Algorithmic Motion Planning (236610)
Lecture 5—Sampling-based planners (2)

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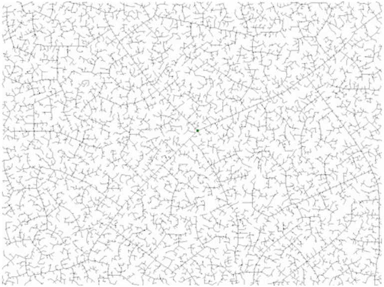
Today's lecture

Rapidly-exploring Random Trees (RRTs)—sampling-based single-query motion planning

- RRT—algorithmic description, implementation details, theoretical properties
- RRT-connect
- High-quality motion planning—RRT*, LBT-RRT



Animation by Javed Hossain, adapted from https://en.wikipedia.org/wiki/Rapidly-exploring_random_tree

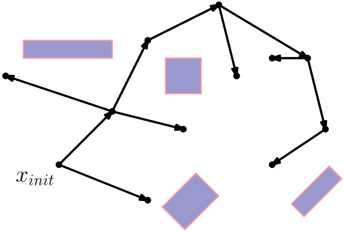
Rapidly-exploring Random Tree—RRT [LaValle, Kuffner01]

- RRTs have been the proven to be an effective, conceptually simple algorithm for single-query planning in high-dimensional C-spaces
- Variants of the basic algorithm have been used for
  - Robotic applications—mobile robotics, manipulation, Mars rovers, humanoid etc.
  - Biological application—drug design
  - Manufacturing and virtual prototyping (assembly analysis)
  - ...
- Variants include
  - High-quality planning
  - Planning for non-holonomic systems
  - Planning on low-dimensional manifolds
  - Parallel RRTs
  - ...

RRT—algorithmic description

Input: The C-space $\mathcal{X}$; start configuration x_{start} goal region $\mathcal{X}_{\text{goal}}$; no. of iterations n; steering param η
Output: Tree $\mathcal{T}$ rooted at x_{start}

```
𝒯.init(x_start)
for i = 1 to n do
    x_rand ← sample_random_state(𝒳)
    x_near ← nearest_neighbor(𝒯, x_rand)
    x_new ← extend(x_rand, x_near, η)
    if collision_free(x_near, x_new) then
        𝒯.add_vertex(x_new)
        𝒯.add_edge(x_near, x_new)
return 𝒯
```



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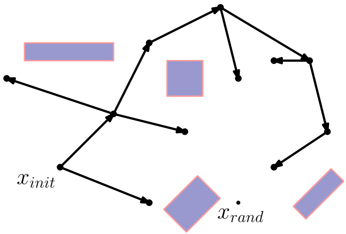
Input: The C-space $\mathcal{X}$; start configuration x_{start} goal region $\mathcal{X}_{\text{goal}}$; no. of iterations n; steering param η
Output: Tree $\mathcal{T}$ rooted at x_{start}

1: $\mathcal{T}$.init(x_{start})
2: **for** $i = 1$ to n **do**
3: $\quad x_{\text{rand}} \leftarrow$ sample_random_state($\mathcal{X}$)
4: $\quad x_{\text{near}} \leftarrow$ nearest_neighbor($\mathcal{T}, x_{\text{rand}}$)
5: $\quad x_{\text{new}} \leftarrow$ extend($x_{\text{rand}}, x_{\text{near}}, \eta$)
6: $\quad$ **if** collision_free($x_{\text{near}}, x_{\text{new}}$) **then**
7: $\quad\quad \mathcal{T}$.add_vertex(x_{new})
8: $\quad\quad \mathcal{T}$.add_edge($x_{\text{near}}, x_{\text{new}}$)
9: **return** $\mathcal{T}$



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    if collision_free(x_near, x_new) then
        T.add_vertex(x_new)
        T.add_edge(x_near, x_new)
return T
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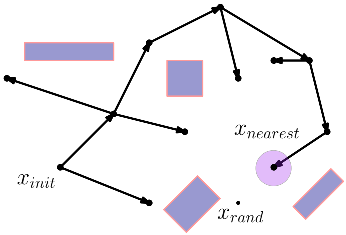

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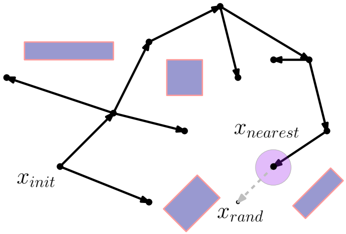

