1

Literature: Weyl 1946 Classical groups, pages 177–229;
Muirhead 1982, Aspects of multivariate statistical theory, chapter 2, pages 50–72;
Katz, Sarnak, Chapter 5, pages 107–121;

1.1 U(n).

For a matrix A, denote

$$\delta A = A^{-1} \mathrm{d}A,$$

whatever this means.

For $A \in U(n)$, represent

$$A = U\,\Lambda\, U^*, \tag{1.1}$$

where

$$\Lambda = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\varphi_1}, \ldots, \mathrm{e}^{\mathrm{i}\varphi_n}), \quad 0 < \varphi_1 < \ldots < \varphi_n < 2\pi, \tag{1.2}$$

and $H^* \in U(n)$ is fixed by the condition that the in every column, the first (from up down over rows) non-zero column is positive. (Otherwise representation (1.1) is not unique, $U \to UD$ with diagonal D). Furthermore, U is parametrized by $n^2 - n = l$ real parameters, Λ is parametrized by n parameters. The representation (1.1) is unique for $A \in U(n)$ with all different eigenvalues, and hence we can consider it as a change of variables. Now, since $UU^* = \mathbf{1}$, and hence $\delta U = -\delta U^*$ is skew-symmetric, we obtain by formal manipulations

$$\delta B \;:=\; \quad U^{-1}\;\delta A\; U = (\Lambda^{-1}\;\delta U\;\Lambda - \delta U) + \delta\Lambda. \tag{1.3}$$

From (1.3), considering separately diagonal and off-diagonal teerms,

$$\begin{aligned} \delta b_{jj} &= \mathrm{i}\mathrm{d}\varphi_j, \\ \delta b_{jk} &= (\mathrm{e}^{\mathrm{i}(\varphi_k - \varphi_j)} - 1)\;\delta u_{jk}, \quad j \neq k. \end{aligned}$$

Now, after all matrix multiplications, we write δB in a row of n^2 real elements:

$$\delta b_{11}, \ldots \delta b_{nn}, \delta b_{12}, \delta b_{21}, \ldots, \delta b_{n-1,n}, \delta b_{n,n-1},$$

and take a (wedge) product of them (as of differentials; we obtain an n^2 dimensional differential). Writing δB as a (scalar rather than matrix) n^2 dimensional differential, and ignoring sign,

$$\delta B = \prod_{j=1}^{n} \mathrm{d}\varphi_j \;\prod_{j\neq k} (\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k})\;\delta b_{j,k} = \prod_{j\neq k} (\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k}) \prod_{j=1}^{n} \mathrm{d}\varphi_j \;\; \mathrm{d}\omega_U.$$

Now, when we integrate over $U(n)$ (or average over group) a function that depends only on eigenvalues, first of all we integrate only over matrices A with all distinct eigenvalues (other A form a sub-manifold of smaller dimension, and hence do not contribute in the integral)

$$\begin{aligned} \int\limits_{U(n)} f(A)\mathrm{d}Haar &= \int\limits_{U(n)} f(A)A^{-1}\mathrm{d}A = \int\limits_{U(n)} f(A)\delta A = \int\limits_{U(n)} f(A)\delta B = \\ &= \int\limits_{U(n)} f(\varphi_1, \ldots, \varphi_n) \prod_{j\neq k} (\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k}) \prod_{j=1}^{n} \mathrm{d}\varphi_j \;\; \mathrm{d}\omega_U = \\ &= \left(\frac{1}{n!} \int\limits_{[0,2\pi]^n} f(\varphi_1, \ldots, \varphi_n) \prod_{j\neq k} |\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k}| \prod_{j=1}^{n} \mathrm{d}\varphi_j \right) \cdot Vol(U) \end{aligned},$$

where we separated contribution in the integral coming from φ_j, and from U. The factor $n!$ appeared since we changed integral from over $0 < \varphi_1 < \ldots < \varphi_n < 2\pi$ to over $\varphi_j \in [0, 2\pi]$. To have it normalized to 1, probably when we integrate over $\mathrm{d}\varphi$ over $[0, 2\pi]$, it is better to divide by 2π. However, Katz Sarnak, chapter 5, divide by 2π not only in the case $U(n)$, when integrate over $[0, 2\pi]$, but also in the cases $O(2n)$, when they integrate over $[0, \pi]$, but still divide by 2π. Furthermore, normalizing, we need to divide by analogous expression with $f = 1$. In that way we get rid of $Vol(U)$.

