Math Pset #2: Inner Product Spaces

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OSM Boot Camp 2018

June 28, 2018

Problem 1.
WTS: $\langle x, y\rangle = \frac{1}{4}(||x+y||^2 - ||x-y||^2)$
PF:

$$= \frac{1}{4}(\langle x+y, x+y\rangle - \langle x-y, x-y\rangle)$$
$$= \frac{1}{4}(\langle x, x+y\rangle + \langle y, x+y\rangle - \langle x, x-y\rangle + \langle y, x-y\rangle)$$
$$= \frac{1}{4}(\langle x, x+y+y-x\rangle + \langle y, x+y+x-y\rangle)$$
$$= \frac{1}{4}(\langle x, 2y\rangle + \langle y, 2x\rangle)$$
$$= \frac{1}{4}(2\langle x, y\rangle + 2\langle y, x\rangle)$$
$$= \frac{1}{4}(2\langle x, y\rangle + 2\overline{\langle x, y\rangle})$$

As we are in $\mathbb{R}$,

$$= \langle x, y\rangle$$

Problem 2.

$(||x+y||^2 - ||x-y||^2 + i||x-iy||^2 - i||x+iy||^2)/4 =$
$(< x+y, x+y > - < x-y, x-y > +i < x-iy, y-iy > -i < x+iy, x+iy >)/4 =$
$(2 < x, y > +2 < y, x > -2 < x, y > +2 < y, x >)/4 =$
$< x, y >$.

Problem 3. $< x, x^5 >= \int_0^1 x^6 dx = x^7/7|_0^1 = 1/7$, $||x|| = \int_0^1 x^2 dx = x^3/3|_0^1 = 1/3$ and $||x^5|| = \int_0^1 x^10 dx = x^11/11|_0^1 = 1/11$. Therefore $\cos\theta = \sqrt{33}/7$ implies $\theta = 34.5$.

*Alberto did most of the Latexing

Problem 4. (i)

$$||\cos(t)|| = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(t)dt = \frac{1}{\pi}\left.\frac{\cos(x)\sin(x)-x}{2}\right|_{-\pi}^{p} i = \frac{\pi}{\pi} = 1,$$

and similarly $||\sin(t)|| = 1$. Also

$$||\cos(2t)|| = \frac{1}{\pi}\int_{-\pi}^{\pi}\cos^2(2t)dt = \frac{1}{\pi}\left.\frac{\sin(4t)+4t}{8}\right|_{-\pi}^{p} i = \frac{\pi}{\pi} = 1,$$

and similarly $||\sin(2t)|| = 1$. Therefore the basis is normalized.

The following integrals:

$$< \cos(t), \sin(t) >= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\sin(t)dt = \frac{1}{\pi}\left.\frac{\sin^2(x)}{x}\right|_{-\pi}^{p} i = 0,$$

$$< \cos(t), \cos(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\cos(2t)dt = \frac{1}{\pi}\ frac3\sin(t) - 2\sin^3(t)3|_{-\pi}^{p} i = 0,$$

$$< \cos(t), \sin(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(t)\sin(2t)dt = \frac{1}{\pi}\ frac{-}2\cos^3(t)3|_{-\pi}^{p} i = 0,$$

$$< \cos(2t), \sin(2t) >= \frac{1}{\pi}\int_{-\pi}^{\pi}\cos(2t)\sin(2t)dt = \frac{1}{\pi}\ frac{-}\cos^2(2t)4|_{-\pi}^{p} i = 0,$$

and so on, shows that S is an orthonormal basis.

(ii)

$$||t|| = \frac{1}{\pi}\int_{-\pi}^{\pi} tdt = \frac{1}{\pi}\left.\frac{t^2}{2}\right|_{-\pi}^{\pi} = 0.$$

(iii) Since $< x, \cos(3x) >= 0$ for any $x \in S$, $\mathrm{proj}_X(\cos(3x)) = 0$.

(i)

$$\begin{aligned}
&< \sin(t), t >= \sin(t) - t\cos(t)|_{-\pi}^{\pi} = 2\pi, \\
&< \cos(t), t >= t\sin(t) - \cos(t)|_{-\pi}^{\pi} = 0, \\
&< \cos(2t), t >= (2t\sin(2t) + \cos(2t))/4|_{-\pi}^{\pi} = 0, \text{ and finally} \\
&< \sin(2t), t > \sin(2x) - 2x\cos(2x)/4|_{-\pi}^{\pi} = -\pi.
\end{aligned}$$

Therefore, $\mathrm{proj}_X(t) = 2\pi\sin(t) - \pi\sin(2t)$

Problem 9. A rotation of angle θ in $\mathbb{R}^2$ represented as a matrix R in the standard basis is an orthonormal transformation since

$$\begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}^T \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} =$$

$$\begin{bmatrix} \cos^2(\theta) + \sin^2(\theta) & 0 \\ 0 & \cos^2(\theta) + \sin^2(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Similarly, one shows that $RR^T = I$. Therefore, a rotation in $\mathbb{R}^2$ is an orthonormal transformation.

