# Analysis of Time Series, L1, comments

March 24, 2020

## slides 5-7

Here we see the global temperature at different time horizons. In slide 5, it seems that the temperaure curve bends upwards in the 70ies, a sign of the global warming.

In slide 6, we can observe the little ice age around the middle of the 1000-2000 millenium. Mount Campito is located in California, close to San Fransisco.

In slide 7, observe that time goes back 800 000 years! In fact, we have had ice ages most of this time!

## slides 8-9

In slide 8, we see a similar trend as for global temperature, and upon this there is a periodicity at about 10 years.

The same periodicity appears in slide 9, but here there is no trend. The sunspot series is used in many time series text books, and we will get back to it in the lectures from time to time.

## slide 9

This is Swedish unemployment. Observe the quarterly season, where unemployment is highest in summer. One might also see the business cycle, about 10 years long.

# slides 10-11

In slide 10, observe the dip at about day 120 (Brexit), which is also seen in slide 11 at the same time spot. If the index is $x_t$, the returns are calculated as

$$r_t = \log\left(\frac{x_t}{x_{t-1}}\right) = \log x_t - \log x_{t-1}.$$

# slide 15

Observe that $x_t = \frac{1}{2}(w_t + w_{t-1})$ and $x_{t-1} = \frac{1}{2}(w_{t-1} + w_{t-2})$. Hence, $x_t$ and $x_{t-1}$ are dependent through $w_{t-1}$. This manifests itself in a smoother curve than for the white noise in slide 14. (The two simulations are built upon the same random seeds.)

# slide 16

Here, we find by recursion that

$$\begin{aligned}
x_t &= x_{t-1} + w_t \\
&= (x_{t-2} + w_{t-1}) + w_t = ... \\
&= x_0 + w_1 + w_2 + ... + w_{t-1} + w_t, \\
x_{t-1} &= x_0 + w_1 + w_2 + ... + w_{t-1},
\end{aligned}$$

so there is a strong dependency between $x_t$ and $x_{t-1}$ through $w_1 + w_2 + ... + w_{t-1}$. This is why the curve here is even smoother than the previous ones. (Again, it is the same random seed.)

# slide 17

For definition 1.2, observe the special case

$$\text{var}(x_t) = \text{cov}(x_t, x_t) = \gamma(t, t).$$

# slide 18

1. We have $\mu_t = E(w_t) = 0$,
$$\gamma(s,t) = \text{cov}(w_s, w_t) = \begin{cases} \sigma_w^2, & \text{if } s = t, \\ 0, & \text{if } s \neq t, \end{cases}$$
and
$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}} = \begin{cases} \frac{\sigma_w^2}{\sqrt{\sigma_w^2\sigma_w^2}} = 1, & \text{if } s = t, \\ \frac{0}{\sqrt{\sigma_w^2\sigma_w^2}} = 0, & \text{if } s \neq t. \end{cases}$$

2. Let $X, Y, U, V$ be random variables, $a, b, c, d$ constants and note the calculation rules
$$\begin{aligned} \mathrm{E}(aX + bY) &= a\mathrm{E}(X) + b\mathrm{E}(Y), \\ \text{cov}(aX + bY, cU + dV) &= ac * \text{cov}(X, U) + ad * \text{cov}(X, V) \\ &\quad + bc * \text{cov}(Y, U) + bd * \text{cov}(Y, V). \end{aligned}$$
With $x_t = \frac{1}{2}w_t + \frac{1}{2}w_{t-1}$, we thus have
$$\mu_t = \mathrm{E}(x_t) = \frac{1}{2}\mathrm{E}(w_t) + \frac{1}{2}\mathrm{E}(w_{t-1}) = 0,$$
$$\begin{aligned} &\gamma(s,t) \\ &= \text{cov}(x_s, x_t) = \text{cov}\left(\frac{1}{2}w_s + \frac{1}{2}w_{s-1}, \frac{1}{2}w_t + \frac{1}{2}w_{t-1}\right) \\ &= \frac{1}{4}\text{cov}(w_s, w_t) + \frac{1}{4}\text{cov}(w_s, w_{t-1}) + \frac{1}{4}\text{cov}(w_{s-1}, w_t) + \frac{1}{4}\text{cov}(w_{s-1}, w_{t-1}) \\ &= \frac{1}{4}\sigma_w^2 I\{s = t\} + \frac{1}{4}\sigma_w^2 I\{s = t-1\} + \frac{1}{4}\sigma_w^2 I\{s-1 = t\} + \frac{1}{4}\sigma_w^2 I\{s-1 = t-1\} \\ &= \frac{1}{2}\sigma_w^2 I\{|s-t| = 0\} + \frac{1}{4}\sigma_w^2 I\{|s-t| = 1\} \\ &= \begin{cases} \sigma_w^2/2 & \text{if } |s-t| = 0, \\ \sigma_w^2/4 & \text{if } |s-t| = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$
where $I\{A\} = 1$ if $A$ is true and zero otherwise.

