Computational Physics (PHYS6350)

Lecture 8: Numerical Derivatives

$$\frac{\mathrm{d}f}{\mathrm{d}x} \simeq \frac{f(x+h)-f(x)}{h}$$

February 9, 2023

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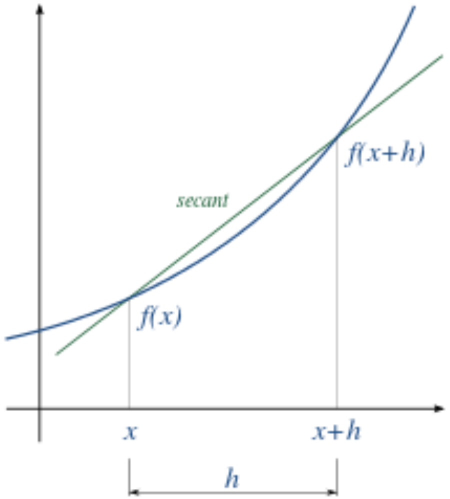
Course materials: https://github.com/vlvovch/PHYS6350-ComputationalPhysics

Numerical differentiation

Generic problem: evaluate

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$



We need numerical differentiation when

- Function f is known at a discrete set of points
- Too expensive/cumbersome to do directly
  - E.g. when f(x) itself is a solution to a complex web of non-linear equations, calculating f'(x) explicitly will require rewriting all the equations

References: Chapter 5 of *Computational Physics* by Mark Newman

Forward difference

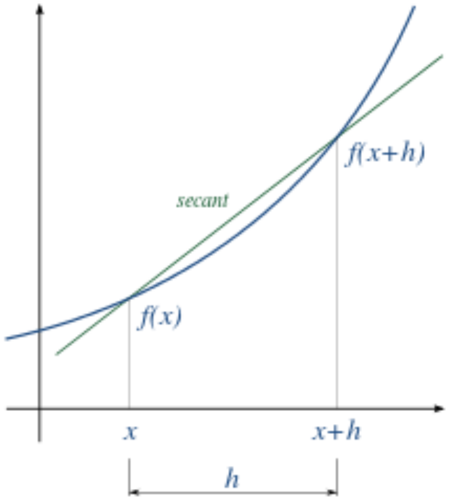
Simply approximate

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

by

$$\frac{\mathrm{d}f}{\mathrm{d}x} \simeq \frac{f(x+h) - f(x)}{h}$$

where *h* is finite



Taylor theorem:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2} f''(x) + \dots$$

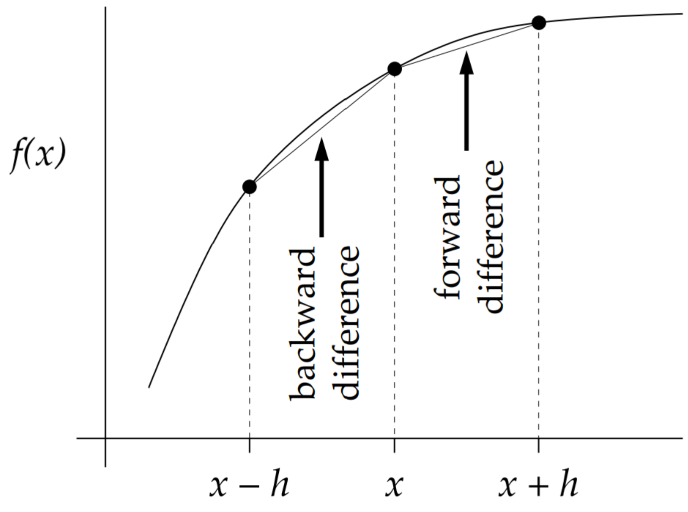
gives the approximation error estimate of

$$R_{\mathrm{forw}} = -\frac{1}{2} h f''(x) + \mathcal{O}(h^2)$$

Backward difference

Backward difference

$$\frac{\mathrm{d}f}{\mathrm{d}x} \simeq \frac{f(x) - f(x-h)}{h}$$



Taylor theorem:

