Linear Optimization Problem Set

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Exercise 3

$$\begin{aligned} \text{maximize} \quad & 4b + 3j \\ \text{subject to} \quad & 15b + 10j \leq 1800 \\ & 2b + 2j \leq 300 \\ & j \leq 200 \\ & b, j \geq 0 \end{aligned}$$

Exercise 4

$$\begin{aligned} \text{maximize} \quad & 2x_{AB} + 5x_{AD} + 5x_{BC} + 2x_{BD} + 7x_{BE} + 9x_{BF} + 2x_{CF} + 4x_{DE} + 3x_{EF} \\ \text{subject to} \quad & x_{AB} + x_{AD} = 10 \\ & x_{BC} + x_{BD} + x_{BE} + x_{BF} - x_{AB} = 1 \\ & x_{CF} - x_{BC} = -2 \\ & x_{DE} - x_{AD} - x_{BD} = -3 \\ & x_{EF} - x_{BE} - x_{DE} = 4 \\ & - x_{BF} - x_{CF} - x_{EF} = -10 \\ & 0 \leq x_{AB}, x_{AD}, x_{BC}, x_{BD}, x_{BE}, x_{BF}, x_{CF}, x_{DE}, x_{EF} \leq 6 \end{aligned}$$

Exercise 5

(i)

$$\begin{aligned} \text{maximize} \quad & 3x_1 + x_2 \\ \text{subject to} \quad & x_1 + 3x_2 + w_1 = 15 \\ & 2x_1 + 3x_2 + w_2 = 18 \\ & x_1 - x_2 + w_3 = 4 \\ & x_1, x_2, w_1, w_2, w_3 \geq 0 \end{aligned}$$

[1] Thank you to Reiko for this beautiful Latex

$$
\begin{array}{ccccccc}
\zeta & = & & & 3x_1 & + & x_2 \\
\hline
w_1 & = & 15 & - & x_1 & - & 3x_2 \\
w_2 & = & 18 & - & 2x_1 & - & 3x_2 \\
w_3 & = & 4 & - & x_1 & + & x_2 \\
\hline\hline
\zeta & = & 12 & + & 4x_2 & - & 3w_3 \\
\hline
w_1 & = & 11 & - & 4x_2 & + & w_3 \\
w_2 & = & 10 & - & 5x_2 & + & 2w_3 \\
x_1 & = & 4 & + & x_2 & - & w_3 \\
\hline\hline
\zeta & = & 20 & - & \frac{4}{5}w_2 & - & \frac{7}{5}w_3 \\
\hline
w_1 & = & 3 & + & \frac{4}{5}w_2 & - & \frac{3}{5}w_3 \\
x_2 & = & 2 & - & \frac{1}{5}w_2 & + & \frac{2}{5}w_3 \\
x_1 & = & 6 & - & \frac{1}{5}w_2 & - & \frac{3}{5}w_3 \\
\hline
\end{array}
$$

Optimizer: $(6, 2)$
Optimum value: 20

(ii)

$$
\begin{array}{ll}
\text{maximize} & 4x + 6y \\
\text{subject to} & -x + 3x_2 + w_1 = 11 \\
& x + y + w_2 = 27 \\
& 2x + 5y + w_3 = 90 \\
& x, y, w_1, w_2, w_3 \geq 0
\end{array}
$$

