



Computational Physics (PHYS6350)

Lecture 11: Classical mechanics problems

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0$$

February 23, 2023

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Course materials: <https://github.com/vlvovch/PHYS6350-ComputationalPhysics>

Three-body problem

Exercise 8.10 (M. Newman, *Computational Physics*)

Three stars interacting through gravitational force

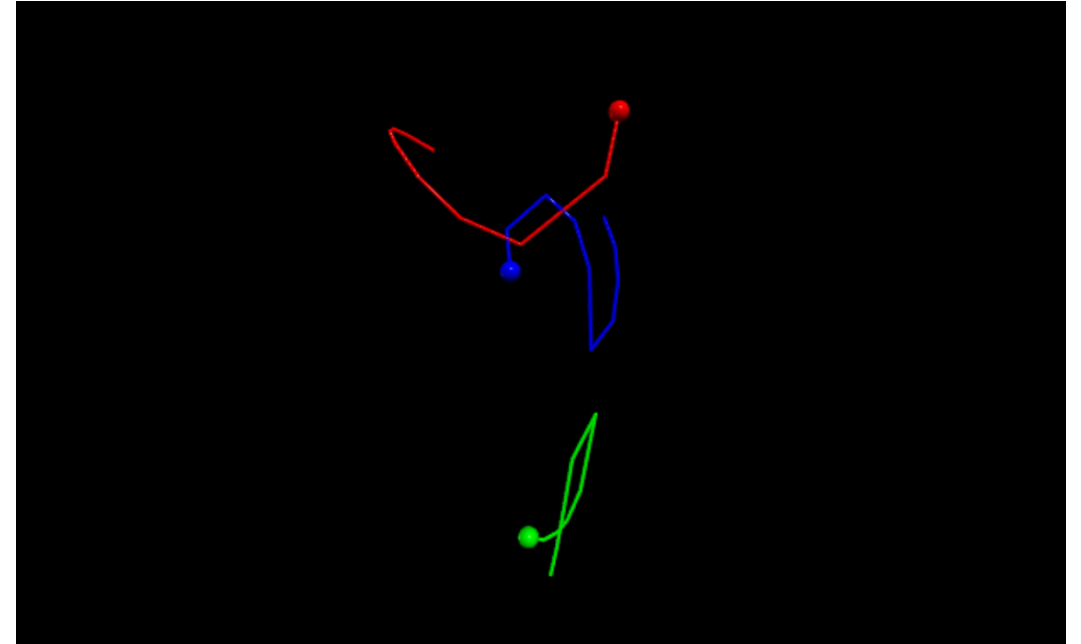
The equations of motion are

$$\frac{d^2 \mathbf{r}_1}{dt^2} = Gm_2 \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} + Gm_3 \frac{\mathbf{r}_3 - \mathbf{r}_1}{|\mathbf{r}_3 - \mathbf{r}_1|^3},$$

$$\frac{d^2 \mathbf{r}_2}{dt^2} = Gm_1 \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} + Gm_3 \frac{\mathbf{r}_3 - \mathbf{r}_2}{|\mathbf{r}_3 - \mathbf{r}_2|^3},$$

$$\frac{d^2 \mathbf{r}_3}{dt^2} = Gm_1 \frac{\mathbf{r}_1 - \mathbf{r}_3}{|\mathbf{r}_1 - \mathbf{r}_3|^3} + Gm_2 \frac{\mathbf{r}_2 - \mathbf{r}_3}{|\mathbf{r}_2 - \mathbf{r}_3|^3}.$$

Name	Mass	x	y
Star 1	150.	3	1
Star 2	200.	-1	-2
Star 3	250.	-1	1



Cannot be solved analytically!