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2} f''(x) + \dots$$

gives the approximation error estimate of

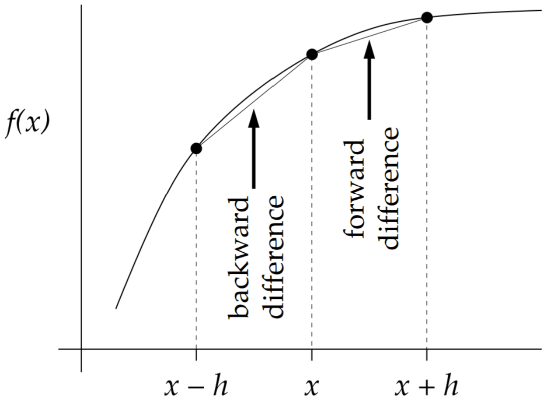
$$R_{\mathrm{back}} = \frac{1}{2} h f''(x) + \mathcal{O}(h^2)$$

Central difference

Recall the forward and backward difference and their errors

$$\frac{\mathrm{d}f}{\mathrm{d}x} \simeq \frac{f(x+h)-f(x)}{h} \qquad \frac{\mathrm{d}f}{\mathrm{d}x} \simeq \frac{f(x)-f(x-h)}{h}$$

$$R_{\text{forw}} = -\frac{1}{2}hf''(x) + \mathcal{O}(h^2) \qquad R_{\text{back}} = \frac{1}{2}hf''(x) + \mathcal{O}(h^2)$$



Taking the average of the two cancels out the *O(h)* error term

central difference $$\frac{df}{dx} \simeq \frac{f(x+h)-f(x-h)}{2h}$$

Error estimate:

$$R_{\text{cent}} = -\frac{f'''(x)}{6}h^2 + \mathcal{O}(h^3)$$

High-order central difference

To improve the approximation error use more than two function evaluations, e.g.

$$\frac{df}{dx} \simeq \frac{Af(x+2h) + Bf(x+h) + Cf(x) + Df(x-h) + Ef(x-2h)}{h} + O(h^4)$$

Determine A, B, C, D, E using Taylor expansion to cancel all terms up to h^4

$$\frac{df}{dx} \simeq \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + \frac{h^4}{30} f^{(5)}(x)$$

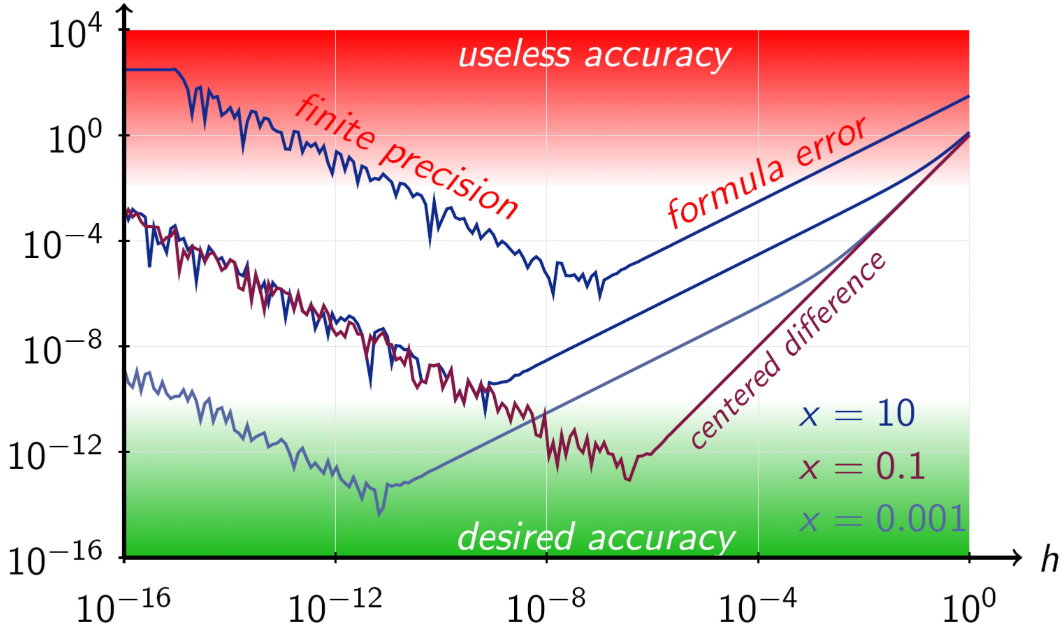
High-order terms:

Derivative	Accuracy	−5	−4	−3	−2	−1	0	1	2	3	4	5
1	2					−1/2	0	1/2				
	4				1/12	−2/3	0	2/3	−1/12			
	6			−1/60	3/20	−3/4	0	3/4	−3/20	1/60		
	8		1/280	−4/105	1/5	−4/5	0	4/5	−1/5	4/105	−1/280	

Balancing truncation and round-off errors

If h is too small, round-off errors become important

- cannot distinguish x and $x+h$ and/or $f(x+h)$ and $f(x)$ with enough accuracy



Credit: Wikipedia

As a rule of thumb, if ε is machine precision and the truncation error is of order $O(h^n)$, then h should not be much smaller than $h \sim \sqrt[n+1]{\varepsilon}$

The higher the finite difference order is, the larger h should be

Balancing truncation and round-off errors

Let f(x) = exp(x)

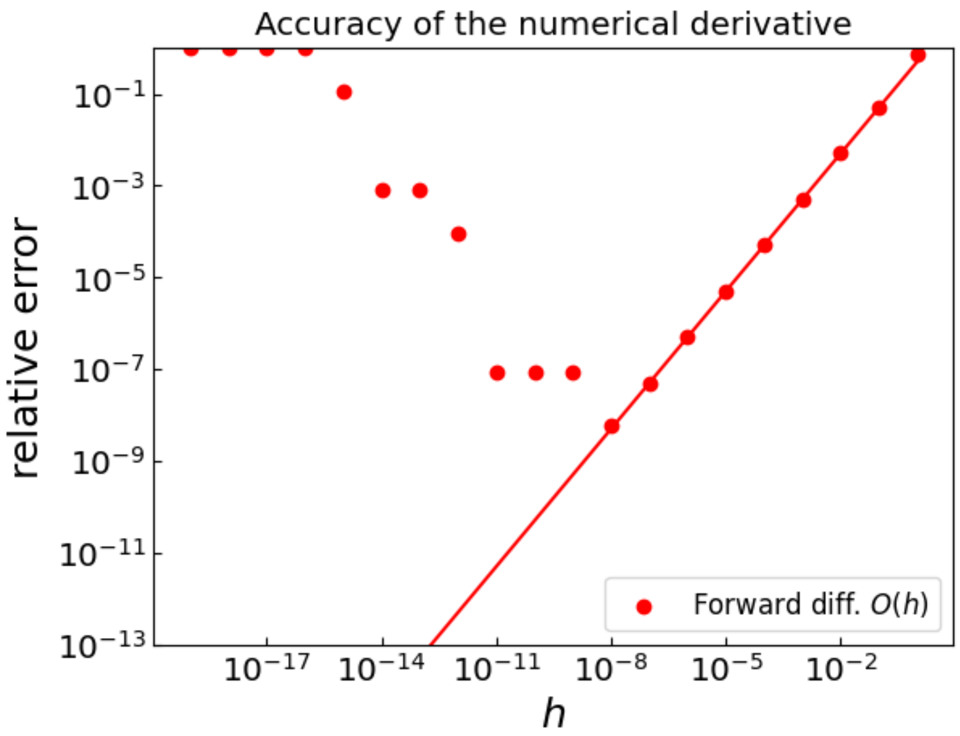
Calculate the derivatives at x = 0

```python
def f(x):
    return np.exp(x)

def df(x):
    return np.exp(x)
```

Forward difference O(h):

Optimal $h \sim \sqrt[2]{10^{-16}} \sim 10^{-8}$



Balancing truncation and round-off errors

Let f(x) = exp(x)

