1

Literature: Weyl 1946 Classical groups, pages 177–229;
Muirhead 1982, Aspects of multivariate statistical theory, chapter 2, pages 50–72;
Katz, Sarnak, Chapter 5, pages 107–121;

1.1 U(n).

For a matrix A, denote

$$\delta A = A^{-1}\mathrm{d}A,$$

whatever this means.

For $A \in U(n)$, represent

$$A = U\,\Lambda\,U^*, \tag{1.1}$$

where

$$\Lambda = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\varphi_1}, \ldots, \mathrm{e}^{\mathrm{i}\varphi_n}), \quad 0 < \varphi_1 < \ldots < \varphi_n < 2\pi, \tag{1.2}$$

and $H^* \in U(n)$ is fixed by the condition that the in every column, the first (from up down over rows) non-zero column is positive. (Otherwise representation (1.1) is not unique, $U \to UD$ with diagonal D). Furthermore, U is parametrized by $n^2 - n = l$ real parameters, Λ is parametrized by n parameters. The representation (1.1) is unique for $A \in U(n)$ with all different eigenvalues, and hence we can consider it as a change of variables. Now, since $UU^* = \mathbf{1}$, and hence $\delta U = -\delta U^*$ is skew-symmetric, we obtain by formal manipulations

$$\delta B \;:=\; \qquad U^{-1}\;\delta A\;U = (\Lambda^{-1}\;\delta U\;\Lambda - \delta U) + \delta\Lambda. \tag{1.3}$$

From (1.3), considering separately diagonal and off-diagonal teerms,

$$\begin{aligned}
\delta b_{jj} &= \mathrm{i}\mathrm{d}\varphi_j,\\
\delta b_{jk} &= (\mathrm{e}^{\mathrm{i}(\varphi_k - \varphi_j)} - 1)\;\delta u_{jk}, \quad j \neq k.
\end{aligned}$$

Now, after all matrix multiplications, we write δB in a row of n^2 real elements:

$$\delta b_{11}, \ldots \delta b_{nn}, \delta b_{12}, \delta b_{21}, \ldots, \delta b_{n-1,n}, \delta b_{n,n-1},$$

and take a (wedge) product of them (as of differentials; we obtain an n^2 dimensional differential). Writing δB as a (scalar rather than matrix) n^2 dimensional differential, and ignoring sign,

$$\delta B = \prod_{j=1}^{n} \mathrm{d}\varphi_j \;\prod_{j\neq k}(\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k})\;\delta b_{j,k} = \prod_{j\neq k}(\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k})\prod_{j=1}^{n}\mathrm{d}\varphi_j \;\; \mathrm{d}\omega_U.$$

Now, when we integrate over $U(n)$ (or average over group) a function that depends only on eigenvalues, first of all we integrate only over matrices A with all distinct eigenvalues (other A form a sub-manifold of smaller dimension, and hence do not contribute in the integral)

$$\begin{aligned}
\int_{U(n)} f(A)\mathrm{d}Haar &= \int_{U(n)} f(A)A^{-1}\mathrm{d}A = \int_{U(n)} f(A)\delta A = \int_{U(n)} f(A)\delta B = \\
&= \int_{U(n)} f(\varphi_1, \ldots, \varphi_n)\prod_{j\neq k}(\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k})\prod_{j=1}^{n}\mathrm{d}\varphi_j \;\; \mathrm{d}\omega_U = \qquad ,\\
&= \left(\frac{1}{n!}\int_{[0,2\pi]^n} f(\varphi_1, \ldots, \varphi_n)\prod_{j\neq k}|\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k}|\prod_{j=1}^{n}\mathrm{d}\varphi_j\right)\cdot Vol(U)
\end{aligned}$$

where we separated contribution in the integral coming from φ_j, and from U. The factor $n!$ appeared since we changed integral from over $0 < \varphi_1 < \ldots < \varphi_n < 2\pi$ to over $\varphi_j \in [0, 2\pi]$. To have it normalized to 1, probably when we integrate over $\mathrm{d}\varphi$ over $[0, 2\pi]$, it is better to divide by 2π. However, Katz Sarnak, chapter 5, divide by 2π not only in the case $U(n)$, when integrate over $[0, 2\pi]$, but also in the cases $O(2n)$, when they integrate over $[0, \pi]$, but still divide by 2π. Furthermore, normalizing, we need to divide by analogous expression with $f = 1$. In that way we get rid of $Vol(U)$.

$$\boxed{\int_{U(n)} f(A)\mathrm{d}Haar = \left(\frac{1}{n!} \int_{[0,2\pi]^n} f(\varphi_1, \ldots, \varphi_n) \prod_{j \neq k} |\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k}| \prod_{j=1}^{n} \frac{\mathrm{d}\varphi_j}{2\pi} \right). \qquad (1.4)}$$

1.2 $O^+(2n)$.