Take $G = 1$ (dimensionless)

Initially at rest and move in plane $z = 0$

Initial coordinates: $\mathbf{r}_1 = (3, 1)$, $\mathbf{r}_2 = (-1, -2)$, $\mathbf{r}_3 = (-1, 1)$

Three-body problem

```
def fthreebody(xin, t):
    global f_evaluations
    f_evaluations += 1

    x1 = xin[0]
    y1 = xin[1]
    x2 = xin[2]
    y2 = xin[3]
    x3 = xin[4]
    y3 = xin[5]

    r12 = np.sqrt((x1-x2)**2 + (y1-y2)**2)
    r13 = np.sqrt((x1-x3)**2 + (y1-y3)**2)
    r23 = np.sqrt((x2-x3)**2 + (y2-y3)**2)

    return np.array([xin[6],xin[7],xin[8],xin[9],xin[10],xin[11],
                    G * m2 * (x2 - x1) / r12**3 + G * m3 * (x3 - x1) / r13**3,
                    G * m2 * (y2 - y1) / r12**3 + G * m3 * (y3 - y1) / r13**3,
                    G * m1 * (x1 - x2) / r12**3 + G * m3 * (x3 - x2) / r23**3,
                    G * m1 * (y1 - y2) / r12**3 + G * m3 * (y3 - y2) / r23**3,
                    G * m1 * (x1 - x3) / r13**3 + G * m2 * (x2 - x3) / r23**3,
                    G * m1 * (y1 - y3) / r13**3 + G * m2 * (y2 - y3) / r23**3
                    ])
)
```

Run from $t = 0$ to 5

Three-body problem

```
def fthreebody(xin, t):
    global f_evaluations
    f_evaluations += 1

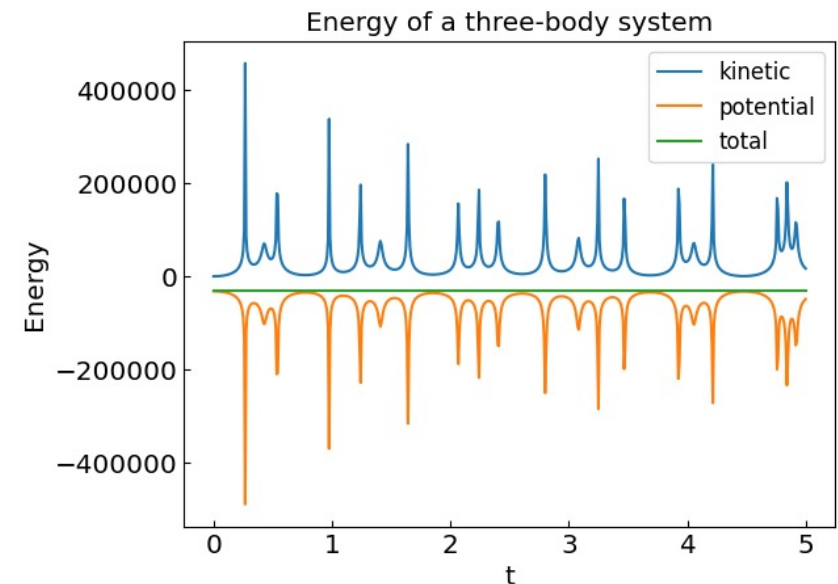
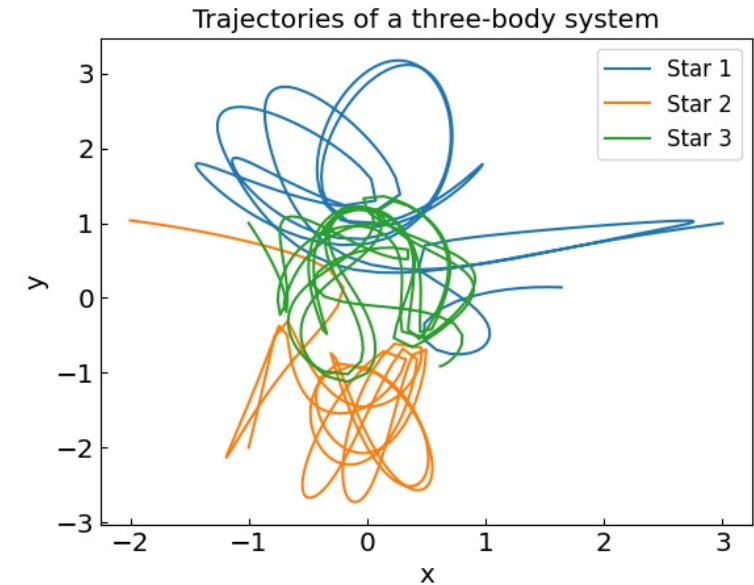
    x1 = xin[0]
    y1 = xin[1]
    x2 = xin[2]
    y2 = xin[3]
    x3 = xin[4]
    y3 = xin[5]

    r12 = np.sqrt((x1-x2)**2 + (y1-y2)**2)
    r13 = np.sqrt((x1-x3)**2 + (y1-y3)**2)
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    return np.array([xin[6],xin[7],xin[8],xin[9],xin[10],xin[11],
                    G * m2 * (x2 - x1) / r12**3 + G * m3 * (x3 - x1) / r13**3,
                    G * m2 * (y2 - y1) / r12**3 + G * m3 * (y3 - y1) / r13**3,
                    G * m1 * (x1 - x2) / r12**3 + G * m3 * (x3 - x2) / r23**3,
                    G * m1 * (y1 - y2) / r12**3 + G * m3 * (y3 - y2) / r23**3,
                    G * m1 * (x1 - x3) / r13**3 + G * m2 * (x2 - x3) / r23**3,
                    G * m1 * (y1 - y3) / r13**3 + G * m2 * (y2 - y3) / r23**3
                    ],
                    )
```