$$\boxed{\int\limits_{U(n)} f(A) \mathrm{d}Haar = \left(\frac{1}{n!} \int\limits_{[0,2\pi]^n} f(\varphi_1, \ldots, \varphi_n) \prod_{j \neq k} |\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k}| \prod_{j=1}^{n} \frac{\mathrm{d}\varphi_j}{2\pi} \right). \qquad (1.4)}$$

1.2 $O^+(2n)$.

We will take $n = 2$, so we consider $O^+(4)$. Here (1.1) and (1.3) still hold,

$$A = U \Lambda U^*, \qquad \delta B := \quad U^{-1} \, \delta A \, U = (\Lambda^{-1} \, \delta U \, \Lambda - \delta U) + \delta \Lambda,$$

but Λ changes: instead of (1.2), we have

$$\Lambda = \begin{bmatrix} \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{bmatrix} \end{bmatrix}, \qquad 0 < \varphi_1 < \varphi_2 < \pi. \tag{1.5}$$

Again, U should be fixed somehow to ensure uniqueness. It is parametrized by

$$\frac{2n(2n-1)}{2} - n = 2n(n-1)$$

parameters. Λ is parametrized by n real parameters. What is δU? It is a skew-symmetric matrix, parametrized by low-diagonal elements, whose are $2n(2n-1)/2$, where we need to subtract n, since we fixed somehow U. We have

$$\delta U = \begin{bmatrix} \begin{bmatrix} 0 & -\delta u_{21} \\ \delta u_{21} & 0 \end{bmatrix} & \begin{bmatrix} -\delta u_{31} & -\delta u_{41} \\ -\delta u_{32} & -\delta u_{42} \end{bmatrix} \\ \begin{bmatrix} \delta u_{31} & \delta u_{32} \\ \delta u_{41} & \delta u_{42} \end{bmatrix} & \begin{bmatrix} 0 & -\delta u_{43} \\ \delta u_{43} & 0 \end{bmatrix} \end{bmatrix}, \quad \text{where now the scalar meaning is } \delta U = \prod_{j>k} \delta u_{jk}.$$

What is $\Lambda^{-1} \, \delta U \, \Lambda$? But first, what is

$$\delta\Lambda = \Lambda^{-1} \, \mathrm{d}\Lambda = \begin{bmatrix} \begin{bmatrix} 0 & -\mathrm{d}\varphi_1 \\ \mathrm{d}\varphi_1 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & -\mathrm{d}\varphi_2 \\ \mathrm{d}\varphi_2 & 0 \end{bmatrix} \end{bmatrix}, \quad \text{with the scalar meaning} \quad \delta\Lambda = \mathrm{d}\varphi_1 \mathrm{d}\varphi_2.$$

Second, diagonal block terms in $\Delta^{-1} \, \delta U \, \Delta$ are conjugated by the same φ :

$$\begin{bmatrix} \cos\varphi_1 & \sin\varphi_1 \\ -\sin\varphi_1 & \cos\varphi_1 \end{bmatrix} \begin{bmatrix} 0 & -\delta u_{21} \\ \delta u_{21} & 0 \end{bmatrix} \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} = \begin{bmatrix} 0 & -\delta u_{21} \\ \delta u_{21} & 0 \end{bmatrix}, \quad \text{with meaning } \delta_{21}.$$

