Formal Analysis of Bitwise Stability in the Prime Resonance Framework

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Abstract

We formalize and rigorously analyze the bitwise stability observed in the *Prime Resonance Framework*, in which an analytic function over primes yields floating-point representations exhibiting unexpected invariance in their binary (IEEE-754) form. We define precise metrics for bitwise stability, prove convergence and stability theorems, and analytically derive the observed empirical law governing stable bits under input perturbations and sequence extension. We distinguish mathematical properties of the resonance structure from generic floating-point artifacts, generalize to other numeric bases and precisions, and outline robust experimental protocols. This work exposes and explains a deep connection between analytic number-theoretic construction and digital computation, with implications for robust encoding and numerical analysis.

1 Introduction

Analytic constructions over the sequence of prime numbers, such as Euler products and resonance functions, occupy a central place in number theory and mathematical physics. Recently, the *Prime Resonance Framework* has revealed an unexpected computational phenomenon: when analytic cascades over primes are computed numerically, the resulting binary representations in IEEE-754 double-precision floating point exhibit remarkable *bitwise stability*—large common prefixes of unchanging bits that persist as the sequence is extended or inputs are perturbed. This paper provides a rigorous foundation for understanding this phenomenon.

We formalize "bitwise stability," derive analytic conditions that guarantee its emergence, and distinguish true resonance structure from effects that are generic to fixed precision arithmetic. We analyze the featured empirical law relating perturbation size to stable bit count, generalize results beyond double-precision binary, and provide recommendations for experimental validation in support of future work. Our results clarify the boundary between mathematical convergence, digital representation, and robust computation.

2 Bitwise Stability: Definitions and Metrics

2.1 IEEE-754 Double-Precision Representation

Definition 2.1 (Binary64 Representation)**.** *An IEEE-754 double-precision (binary64) floating-point number x is encoded as a 64-bit binary string $B(x) = b_1 b_2 \dots b_{64}$, comprising 1 sign bit (b_1), 11 exponent bits (b_2 to b_{12}), and 52 fraction bits (b_{13} to b_{64}, with implicit leading 1 for normalized numbers).*[1]

2.2 Stability Metrics

Definition 2.2 (Common Bit Prefix, m-Bit Stability)**.** *Let $\{x_i\}$ be a collection of real numbers and $B(x_i)$ their binary64 representations. The* first m bits *are **invariant** (*m-bit stable*) if $B(x_i)$ agree on their prefix of length m: for each $j = 1, \dots, m$, $b_j(x_i)$ is identical for all i.*

Definition 2.3 (Hamming Stability)**.** *Let $d_H(B(x), B(y))$ denote Hamming distance (the number of differing bits) between two binary64 encodings. If $d_H(B(x), B(y)) \leq 64 - m$, then x and y share at least an m-bit common prefix.*

Definition 2.4 (Probabilistic Bit Stability)**.** *Given a random perturbation model or transformation T, bit j is stable with confidence $1 - \alpha$ if $\mathbb{P}(b_j(T(x))$ flips$) \leq \alpha$. The expected number of stable bits is $E[m]$ over the ensemble.*

These complementary perspectives enable deterministic, quantitative, and probabilistic analysis of bitwise invariance.

Remark 2.1. *If two values x, y share an m-bit common prefix (with identical exponent), their numerical difference $|x - y|$ is $< 2^{-m}$ times the value's scale. For double-precision, each differing bit corresponds to a unit-in-the-last-place (ULP) at the given exponent.*

3 Prime Resonance Cascade and Convergence

Consider the resonance cascade:

$$C_n(z) = \prod_{k=1}^{n} R(z, p_k)$$

where p_k is the k-th prime, and $R(z, p)$ is a fixed, analytic resonance function. Throughout, we focus on the canonical case $z = 0.5$ unless stated otherwise.

Assumptions on $R(z, p)$:

[1]See: `https://en.wikipedia.org/wiki/Double-precision_floating-point_format`

(A1) **Boundedness:** $0 < R(z,p) \le 1$ for all z, p.

(A2) **Decay/Convergence:** $R(z,p) \to 1$ as $p \to \infty$, so that $\prod_{k=1}^{\infty} R(z,p_k)$ converges.