Run from $t = 0$ to 5

Final project idea: simulate the solar system



Lagrangian mechanics

In Lagrangian mechanics, a classical system with N degrees of freedom is characterized by generalized coordinates q_j that obey Euler-Lagrange equations of motion:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = 0, \quad j = 1 \dots N$$

L is the non-relativistic Lagrangian which is typically defined as the difference of kinetic and potential energies, $L = T - V$.

One can rewrite these equations using the chain rule:

$$\sum_{i=1}^N \frac{\partial^2 L}{\partial \dot{q}_j \partial \dot{q}_i} \ddot{q}_i = - \sum_{i=1}^N \frac{\partial^2 L}{\partial \dot{q}_j \partial q_i} \dot{q}_i - \frac{\partial^2 L}{\partial \dot{q}_j \partial t} + \frac{\partial L}{\partial q_j}, \quad j = 1 \dots N.$$

This is a system of N linear equations for \ddot{q}_j . When solved we have

$$\ddot{q}_i = f_i(\{q_j\}, \{\dot{q}_j\}, t).$$

or

$$\begin{aligned} \frac{dq_i}{dt} &= \dot{q}_i, \\ \frac{d\dot{q}_i}{dt} &= f_i(\{q_j\}, \{\dot{q}_j\}, t), \end{aligned}$$

which can be solved using standard methods.