Calculate the derivatives at x = 0

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def f(x):
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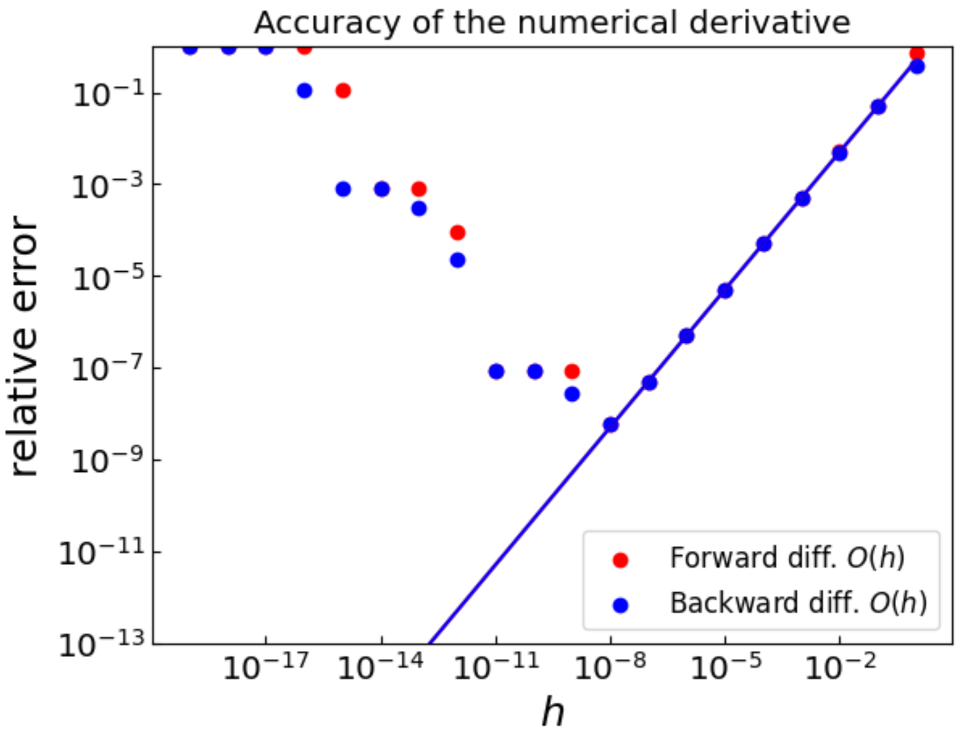
```

Backward difference O(h):

Optimal $h \sim \sqrt[2]{10^{-16}} \sim 10^{-8}$



Balancing truncation and round-off errors

Let f(x) = exp(x)

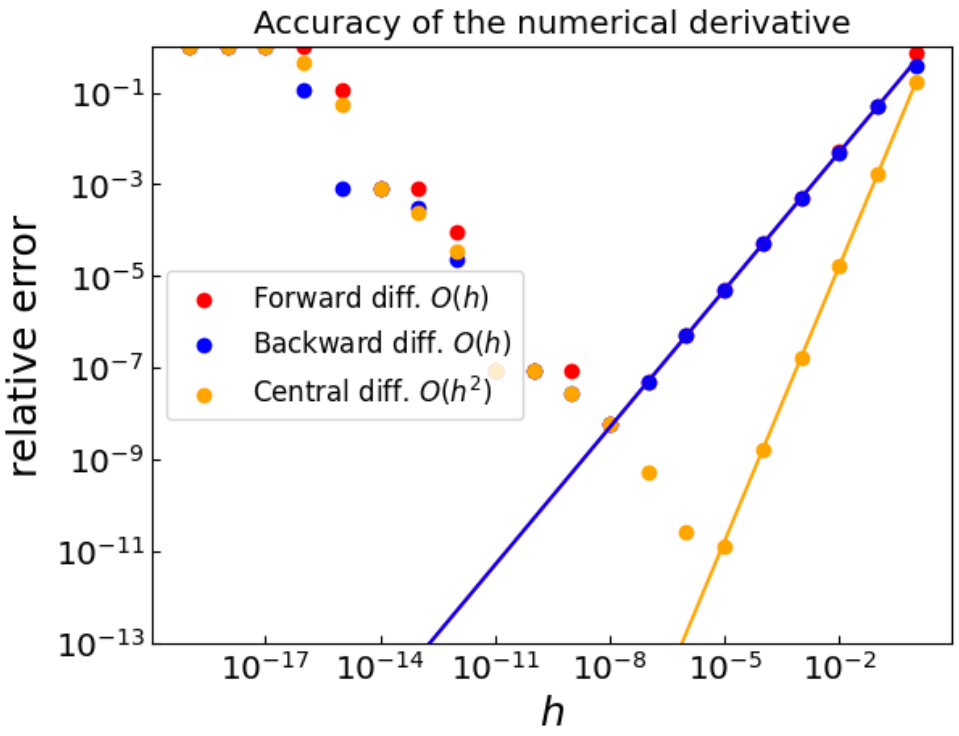
Calculate the derivatives at x = 0