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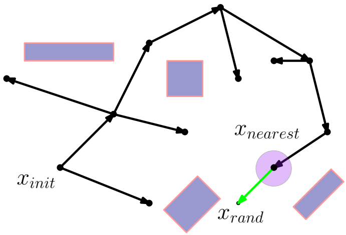

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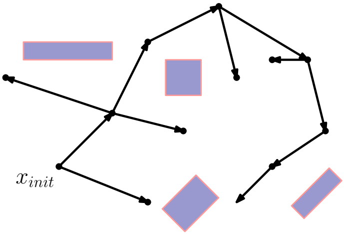

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$\mathcal{T}$.init(x_{start})
for $i = 1$ to n **do**
    $x_{\text{rand}} \leftarrow$ sample_random_state($\mathcal{X}$)
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    $x_{\text{new}} \leftarrow$ extend($x_{\text{rand}}, x_{\text{near}}, \eta$)
    if collision_free($x_{\text{near}}, x_{\text{new}}$) **then**
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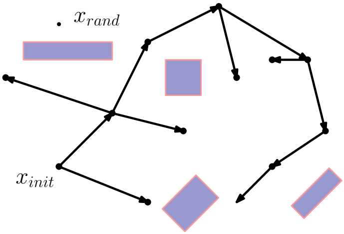

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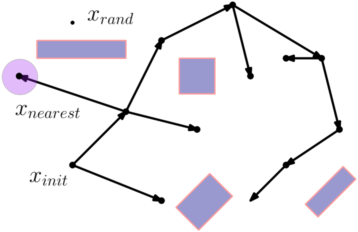

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$\mathcal{T}$.`init`(x_{start})
for $i = 1$ to n **do**
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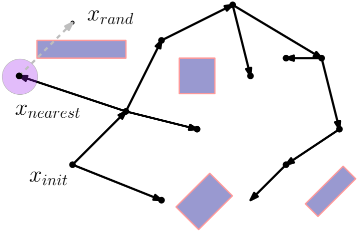

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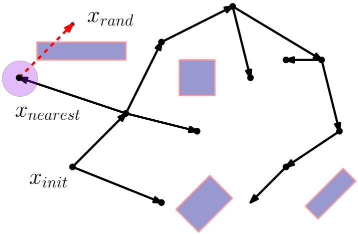

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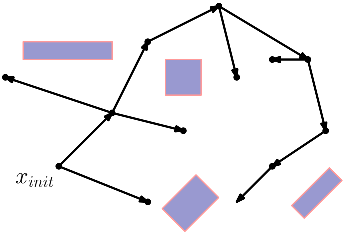

RRT—algorithmic description (extend)

Input: The C-space $\mathcal{X}$; start configuration x_{start} goal region $\mathcal{X}_{\text{goal}}$; no. of iterations n; steering param η
Output: Tree $\mathcal{T}$ rooted at x_{start}

$\mathcal{T}$.init(x_{start})
for $i = 1$ to n **do**
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  if collision_free($x_{\text{near}}, x_{\text{new}}$) **then**
    $\mathcal{T}$.add_vertex(x_{new})
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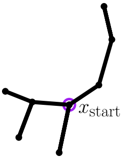

Figures adapted from https://www.cs.cmu.edu/~motionplanning/lecture/Chap7-Prob-Planning_howie.pdf

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Input: The C-space $\mathcal{X}$; start configuration x_{start} goal region $\mathcal{X}_{\text{goal}}$; no. of iterations n; steering param η
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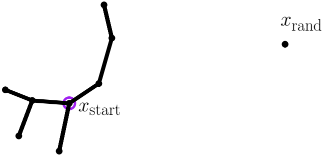

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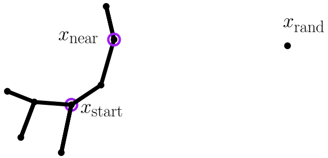

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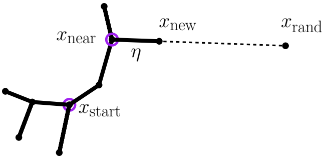

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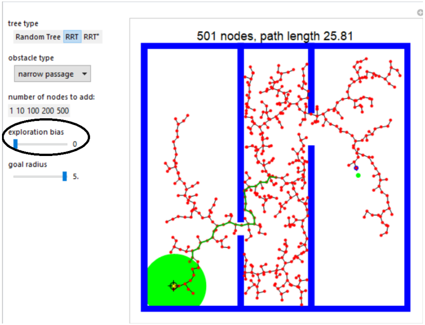
RRT—implementation details

- Goal biasing
  - Sample x_{rand} uniformly from $\mathcal{X}$ with prob. $1 - p_{\text{bias}}$ and uniformly from $\mathcal{X}_{\text{goal}}$ with prob. p_{bias}
  - Rule of thumb—use $p_{\text{bias}} = 0.05$
- Step size η—what if it is too big? too small?
- Metric—typical C-spaces are non-Eucl. How can we compute NN?