Non-linear pendulum

The generalized coordinate is the displacement angle θ

$$x = l \sin(\theta),$$

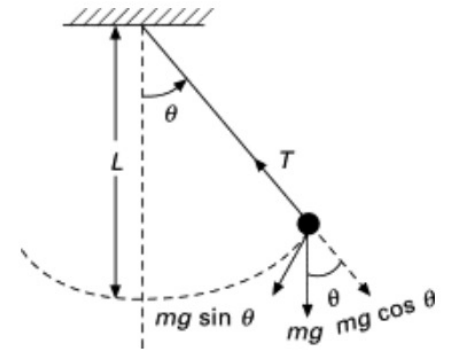
$$y = -l \cos(\theta),$$

Lagrangian

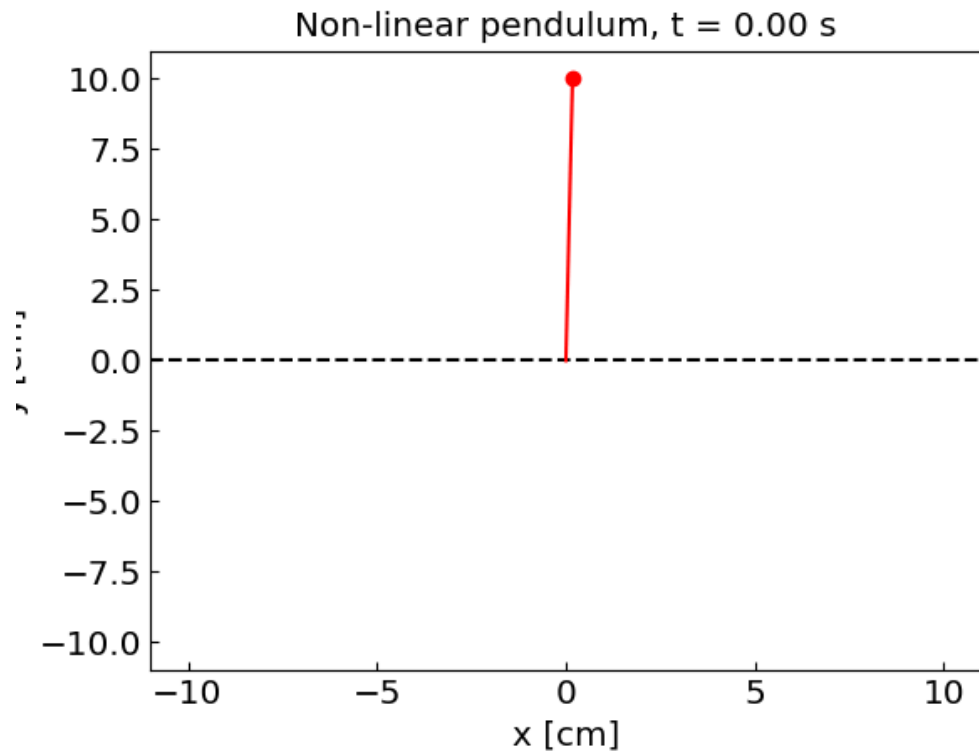
$$L = T - V = \frac{ml^2 \dot{\theta}^2}{2} - mgl \cos(\theta),$$