```
def f(x):
    return np.exp(x)

def df(x):
    return np.exp(x)
```

Central difference O(h²):

Optimal $h \sim \sqrt[3]{10^{-16}} \sim 10^{-5}$



Balancing truncation and round-off errors

Let f(x) = exp(x)

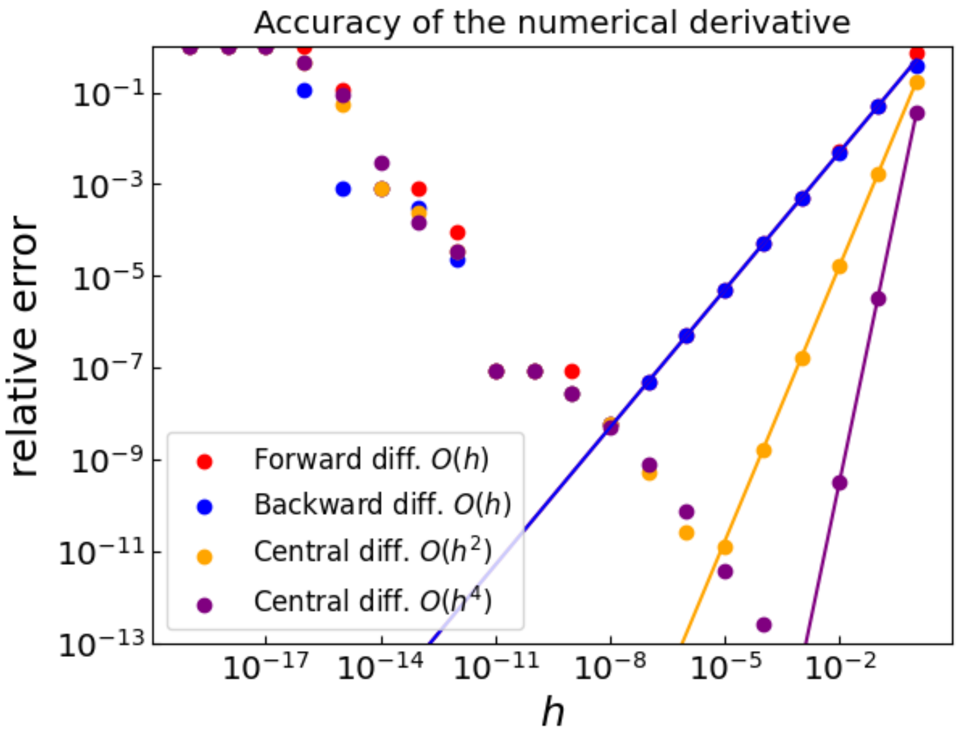
Calculate the derivatives at x = 0

```python
def f(x):
    return np.exp(x)

def df(x):
    return np.exp(x)
```

Central difference O(h^4):

Optimal $h \sim \sqrt[5]{10^{-16}} \sim 10^{-3}$



High-order derivatives

Central difference

$$\frac{df}{dx}(x) \simeq \frac{f(x+h/2) - f(x-h/2)}{h}$$

Now apply the central difference again to *f'(x+h/2)* and *f'(x-h/2)*

$$\begin{aligned} f''(x) &\simeq \frac{f'(x+h/2) - f'(x-h/2)}{h} \\ &= \frac{[f(x+h) - f(x)]/h - [f(x) - f(x-h)]/h}{h} \\ &= \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}. \end{aligned}$$

General formula (to order *h*)

$$f^{(n)}(x) = \frac{1}{h^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} f\left[x + (n/2 - k)h\right] + O(h^2)$$

Second derivative