$$
\begin{array}{ccccccc}
\zeta & = & & & 4x & + & 6y \\
\hline
w_1 & = & 11 & + & x & - & y \\
w_2 & = & 27 & - & x & - & y \\
w_3 & = & 90 & - & 2x & - & 5y \\
\hline\hline
\zeta & = & 66 & + & 10x & - & 6w_1 \\
\hline
y & = & 11 & + & x & - & w_1 \\
w_2 & = & 16 & - & 2x & + & w_1 \\
w_3 & = & 35 & - & 7x & + & 5w_1 \\
\hline\hline
\zeta & = & 116 & + & \frac{8}{7}w_1 & - & \frac{10}{7}w_3 \\
\hline
y & = & 16 & - & \frac{2}{7}w_1 & - & \frac{1}{7}w_3 \\
w_2 & = & 6 & - & \frac{3}{7}w_1 & + & \frac{2}{7}w_3 \\
x & = & 5 & + & \frac{5}{7}w_1 & - & \frac{1}{7}w_3 \\
\hline\hline
\zeta & = & 132 & - & \frac{8}{3}w_2 & - & \frac{2}{7}w_3 \\
\hline
y & = & 12 & + & \frac{2}{3}w_2 & - & \frac{1}{3}w_3 \\
w_1 & = & 14 & - & \frac{7}{3}w_2 & + & \frac{2}{3}w_3 \\
x & = & 15 & - & \frac{5}{3}w_2 & + & \frac{1}{3}w_3 \\
\hline
\end{array}
$$

Optimizer: $(15, 12)$
Optimum value: 132

Exercise 6

$$
\begin{array}{ll}
\text{maximize} & 4b + 3j \\
\text{subject to} & 15b + 10j + w_1 = 1800 \\
 & 2b + 2j + w_2 = 300 \\
 & j + w_3 = 200 \\
 & b, j, w_1, w_2, w_3 \geq 0
\end{array}
$$

$$
\begin{array}{rcrcrcr}
\zeta & = & & & 4b & + & 3j \\
\hline
w_1 & = & 1800 & - & 15b & - & 10j \\
w_2 & = & 300 & - & 2b & - & 2j \\
w_3 & = & 200 & - & j & & \\
\hline\hline
\zeta & = & 450 & + & b & - & \frac{3}{2}w_2 \\
\hline
w_1 & = & 300 & - & 5b & + & 5w_2 \\
j & = & 150 & - & b & - & \frac{1}{2}w_2 \\
w_3 & = & 50 & + & b & + & \frac{1}{2}w_2 \\
\hline\hline
\zeta & = & 510 & - & \frac{1}{5}w_1 & - & \frac{1}{2}w_2 \\
\hline
b & = & 60 & - & \frac{1}{5}w_1 & + & w_2 \\
j & = & 90 & + & \frac{1}{5}w_1 & - & \frac{3}{2}w_2 \\
w_3 & = & 110 & - & \frac{1}{5}w_1 & + & \frac{3}{2}w_2 \\
\hline
\end{array}
$$

Optimal choice: 60 GI Barb soldiers, 90 Joey dolls
Maximal profit: \$510

Exercise 7
(i)

$$
\begin{aligned}
\text{maximize}\quad & x_1 + 2x_2 \\
\text{subject to}\quad & -4x_1 - 2x_2 + x_3 = -8 \\
& -2x_1 + 3x_2 + x_4 = 6 \\
& x_1 + x_5 = 3 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

Auxiliary problem:

$$
\begin{aligned}
\text{maximize}\quad & -x_0 \\
\text{subject to}\quad & -4x_1 - 2x_2 + x_3 - x_0 = -8 \\
& -2x_1 + 3x_2 + x_4 - x_0 = 6 \\
& x_1 + x_5 - x_0 = 3 \\
& x_0, x_1, x_2, x_3, x_4, x_5 \geq 0
\end{aligned}
$$

