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Dynamic Programming with Sparse Grids

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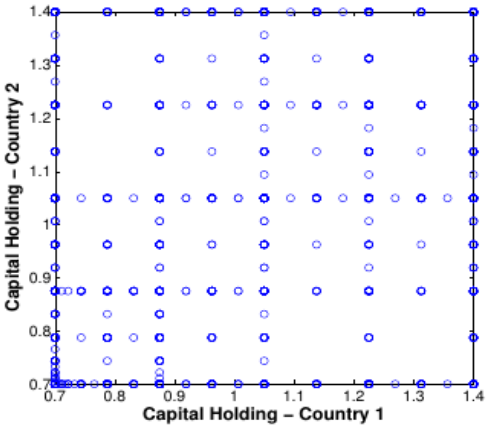


Growth Model & Dynamic Programming & ASG

To demonstrate the capabilities of sparse grids, we consider an **infinite-horizon discrete-time multi-dimensional optimal growth model** (see, e.g, Cai & Judd (2014), Scheidegger & Bilionis (2017), and references therein).

The model has few parameters and is relatively easy to explain, whereas the dimensionality of the problem can be scaled up in a straightforward but meaningful way

→ state-space depends linearly on the number of ***D* sectors** considered.

→ there are D sectors with **capital** $\mathbf{k}_t = (k_{t,1}, ..., k_{t,D})$

and elastic **labour supply** $\mathbf{l}_t = (l_{t,1}, ..., l_{t,D})$

Stochastic growth model

The production function of sector ***i*** at time ***t*** is $f(k_{t,i}, l_{t,i}),\ \text{for}\ i = 1, ..., D.$

Consumption: $\mathbf{c}_t = (c_{t,1}, ..., c_{t,D})$

Investment of the sectors at time *t:* $\mathbf{I}_t = (I_{t,1}, ..., I_{t,D})$

→ The goal now is to find **optimal consumption** and **labour supply decisions** such that expected total utility over an infinite time horizon is maximized.

Model

$$V_0(\mathbf{k}_0) = \max_{\mathbf{k}_t, \mathbf{I}_t, \mathbf{c}_t, \mathbf{l}_t, \mathbf{\Gamma}_t} \left\{ \sum_{t=0}^{\infty} \beta^t \cdot u(\mathbf{c}_t, \mathbf{l}_t) \right\},$$

$$s.t.$$

$$k_{t+1,j} = (1-\delta) \cdot k_{t,j} + I_{t,j} \qquad j = 1, ..., D$$

$$\Gamma_{t,j} = \frac{\zeta}{2} k_{t,j} \left(\frac{I_{t,j}}{k_{t,j}} - \delta \right)^2, \quad j = 1, ..., D$$

$$\sum_{j=1}^{D} (c_{t,j} + I_{t,j} - \delta \cdot k_{t,j}) = \sum_{j=1}^{D} (f(k_{t,j}, l_{t,j}) - \Gamma_{t,j})$$

Model (II)

Convex adjustment cost of sector *j*: $\mathbf{\Gamma}_t = (\Gamma_{t,1},,...,\Gamma_{t,D})$

Capital depreciation: δ

Discount factor: β

Recursive formulation

$$V(\mathbf{k}) = \max_{\mathbf{I},\mathbf{c},\mathbf{l}} \left(u(c,l) + \beta \quad \left\{ V_{next}(k^+) \right\} \right),$$

$$s.t.$$

$$k_j^+ = (1-\delta)\cdot k_j + I_j \qquad j = 1, ..., D$$

$$\Gamma_j = \frac{\zeta}{2} k_j \left(\frac{I_j}{k_j} - \delta \right)^2, \quad j = 1, ..., D$$

$$\sum_{j=1}^{D} (c_j + I_j - \delta \cdot k_j) = \sum_{j=1}^{D} (f(k_j, l_j) - \Gamma_j)$$

where we indicate the next period's variables with a superscript "+". $\mathbf{k} = (k_1, ..., k_D)$ represents the state vector, $\mathbf{l} = (l_1, ..., l_D)$, $\mathbf{c} = (c_1, ..., c_D)$, and $\mathbf{I} = (I_1, ..., I_D)$ are $3D$ control variables. $\mathbf{k}^+ = \left(k_1^+, ..., k_D^+\right)$ is the vector of next period's variables. Today's and tomorrow's states are restricted to the finite range $[\underline{\mathbf{k}}, \overline{\mathbf{k}}]^D$, where the lower edge of the computational domain is given by $\underline{\mathbf{k}} = (\underline{k_1}, ..., \underline{k_D})$, and the upper bound is given by $\overline{\mathbf{k}} = (\overline{k_1}, ..., \overline{k_D})$. Moreover, $\mathbf{c} > 0$ and $\mathbf{l} > 0$ holds component-wise.