Figures constructed using http://demonstrations.wolfram.com/RapidlyExploringRandomTreeRRTAndRRT/

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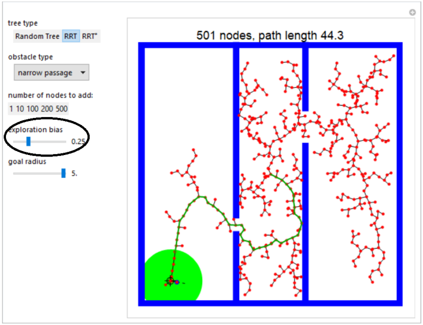

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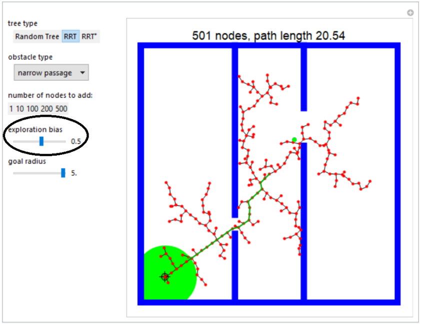

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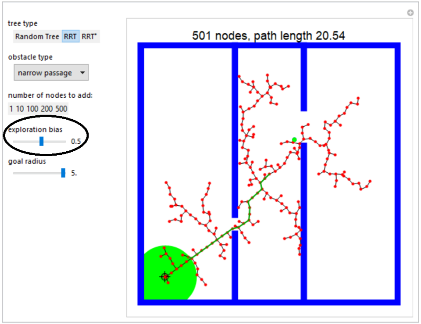

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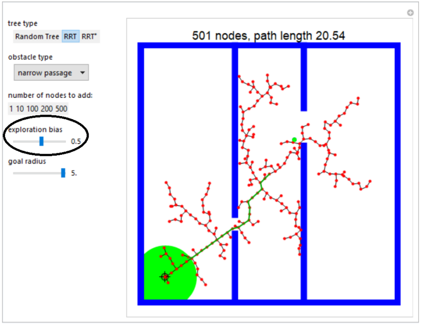

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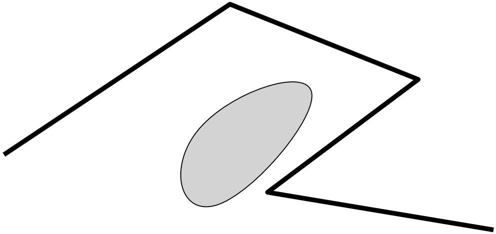
- Smoothing / post-processing—Paths produced by RRT are long and have unnecessary turns. What can we do?



RRT—implementation details (cont.)

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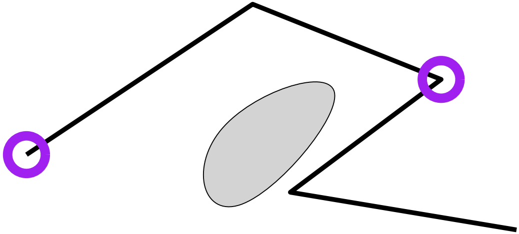

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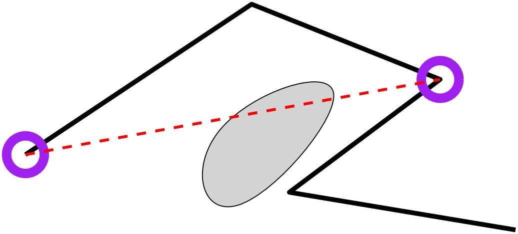

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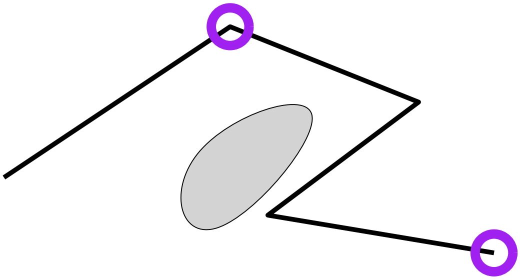