Problem 1. 0 (i) Suppose Q represents an orthonormal operator on $\mathbb{F}^n$. Then $< x, y > = < Q(x), Q(y) >$ for each $x, y \in \mathbb{F}^n$. Since $< Q(x), Q(y) > = (Qx)^H(Qy) = x^H Q^H Q y$, it equals $x^H y$ for all $x, y \in \mathbb{F}^n$ only if $Q^H Q = I$. On the other hand if $Q^H Q = QQ^H = I$, then $< Q(x), Q(y) > = (Qx)^H(Qy) = x^H Q^H Q y = x^H y = < x, y >$.

(ii)

$$||Qx|| = \sqrt{< Qx, Qx >} = \sqrt{x^H Q^H Q x} = \sqrt{< x, x >} = ||x||.$$

(iii) If $Q^H Q = QQ^H = I$, then $Q^{-1} = Q^H$. Since $(Q^H)^H = Q$, Q^H is also orthonormal:

$$(Q^H)^H Q^H = QQ^H = I = Q^H Q = Q^H (Q^H)^H.$$

(iv) Let q_i denote the i^th column of Q. Since Q is orthonormal, $(Q^H Q)_{ij} = q_i^H q_j = < q_i, q_j >$ is 1 if $i = j$ and 0 if $i \neq j$. Thus, the columns of Q are orthonormal.

(v) The matrix

$$\begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix}$$

shows that the converse is not true.

(vi)

$$(Q_1 Q_2)^H Q_1 Q_2 = Q_2^H Q_1^H Q_1 Q_2 = Q_2^H Q_2 = I$$

and

$$Q_1 Q_2 (Q_1 Q_2)^H = Q_1 Q_2 Q_2^H Q_1^H = Q_1 Q_1^H = I.$$

Therefore, $Q_1 Q_2$ is orthonormal.

Problem 1. 1 Fix $N \in \mathbb{N}$, $N > 0$, and suppose $\{x_i\}_{i=1}^N$ is a set of linearly dependent vectors in V. Also, suppose, without loss of generality, that for $2 < k < N$, $\{x_i\}_{i=1}^{k-1}$ is a linearly independent set and $\{x_i\}_{i=1}^k$ is a linearly dependent set. Then $\{q_i\}_{i=1}^{k-1}$ (as they are defined in the book) is also a linearly independent set. However, since $x_k \in \text{span}(\{x_i\}_{i=1}^{k-1})$, we have that $q_k = 0$. Therefore the Gram-Schmidt orthonormalization process brakes down.

Problem 1. 6 (i) Let $A \in \mathbb{M}_{mxn}$ where $\text{rank}(A) = n \leq m$. Then there exist orthonormal $Q \in \mathbb{M}_{mxm}$ and upper triangular $R \in \mathbb{M}_{mxn}$ such that $A = QR$. Since $\tilde{Q} = -Q$ is still orthonormal $(-Q(-Q)^H = -Q(-Q^H) = QQ^H = I$ and similarly one shows $(-Q)^H(-Q) = I)$ and $\tilde{R} = -R$ is still upper triangular, $A = QR = \tilde{Q}\tilde{R}$. Therefore QR-decomposition is not unique.

(ii) Now take a reduced QR-decomposition $A = \hat{Q}\hat{R}$, where $\hat{Q} \in \mathbb{M}_{mxn}$ is orthonormal and $\hat{R} \in \mathbb{M}_{nxn}$ is upper triangular. Since A has full column rank, $\hat{R}$ has full rank and is therefore nonsingular. Then,

$$
\begin{aligned}
&A^H Ax = A^H b \implies \\
&(\hat{Q}\hat{R})^H \hat{Q}\hat{R}x = (\hat{Q}\hat{R})^H b \implies \\
&\hat{R}^H \hat{Q}^H \hat{Q}\hat{R}x = \hat{R}^H \hat{Q}^H b,
\end{aligned}
$$

and premultiplying both LHS and RHS of the last equation by $\hat{R}^{-1}$ gives $\hat{R}x = \hat{Q}^H b$.