We will take $n = 2$, so we consider $O^+(4)$. Here (1.1) and (1.3) still hold,

$$A = U\Lambda U^*, \qquad \delta B := \quad U^{-1}\,\delta A\, U = (\Lambda^{-1}\,\delta U\,\Lambda - \delta U) + \delta\Lambda,$$

but Λ changes: instead of (1.2), we have

$$\Lambda = \begin{bmatrix} \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} \cos\varphi_2 & -\sin\varphi_2 \\ \sin\varphi_2 & \cos\varphi_2 \end{bmatrix} \end{bmatrix}, \qquad 0 < \varphi_1 < \varphi_2 < \pi. \tag{1.5}$$

Again, U should be fixed somehow to ensure uniqueness. It is parametrized by

$$\frac{2n(2n-1)}{2} - n = 2n(n-1)$$

parameters. Λ is parametrized by n real parameters. What is δU? It is a skew-symmetric matrix, parametrized by low-diagonal elements, whose are $2n(2n-1)/2$, where we need to subtract n, since we fixed somehow U. We have

$$\delta U = \begin{bmatrix} \begin{bmatrix} 0 & -\delta u_{21} \\ \delta u_{21} & 0 \end{bmatrix} & \begin{bmatrix} -\delta u_{31} & -\delta u_{41} \\ -\delta u_{32} & -\delta u_{42} \end{bmatrix} \\ \begin{bmatrix} \delta u_{31} & \delta u_{32} \\ \delta u_{41} & \delta u_{42} \end{bmatrix} & \begin{bmatrix} 0 & -\delta u_{43} \\ \delta u_{43} & 0 \end{bmatrix} \end{bmatrix}, \quad \text{where now the scalar meaning is } \delta U = \prod_{j>k} \delta u_{jk}.$$

What is $\Lambda^{-1}\,\delta U\,\Lambda$? But first, what is

$$\delta\Lambda = \Lambda^{-1}\,\mathrm{d}\Lambda = \begin{bmatrix} \begin{bmatrix} 0 & -\mathrm{d}\varphi_1 \\ \mathrm{d}\varphi_1 & 0 \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & -\mathrm{d}\varphi_2 \\ \mathrm{d}\varphi_2 & 0 \end{bmatrix} \end{bmatrix}, \quad \text{with the scalar meaning} \quad \delta\Lambda = \mathrm{d}\varphi_1\mathrm{d}\varphi_2.$$

Second, diagonal block terms in $\Delta^{-1}\,\delta U\,\Delta$ are conjugated by the same φ :

$$\begin{bmatrix} \cos\varphi_1 & \sin\varphi_1 \\ -\sin\varphi_1 & \cos\varphi_1 \end{bmatrix} \begin{bmatrix} 0 & -\delta u_{21} \\ \delta u_{21} & 0 \end{bmatrix} \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} = \begin{bmatrix} 0 & -\delta u_{21} \\ \delta u_{21} & 0 \end{bmatrix}, \quad \text{with meaning } \delta_{21}.$$

Third, off-diagonal block terms in $\Delta^{-1}\,\delta U\,\Delta$ are conjugated by different φ :

$$\begin{bmatrix} \cos\varphi_2 & \sin\varphi_2 \\ -\sin\varphi_2 & \cos\varphi_2 \end{bmatrix} \begin{bmatrix} \delta u_{31} & \delta u_{32} \\ \delta u_{41} & \delta u_{42} \end{bmatrix} \begin{bmatrix} \cos\varphi_1 & -\sin\varphi_1 \\ \sin\varphi_1 & \cos\varphi_1 \end{bmatrix} - \begin{bmatrix} \delta u_{31} & \delta u_{32} \\ \delta u_{41} & \delta u_{42} \end{bmatrix} =: \begin{bmatrix} \delta v_1 & \delta v_2 \\ \delta v_3 & \delta v_4 \end{bmatrix}. \tag{1.6}$$

The scalar meaning of $\begin{bmatrix} \delta v_1 & \delta v_2 \\ \delta v_3 & \delta v_4 \end{bmatrix}$ itself is just the (wedge) product of all the elements: $\delta v_1\,\delta v_2\,\delta v_3\,\delta v_4$. Formula (1.6) can be rewritten, elementwise, as (we denote for shortness $c = \cos, s = \sin$)

$$\begin{bmatrix} \begin{bmatrix} c_1c_2 - 1 & s_1c_2 \\ -s_1c_2 & c_1c_2 - 1 \end{bmatrix} & \begin{bmatrix} c_1s_2 & s_1s_2 \\ -s_1s_2 & c_1s_2 \end{bmatrix} \\ \begin{bmatrix} -c_1s_2 & -s_1s_2 \\ s_1s_2 & -c_1s_2 \end{bmatrix} & \begin{bmatrix} c_1c_2 - 1 & s_1c_2 \\ -s_1c_2 & c_1c_2 - 1 \end{bmatrix} \end{bmatrix} \cdot \begin{bmatrix} \delta u_{31} \\ \delta u_{32} \\ \delta u_{41} \\ \delta u_{42} \end{bmatrix}.$$

We need to take a covariant product of those differential forms, or, equivalently, take the determinant of the matrix, which equals

$$\det = 4(\cos\varphi_1 - \cos\varphi_2)^2.$$

Hence, the contribution from off-diagonal blocks are

$$4(\cos\varphi_j - \cos\varphi_k)^2 \prod \delta u_{...},$$

and the contribution from each diagonal block is

$$\delta u_{...}.$$

Finally,

$$\int\limits_{O^+(2n)} f(A)\mathrm{d}Haar = \int\limits_{O^+(2n)} f(A)\delta A = \int\limits_{O^+(2n)} f(A)\delta B =$$
$$= \left(\frac{1}{n!} \int\limits_{[0,\pi]^n} f(\varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j \mathrm{d}\varphi_j \right) \cdot Vol(U).$$

or, cheating with constant, using mathematical packages to check the correct constant for the first several n,

$$\boxed{\int\limits_{O^+(2n)} f(A)\mathrm{d}Haar = \frac{2}{n!} \int\limits_{[0,\pi]^n} f(\varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j \frac{\mathrm{d}\varphi_j}{2\pi}.} \tag{1.7}$$

1.3 $O^-(2n+2)$.