Hence, $\gamma(t,t) = \sigma_w^2/2$ and

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}} = \begin{cases} \frac{\sigma_w^2/2}{\sqrt{\sigma_w^2/2\sigma_w^2/2}} = 1, & \text{if } |s-t| = 0, \\ \frac{\sigma_w^2/4}{\sqrt{\sigma_w^2/2\sigma_w^2/2}} = 1/2, & \text{if } |s-t| = 1, \\ \frac{0}{\sqrt{\sigma_w^2/2\sigma_w^2/2}} = 0, & \text{otherwise.} \end{cases}$$

3. Write $x_t = w_1 + ... + w_t$. Now,

$$\mu_t = \mathrm{E}(x_t) = \mathrm{E}(w_1) + ... + \mathrm{E}(w_t) = 0.$$

Moreover, take $s \leq t$. Then,

$\gamma(s,t)$
$= \mathrm{cov}(w_1 + ... + w_s, w_1 + ... + w_s + w_{s+1} + ... + w_t)$
$= \mathrm{cov}(w_1 + ... + w_s, w_1 + ... + w_s) + \mathrm{cov}(w_1 + ... + w_s, w_{s+1} + ... + w_t)$
$= \mathrm{var}(w_1 + ... + w_s, w_1 + ... + w_s) + 0$
$= \mathrm{var}(w_1) + ... + \mathrm{var}(w_s) = s\sigma_w^2.$

Hence, $\gamma(s,t) = \min(s,t)\sigma_w^2$, $\gamma(t,t) = \sigma_w^2$, and so,

$$\rho(s,t) = \frac{\min(s,t)\sigma_w^2}{\sqrt{s\sigma_w^2 t\sigma_w^2}} = \frac{\min(s,t)}{\sqrt{st}}.$$

In general,

$$\gamma(s,t) = \mathrm{cov}(x_s, x_t) = \mathrm{cov}(x_t, x_s) = \gamma(t,s),$$

and similarly, $\rho(s,t) = \rho(t,s)$.

## slides 19-21

In these figures, depicted are the estimated autocorrelation functions (ACF). More on estimation later.

Compare to the theoretical ACF, given above as

$$\rho(s,t) = \frac{\gamma(s,t)}{\sqrt{\gamma(s,s)\gamma(t,t)}} = \begin{cases} 1, & \text{if } |s-t| = 0, \\ 1/2, & \text{if } |s-t| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

The point with these figures is to illustrate that for different simulations, the realizations are quite different while the estimated ACFs are fairly similar.

The dotted interval around the x axis in the ACF figures shows $\pm 2/\sqrt{n} = \pm 0.2$. More about this later.

# slide 23

1.

$$
\begin{aligned}
&\gamma_{xy}(s,t)\\
&= \operatorname{cov}\left(\frac{1}{2}w_s + \frac{1}{2}w_{s-1}, w_t\right) = \frac{1}{2}\operatorname{cov}(w_s, w_t) + \frac{1}{2}\operatorname{cov}(w_{s-1}, w_t)\\
&= \frac{\sigma_w^2}{2} I\{s=t\} + \frac{\sigma_w^2}{2} I\{s-1=t\}\\
&= \begin{cases} \sigma_w^2/2, & \text{if } s=t \text{ } or \text{ } s=t+1,\\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
$$

Since $\gamma_x(t,t) = \sigma_w^2/2$ and $\gamma_y(t,t) = \sigma_w^2$, this gives

$$
\rho_{xy}(s,t) = \frac{\gamma_{xy}(s,t)}{\sqrt{\gamma_x(s,s)\gamma_y(t,t)}} = \begin{cases} \frac{\sigma_w^2/2}{\sqrt{\sigma_w^2/2 * \sigma_w^2}} = 1/\sqrt{2}, & \text{if } s=t \text{ } or \text{ } s=t+1,\\ 0 & \text{otherwise.} \end{cases}
$$

2. Similarly,

$$
\gamma_{yx}(s,t) = \operatorname{cov}(y_s, x_t) = \operatorname{cov}(x_t, y_s) = \begin{cases} \sigma_w^2/2, & \text{if } t=s \text{ } or \text{ } t=s+1,\\ 0, & \text{otherwise,} \end{cases}
$$

and

$$
\rho_{yx}(s,t) = \begin{cases} 1/\sqrt{2}, & \text{if } t=s \text{ } or \text{ } t=s+1,\\ 0 & \text{otherwise.} \end{cases}
$$

Hence, these are not identical to the corresponding functions under 1.

In general,

$$
\gamma_{xy}(s,t) = \operatorname{cov}(x_s, y_t) = \operatorname{cov}(y_t, x_s) = \gamma_{yx}(t,s),
$$

and similarly $\rho_{xy}(s,t) = \rho_{yx}(t,s)$.