Equations of motion

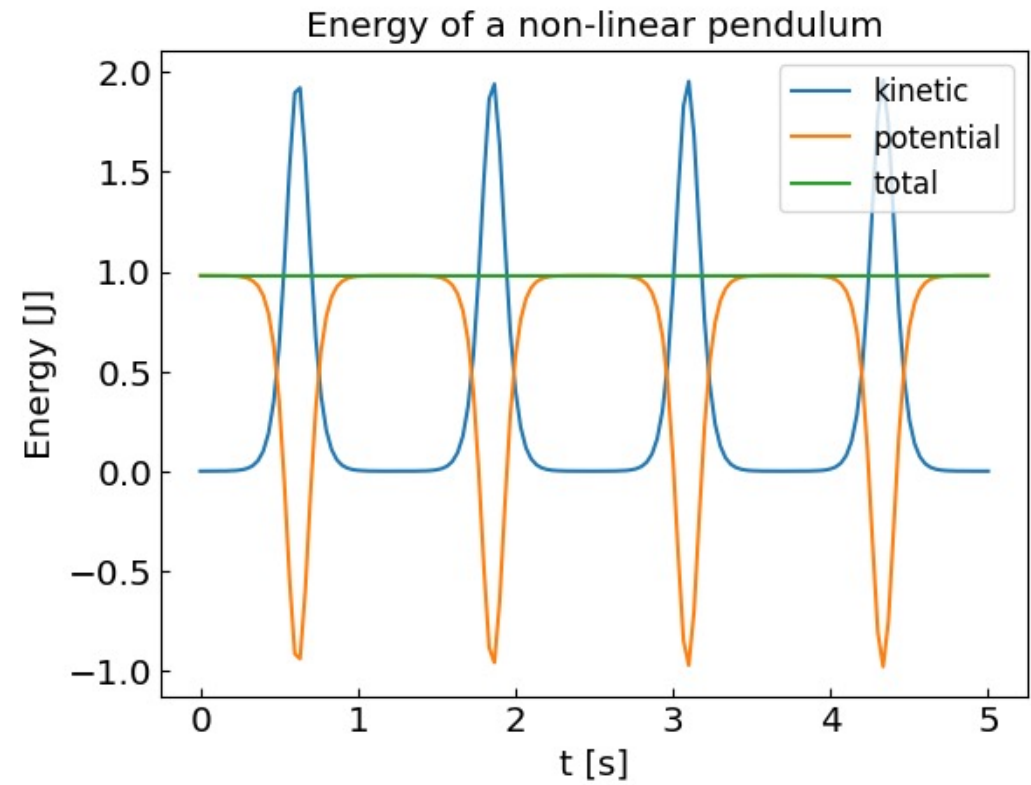
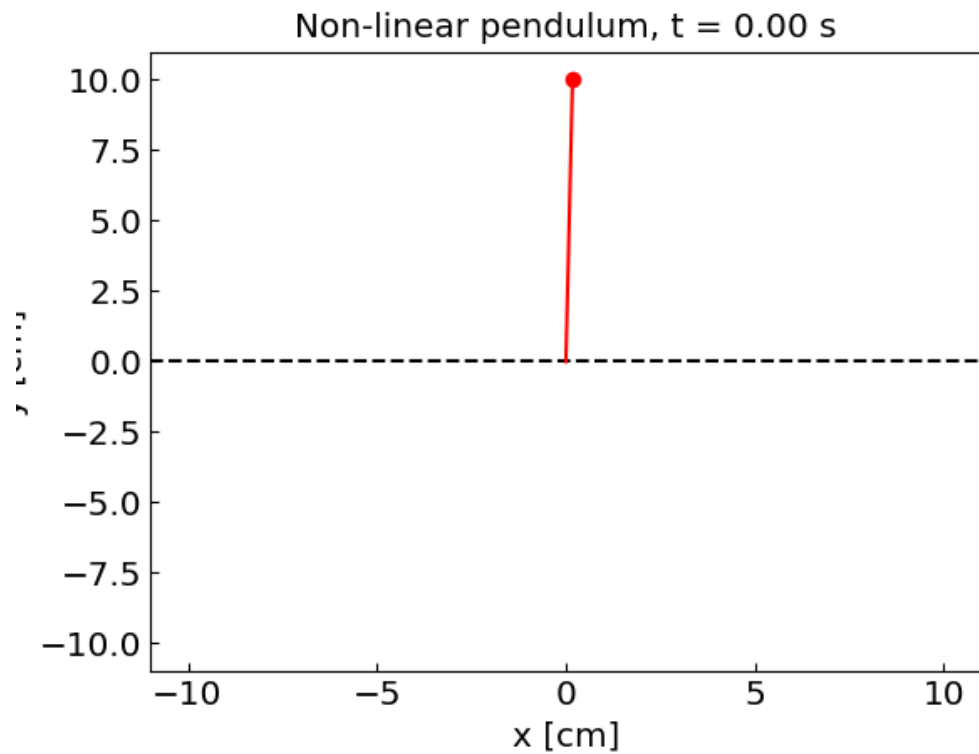
$$ml\ddot{\theta} = -mgl \sin \theta.$$



Non-linear pendulum



Non-linear pendulum



Double pendulum

Double pendulum is the simplest system exhibiting chaotic motion – *deterministic chaos*

Two degrees of freedom: the displacement angles θ_1 and θ_2 .

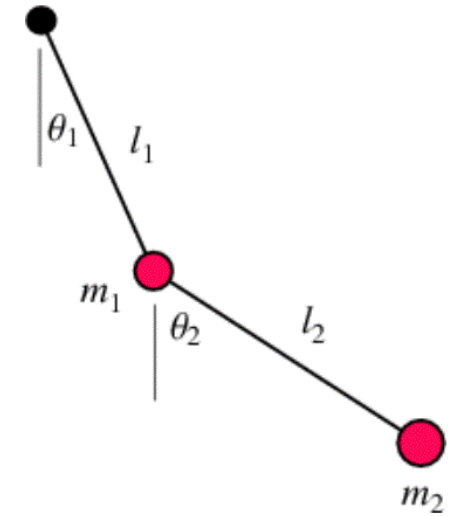
$$\begin{aligned}x_1 &= l_1 \sin(\theta_1), \\y_1 &= -l_1 \cos(\theta_1), \\x_2 &= l_1 \sin(\theta_1) + l_2 \sin(\theta_2), \\y_2 &= -l_1 \cos(\theta_1) - l_2 \cos(\theta_2).\end{aligned}$$