Utility function etc.

Productivity: $$f(k_j, l_j) \quad = \quad A \cdot k_i^{\psi} \cdot l_i^{1-\psi}$$

Utility: $$u\left(\mathbf{c}, \mathbf{l}\right) = \sum_{i=1}^{d} \left[\frac{\left(c_i / A\right)^{1-\gamma} - 1}{1-\gamma} - \left(1-\psi\right) \frac{l_i^{1+\eta} - 1}{1+\eta} \right]$$

Terminal Value function: $$V^{\infty}\left(\mathbf{k}\right) = u\left(f(k, \mathbf{e}), \mathbf{e}\right) / (\mathbf{1} - \beta)$$

where $\mathbf{e}$ is the unit vector

Parametrization

Parameter	Value
β	0.8
δ	0.025
ζ	0.5
$\left[\underline{\mathbf{k}}, \overline{\mathbf{k}}\right]^D$	$[0.2, 3.0]^D$
ψ	0.36
A	$(1-\beta)/(\psi \cdot \beta)$
γ	2
η	1

Value function iteration

$$V(\mathbf{k}) = \max_{\mathbf{I},\mathbf{c},\mathbf{l}} \left(u(c,l) + \beta \quad \left\{ V_{next}(k^+) \right\} \right),$$

$$s.t.$$

$$k_j^+ = (1-\delta)\cdot k_j + I_j \qquad , \quad j = 1, ..., D$$

$$\Gamma_j = \frac{\zeta}{2} k_j \left(\frac{I_j}{k_j} - \delta \right)^2 , \quad j = 1, ..., D$$

$$\sum_{j=1}^{D} (c_j + I_j - \delta \cdot k_j) = \sum_{j=1}^{D} (f(k_j, l_j) - \Gamma_j)$$

State k: sparse grid coordinates

V_{next} : sparse grid interpolator from the previous iteration step

Solve this optimization problem at every point in the sparse grid!

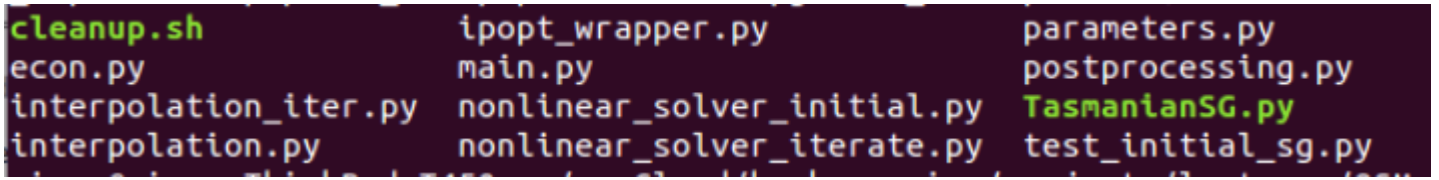
Attention: Take care of the econ domain/ sparse grid domain

Convergence measures (due to contraction mapping)

Average error: $$e^s = \frac{1}{N} \sum_{i=1}^{N} |V^s(\mathbf{x^i}) - V^{s-1}(\mathbf{x^i})|$$

Max. error: $$a^s = \max_{i=1,N} |V^s(\mathbf{x^i}) - V^{s-1}(\mathbf{x^i})|$$

Setup of Code



main.py: driver routine

econ.py: contains production function, utility,...

nonlinear_solver_initial/iterate.py: interface SG ↔ IPOPT (optimizer).

ipopt_wrapper.py: specifies the optimization problem (objective function,...).

interpolation.py: interface value function iteration ↔ sparse grid.

postprocessing.py: auxiliary routines, e.g., to compute the error.

