Problem Set #1

Measure Theory, Jan Ertl
Alex Weinberg

Solution 1. (1.3)

1. Suppose $A \in \mathbb{R}$ open. So $A \in \mathbb{G}_1$ by definition. Take $A^c \in \mathbb{R}$ which is closed by properties of open/closed sets.
$\Rightarrow A^c \notin \mathcal{G}_1$ by definition of $\mathcal{G}_1$
$\Rightarrow$ G1 not closed under complements. So G1 is not an algebra. $\square$

2. **WTS:** G2 algebra
   - $\emptyset \in \mathcal{G}_2$
   $\Rightarrow \emptyset \in \mathcal{G}_2$
   - Suppose $A_j = \bigcup_{i=1}^{N_j}(a_i, b_i]$
   So then $\bigcup_j^M A_j = \bigcup_j^M (\bigcup_{i=1}^{N_j}((a_i, b_i]) \in \mathcal{G}_2$
   - Suppose $A_j = \bigcup_{i=1}^{N_j}(a_i, b_i]$
   So then complement $(-\infty, a_1] \bigcup (b_n, \infty) \in \mathcal{G}_2$ $\square$

   So G2 is an algebra.

3. **WTS:** G3 sig-alg
   - $\emptyset \in \mathcal{G}_3$
   - Suppose $A_j = \cup_{i=1}^{N_j}((a_i, b_i] \cup (-\infty, b] \cup (a, \infty))$
   So then $\cup_j^\infty A_j = \cup_j^\infty(\cup_{i=1}^\infty((a_i, b_i] \cup (-\infty, b] \cup (a, \infty))) \in \mathbb{G}_3$ $\square$

   So, G3 is a sigma algebra

Solution 2. (1.7) Suppose $\mathcal{A}$ is an sigma algebra.
WTS: $\{\emptyset, X\} \subset \mathcal{A} \subset \mathcal{P}(X)$
Pf: $\emptyset \in S$ for every S sigma algebra. Also, S must be closed under complements therefore $\emptyset^c = X \in S$. So the smallest possible sigma algebra is $\{\emptyset, X\}$. Also, suppose $A \in \mathcal{A}$ so $A \in \mathcal{P}(X)$ because $\mathcal{A} \subset X$
$\therefore \mathcal{A} \subset \mathcal{P}(X)$

Solution 3. (1.10)
i) $\emptyset \in S_\alpha \quad \forall\alpha$ by definition of sig-alg. $\Rightarrow \emptyset \in \cap^\alpha S_\alpha$
ii) suppose $A_1, ... \in \cap^\alpha S_\alpha$ this implies $A_1, ... \in S_\alpha \forall\alpha$
So the union of $A_i \in S_\alpha$ for every alpha.

So $\cup A_1, ... \in \cap^\infty S_\alpha$
Also, suppose $A \in \bigcap S_\alpha$ now we know that $A \in S_\alpha \quad \forall\alpha$ by definition of sigma-algebra.
$A^c \in S_\alpha \quad \forall\alpha$ so, $A^c \in \bigcap S_\alpha$ Therefore intersection is a sigma algebra.

Solution 4. (1.17)
i) We know

$$\mu(A \cup B)) = \mu(A) + \mu(B)$$

if

$$A \cap B = \emptyset$$

. Now suppose $A \subset B$ and $B = A \cup U$.
So,

$$\mu(A \cup U) = \mu(A) + \mu(U) \geq \mu(A)$$

because measure is valued on positive reals.

ii) We know,

$$\mu(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i) - \mu(\cap_i A_i)$$

So,

$$\mu(\cup_{i=1}^{n} A_i) + \underbrace{\mu(\cap_i^{\infty} A_i)}_{\geq 0} = \sum_{i=1}^{\infty} \mu(A_i)$$

Solution 5. (1.18)
WTS:

$$\lambda(A) = \mu(A \cap B)$$

Pf:

$$A, B \subset S \Rightarrow (A \cap B) \subset S. \Rightarrow \emptyset \cap B = \emptyset$$

So, i)

$$\lambda(\emptyset) = \mu(\emptyset) = 0$$

And because intersection is in S and ,

$$\lambda(A) = \mu(A \cap B) \Rightarrow \lambda(\cup^{\infty} A_i) = \mu(\cup^{\infty}(A_i \cap B)) = \sum^{\infty} \underbrace{\mu(A_i \cap B)}_{\lambda(A_i)} = \sum^{\infty} \lambda(A_i)$$

Solution 6. (1.20)

$$\mu(A_1) - \mu(\lim_{n\to\infty}(A_n))$$
$$= \mu(A_1 \bigcap \lim_{n\to\infty} A_n)$$
$$= \mu(\lim_{n\to\infty}(A_1 \bigcap A_n))$$
$$= \lim_{n\to\infty} \mu(A_1 \bigcap A_n)$$
$$= \lim_{n\to\infty} \mu(A_1) - \mu(A_n))$$
$$= \mu(A_1) - \lim_{n\to\infty} \mu(A_n)$$
$$\therefore \mu(\lim_{n\to\infty}(A_n)) = \lim_{n\to\infty} \mu(A_n)$$