Third, off-diagonal block terms in $\Delta^{-1}\,\delta U\,\Delta$ are conjugated by different φ :

$$\begin{bmatrix}\cos\varphi_2 & \sin\varphi_2\\ -\sin\varphi_2 & \cos\varphi_2\end{bmatrix}\begin{bmatrix}\delta u_{31} & \delta u_{32}\\ \delta u_{41} & \delta u_{42}\end{bmatrix}\begin{bmatrix}\cos\varphi_1 & -\sin\varphi_1\\ \sin\varphi_1 & \cos\varphi_1\end{bmatrix}-\begin{bmatrix}\delta u_{31} & \delta u_{32}\\ \delta u_{41} & \delta u_{42}\end{bmatrix}=:\begin{bmatrix}\delta v_1 & \delta v_2\\ \delta v_3 & \delta v_4\end{bmatrix}. \tag{1.6}$$

The scalar meaning of $\begin{bmatrix}\delta v_1 & \delta v_2\\ \delta v_3 & \delta v_4\end{bmatrix}$ itself is just the (wedge) product of all the elements: $\delta v_1\,\delta v_2\,\delta v_3\,\delta v_4$. Formula (1.6) can be rewritten, elementwise, as (we denote for shortness $c=\cos, s=\sin$)

$$\begin{bmatrix}\begin{bmatrix}c_1c_2-1 & s_1c_2\\ -s_1c_2 & c_1c_2-1\end{bmatrix} & \begin{bmatrix}c_1s_2 & s_1s_2\\ -s_1s_2 & c_1s_2\end{bmatrix}\\ \begin{bmatrix}-c_1s_2 & -s_1s_2\\ s_1s_2 & -c_1s_2\end{bmatrix} & \begin{bmatrix}c_1c_2-1 & s_1c_2\\ -s_1c_2 & c_1c_2-1\end{bmatrix}\end{bmatrix}\cdot\begin{bmatrix}\delta u_{31}\\ \delta u_{32}\\ \delta u_{41}\\ \delta u_{42}\end{bmatrix}.$$

We need to take a covariant product of those differential forms, or, equivalently, take the determinant of the matrix, which equals

$$\det = 4(\cos\varphi_1-\cos\varphi_2)^2.$$

Hence, the contribution from off-diagonal blocks are

$$4(\cos\varphi_j-\cos\varphi_k)^2\prod\delta u_{\ldots},$$

and the contribution from each diagonal block is

$$\delta u_{\ldots}.$$

Finally,

$$\int\limits_{O^+(2n)} f(A)\mathrm{d}Haar=\int\limits_{O^+(2n)} f(A)\delta A=\int\limits_{O^+(2n)} f(A)\delta B=$$
$$=\left(\frac{1}{n!}\int\limits_{[0,\pi]^n} f(\varphi_1,...,\varphi_n)\prod_{j<k}4(\cos\varphi_j-\cos\varphi_k)^2\prod_j\mathrm{d}\varphi_j\right)\cdot Vol(U).$$

or, cheating with constant, using mathematical packages to check the correct constant for the first several n,

$$\boxed{\int\limits_{O^+(2n)} f(A)\mathrm{d}Haar=\frac{2}{n!}\int\limits_{[0,\pi]^n} f(\varphi_1,...,\varphi_n)\prod_{j<k}4(\cos\varphi_j-\cos\varphi_k)^2\prod_j\frac{\mathrm{d}\varphi_j}{2\pi}.}$$

1.3 $O^-(2n+2)$.

