Numerical methods for solving PDEs
Practicum 1: numerical solution of the heat equation

1 Aims

This practicum has the following **general** aims:

- To learn, by means of a specific model problem, to convert a partial differential equation into a discrete problem and to solve this problem on a computer.
- To get insight in the concepts accuracy and numerical approximation.
- To get insight in the correct implementation and the differences between explicit and implicit difference schemes.

The **specific** aims of this practicum are:

- To learn to implement the explicit Euler scheme, implicit Euler scheme and the Crank-Nicolson scheme.
- To understand the importance of satisfying the stability criteria for explicit schemes.
- To get insight in the relation between the accuracy and the order of consistence of these two schemes.
- To learn to implement (correctly) an explicit difference scheme for a nonlinear PDE.

2 Task

Implement the explicit Euler, implicit Euler en Crank-Nicolson methods for the numerical solution of the one-dimensional, time dependent heat problem:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad \text{with} \quad x \in [0,1], \quad t > 0,$$

with given initial and boundary conditions:

$$u(0,t) = f_1(t), \quad u(1,t) = f_2(t), \quad u(x,0) = g(x).$$

Use an equidistant grid with distance Δx in the $x-$direction, and Δt in the $t-$direction. Write the programme preferentially in Matlab. You can use the provided example code as an inspiration source for a possible programming style.

1. Discuss the tests you applied to check if your code is correct. Include a listing of your code.
2. Consider the problem with $f_1(t) = 0,\ f_2(t) = 1,\ g(x) = \sin(5\pi x/2)$. Compute a numerical approximation to this solution on $t = 0.05$ and $t = 0.5$ with the explicit Euler method. Take $\Delta x = 1/20$ and select a time step Δt that satisfies the stability conditions. Make a plot of the numerical solution $u(x, 0.05)$ and $u(x, 0.5)$. Illustrate the use of a time step that is just above the stability limit.

3. For $f_1(t) = 0$, $f_2(t) = 0$, $g(x) = \sin(\pi x)$, we find as *exact* solution of the PDE
$$\phi(x,t) = \exp^{-\pi^2 t}\sin(\pi x).$$
Determine the numerical approximation of this solution at $t = 0.5$ with the implicit Euler method and the Crank-Nicolson method. Use grid distances $\Delta x = 1/10, 1/20, 1/40, 1/80$ and $\Delta t = 1/20, 1/40, 1/80, 1/160, 1/320, 1/640$. Compare the accuracy of both methods by providing a table with the error for each $(\Delta x, \Delta t)$-combination. Show how the consistency order determines the accuracy.

4. Consider the following nonlinear problem, the viscous Burgers' equation, an important PDE in fluid mechanics to model the onset of a wavefront:
$$\frac{\partial u}{\partial t} = b\frac{\partial^2 u}{\partial x^2} - u\frac{\partial u}{\partial x}\,.$$
Adapt your explicit code to solve this problem. Determine the numerical solution for a constant value $b = 0.005$, and $f_1(t) = 0$, $f_2(t) = 0$, $g(x) = \mathrm{abs}(\sin(2\pi x))$. Illustrate with a few plots of the solution over the time-interval $[0, 1]$.

5. Consider another nonlinear problem, the phase-field equation, a reaction-diffusion PDE from material science that models grain growth in polycrystalline materials:
$$\frac{\partial u}{\partial t} = b\frac{\partial^2 u}{\partial x^2} - (u^3 - u)\,.$$
Adapt your explicit code to solve this problem. Determine the numerical solution for a constant value $b = 10^{-4}$, and $f_1(t) = 0$, $f_2(t) = 0$, $g(x) = \mathrm{sinc}(3\pi x^2)$. Illustrate with a few plots of the solution over the time-interval $[0, 5]$.

3 Modalities

- The *maximal length of the report is 10 pages* (incl. listing and figures).
- The report has to be handed in before 30/11/2017.
- This practicum will be marked and the score is part of the total score for this course. The report will be evaluated on correctness of the implementation, the delivery of the requested results and figures, and the correctness of the interpretation. The score accounts for 2 points (out of the total of 20 for this course).
- It is allowed (if wanted) to collaborate in groups of maximal two students. In this case only one report has to be handed in and both students get the same mark.

4 Help line and feedback

In case of unclearness about the task or the modalities you can contact us for more information. You can also contact us in case you get stuck with the implementation in Matlab. After 30/11/2017 a feedback session will be planned to provide input about the good points and the bad points and possible improvements or suggestions for the next assignment.

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