$$
\begin{array}{ccccccccc}
\zeta & = & & & & & & - & x_0 \\
\hline
x_3 & = & -8 & + & 4x_1 & + & 2x_2 & + & x_0 \\
x_4 & = & 6 & + & 2x_1 & - & 3x_2 & + & x_0 \\
x_5 & = & 3 & - & x_1 & & & + & x_0 \\
\hline\hline
\zeta & = & -8 & + & 4x_1 & + & 2x_2 & - & x_3 \\
\hline
x_0 & = & 8 & - & 4x_1 & - & 2x_2 & + & x_3 \\
x_4 & = & 14 & - & 2x_1 & - & 5x_2 & + & x_3 \\
x_5 & = & 11 & - & 5x_1 & - & 2x_2 & + & x_3 \\
\hline\hline
\zeta & = & & & & & & - & x_0 \\
\hline
x_1 & = & 2 & - & \frac{1}{2}x_2 & + & \frac{1}{4}x_3 & - & \frac{1}{4}x_0 \\
x_4 & = & 10 & - & 4x_2 & + & \frac{1}{2}x_3 & + & \frac{1}{2}x_0 \\
x_5 & = & 1 & + & \frac{1}{2}x_2 & - & \frac{1}{4}x_3 & + & \frac{5}{4}x_0 \\
\hline\hline
\zeta & = & 2 & + & \frac{3}{2}x_2 & + & \frac{1}{4}x_3 & & \\
\hline
x_1 & = & 2 & - & \frac{1}{2}x_2 & + & \frac{1}{4}x_3 & & \\
x_4 & = & 10 & - & 4x_2 & + & \frac{1}{2}x_3 & & \\
x_5 & = & 1 & + & \frac{1}{2}x_2 & - & \frac{1}{4}x_3 & & \\
\hline\hline
\zeta & = & 3 & + & 2x_2 & - & x_5 & & \\
\hline
x_1 & = & 3 & & & - & x_5 & & \\
x_4 & = & 12 & - & 3x_2 & - & 2x_5 & & \\
x_3 & = & 4 & + & 2x_2 & - & 4x_5 & & \\
\hline\hline
\zeta & = & 11 & - & \frac{2}{3}x_4 & - & \frac{7}{3}x_5 & & \\
\hline
x_1 & = & 3 & & & - & x_5 & & \\
x_2 & = & 4 & - & \frac{1}{3}x_4 & - & \frac{2}{3}x_5 & & \\
x_3 & = & 4 & - & \frac{2}{3}2x_4 & - & \frac{16}{3}x_5 & & \\
\hline
\end{array}
$$

Optimal point: $(3, 4)$
Optimal value: 11

(ii)

$$
\begin{array}{ll}
\text{maximize} & 5x_1 + 2x_2 \\
\text{subject to} & 5x_1 + 3x_2 + x_3 = 15 \\
& 3x_1 + 5x_2 + x_4 = 15 \\
& 4x_1 - 3x_2 + x_5 = -12 \\
& x_1, x_2, x_3, x_4, x_5 \geq 0
\end{array}
$$

Auxiliary problem:

$$\begin{aligned} \text{maximize} \quad & -x_0 \\ \text{subject to} \quad & 5x_1 + 3x_2 + x_3 - x_0 = 15 \\ & 3x_1 + 5x_2 + x_4 - x_0 = 15 \\ & 4x_1 - 3x_2 + x_5 - x_0 = -12 \\ & x_0, x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

$$\begin{array}{ccccccccc}
\zeta & = & & & & & & - & x_0 \\ \hline
x_3 & = & 15 & - & 5x_1 & - & 3x_2 & + & x_0 \\
x_4 & = & 15 & - & 3x_1 & - & 5x_2 & + & x_0 \\
x_5 & = & -12 & - & 4x_1 & + & 3x_2 & + & x_0 \\ \hline\hline
\zeta & = & -12 & - & 4x_1 & + & 3x_2 & - & x_5 \\ \hline
x_3 & = & 27 & - & x_1 & - & 6x_2 & + & x_5 \\
x_4 & = & 27 & + & x_1 & - & 8x_2 & + & x_5 \\
x_0 & = & 12 & + & 4x_1 & - & 3x_2 & + & x_5 \\ \hline\hline
\zeta & = & -\frac{15}{8} & - & \frac{29}{8}x_1 & - & \frac{3}{8}x_4 & - & \frac{5}{8}x_5 \\ \hline
x_3 & = & \frac{27}{4} & - & \frac{7}{4}x_1 & + & \frac{3}{4}x_4 & + & \frac{1}{4}x_5 \\
x_2 & = & \frac{27}{8} & + & \frac{1}{8}x_1 & - & \frac{1}{8}x_4 & + & \frac{1}{8}x_5 \\
x_0 & = & \frac{15}{8} & + & \frac{29}{8}x_1 & + & \frac{3}{8}x_4 & + & \frac{5}{8}x_5 \\ \hline
\end{array}$$

The original problem has no feasible solutions.