Problem 2. 3 Let $x, y \in V$ and define $v := -y$. Since a norm is nonnegative and satisfies the triangular property, $||x|| - ||v|| \leq ||x|| + ||v|| \leq ||x + v||$. Then our definition of v implies $||x|| - ||y|| = ||x|| - || - y|| \leq ||x - y||$. Interchanging the role of x and y and using the homogeneity property of norms we have $||y|| - ||x|| \leq ||y - x|| = || - (y - x)|| = ||x - y||$, and the result follows.

Problem 2. 4 (i) Since $|f(t)| \geq 0$ for every t, so is $\int_a^b |f(t)|dt$. In addition, if $f = 0$, then $\int_a^b |f(t)|dt = 0$. On the other hand, if $\int_a^b |f(t)|dt = 0$ and $|f(t)| \geq 0$, it must be that $|f(t)| = 0$ for all t, implying that $f = 0$. Now take a constant $c \in \mathbb{F}$, then $\int_a^b |cf(t)|dt = \int_a^b |c||f(t)|dt = |c| \int_a^b |f(t)|dt$, since c does not depend on t. Finally, take $g \in C([a, b]; \mathbb{F})$. Since $|f(t) + g(t)| \leq |f(t)| + |g(t)|$ for all t and the integral is a linear operator, we have that $\int_a^b |f(t) + g(t)|dt \leq \int_a^b |f(t)|dt + \int_a^b |g(t)|dt$.

(ii) Since $|f(t)|^2 \geq 0$ for every t, so is $\int_a^b |f(t)|^2 dt$ and its square root. In addition, if $f = 0$, then $|f(t)|^2 = 0$ for all t and $\sqrt{\int_a^b |f(t)|^2 dt} = 0$. On the other hand, if $\sqrt{\int_a^b |f(t)|^2 dt} = 0$, then $\int_a^b |f(t)|^2 dt = 0$ and since $|f(t)|^2 \geq 0$ for all t, it must be that $|f(t)|^2 = 0$ for all t, implying that $f = 0$. Now take a constant $c \in \mathbb{F}$, then $\sqrt{\int_a^b |cf(t)|^2 dt} = \sqrt{\int_a^b |c|^2 |f(t)|^2 dt} = |c|\sqrt{\int_a^b |f(t)|^2 dt}$, since c does not depend on t. Finally, take $g \in C([a, b]; \mathbb{F})$. Since $|f(t) + g(t)| \leq |f(t)| + |g(t)|$ for all t, $x \mapsto x^2$ and $x \mapsto \sqrt{x}$ are monotonically increasing for nonnegative x and the integral is a linear operator, we have that $\sqrt{\int_a^b |f(t) + g(t)|^2 dt} \leq \sqrt{\int_a^b |f(t)|^2 dt + \int_a^b |g(t)|^2 dt} \leq ||f||_{L2} + ||g||_{L2}$.

(iii) Since $|f(x)| \geq 0$ for all x, so is the $\sup_{x \in [a,b]} |f(x)|$. In addition, if $f = 0$, then $\sup_{x \in [a,b]} |f(x)|$ is also zero. On the other hand, since $|f(x)| \geq 0$ for all x, $0 \leq \sup_{x \in [a,b]} |f(x)| = 0$ implies that we must have $f = 0$. Now take a constant $c \in \mathbb{F}$, then $\sup_{x \in [a,b]} |cf(x)| = \sup_{x \in [a,b]} |c||f(x)| = |c| \sup_{x \in [a,b]} |f(x)|$. Finally, take $g \in C([a, b]; \mathbb{F})$. Since $|f(x)+g(x)| \leq |f(x)|+|g(x)|$ for all x, we have that $\sup_{x \in [a,b]} |f(x)+g(x)| \leq \sup_{x \in [a,b]}\{|f(x)| + |g(x)|\} \leq \sup_{x \in [a,b]} |f(x)| + \sup_{x \in [a,b]} |g(x)|$.

Problem 2. 6 We show that topological equivalence is an equivalence relation. Let $|| \cdot ||_r$ be a norm on X for $r \in \{a, b, c\}$. Clearly $|| \cdot ||_r$ is in topologically equivalent with itself, just pick any $0 < m \leq 1$ and any $M \geq 1$ to show this. Also, suppose that $|| \cdot ||_a$ is topologically equivalent to $|| \cdot ||_b$ with constants $0 < m \leq M$. Then, $|| \cdot ||_b$ is topologically equivalent to $||\cdot||_a$ with constants $0 < 1/M' \leq 1/m'$. Finally, if $||\cdot||_a$ is topologically equivalent to $||\cdot||_b$ with constants $0 < m \leq M$ and so is $||\cdot||_b$ with $||\cdot||_c$ with constants $0 < m' \leq M'$, then $||\cdot||_a$ is topologically equivalent to $||\cdot||_b$ with constants $0 < mm' \leq MM'$.