Code snippet – main.py

```
#=====================================================================
# Start with Value Function Iteration

# terminal value function
valnew=TasmanianSG.TasmanianSparseGrid()
if (numstart==0):
    valnew=interpol.sparse_grid(n_agents, iDepth)
    valnew.write("valnew_1." + str(numstart) + ".txt") #write file to disk for restart

# value function during iteration
else:
    valnew.read("valnew_1." + str(numstart) + ".txt")  #write file to disk for restart

valold=TasmanianSG.TasmanianSparseGrid()
valold=valnew

for i in range(numstart, numits):
    valnew=TasmanianSG.TasmanianSparseGrid()
    valnew=interpol_iter.sparse_grid_iter(n_agents, iDepth, valold)
    valold=TasmanianSG.TasmanianSparseGrid()
    valold=valnew
    valnew.write("valnew_1." + str(i+1) + ".txt")

    #=================================================================
print "============================================================"
print " "
print " Computation of a growth model of dimension ", n_agents ," finished after ", numits, " steps"
print " "
print "============================================================"
#=====================================================================

# compute errors
avg_err=post.ls_error(n_agents, numstart, numits, No_samples)

#=====================================================================
print "============================================================"
print " "
print " Errors are computed -- see error.txt"
print " "
print "============================================================"
#=====================================================================
```

Code snippet – parameters.py

```
#=====================================================================

# Depth of "Classical" Sparse grid
iDepth=1
iOut=1          # how many outputs
which_basis = 1 #linear basis function (2: quadratic local basis)

# control of iterations
numstart = 0    # which is iteration to start (numstart = 0: start from scratch, number=/0: restart)
numits = 10     # which is the iteration to end

# How many random points for computing the errors
No_samples = 1000

#=====================================================================

# Model Paramters

n_agents=2  # number of continuous dimensions of the model

beta=0.8
rho=0.95
zeta=0.5
psi=0.36
gamma=2.0
delta=0.025
eta=1
big_A=(1.0-beta)/(psi*beta)

# Ranges For States
range_cube=1 # range of [0..1]^d in 1D
k_bar=0.2
k_up=3.0

# Ranges for Controls
c_bar=1e-2
c_up=10000.0

l_bar=1e-2
l_up=1.0

inv_bar=1e-2
inv_up=10000.0
```

Code snippet – econ.py

```
#=====================================================================
#utility function u(c,l)

def utility(cons=[], lab=[]):
    sum_util=0.0
    n=len(cons)
    for i in range(n):
        nom1=(cons[i]/big_A)**(1.0-gamma) -1.0
        den1=1.0-gamma

        nom2=(1.0-psi)*((lab[i]**(1.0+eta)) -1.0)
        den2=1.0+eta

        sum_util+=(nom1/den1 - nom2/den2)

    util=sum_util

    return util


#=====================================================================
# output_f

def output_f(kap=[], lab=[]):
    fun_val = big_A*(kap**psi)*(lab**(1.0 - psi))
    return fun_val
```

Code snippet – ipopt_wrapper.py

```
#=======================================================================
#   Objective Function to start VFI (in our case, the value function)

def EV_F(X, k_init, n_agents):

    # Extract Variables
    cons=X[0:n_agents]
    lab=X[n_agents:2*n_agents]
    inv=X[2*n_agents:3*n_agents]

    knext= (1-delta)*k_init + inv
    # Compute Value Function

    VT_sum=utility(cons, lab) + beta*V_INFINITY(knext)

    return VT_sum

# V infinity
def V_INFINITY(k=[]):
    e=np.ones(len(k))
    c=output_f(k,e)
    v_infinity=utility(c,e)/(1-beta)
    return v_infinity

#=======================================================================
#   Objective Function during VFI (note - we need to interpolate on an "old" sprase grid)

def EV_F_ITER(X, k_init, n_agents, grid):

    # Extract Variables
    cons=X[0:n_agents]
    lab=X[n_agents:2*n_agents]
    inv=X[2*n_agents:3*n_agents]

    knext= (1-delta)*k_init + inv

    # Compute Value Function

    VT_sum=utility(cons, lab) + beta*grid.evaluate(knext)

    return VT_sum

#=======================================================================
```

Run the Growth model code

→ Model implemented in Python (TASMANIAN)

→ Optimizer used: IPOPT & PYIPOPT (python interface)

→ OSM2018/day1_SparseGrid/SparseGridCode/growth_model

→ run with

>python main.py


Help!
Shift

Installed software on Midway

1) PYIPOPT (https://github.com/xuy/pyipopt)
→ Python Interface to IPOPT

An example is given here:

```
> cd OSM2018/day1_SparseGrid/SparseGridCode/pyipopt_midway/pyipopt/examples
> python hs071.py
```

2) TASMANIAN (http://tasmanian.ornl.gov/)

```
>cd OSM2018/day1_SparseGrid/SparseGridCode/TasmanianSparseGrids/InterfacePython/
>python example.py
```