In this case there are 1, -1 among eigenvalues. Let again $n = 2$. Denote by

$$T = \begin{bmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{bmatrix}$$

the block matrix. Then Λ can be parametrized in the form

$$\Lambda = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \end{pmatrix}, \qquad 0 < \varphi_1 < \varphi_2 < \pi.$$

What changes in our fundamental relation (1.3)?

$$\delta B = U^{-1}\,\delta A\,U = (\Lambda^{-1}\,\delta U\,\Lambda - \delta U) + \delta\Lambda.$$

We have

$$\delta U = \begin{bmatrix} \begin{bmatrix} 0 & -\delta u_{21} \\ \delta u_{21} & 0 \end{bmatrix} & \begin{bmatrix} -\delta u_{31} & -\delta u_{41} \\ -\delta u_{32} & -\delta u_{42} \end{bmatrix} & \begin{bmatrix} -\delta u_{51} & -\delta u_{61} \\ -\delta u_{52} & -\delta u_{62} \end{bmatrix} \\ \begin{bmatrix} \delta u_{31} & \delta u_{32} \\ \delta u_{41} & \delta u_{42} \end{bmatrix} & \begin{bmatrix} 0 & -\delta u_{43} \\ \delta u_{43} & 0 \end{bmatrix} & \begin{bmatrix} -\delta u_{53} & -\delta u_{63} \\ -\delta u_{54} & -\delta u_{64} \end{bmatrix} \\ \begin{bmatrix} \delta u_{51} & \delta u_{52} \\ \delta u_{61} & \delta u_{62} \end{bmatrix} & \begin{bmatrix} \delta u_{53} & \delta u_{54} \\ \delta u_{63} & \delta u_{64} \end{bmatrix} & \begin{bmatrix} 0 & -\delta u_{65} \\ \delta u_{65} & 0 \end{bmatrix} \end{bmatrix},$$

where now the scalar meaning is $\delta U = \prod_{j>k} \delta u_{jk}$. New pieces comparing to the previous case are now lying on the last block row (i.e., in last 2 rows).

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \delta u_{51} & \delta u_{52} \\ \delta u_{61} & \delta u_{62} \end{bmatrix} \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} - \begin{bmatrix} \delta u_{51} & \delta u_{52} \\ \delta u_{61} & \delta u_{62} \end{bmatrix},$$

and the latter can be written line-by-line in the form of two decomposed relations, for δ_{51}, δ_{52}, and distinctively for δ_{61}, δ_{62}. We have

$$\begin{pmatrix} c_1 - 1 & s_1 \\ -s_1 & c_1 - 1 \end{pmatrix} \begin{pmatrix} \delta u_{51} \\ \delta u_{52} \end{pmatrix}, \qquad \begin{pmatrix} c_1 + 1 & s_1 \\ -s_1 & c_1 + 1 \end{pmatrix} \begin{pmatrix} \delta u_{61} \\ \delta u_{62} \end{pmatrix}$$

and the determinant equals

$$(2 - 2\cos\varphi_1)(2 + 2\cos\varphi_1) = 4\sin^2\varphi_1.$$

Finally,

$$\int\limits_{O^-(2n+2)} f(A)\mathrm{d}Haar = \int\limits_{O-(2n+2)} f(A)\delta A = \int\limits_{O^-(2n+2)} f(A)\delta B =$$
$$= \left(\frac{1}{n!} \int\limits_{[0,\pi]^n} f(1, -1, \varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j 4\sin^2\varphi_j \prod_j \mathrm{d}\varphi_j \right) \cdot Vol(U).$$

and, cheating with constant,

$$\int\limits_{O^-(2n+2)} f(A)\mathrm{d}Haar = \int\limits_{O-(2n+2)} f(A)\delta A = \int\limits_{O^-(2n+2)} f(A)\delta B =$$
$$= \left(\frac{1}{n!} \int\limits_{[0,\pi]^n} f(1, -1, \varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j 4\sin^2\varphi_j \prod_j \frac{\mathrm{d}\varphi_j}{2\pi} \right). \tag{1.8}$$

1.4 $O^+(2n+1)$.

In this case there is an eigenvalue 1. We have (again $n=2$)

$$\Lambda = \begin{pmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad 0 < \varphi_1 < \varphi_2 < \pi.$$

Furthermore, δU is

$$\delta U = \begin{bmatrix} \begin{bmatrix} 0 & -\delta u_{21} \\ \delta u_{21} & 0 \end{bmatrix} & \begin{bmatrix} -\delta u_{31} & -\delta u_{41} \\ -\delta u_{32} & -\delta u_{42} \end{bmatrix} & \begin{bmatrix} -\delta u_{51} \\ -\delta u_{52} \end{bmatrix} \\ \begin{bmatrix} \delta u_{31} & \delta u_{32} \\ \delta u_{41} & \delta u_{42} \end{bmatrix} & \begin{bmatrix} 0 & -\delta u_{43} \\ \delta u_{43} & 0 \end{bmatrix} & \begin{bmatrix} -\delta u_{53} \\ -\delta u_{54} \end{bmatrix} \\ \begin{bmatrix} \delta u_{51} & \delta u_{52} \end{bmatrix} & \begin{bmatrix} \delta u_{53} & \delta u_{54} \end{bmatrix} & 0 \end{bmatrix}.$$