(iii)

$$\begin{aligned} \text{maximize} \quad & -3x_1 + x_2 \\ \text{subject to} \quad & x_2 + x_3 = 4 \\ & -2x_1 + 3x_2 + x_4 = 6 \\ & x_1, x_2, x_3, x_4 \geq 0 \end{aligned}$$

$$\begin{array}{ccccccc}
\zeta & = & & - & 3x_1 & + & x_2 \\ \hline
x_3 & = & 4 & & & - & x_2 \\
x_4 & = & 6 & + & 2x_1 & - & 3x_2 \\ \hline\hline
\zeta & = & 2 & - & \frac{7}{3}x_1 & - & \frac{1}{3}x_4 \\ \hline
x_3 & = & 2 & - & \frac{2}{3}x_1 & + & \frac{1}{3}x_4 \\
x_2 & = & 2 & + & \frac{2}{3}x_1 & - & \frac{1}{3}x_4 \\ \hline
\end{array}$$

Optimal point: $(0, 2)$
Optimal value: 2

Exercise 8
Give an example of a three-dimensional linear problem where the feasible region is closed and unbounded, but where the objective function still has a unique feasible maximizer.

$$\begin{aligned} \text{maximize} \quad & -x-y-z \\ \text{subject to} \quad & x,y,z \geq 0 \end{aligned}$$

Maximizer: $(0,0,0)$

Exercise 9
Give an example of a three-dimensional linear problem where the feasible region is closed and unbounded and where the objective function has no maximizer.

$$\begin{aligned} \text{maximize} \quad & x+y+z \\ \text{subject to} \quad & x,y,z \geq 0 \end{aligned}$$

Exercise 10
Give an example of a three-dimensional linear problem where the feasible region is empty.

$$\begin{aligned} \text{maximize} \quad & x+y+z \\ \text{subject to} \quad & x+y+z \leq -1 \\ & x,y,z \geq 0 \end{aligned}$$

Exercise 11
Give an example of a three-dimensional linear problem where the feasible region is nonempty, closed, and bounded, but $(0,0,0)$ is not feasible.

$$\begin{aligned} \text{maximize} \quad & x+y+z \\ \text{subject to} \quad & x+y+z \geq 1 \\ & x+y+z \leq 4 \\ & x,y,z \geq 0 \end{aligned}$$

Auxiliary problem:

$$\begin{aligned} \text{maximize} \quad & -w \\ \text{subject to} \quad & -x-y-z-w \leq -1 \\ & x+y+z-w \leq 4 \\ & x,y,z,w \geq 0 \end{aligned}$$

Exercise 12

$$\begin{aligned} \text{maximize} \quad & 10x_1 - 57x_2 - 9x_3 - 24x_4 \\ \text{subject to} \quad & 0.5x_1 - 1.5x_2 - 0.5x_3 + x_4 + x_5 = 0 \\ & 0.5x_1 - 5.5x_2 - 2.5x_3 + 9x_4 + x_6 = 0 \\ & x_1 + x_7 = 0 \\ & x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \end{aligned}$$

