Homework 3

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Analysis of Linear Systems

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Theorem 1. *Let $\mathcal{A}\colon X \to Y$ be a linear transformation between vector spaces X and Y over the field $\mathcal{F}$. $\mathcal{A}$ is one-to-one if and only if $\mathcal{N}(\mathcal{A}) = \{\underset{X}{\odot}\}$.*

Proof.

1. *Subproof.* $\mathcal{A}$ is 1:1 $\implies \mathcal{N}(\mathcal{A}) = \{\odot_X\}$

   Assume $\mathcal{A}$ one-to-one. Let $x \in \mathcal{N}(\mathcal{A})$

$$
\begin{aligned}
\mathcal{A}(\underset{X}{\bigodot}) &= \underset{Y}{\bigodot} && \text{(definition of null space)} \\
x &= \underset{X}{\bigodot} && \text{(by definition of 1:1)}
\end{aligned}
$$

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2. *Subproof.* $\mathcal{N}(\mathcal{A}) = \{\odot_X\} \implies \mathcal{A}$ is 1:1

   Assume $\mathcal{N}(\mathcal{A}) = \{\odot_X\}$. Let $x_1, x_2 \in X$ such that $\mathcal{A}(x_1) = \mathcal{A}(x_2)$

$$
\begin{aligned}
\mathcal{A}(x_1) - \mathcal{A}(x_2) &= \underset{Y}{\bigodot} \\
\mathcal{A}(x_1) + (-1)\mathcal{A}(x_2) &= \underset{Y}{\bigodot} \\
\mathcal{A}(x_1 + (-1)x_2) &= \underset{Y}{\bigodot} && \text{(by lineararity)} \\
\mathcal{A}(x_1 - x_2) &= \underset{Y}{\bigodot} \\
x_1 - x_2 &= \underset{X}{\bigodot} && \text{(by definition of null space)} \\
x_1 &= x_2
\end{aligned}
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