In $\Lambda^{-1}\,\delta U\,\Lambda - \delta U$, comparing to the case case $O^+(2n)$, the new pieces are on the last row. We have

$$(\delta u_{51}, \delta u_{52}) \begin{pmatrix} c_1 - 1 & s_1 \\ -s_1 & c_1 - 1 \end{pmatrix}, \quad \text{or} \quad \begin{pmatrix} c-1 & s \\ -s & c-1 \end{pmatrix} \begin{pmatrix} \delta u_{51} \\ \delta u_{52} \end{pmatrix},$$

from where the denominator is $2(1-\cos\varphi_1)$. The result now is

$$\int\limits_{O^+(2n+1)} f(A)\mathrm{d}Haar = \int\limits_{O^+(2n+1)} f(A)\delta A = \int\limits_{O^+(2n+1)} f(A)\delta B =$$

$$= \left(\frac{1}{n!} \int\limits_{[0,\pi]^n} f(1, \varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j 2(1-\cos\varphi_j) \prod_j \mathrm{d}\varphi_j \right) \cdot Vol(U).$$

and cheating with a constant,

$$\int\limits_{O^+(2n+1)} f(A)\mathrm{d}Haar = \int\limits_{O^+(2n+1)} f(A)\delta A = \int\limits_{O^+(2n+1)} f(A)\delta B =$$

$$= \left(\frac{1}{n!} \int\limits_{[0,\pi]^n} f(1, \varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j 2(1-\cos\varphi_j) \prod_j \frac{\mathrm{d}\varphi_j}{2\pi} \right). \tag{1.9}$$

1.5 $O^-(2n+1)$.

In this case the representation Λ will have -1 instead of 1 in the last row. The rest is the same as for $O^+(2n+1)$.

$$\int\limits_{O^-(2n+1)} f(A)\mathrm{d}Haar =$$

$$= \left(\frac{1}{n!} \int\limits_{[0,\pi]^n} f(-1, \varphi_1, ..., \varphi_n) \prod_{j<k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j 2(1-\cos\varphi_j) \prod_j \frac{\mathrm{d}\varphi_j}{2\pi} \right). \tag{1.10}$$

All the wrapped formulas coincide with Katz, Sarnak, Chapter 5, pages 107-108, up to a constant, with the difference that he divides all $\mathrm{d}\varphi$ by 2π, and discard $Vol(U)$. Indeed, the latter will cancel if we want our integral be normalized. In the case $O^+(2n)$, Katz Sarnak also multiply by 2.

2 Meet Andreeiv.

Now a general statement,

$$\frac{1}{N!}\int\limits_{X^N} \det(g_k(\theta_j))_{k,j=1}^N \cdot \det(h_k(\theta_j))_{k,j=1}^N \cdot \prod_{j=1}^N \mathrm{d}\sigma(\theta_j) = \det\left[\int\limits_X g_k(\theta)h_j(\theta)\mathrm{d}\sigma(\theta)\right]_{j,k=1}^N .$$

2.1 U(N).

In formula (1.4), set

$$f(A) = \prod_j f(\mathrm{e}^{\mathrm{i}\varphi_j}), \qquad g_k(\varphi_j) = \mathrm{e}^{\mathrm{i}\varphi_j(k-1)}, \quad h_k(\varphi_j) = \mathrm{e}^{-\mathrm{i}\varphi_j(k-1)}, \quad \mathrm{d}\sigma(\varphi) = f(\mathrm{e}^{\mathrm{i}\varphi})\frac{\mathrm{d}\varphi}{2\pi},$$

and since

$$g_k(\varphi)h_j(\varphi)\mathrm{d}\sigma(\varphi) = \mathrm{e}^{\mathrm{i}\varphi(k-j)}f(\mathrm{e}^{\mathrm{i}\varphi})\frac{\mathrm{d}\varphi}{2\pi}, \qquad \int\limits_0^{2\pi} g_k(\varphi)h_j(\varphi)\mathrm{d}\sigma(\varphi) = \int\limits_0^{2\pi} \mathrm{e}^{\mathrm{i}\varphi(k-j)}f(\mathrm{e}^{\mathrm{i}\varphi})\frac{\mathrm{d}\varphi}{2\pi} =: m_{k-j},$$

and (1.4) equals

$$\int\limits_{U(n)} f(A)\mathrm{d}Haar = \left(\frac{1}{n!}\int\limits_{[0,2\pi]^n} f(\varphi_1,\dots,\varphi_n)\prod_{j\neq k}|\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k}|\prod_{j=1}^n \frac{\mathrm{d}\varphi_j}{2\pi}\right) = \det(m_{k-j})_{j,k=1}^n.$$

2.2 $O^+(2n)$.