```python
def d2f_central(f,x,h):
    return (f(x+h) - 2*f(x) + f(x-h)) / (h**2)
```

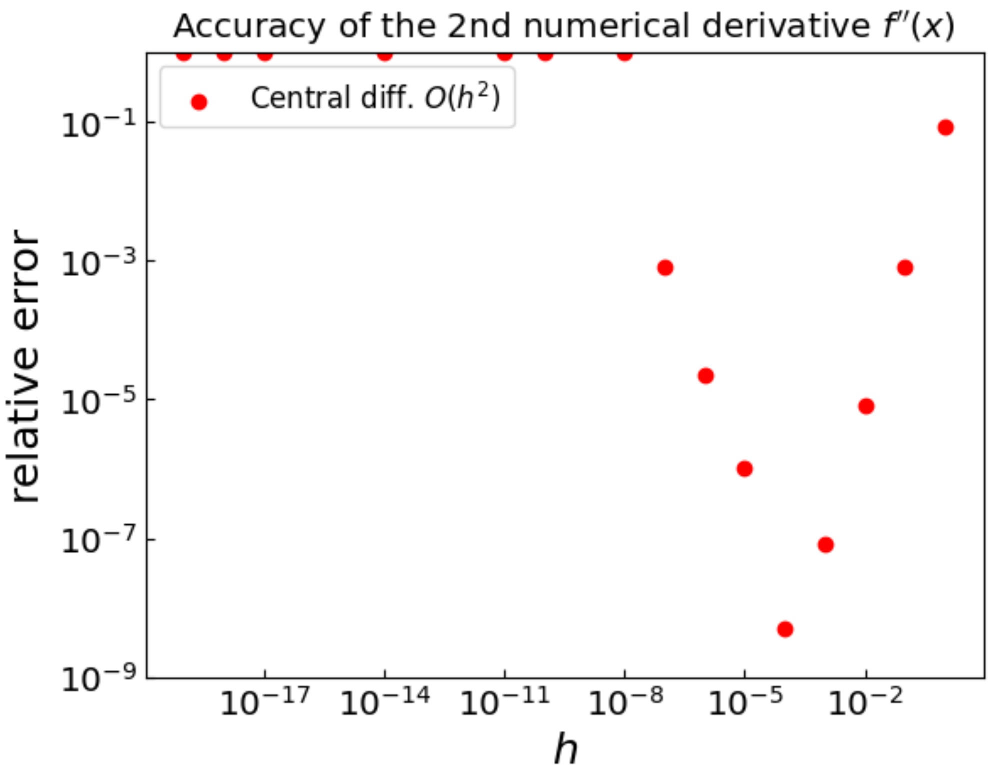
f(x) = exp(x)

```python
def f(x):
    return np.exp(x)

def df(x):
    return np.exp(x)

def d2f(x):
    return np.exp(x)
```

Optimal $h \sim \sqrt[4]{10^{-16}} \sim 10^{-4}$



Partial derivatives

Let us have *f(x,y)*
Use central difference to calculate first-order derivatives

$$\frac{\partial f}{\partial x} = \frac{f(x+h/2,y) - f(x-h/2,y)}{h}$$
$$\frac{\partial f}{\partial y} = \frac{f(x,y+h/2) - f(x,y-h/2)}{h}$$

Reapply the central difference to calculate $\partial^2 f(x,y)/\partial x \partial y$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{f(x+h/2,y+h/2) - f(x-h/2,y+h/2) - f(x+h/2,y-h/2) + f(x-h/2,y-h/2)}{h^2}$$

Summary: Numerical differentiation

- Forward/backward differences
  - Useful when we are given a grid of function values
  - Have limited accuracy (linear in h)

- Central difference
  - More precise than forward/backward differences (quadratic in h)
  - Gives f'(x) estimate at the midpoint of function evaluation points

- Higher-order formulas are obtained by using more than two function evaluations
  - Can be used when limited number of function evaluations available

- Straightforwardly extendable to high-order and partial derivatives

- Balance between truncation and round-off error must be respected
  - h should not be taken too small

Numerical derivative and ordinary differential equations

Ordinary differential equation

$$\frac{dx}{dt} = f(x, t),$$

with initial condition

$$f(x, t_0) = f_0$$

Use the forward difference to approximate dx/dt

$$\frac{dx}{dt} \approx \frac{x(t+h) - x(t)}{h}$$

gives the **Euler method** of solving the equation for x(t)

$$x(t+h) = x(t) + h\, f[x(t), t]$$