(A3) **Smoothness:** For each fixed p, $R(z,p)$ is continuous and Lipschitz in z, i.e., $|R(z,p) - R(z',p)| \le L_p|z - z'|$.

Remark 3.1. *For the explicit $R(z,p)$ used in prior empirical work—e.g., $R(0.5,p) = \sin(\pi/p)\cos(\pi/(2p))\sin(\pi/p)$—direct calculation reveals $R(0.5,p) \sim \mathcal{O}(1/p^3)$ for large p. Hence, convergence to a nonzero limit is ensured by comparison with the Euler product for $\zeta(s)$, $s > 1$.*

4 Main Theorems: Bitwise Stability in the Cascade

4.1 Stability under Sequence Extension

Theorem 4.1 (Bit Stability under Increasing n)**.** *Suppose $C_n(z)$ as above, with $R(z,p)$ satisfying (A1)-(A2), and $\lim_{n\to\infty} C_n(z) = L(z) > 0$. Then, for any fixed $m \ge 1$, there exists N such that for all $n \ge N$, $B(C_n(z))$ has an m-bit stable prefix invariant with $L(z)$.*

Proof. Convergence implies for any $\epsilon > 0$, $\exists N$ such that $|C_n(z) - L(z)| < \epsilon$ for $n \ge N$. For floating-point, having $|C_n(z) - L(z)| < 2^{-m}L(z)$ ensures first m bits are unchanged (a difference below an ULP at the m-th bit). Therefore for all $n \ge N$, $B(C_n(z))$ and $B(L(z))$ have an m-bit common prefix. □

Corollary 4.1 (Empirical Bit Stability Law)**.** *If $|C_{n+1}(z) - C_n(z)| < \epsilon$, then*

$$S(\epsilon) \ge 64 - \lfloor \log_2(1/\epsilon) \rfloor,$$

where $S(\epsilon)$ is the number of stable bits, up to a small constant for exponent/sign alignment.

4.2 Stability under Input Perturbation

Theorem 4.2 (Lipschitz Perturbation-Stability)**.** *Let $C_n(z)$ as above and suppose $R(z,p)$ satisfies (A3), with Lipschitz constants L_p. Set $K_n = \sum_{k=1}^{n} L_p$. For any $\epsilon > 0$, if $|z - z'| < \frac{\epsilon}{K_n}$, then $|C_n(z) - C_n(z')| < \epsilon$, and thus at least $m = 64 - \lfloor \log_2(1/\epsilon) \rfloor$ leading bits in $B(C_n(z))$ are stable under this perturbation.*

Proof. Write $C_n(z) = \prod_{k=1}^{n} R(z,p_k)$, $f_k(z) = \ln R(z,p_k)$.

$$|\ln C_n(z) - \ln C_n(z')| = \left| \sum_{k=1}^{n} (f_k(z) - f_k(z')) \right| \le K_n|z - z'|$$

Exponentiating and invoking **??** yields the result. □

Remark 4.1. *This analytic derivation explains the empirically observed law*

$$S(\epsilon) \approx 64 - \lfloor \log_2(1/\epsilon) \rfloor ,$$

where ϵ is the relative change in output (comes from the smooth, cascaded effect of the analytic R function).

4.3 Role of Sequence: Primes versus General Sequences

Proposition 4.1 (Role of Sequence Properties). *Let $\{q_k\}$ be any increasing sequence; set $C'_n(z) = \prod_{k=1}^{n} R(z, q_k)$.*

(i) If $R(z, q_k) \to 1$ and the product converges, Theorems **??** *and* **??** *hold for C'_n as for primes.*

(ii) If $R(z, q_k)$ does not *approach 1 (e.g., infinitely often $R(z, q_k) < 1 - \gamma$), then no nontrivial bit prefix remains invariant as $n \to \infty$; frequent large "drops" cause cascading bit-flips, including in exponent bits.*

Proof. Follow the argument of Theorems **??** and **??**. For (i), convergence gives stabilization. For (ii), repeated multiplicative drops shift the exponent, causing high-order bits to change arbitrarily often. □

4.4 Floating-Point Effects Versus Analytic Structure

Proposition 4.2 (Generic Floating-Point Stabilization). *Let (x_n) be any convergent sequence of real numbers. Then in* fixed P-bit floating-point, *for large enough n, all P significant bits of $B(x_n)$ stabilize. This is a generic property of finite-precision.*

Proof. As $|x_n - L| < 2^{-P}|L|$, x_n rounds to L at machine precision; bits lock in. □

Remark 4.2. *For arbitrary-precision ($P \to \infty$), stabilization only occurs if the sequence is stationary. Thus, the observed m-bit stability in IEEE-754 is shaped both by mathematical convergence and precision limitation.*

5 Extensions: General Numeric Bases and Formats

5.1 Other Floating-Point Precisions

Let P be the precision (e.g., $P = 24$ for single, $P = 113$ for quadruple).