$$
\begin{array}{rclcrcrcrcr}
\zeta & = & & & 10x_1 & - & 57x_2 & - & 9x_3 & - & 24x_4 \\
\hline
x_5 & = & & - & 0.5x_1 & + & 1.5x_2 & + & 0.5x_3 & - & x_4 \\
x_6 & = & & - & 0.5x_1 & + & 5.5x_2 & + & 2.5x_3 & - & 9x_4 \\
x_7 & = & 1 & - & x_1 & & & & & & \\
\hline\hline
\zeta & = & & - & 27x_2 & + & x_3 & - & 44x_4 & - & 20x_5 \\
\hline
x_1 & = & & & 3x_2 & + & x_3 & - & 2x_4 & - & 2x_5 \\
x_6 & = & & & 4x_2 & + & 2x_3 & - & 8x_4 & + & x_5 \\
x_7 & = & 1 & - & 3x_2 & - & x_3 & + & 2x_4 & + & 2x_5 \\
\hline\hline
\zeta & = & 1 & - & 30x_2 & - & 42x_4 & - & 18x_5 & - & x_7 \\
\hline
x_1 & = & 1 & & & & & & & - & x_7 \\
x_6 & = & 2 & - & 2x_2 & - & 4x_4 & + & 5x_5 & - & 2x_7 \\
x_3 & = & 1 & - & 3x_2 & + & 2x_4 & + & 2x_5 & - & x_7 \\
\hline
\end{array}
$$

Optimal point: $(1, 0, 1, 0)$ Optimum value: 1

Exercise 15

If $x \in \mathbb{R}^n$ is feasible for the primal and $y \in \mathbb{R}^m$ is feasible for the dual, then $c^T x \leq b^T y$.

Proof:

Notice that

$$
\begin{aligned}
c^T x &= x^T c \\
&\leq x^T A^T y \\
&= y^T A x \\
&\leq y^T b \\
&= b^T y.
\end{aligned}
$$

Exercise 17

For a linear optimization problem in standard form, the dual of the dual optimization problem is again the primal problem.

Proof:

Consider the primal problem:

$$
\begin{aligned}
\text{maximize } & c^T x \\
\text{subject to } & Ax \preceq b \\
& \mathbf{x} \succeq 0
\end{aligned}
$$

with the dual problem:

$$
\begin{aligned}
\text{minimize } & b^T y \\
\text{subject to } & A^T y \succeq c \\
& y \succeq 0
\end{aligned}
$$

The dual in standard form becomes:

$$
\begin{aligned}
\text{maximize } & (-b^T)y \\
\text{subject to } & (-A)^T y \preceq -c \\
& y \succeq 0
\end{aligned}
$$

and the dual of the dual is:

$$
\begin{aligned}
\text{minimize } & (-c^T)\mathbf{x} \\
\text{subject to } & ((-A)^T)^T x \succeq -b \\
& x \succeq 0
\end{aligned}
$$

In standard form, this is:

$$
\begin{aligned}
\text{maximize } & c^T x \\
\text{subject to } & Ax \preceq b \\
& x \succeq 0
\end{aligned}
$$

which is the original primal problem.

Exercise 18
Primal:

$$
\begin{aligned}
\text{maximize } & x_1 + x_2 \\
\text{subject to } & 2x_1 + x_2 + w_1 = 3 \\
& x_1 + 3x_2 + w_2 = 5 \\
& 2x_1 + 3x_2 + w_3 = 4 \\
& x_1, x_2, w_1, w_2, w_3 \geq 0
\end{aligned}
$$