In this case there are 1, -1 among eigenvalues. Let again $n=2$. Denote by

$$T=\begin{bmatrix}\cos\varphi & -\sin\varphi\\ \sin\varphi & \cos\varphi\end{bmatrix}$$

the block matrix. Then Λ can be parametrized in the form

$$\Lambda = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{pmatrix}, \qquad 0 < \varphi_1 < \varphi_2 < \pi.$$

What changes in our fundamental relation (1.3)?

$$\delta B = U^{-1}\, \delta A\, U = (\Lambda^{-1}\, \delta U\, \Lambda - \delta U) + \delta\Lambda.$$

We have

$$\delta U = \begin{bmatrix} \begin{bmatrix} 0 & -\delta u_{21} \\ \delta u_{21} & 0 \end{bmatrix} & \begin{bmatrix} -\delta u_{31} & -\delta u_{41} \\ -\delta u_{32} & -\delta u_{42} \end{bmatrix} & \begin{bmatrix} -\delta u_{51} & -\delta u_{61} \\ -\delta u_{52} & -\delta u_{62} \end{bmatrix} \\ \begin{bmatrix} \delta u_{31} & \delta u_{32} \\ \delta u_{41} & \delta u_{42} \end{bmatrix} & \begin{bmatrix} 0 & -\delta u_{43} \\ \delta u_{43} & 0 \end{bmatrix} & \begin{bmatrix} -\delta u_{53} & -\delta u_{63} \\ -\delta u_{54} & -\delta u_{64} \end{bmatrix} \\ \begin{bmatrix} \delta u_{51} & \delta u_{52} \\ \delta u_{61} & \delta u_{62} \end{bmatrix} & \begin{bmatrix} \delta u_{53} & \delta u_{54} \\ \delta u_{63} & \delta u_{64} \end{bmatrix} & \begin{bmatrix} 0 & -\delta u_{65} \\ \delta u_{65} & 0 \end{bmatrix} \end{bmatrix},$$

where now the scalar meaning is $\delta U = \prod_{j>k} \delta u_{jk}$. New pieces comparing to the previous case are now lying on the last block row (i.e., in last 2 rows).

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \delta u_{51} & \delta u_{52} \\ \delta u_{61} & \delta u_{62} \end{bmatrix} \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} - \begin{bmatrix} \delta u_{51} & \delta u_{52} \\ \delta u_{61} & \delta u_{62} \end{bmatrix},$$

and the latter can be written line-by-line in the form of two decomposed relations, for δ_{51}, δ_{52}, and distinctively for δ_{61}, δ_{62}. We have

$$\begin{pmatrix} c_1 - 1 & s_1 \\ -s_1 & c_1 - 1 \end{pmatrix} \begin{pmatrix} \delta u_{51} \\ \delta u_{52} \end{pmatrix}, \qquad \begin{pmatrix} c_1 + 1 & s_1 \\ -s_1 & c_1 + 1 \end{pmatrix} \begin{pmatrix} \delta u_{61} \\ \delta u_{62} \end{pmatrix}$$

and the determinant equals

$$(2 - 2\cos\varphi_1)(2 + 2\cos\varphi_1) = 4\sin^2\varphi_1.$$

Finally,

$$\int\limits_{O^-(2n+2)} f(A)\mathrm{d}Haar = \int\limits_{O-(2n+2)} f(A)\delta A = \int\limits_{O^-(2n+2)} f(A)\delta B =$$
$$= \left(\frac{1}{n!} \int\limits_{[0,\pi]^n} f(1, -1, \varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j 4\sin^2\varphi_j \prod_j \mathrm{d}\varphi_j \right) \cdot Vol(U).,$$

and, cheating with constant,

$$\boxed{\begin{aligned} &\int\limits_{O^-(2n+2)} f(A)\mathrm{d}Haar = \int\limits_{O-(2n+2)} f(A)\delta A = \int\limits_{O^-(2n+2)} f(A)\delta B = \\ &= \left(\frac{1}{n!} \int\limits_{[0,\pi]^n} f(1, -1, \varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j 4\sin^2\varphi_j \prod_j \frac{\mathrm{d}\varphi_j}{2\pi} \right). \end{aligned}}$$

1.4 $O^+(2n+1)$.