In (1.7), we set (then we can integrate over $[0,2\pi]$, dividing the integral by 2.)

$$f(A) = \prod_{j=1}^n f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi}),$$

and denote (pay attention to π or 2π in two different integrals)

$$d_j := \int\limits_0^{\pi} (2\cos\varphi)^j f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi})\frac{\mathrm{d}\varphi}{2\pi}, \qquad m_j := \int\limits_0^{2\pi} \mathrm{e}^{\mathrm{i}j\varphi} f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi})\frac{\mathrm{d}\varphi}{2\pi}.$$

Formula (1.7) can be written in 2 different ways to meet Andreeiv's requirements. First of all,

$$\begin{aligned}
&\int\limits_{O^+(2n)} f(A)\mathrm{d}Haar = \frac{2}{n!}\int\limits_{[0,\pi]^n}\prod_{j>k}4(\cos\varphi_j-\cos\varphi_k)^2\prod_j f(\mathrm{e}^{\mathrm{i}\varphi_j})f(\mathrm{e}^{-\mathrm{i}\varphi_j})\frac{\mathrm{d}\varphi}{2\pi} = \\
&= \frac{2}{n!}\int\limits_{[0,\pi]^n} \det\left[(2\cos\varphi_j)^{k-1}\right]_{j,k=1}^n\cdot\det\left[(2\cos\varphi_j)^{k-1}\right]_{j,k=1}^n\prod_j f(\mathrm{e}^{\mathrm{i}\varphi_j})f(\mathrm{e}^{-\mathrm{i}\varphi_j})\frac{\mathrm{d}\varphi}{2\pi} = \\
&= 2\det\left[\int\limits_0^\pi(2\cos\varphi)^{k+j-2}f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi})\frac{\mathrm{d}\varphi}{2\pi}\right]_{j,k=1}^n = 2\det\left[d_{k+j-2}\right]_{j,k=1}^n.
\end{aligned} \tag{2.11}$$

There is also another way to simplify (1.7), namely, to write the kernel as a product of 2 other determinants. Namely (we denote below $z_j=\mathrm{e}^{\mathrm{i}\varphi_j}$),

$$\begin{aligned}
&\prod_{j>k}2(\cos\varphi_j-\cos\varphi_k) = \prod_{j>k}\left(z_j+z_j^{-1}-z_k-z_k^{-1}\right) = |\text{Vandermonde}| = \\
&= \det\begin{pmatrix}
1 & 1 & \dots & 1\\
z_1+\frac{1}{z_1} & z_2+\frac{1}{z_2} & \dots & z_n+\frac{1}{z_n}\\
\left(z_1+\frac{1}{z_1}\right)^2 & \left(z_2+\frac{1}{z_2}\right)^2 & \dots & \left(z_n+\frac{1}{z_n}\right)^2\\
\dots & \dots & \dots & \dots\\
\left(z_1+\frac{1}{z_1}\right)^{n-1} & \left(z_2+\frac{1}{z_2}\right)^{n-1} & \dots & \left(z_n+\frac{1}{z_n}\right)^{n-1}
\end{pmatrix} = \\
&= \det\begin{pmatrix}
1 & 1 & \dots & 1\\
z_1+\frac{1}{z_1} & z_2+\frac{1}{z_2} & \dots & z_n+\frac{1}{z_n}\\
z_1^2+\frac{1}{z_1^2} & z_2^2+\frac{1}{z_2^2} & \dots & z_n^2+\frac{1}{z_n^2}\\
\dots & \dots & \dots & \dots\\
z_1^{n-1}+\frac{1}{z_1^{n-1}} & z_2^{n-1}+\frac{1}{z_2^{n-1}} & \dots & z_n^{n-1}+\frac{1}{z_n^{n-1}}
\end{pmatrix} = \\
&= \frac{1}{2}\det((\mathrm{e}^{\mathrm{i}(k-1)\varphi_j}+\mathrm{e}^{-\mathrm{i}(k-1)\varphi_j}))_{j,k=1}^n
\end{aligned}$$

One half is because the first raw in the latter determinant is 2 instead of 1. Now we have for (1.7) (we change the path of integration to $[0,2\pi]$, and hence divide by 2):

$$\begin{aligned}
&\int\limits_{O^+(2n)} f(A)\mathrm{d}A = \frac{2}{n!}\int\limits_{[0,2\pi]^n}\prod_{j>k}4(\cos\varphi_j-\cos\varphi_k)^2\prod_j f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi})\frac{\mathrm{d}\varphi}{4\pi} = \\
&= \frac{1}{2n!}\int\limits_{[0,2\pi]^n}\det\left[\mathrm{e}^{\mathrm{i}(k-1)\varphi_j}+\mathrm{e}^{-\mathrm{i}(k-1)\varphi_j}\right]_{j,k=1}^n\cdot\det\left[\mathrm{e}^{\mathrm{i}(k-1)\varphi_j}+\mathrm{e}^{-\mathrm{i}(k-1)\varphi_j}\right]_{j,k=1}^n\prod_j f(\mathrm{e}^{\mathrm{i}\varphi_j})f(\mathrm{e}^{-\mathrm{i}\varphi_j})\frac{\mathrm{d}\varphi}{4\pi} = \\
&= \frac{1}{2}\det\left[\int\limits_0^{2\pi}\left(\mathrm{e}^{\mathrm{i}(k-1)\varphi}+\mathrm{e}^{-\mathrm{i}(k-1)\varphi}\right)\cdot\left(\mathrm{e}^{\mathrm{i}(j-1)\varphi}+\mathrm{e}^{-\mathrm{i}(j-1)\varphi}\right)\cdot f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi})\frac{\mathrm{d}\varphi}{4\pi}\right]_{j,k=1}^n = \\
&= \frac{1}{2}\det\left[\int\limits_0^{2\pi}\left(\mathrm{e}^{\mathrm{i}(k+j-2)\varphi}+\mathrm{e}^{-\mathrm{i}(k+j-2)\varphi}+\mathrm{e}^{\mathrm{i}(j-k)\varphi}+\mathrm{e}^{\mathrm{i}(k-j)\varphi}\right)\cdot f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi})\frac{\mathrm{d}\varphi}{4\pi}\right]_{j,k=1}^n = \\
&= \frac{1}{2}\det\left[m_{k+j-2}+m_{k-j}\right]_{j,k=1}^n.
\end{aligned} \tag{2.12}$$