Proposition 5.1 (Scaling of Stability Law). *For floating-point of P-bit precision,*

$$S_P(\epsilon) \approx P + k - \lfloor \log_2(1/\epsilon) \rfloor,$$

where k is the number of sign+exponent bits (8 for single, 15 for quad, 11 for double, etc.).

5.2 Decimal and Other Radices

Remark 5.1. *Analogous results hold for decimal formats: as the function converges within 10^{-D}, D leading decimal digits stabilize.*

$$S_{10}(\epsilon) \approx N - \lfloor \log_{10}(1/\epsilon) \rfloor$$

where N is the number of significant decimal digits in the representation (e.g., 16 for decimal64).

5.3 Effect of Exponent Normalization

Proposition 5.2 (Exponent Normalization and Stability)**.** *Bitwise stability in floating-point applies to fraction bits* within fixed exponent periods. *As C_n crosses a power-of-two threshold, exponent bits change, transiently disrupting the leading prefix, after which new stable patterns emerge until the next threshold.*

6 Experimental Protocols and Recommendations

We propose the following empirical experiments for theoretical validation and further exploration:

- **High-Precision Simulation:** Compute C_n in arbitrary or quadruple precision to observe the progression to bit stability; compare with double precision to isolate floating-point artifacts.
- **z-Parameter Study:** Vary z in $R(z, p)$ and study the effect on stable bits, verifying the Lipschitz perturbation theorem.
- **Sequence Variants:** Replace the prime sequence with composites, random choices, or permutations; measure the resulting bit stability.
- **Lipschitz Estimate:** Empirically calculate the Lipschitz constant for $C_n(z)$ and test if Theorem **??** bounds are tight.
- **Bit-Flip Probability:** For random perturbations, sample the rate at which each bit flips, plotting the stability profile for each bit position.
- **Visualization:** Plot binary strings of C_n for increasing n as a bit matrix to visualize the "front" of bit stability.

7 Discussion and Conclusion

We have rigorously formalized the surprising stability phenomena reported in the Prime Resonance Framework, showing they arise from the confluence of analytic convergence and floating-point discretization. Analytic properties of $R(z, p)$—such as decay and smoothness—guarantee convergence and, thus, eventual bitwise invariance; the number and rapidity of stable bits relates directly to the decay rate and smoothness. The observed empirical law relating bit stability to error magnitude is analytically justified, and is shown to apply generally for convergent cascades.

Floating-point representation, while imposing hard stabilization at the quantum of precision, can "mask" further mathematical differences—thus, the practitioner must carefully distinguish resonance-induced invariance from digital quantization. Our generalizations reveal that stability phenomena are not unique to binary or double precision, but are intrinsic to any fixed-precision positional system.

The Prime Resonance Framework serves as a compelling case study in the interplay between mathematical structure and computational representation, with potential impacts in robust encoding, numerical analysis, and digital signal design.

References

1. D. Goldberg, "What Every Computer Scientist Should Know About Floating-Point Arithmetic," *ACM Computing Surveys*, vol. 23, no. 1, Mar. 1991.

2. N.J. Higham, *Accuracy and Stability of Numerical Algorithms*, SIAM, 2nd Ed., 2002.

3. IEEE Standard 754-2019, *IEEE Standard for Floating-Point Arithmetic*.

4. Wikipedia contributors, "Double-precision floating-point format," *Wikipedia, The Free Encyclopedia*, `https://en.wikipedia.org/wiki/Double-precision_floating-point_format`

5. Your prior papers/preprints on the Prime Resonance Framework.

A Empirical Data and Code Example

A reference Python script for counting matching bits in IEEE-754 double precision, and instructions for generating further tables, will be provided in an online repository (link to be added).