In this case there is an eigenvalue 1. We have (again $n = 2$)

$$\Lambda = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad 0 < \varphi_1 < \varphi_2 < \pi.$$

Furthermore, δU is

$$\delta U = \begin{bmatrix} \begin{bmatrix} 0 & -\delta u_{21} \\ \delta u_{21} & 0 \end{bmatrix} & \begin{bmatrix} -\delta u_{31} & -\delta u_{41} \\ -\delta u_{32} & -\delta u_{42} \end{bmatrix} & \begin{bmatrix} -\delta u_{51} \\ -\delta u_{52} \end{bmatrix} \\ \begin{bmatrix} \delta u_{31} & \delta u_{32} \\ \delta u_{41} & \delta u_{42} \end{bmatrix} & \begin{bmatrix} 0 & -\delta u_{43} \\ \delta u_{43} & 0 \end{bmatrix} & \begin{bmatrix} -\delta u_{53} \\ -\delta u_{54} \end{bmatrix} \\ \begin{bmatrix} \delta u_{51} & \delta u_{52} \end{bmatrix} & \begin{bmatrix} \delta u_{53} & \delta u_{54} \end{bmatrix} & 0 \end{bmatrix}.$$

In $\Lambda^{-1}\,\delta U\,\Lambda - \delta U$, comparing to the case case $O^+(2n)$, the new pieces are on the last row. We have

$$(\delta u_{51}, \delta u_{52}) \begin{pmatrix} c_1 - 1 & s_1 \\ -s_1 & c_1 - 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} c - 1 & s \\ -s & c - 1 \end{pmatrix} \begin{pmatrix} \delta u_{51} \\ \delta u_{52} \end{pmatrix},$$

from where the denominator is $2(1 - \cos\varphi_1)$. The result now is

$$\int\limits_{O^+(2n+1)} f(A)\mathrm{d}Haar = \int\limits_{O^+(2n+1)} f(A)\delta A = \int\limits_{O^+(2n+1)} f(A)\delta B =$$

$$= \left(\frac{1}{n!} \int\limits_{[0,\pi]^n} f(1, \varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j 2(1 - \cos\varphi_j) \prod_j \mathrm{d}\varphi_j \right) \cdot Vol(U).$$

and cheating with a constant,

$$\boxed{\begin{aligned} &\int\limits_{O^+(2n+1)} f(A)\mathrm{d}Haar = \int\limits_{O^+(2n+1)} f(A)\delta A = \int\limits_{O^+(2n+1)} f(A)\delta B = \\ &= \left(\frac{1}{n!} \int\limits_{[0,\pi]^n} f(1, \varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j 2(1 - \cos\varphi_j) \prod_j \frac{\mathrm{d}\varphi_j}{2\pi} \right). \end{aligned}}$$

1.5 $O^-(2n+1)$.

In this case the representation Λ will have -1 instead of 1 in the last row. The rest is the same as for $O^+(2n+1)$.

$$\boxed{\begin{aligned} &\int\limits_{O^-(2n+1)} f(A)\mathrm{d}Haar = \\ &= \left(\frac{1}{n!} \int\limits_{[0,\pi]^n} f(-1, \varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j 2(1 - \cos\varphi_j) \prod_j \frac{\mathrm{d}\varphi_j}{2\pi} \right). \end{aligned}}$$

All the wrapped formulas coincide with Katz, Sarnak, Chapter 5, pages 107-108, up to a constant, with the difference that he divides all $\mathrm{d}\varphi$ by 2π, and discard $Vol(U)$. Indeed, the latter will cancel if we want our integral be normalized.

2 Meet Andreeiv.

Now a general statement,

$$\frac{1}{N!}\int\limits_{X^N} \det(g_k(\theta_j))_{k,j=1}^N \cdot \det(h_k(\theta_j))_{k,j=1}^N \cdot \prod_{j=1}^N \mathrm{d}\sigma(\theta_j) = \det\left[\int\limits_X g_k(\theta)h_j(\theta)\mathrm{d}\sigma(\theta)\right]_{j,k=1}^N.$$

2.1 U(N).