Now we set

$$g_k(\varphi_j) = h_k(\varphi_j) = \mathrm{e}^{\mathrm{i}(k-1)\varphi_j} + \mathrm{e}^{-\mathrm{i}(k-1)\varphi_j},$$

hence

$$g_k(\varphi)h_j(\varphi) = (\mathrm{e}^{\mathrm{i}(k-1)\varphi} + \mathrm{e}^{-\mathrm{i}(k-1)\varphi})(\mathrm{e}^{\mathrm{i}(j-1)\varphi} + \mathrm{e}^{-\mathrm{i}(j-1)\varphi})$$

There is another way, highly non-trivial and non-clear. Another cheating. Try to reverse (44)-(46) from Garcia, Tiers, 19. Observe that

$$\left(x + \frac{1}{x} - y - \frac{1}{y}\right)^2 = \left(\sqrt{\frac{x}{y}} - \sqrt{\frac{y}{x}}\right)^2 \left(\sqrt{xy} - \frac{1}{\sqrt{xy}}\right)^2 = \left(2 - \frac{x}{y} - \frac{y}{x}\right)\left(2 - xy - \frac{1}{xy}\right) =$$
$$= (x-y)\left(\frac{1}{x} - \frac{1}{y}\right)(1-xy)\left(1 - \frac{1}{xy}\right)$$

and so

$$4(\cos\varphi_j - \cos\varphi_k)^2 = \left[\left(\mathrm{e}^{\mathrm{i}\varphi_j} - \mathrm{e}^{\mathrm{i}\varphi_k}\right)\left(1 - \mathrm{e}^{\mathrm{i}\varphi_j}\mathrm{e}^{\mathrm{i}\varphi_k}\right)\right] \cdot \left[\left(\mathrm{e}^{-\mathrm{i}\varphi_j} - \mathrm{e}^{-\mathrm{i}\varphi_k}\right)\left(1 - \mathrm{e}^{-\mathrm{i}\varphi_j}\mathrm{e}^{-\mathrm{i}\varphi_k}\right)\right].$$

Now let us consider the following determinants:

$$W_a^{(1)}(z_1, \ldots, z_N) = \begin{bmatrix} z_1^a - z_1^{-a} & \ldots & z_N^a - z_N^{-a} \\ z_1^{a+1} - z_1^{-a-1} & \ldots & z_N^{a+1} - z_N^{-a-1} \\ \ldots & \ldots & \ldots \\ z_1^{a+N-1} - z_1^{-a-N+1} & \ldots & z_N^{a+N-1} - z_N^{-a-N+1} \end{bmatrix}$$
$$W_a^{(2)}(z_1, \ldots, z_N) = \begin{bmatrix} z_1^a + z_1^{-a} & \ldots & z_N^a + z_N^{-a} \\ z_1^{a+1} + z_1^{-a-1} & \ldots & z_N^{a+1} + z_N^{-a-1} \\ \ldots & \ldots & \ldots \\ z_1^{a+N-1} + z_1^{-a-N+1} & \ldots & z_N^{a+N-1} + z_N^{-a-N+1} \end{bmatrix}$$

2.3 $O^-(2n+2)$

In (1.8), we put (and hence the integral over $[0,\pi]$ is a half of the integral over $[0, 2\pi]$)

$$f(A) = f(1)f(-1)\prod_j f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi}),$$

and then

$$\int\limits_{O^-(2n+2)} f(A)\mathrm{d}Haar = \frac{f(1)f(-1)}{n!} \int\limits_{[0,\pi]^n} \prod_{j>k} 4(\cos\varphi_j - \cos\varphi_k)^2 \prod_j 4\sin^2\varphi_j \cdot f(\mathrm{e}^{\varphi_j})f(\mathrm{e}^{-\varphi_j})\frac{\mathrm{d}\varphi_j}{2\pi} =$$
$$= f(1)f(-1)\frac{1}{n!}\int\limits_{[0,\pi]^n} \left\{\det\left[(2\cos\varphi_j)^{k-1}\right]_{j,k=1}^n\right\}^2 \prod_j 4\sin^2\varphi_j \cdot f(\mathrm{e}^{\varphi_j})f(\mathrm{e}^{-\varphi_j})\frac{\mathrm{d}\varphi}{2\pi} =$$
$$= f(1)f(-1)\det\left[\int\limits_0^\pi (2\cos\varphi)^{j+k-2} \cdot 4\sin^2\varphi \cdot f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi})\frac{\mathrm{d}\varphi}{2\pi}\right]_{j,k=1}^n. \tag{2.13}$$