$$
\begin{array}{ccccccc}
\zeta & = & & & x_1 & + & x_2 \\
\hline
w_1 & = & 3 & - & 2x_1 & - & x_2 \\
x_2 & = & 5 & - & x_1 & - & 3x_2 \\
x_5 & = & 4 & - & 2x_1 & - & 3x_2 \\
\hline\hline
\zeta & = & \frac{3}{2} & + & \frac{1}{2}x_2 & - & \frac{1}{2}w_1 \\
\hline
x_1 & = & \frac{3}{2} & - & \frac{1}{2}x_2 & - & \frac{1}{2}w_1 \\
w_2 & = & \frac{7}{2} & - & \frac{5}{2}x_2 & + & \frac{1}{2}w_1 \\
w_3 & = & 1 & - & 2x_2 & + & w_1 \\
\hline\hline
\zeta & = & \frac{7}{4} & - & \frac{1}{4}w_1 & - & \frac{1}{4}w_1 \\
\hline
x_1 & = & \frac{5}{4} & - & \frac{3}{4}w_1 & + & \frac{1}{4}w_3 \\
w_2 & = & \frac{9}{4} & - & \frac{3}{4}w_1 & + & \frac{5}{4}w_3 \\
x_2 & = & \frac{1}{2} & + & \frac{1}{2}w_1 & - & \frac{1}{2}w_3 \\
\hline
\end{array}
$$

Optimal point: $(\frac{5}{4}, \frac{1}{2})$

Optimum value: $\frac{7}{4}$

Dual:

$$
\begin{aligned}
\text{minimize} \quad & 3y_1 + 5y_2 + 4y_3 \\
\text{subject to} \quad & 2y_1 + y_2 + 2y_3 \geq 1 \\
& y_1 + 3y_2 + 3y_3 \geq 1 \\
& y_1 + 3y_2 + 3y_3 \geq 1 \\
& y_1, y_2, y_3 \geq 0
\end{aligned}
$$

Dual in standard form:

$$
\begin{aligned}
\text{maximize} \quad & -3y_1 - 5y_2 - 4y_3 \\
\text{subject to} \quad & -2y_1 - y_2 - 2y_3 + v_1 - v_0 = -1 \\
& -y_1 - 3y_2 - 3y_3 + v_2 - v_0 = -1 \\
& y_1, y_2, y_3, v_1, v_2 \geq 0
\end{aligned}
$$

$$
\begin{array}{cccccccccccc}
\zeta & = & & & & & & & & - & v_0 \\
\hline
v_1 & = & -1 & + & 2y_1 & + & y_2 & + & 2y_3 & + & v_0 \\
v_2 & = & -1 & + & y_1 & + & 3y_2 & + & 3y_3 & + & v_0 \\
\hline\hline
\zeta & = & -1 & + & 2y_1 & + & y_2 & + & 2y_3 & - & v_1 \\
\hline
v_0 & = & 1 & - & 2y_1 & - & y_2 & - & 2y_3 & + & v_1 \\
v_2 & = & & - & y_1 & + & 2y_2 & + & y_3 & + & v_1 \\
\hline\hline
\zeta & = & & & & & & & & - & v_0 \\
\hline
y_2 & = & 1 & - & 2y_1 & - & 2y_3 & + & v_1 & - & v_0 \\
v_2 & = & 2 & - & 5y_1 & - & 3y_3 & + & 3v_1 & - & 2v_0 \\
\hline\hline
\zeta & = & -2 & + & y_1 & - & 3y_2 & - & 2v_1 & & \\
\hline
y_3 & = & \frac{1}{2} & - & y_1 & - & \frac{1}{2}y_2 & + & \frac{1}{2}v_1 & & \\
v_2 & = & \frac{1}{2} & - & 2y_1 & + & \frac{3}{2}y_2 & + & \frac{3}{2}v_1 & & \\
\hline\hline
\zeta & = & -\frac{7}{4} & - & \frac{3}{2}y_2 & - & \frac{5}{4}v_1 & - & \frac{1}{2}v_2 & & \\
\hline
y_3 & = & \frac{1}{4} & - & 2\frac{3}{2}y_2 & - & \frac{1}{4}v_1 & + & \frac{1}{2}v_2 & & \\
y_1 & = & \frac{1}{4} & + & \frac{3}{2}y_2 & + & \frac{3}{4}v_1 & - & \frac{1}{2}v_2 & & \\
\hline
\end{array}
$$

Optimal point: $(\frac{1}{4}, 0, \frac{1}{4})$

Optimum value: $\frac{7}{4}$