$$T = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right]$$

kinetic energy

$$V = m_1 g y_1 + m_2 g y_2 = -(m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2).$$

potential energy



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$$\begin{aligned}x_1 &= l_1 \sin(\theta_1), \\y_1 &= -l_1 \cos(\theta_1), \\x_2 &= l_1 \sin(\theta_1) + l_2 \sin(\theta_2), \\y_2 &= -l_1 \cos(\theta_1) - l_2 \cos(\theta_2).\end{aligned}$$

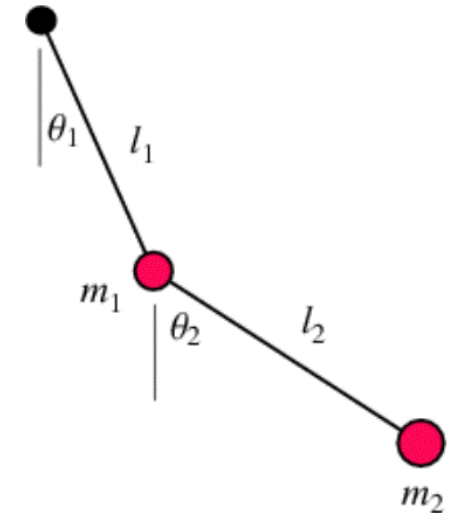
$$T = \frac{m_1 \dot{x}_1^2}{2} + \frac{m_2 \dot{x}_2^2}{2} = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left[l_1^2 \dot{\theta}_1^2 + l_2^2 \dot{\theta}_2^2 + 2l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right] \quad \text{kinetic energy}$$

$$V = m_1 g y_1 + m_2 g y_2 = -(m_1 + m_2) g l_1 \cos(\theta_1) - m_2 g l_2 \cos(\theta_2). \quad \text{potential energy}$$

The Lagrange equations of motion read

$$\begin{aligned}(m_1 + m_2) l_1 \ddot{\theta}_1 + m_2 l_2 \cos(\theta_1 - \theta_2) \ddot{\theta}_2 &= -m_2 l_1 \dot{\theta}_2^2 \sin(\theta_1 - \theta_2) - (m_1 + m_2) g \sin(\theta_1), \\m_2 l_1 \cos(\theta_1 - \theta_2) \ddot{\theta}_1 + m_2 l_2 \ddot{\theta}_2 &= m_2 l_1 \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) - m_2 g \sin(\theta_2).\end{aligned}$$

This is a system of two linear equation for $\ddot{\theta}_{1,2}$ that can be solved easily.