To have another expression, we again need to represent the kernel as the product of two determinants in another way. The square root of the kernel is (we will multiply the integral by $(-1)^n$ because of that i)

$$\prod_{j>k} 2(\cos\varphi_j - \cos\varphi_k)\cdot\prod_j (2\mathrm{i}\sin\varphi_j) = \prod_{j>k}(z_j + z_j^{-1} - z_k - z_k^{-1})\prod_j (z_j - z_j^{-1}) =$$

$$= \prod_j (z_j - z_j^{-1})\cdot\det\begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1+z_1^{-1} & z_2+z_2^{-1} & \dots & z_n+z_n^{-1} \\ (z_1+z_1^{-1})^2 & (z_2+z_2^{-1})^2 & \dots & (z_n+z_n^{-1})^2 \\ \dots & \dots & \dots & \dots \\ (z_1+z_1^{-1})^{n-1} & (z_2+z_2^{-1})^{n-1} & \dots & (z_n+z_n^{-1})^{n-1} \end{bmatrix} ==$$

$$= \prod_j (z_j - z_j^{-1})\cdot\det\begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1+z_1^{-1} & z_2+z_2^{-1} & \dots & z_n+z_n^{-1} \\ z_1^2+z_1^{-2} & z_2^2+z_2^{-2} & \dots & z_n^2+z_n^{-2} \\ \dots & \dots & \dots & \dots \\ z_1^{n-1}+z_1^{-n+1} & z_2^{n-1}+z_2^{-n+1} & \dots & z_n^{n-1}+z_n^{-n+1} \end{bmatrix} =$$

now we can add any symmetric part to $z_j^{k-1}+z_j^{-k+1}$, to obtain $z_j^{k-1}+c_{j,k}z_j^{k-3}+\ldots+c_{j,k}z_j^{-k+3}+z_j^{-k+1}$

$$= \prod_j (z_j - z_j^{-1})\cdot\det\begin{bmatrix} 1 & 1 & \dots & 1 \\ z_1+z_1^{-1} & z_2+z_2^{-1} & \dots & z_n+z_n^{-1} \\ z_1^2+z_1^{-2} & z_2^2+z_2^{-2} & \dots & z_n^2+z_n^{-2} \\ \dots & \dots & \dots & \dots \\ z_1^{n-1}+z_1^{-n+1} & z_2^{n-1}+z_2^{-n+1} & \dots & z_n^{n-1}+z_n^{-n+1} \end{bmatrix} =$$

$$= \det\begin{bmatrix} z_1-z_1^{-1} & z_2-z_2^{-1} & \dots & z_n-z_n^{-1} \\ z_1^2-z_1^{-2} & z_2^2-z_2^{-2} & \dots & z_n^2-z_n^{-2} \\ z_1^3-z_1^{-3} & z_2^3-z_2^{-3} & \dots & z_n^3-z_n^{-3} \\ \dots & \dots & \dots & \dots \\ z_1^n-z_1^{-n} & z_2^n-z_2^{-n} & \dots & z_n^n-z_n^{-n} \end{bmatrix} = \det\left[(z_j^k - z_j^{-k})\right]_{j,k=1}^n .$$

Formula (1.8) now can be rewritten as (we change integration from $[0,\pi]$ to $[0,2\pi]$)

$$\int\limits_{O^-(2n+2)} f(A)\mathrm{d}Haar = \frac{f(1)f(-1)}{n!}\int\limits_{[0,2\pi]}\prod_{j>k}4(\cos\varphi_j-\cos\varphi_k)^2\prod_j 4\sin^2\varphi_j\cdot f(\mathrm{e}^{\mathrm{i}\varphi_j})f(\mathrm{e}^{-\mathrm{i}\varphi_j})\frac{\mathrm{d}\varphi}{4\pi} =$$

$$= (-1)^n\frac{f(1)f(-1)}{n!}\int\limits_{[0,2\pi]^n}\left\{\det\left[\mathrm{e}^{\mathrm{i}k\varphi_j}-\mathrm{e}^{-\mathrm{i}k\varphi_j}\right]_{j,k=1}^n\right\}^2\prod_j f(\mathrm{e}^{\mathrm{i}\varphi_j})f(\mathrm{e}^{-\mathrm{i}\varphi_j})\frac{\mathrm{d}\varphi}{4\pi} =$$

$$= (-1)^n f(1)f(-1)\det\left[\int\limits_0^{2\pi}(\mathrm{e}^{\mathrm{i}k\varphi}-\mathrm{e}^{-\mathrm{i}k\varphi})(\mathrm{e}^{\mathrm{i}j\varphi}-\mathrm{e}^{-\mathrm{i}j\varphi})\cdot f(\mathrm{e}^{\mathrm{i}\varphi_j})f(\mathrm{e}^{-\mathrm{i}\varphi_j})\frac{\mathrm{d}\varphi}{4\pi}\right]_{j,k=1}^n =$$

$$= (-1)^n f(1)f(-1)\det\left[m_{k+j}-m_{k-j}\right]_{j,k=1}^n = f(1)f(-1)\det\left[m_{k-j}-m_{k+j}\right]_{j,k=1}^n . \tag{2.14}$$

2.4 $O^\pm(2n+1)$.

We set in (1.9), (1.10)

$$f(A) = f(\pm1)\prod_j f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi}),\qquad g_k(\varphi)=h_k(\varphi)=(2\cos\varphi)^{k-1},\quad \mathrm{d}\sigma(\varphi)=2(1-\cos\varphi)\frac{\mathrm{d}\varphi}{2\pi},$$

hence

$$\int_{O^\pm(2n+1)} f(A)\mathrm{d}Haar = f(\pm1)\cdot\det\left(\int_0^\pi (2\cos\varphi)^{k+j-2} f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi})2(1-\cos\varphi)\frac{\mathrm{d}\varphi}{2\pi}\right)_{j,k=1}^n .$$

