.

**Solution 15.** (4.15) Suppose $f, g \in \mathcal{L}^1(\mu, E)$.
Define the set of simple functions $B(f) = \{s : 0 \leq s \leq f, s \text{ simple, measurable}\}$.
Suppose WLOG $f \leq g$. If follows that $f^+ \leq g^+$ and $f^- \geq g^-$.
So we have that $B(f^+) \subset B(g^+)$ and $B(g^-) \subset B(f^-)$
Which gives us that $\int_E f^+ d\mu \leq \int_E g^+ d\mu$ and $\int_E f^- d\mu \geq \int_E g^- d\mu$.
From definition of the lebesque integral we know,

$$\int_E f d\mu = \int_E f^+ d\mu - \int_E f^- d\mu \leq \int_E g^+ d\mu - \int_E g^- d\mu = \int_E g d\mu$$

So,

$$\int_E f d\mu \leq \int_E g d\mu$$

**Solution 16.** (4.16)
Suppose that $f \in \mathcal{L}(E, \mu)$.
So, $\int_E f^+ d\mu < \infty, \int_E f^- d\mu < \infty$.
As $A \subset E$,
$\int_A f^+ d\mu < \infty$ and $\int_A f^- d\mu < \infty$.
So, $f \in \mathcal{L}(A, \mu)$.

**Solution 17.** (4.21) From above, $\lambda() = \int f d\mu$ is a measure on $\mathcal{M}$.
Therefore,

$$\lambda(A) = \lambda(A \setminus B) + \lambda(A \cap B) = \lambda(A \setminus B) + \lambda(B)$$

So,

$$\int_A f d\mu = \int_{A \setminus B} f d\mu + \int_B f d\mu = \int_B f d\mu$$