Take $x \in \mathbb{R}^n$ Notice that

$$\sum_{i=1}^n |x_i|^2 \leq \left(\sum_{i=1}^n |x_i|^2 + 2 \sum_{i \neq j} |x_i||x_j| \right) = \left(\sum_{i=1}^n |x_i| \right)^2$$

and that

$$\sum_{i=1}^n |x_i| \cdot 1 \leq \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} \left(\sum_{i=1}^n 1^2 \right)^{1/2} = \sqrt{n} \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2}$$

prove that $||x||_2 \leq ||x||_1 \leq \sqrt{n}||x||_2$.

Also notice that

$$\max_i |x_i| = \left(\max_i |x_i|^2 \right)^{1/2} \leq \left(\sum_{i=1}^n |x_i|^2 \right)^{1/2} =$$

and

$$\sum_{i=1}^n |x_i|^2 \leq n \cdot \max_i |x_i|^2$$

prove that $||x||_\infty \leq ||x||_2 \leq \sqrt{n}||x||_\infty$.

Problem 2. 8 (i) Notice that (applying the results of the previous exercise)

$$\sup_{x \neq 0} \frac{||Ax||_1}{||x||_1} \leq \sup_{x \neq 0} \frac{||Ax||_1}{||x||_1} \leq \sqrt{n} \sup_{x \neq 0} \frac{||Ax||_2}{||x||_2},$$

and

$$\sup_{x \neq 0} \frac{||Ax||_1}{||x||_1} \geq \sup_{x \neq 0} \frac{||Ax||_2}{||x||_1} \geq \frac{1}{\sqrt{n}} \sup_{x \neq 0} \frac{||Ax||_2}{||x||_2}$$

imply that $\frac{1}{\sqrt{n}}||A||_2 \leq ||A||_1 \leq ||A||_2$.

(ii) Notice that

$$\sup_{x \neq 0} \frac{||Ax||_2}{||x||_2} \leq \sup_{x \neq 0} \frac{\sqrt{n}||Ax||_\infty}{||x||_\infty},$$

and

$$\sup_{x \neq 0} \frac{||Ax||_2}{||x||_2} \geq \sup_{x \neq 0} \frac{||Ax||_\infty}{\sqrt{n}||x||_\infty}.$$

Problem 2. 9 Take an arbitrary $x \neq 0$ and suppose $||\cdot||$ is an inner product induced norm. Since

$$||Qx|| = (< Qx, Qx >)^{1/2} = \left(< Q^H Qx, x >\right)^{1/2} = (< x, x >)^{1/2} = ||x||,$$

then

$$||Q|| = \sup_{x \neq 0} \frac{||Qx||}{||x||} = 1.$$

Now let $R_x : \mathbb{M}_n(\mathbb{F}) \to \mathbb{F}^n, A \mapsto Ax$ for every $x \in \mathbb{F}^n$. Notice that

$$||R_x|| = \sup_{A \neq 0} \frac{||Ax||}{||A||} = \sup_{A \neq 0} \frac{||Ax||||x||}{||A||||x||} \leq \sup_{A \neq 0} \left(\frac{||Ax||||x||}{||Ax||} \right) = ||x||.$$

Problem 3. 0 Take an arbitrary $A \in \mathbb{M}_n(\mathbb{F})$. First, $||A||_S = ||SAS^{-1}|| \geq 0$ for any A because $||\cdot||$ is a norm on $\mathbb{M}_n(\mathbb{F})$ and $SAS^{-1} \in \mathbb{M}_n(\mathbb{F})$. In addition, $||0||_S = ||S0S{-}1|| = ||0|| = 0$ and if $0 = ||A||_S = ||SAS^{-1}||$, then $SAS^{-1} = 0$ which implies $A = 0$. Second, take $a \in \mathbb{F}$, then

$$||aA||_S = ||SaAS^{-1}|| = ||aSAS^{-1}|| = |a|||SAS^{-1}|| = |a|||A||_S.$$

Finally, let $B \in \mathbb{M}_n(\mathbb{F})$ and notice that

$$||A{+}B||_S = ||S(A{+}B)S^{-1}|| = ||SAS^{-1}{+}SBS^{-1}|| \leq ||SAS^{-1}||{+}||SBS^{-1}|| = ||A||_S{+}||B||_S.$$

Therefore $||\cdot||_S$ is a norm on $\mathbb{M}_n(\mathbb{F})$.