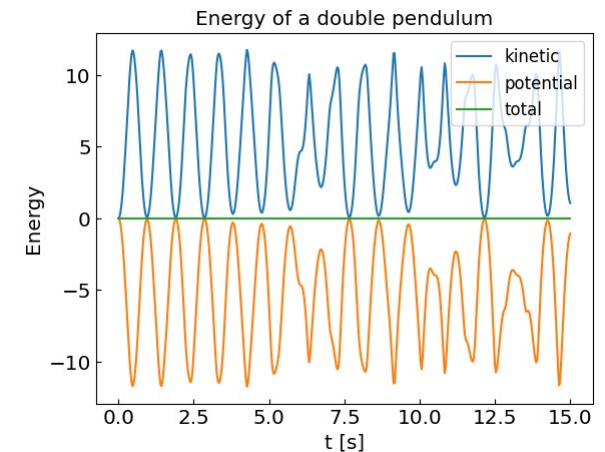
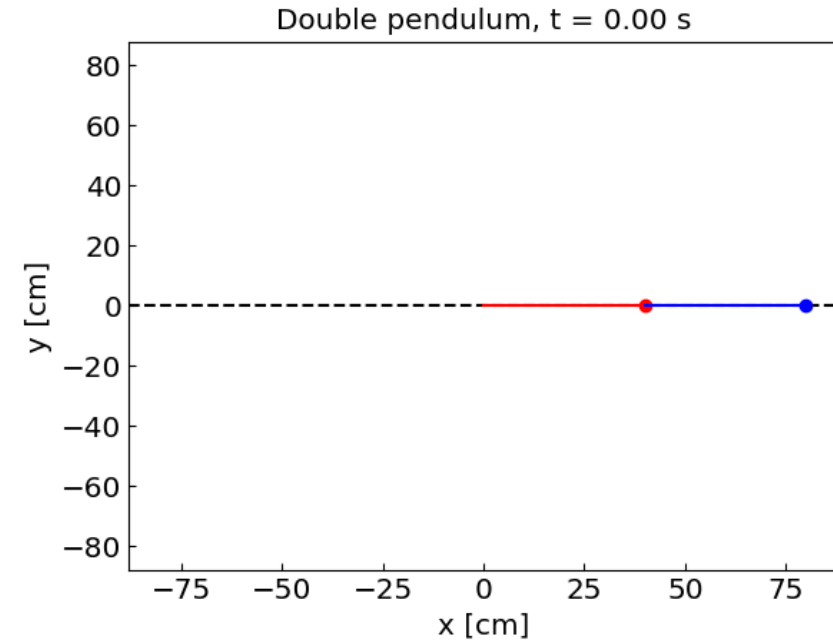
Double pendulum

```
g = 9.81
l1 = 0.4
l2 = 0.4
m1 = 1.0
m2 = 1.0

def fdoublependulum(xin, t):
    global f_evaluations
    f_evaluations += 1

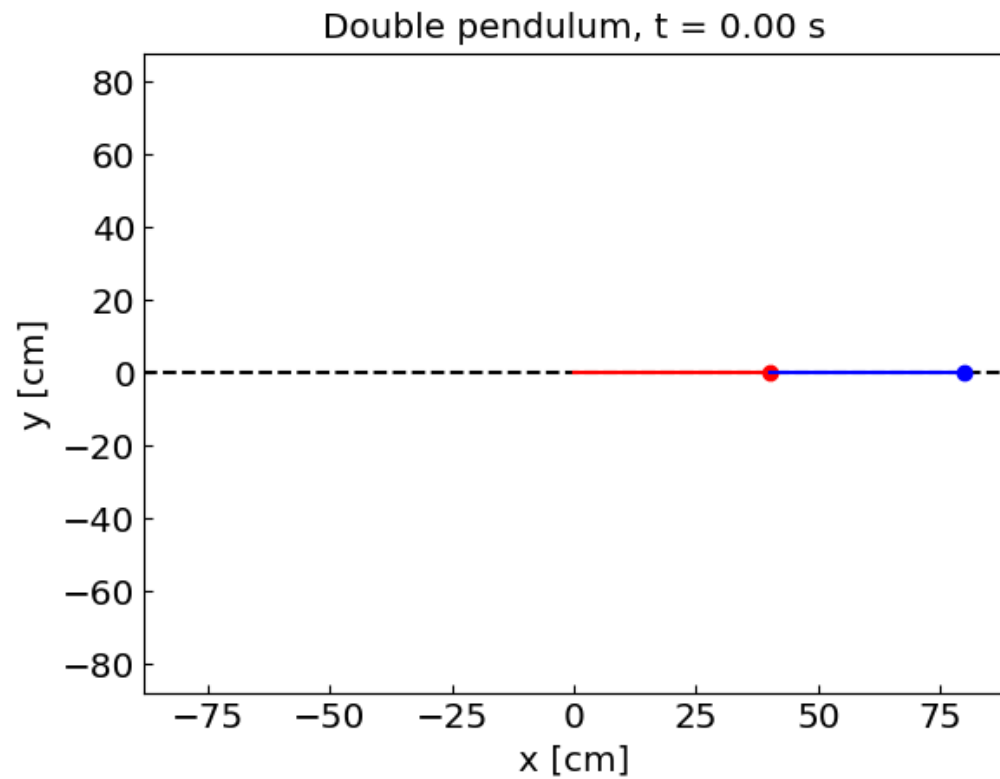
    theta1 = xin[0]
    theta2 = xin[1]
    omega1 = xin[2]
    omega2 = xin[3]
    a1 = (m1 + m2)*l1
    b1 = m2*l2*np.cos(theta1 - theta2)
    c1 = m2*l2*omega2*omega2*np.sin(theta1 - theta2) + (m1 + m2)*g*np.sin(theta1)
    a2 = m2*l1*np.cos(theta1 - theta2)
    b2 = m2*l2;
    c2 = -m2*l1*omega1*omega1*np.sin(theta1 - theta2) + m2*g*np.sin(theta2) # + k
    domega1 = - ( c2/b2 - c1/b1 ) / ( a2/b2 - a1/b1 )
    domega2 = - ( c2/a2 - c1/a1 ) / ( b2/a2 - b1/a1 )
    return np.array([omega1,
                     omega2,
                     domega1,
                     domega2
                    ])
```

```
sol = bulirsch_stoer(fpendulum, x0, t0, 1, tend, eps, error_definition_pendulum, 10)
```