For alternative formula, rewrite the kernel in (1.9), (1.10)

$$\int_{O^\pm(2n+1)} f(A)\mathrm{d}Haar = \frac{f(\pm1)}{n!}\int_{[0,\pi]^n}\prod_{j>k}4(\cos\varphi_j-\cos\varphi_k)^2\prod_j 2(1-\cos\varphi_j)\cdot f(\mathrm{e}^{\mathrm{i}\varphi_j})f(\mathrm{e}^{-\mathrm{i}\varphi_j})\frac{\mathrm{d}\varphi_j}{2\pi}$$

in the following way ($z_j=\mathrm{e}^{\mathrm{i}\varphi_j}$, and we get $(-1)^n$ to multiply the integral because of i):

$$\prod_{j>k}2(\cos\varphi_j-\cos\varphi_k)\prod_j 2\mathrm{i}\sin\frac{\varphi_j}{2} = \prod_{j>k}(z_j+z_j^{-1}-z_k-z_k^{-1})\prod_j\left(z_j^{1/2}-z_j^{-1/2}\right) =$$

$$= \prod_j\left(z_j^{1/2}-z_j^{-1/2}\right)\cdot\det\begin{bmatrix} 1 & 1 & \dots & 1\\ z_1+z_1^{-1} & z_2+z_2^{-1} & \dots & z_n+z_n^{-1}\\ (z_1+z_1^{-1})^2 & (z_2+z_2^{-1})^2 & \dots & (z_n+z_n^{-1})^2\\ \dots & \dots & \dots & \dots\\ (z_1+z_1^{-1})^{n-1} & (z_2+z_2^{-1})^{n-1} & \dots & (z_n+z_n^{-1})^{n-1}\end{bmatrix} =$$

$$= \prod_j\left(z_j^{1/2}-z_j^{-1/2}\right)\cdot\det\begin{bmatrix} 1 & 1 & \dots & 1\\ z_1+z_1^{-1} & z_2+z_2^{-1} & \dots & z_n+z_n^{-1}\\ z_1^2+z_1^{-2} & z_2^2+z_2^{-2} & \dots & z_n^2+z_n^{-2}\\ \dots & \dots & \dots & \dots\\ z_1^{n-1}+z_1^{-n+1} & z_2^{n-1}+z_2^{-n+1} & \dots & z_n^{n-1}+z_n^{-n+1}\end{bmatrix} =$$

we can add arbitrary symmetric middle part

$$= \prod_j\left(z_j^{1/2}-z_j^{-1/2}\right)\cdot\det\begin{bmatrix} 1 & 1 & \dots & 1\\ z_1+\ldots+z_1^{-1} & z_2+\ldots+z_2^{-1} & \dots & z_n+\ldots+z_n^{-1}\\ z_1^2+\ldots+z_1^{-2} & z_2^2+\ldots+z_2^{-2} & \dots & z_n^2+\ldots+z_n^{-2}\\ \dots & \dots & \dots & \dots\\ z_1^{n-1}+\ldots+z_1^{-n+1} & z_2^{n-1}+\ldots+z_2^{-n+1} & \dots & z_n^{n-1}+\ldots+z_n^{-n+1}\end{bmatrix} =$$

$$= \det\begin{bmatrix} z_1^{1/2}-z_1^{-1/2} & z_2^{1/2}-z_2^{-1/2} & \dots & z_n^{1/2}-z_n^{-1/2}\\ z_1^{3/2}-z_1^{-3/2} & z_2^{3/2}-z_2^{-3/2} & \dots & z_n^{3/2}-z_n^{-3/2}\\ z_1^{5/2}-z_1^{-5/2} & z_2^{5/2}-z_2^{-5/2} & \dots & z_n^{5/2}-z_n^{-5/2}\\ \dots & \dots & \dots & \dots\\ z_1^{n-1/2}+\ldots+z_1^{-n+1/2} & z_2^{n-1/2}+\ldots+z_2^{-n+1/2} & \dots & z_n^{n-1/2}+\ldots+z_n^{-n+1/2}\end{bmatrix} =$$

$$= \det\left[z_j^{k-1/2}-z_j^{-k+1/2}\right]_{j,k=1}^n .$$

Now formulas (1.9), (1.10) read as

$$\begin{aligned}\int_{O^\pm(2n+1)} f(A)\mathrm{d}Haar &= \frac{(-1)^n f(\pm1)}{n!}\int_{[0,2\pi]^n}\left\{\det\left[\mathrm{e}^{\mathrm{i}(k-1/2)\varphi_j}-\mathrm{e}^{-\mathrm{i}(k-1/2)\varphi_j}\right]_{j,k=1}^n\right\}^2\prod_j f(\mathrm{e}^{\mathrm{i}\varphi_j})f(\mathrm{e}^{-\mathrm{i}\varphi_j})\frac{\mathrm{d}\varphi_j}{4\pi} =\\ &= (-1)^n f(\pm1)\det\left[\int_0^{2\pi}(\mathrm{e}^{\mathrm{i}(k-1/2)\varphi}-\mathrm{e}^{-\mathrm{i}(k-1/2)\varphi})(\mathrm{e}^{\mathrm{i}(j-1/2)\varphi}-\mathrm{e}^{-\mathrm{i}(j-1/2)\varphi})\cdot f(\mathrm{e}^{\mathrm{i}\varphi})f(\mathrm{e}^{-\mathrm{i}\varphi})\frac{\mathrm{d}\varphi}{4\pi}\right]_{j,k=1}^n =\\ &= (-1)^n f(\pm1)\det\left[m_{k+j-1}-m_{k-j}\right]_{j,k=1}^n = f(\pm1)\det\left[m_{k-j}-m_{k+j-1}\right]_{j,k=1}^n .\end{aligned} \tag{2.15}$$