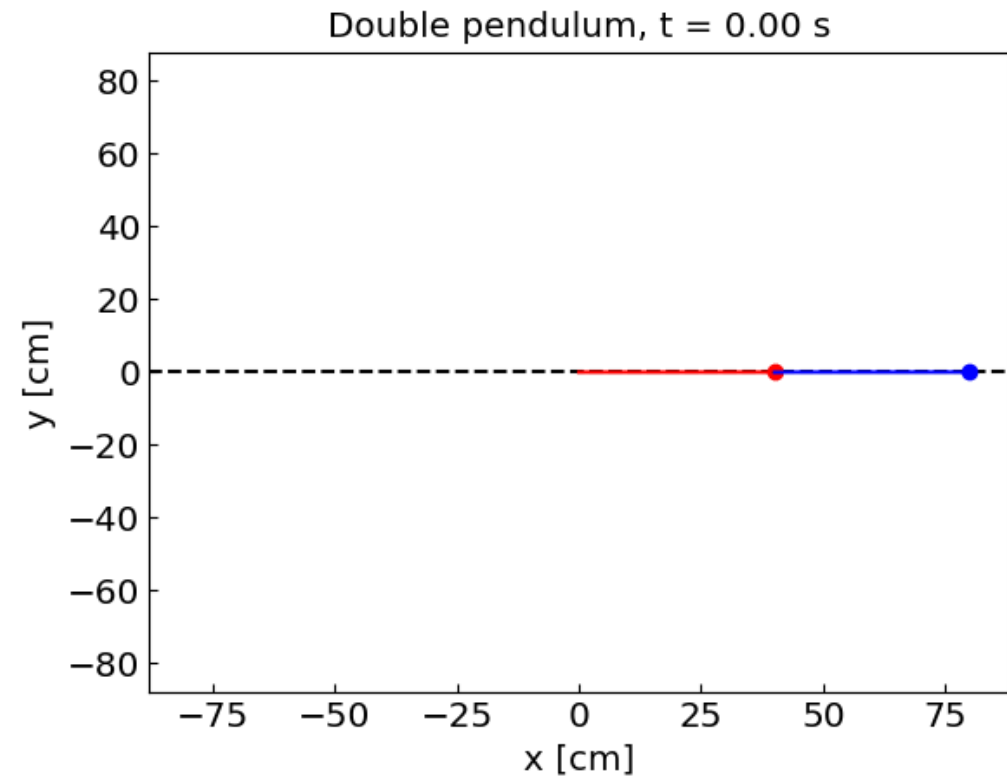


Double pendulum: chaotic behavior

$$\theta_1^0 = \theta_2^0 = \pi/2$$



$$\theta_1^0 = \theta_2^0 = \pi/2 + 10^{-4}$$



Double pendulum: chaotic behavior

$$\theta_1^0 = \theta_2^0 = \pi/2$$

$$\theta_1^0 = \theta_2^0 = \pi/2 + 10^{-4}$$