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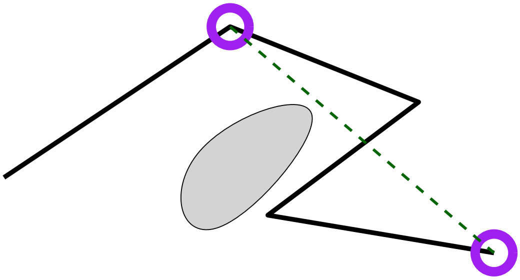

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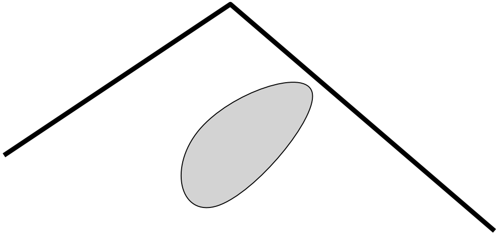

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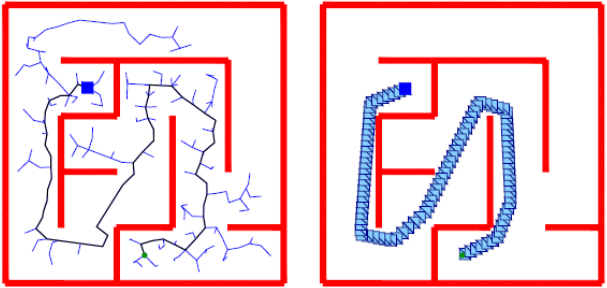

Figure adapted from [Kuffner, LaValle00]

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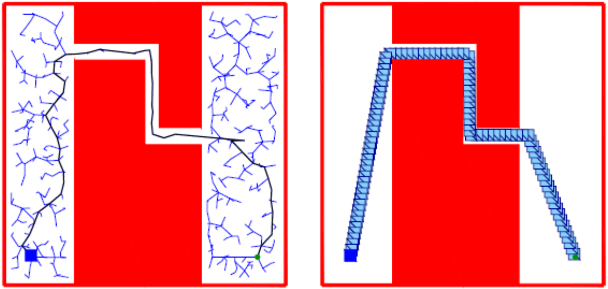

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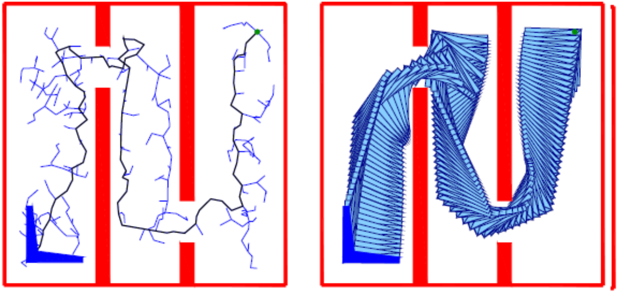


Figure adapted from [Kuffner, LaValle00]

RRT-connect

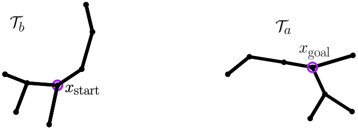
- It is often beneficial to search both from the start and from the goal
- This has been used in graph-search (e.g., bidirectional Dijkstra)

Animation adapted from https://meyavuz.wordpress.com/2017/05/14/dijkstra-vs-bi-directional-dijkstra-comparison-on-sample-us-road-network/

RRT-connect—algorithmic description

Input: The C-space $\mathcal{X}$; start configuration x_{start} goal configuration x_{goal}; no. of iterations n; steering param η
Output: Path connecting x_{start} to x_{goal}

$\mathcal{T}_a$.init(x_{start}), $\mathcal{T}_b$.init(x_{goal})
for $i = 1$ to n **do**
  $x_{\text{rand}} \leftarrow$ sample_random_state($\mathcal{X}$)
  $x_{\text{near}} \leftarrow$ nearest_neighbor($\mathcal{T}_a, x_{\text{rand}}$)
  $x_{\text{new}} \leftarrow$ extend($x_{\text{rand}}, x_{\text{near}}, \eta$)
  if collision_free($x_{\text{near}}, x_{\text{new}}$) **then**
    $\mathcal{T}_a$.add_vertex(x_{new})
    $\mathcal{T}_a$.add_edge($x_{\text{near}}, x_{\text{new}}$)
    if connect ($\mathcal{T}_b, x_{\text{new}}$) **then**
      return path($\mathcal{T}_a, \mathcal{T}_b$)
  swap($\mathcal{T}_a, \mathcal{T}_b$)
return failure

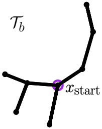


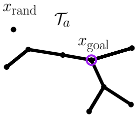
Figures adapted from https://www.cs.cmu.edu/~motionplanning/lecture/Chap7-Prob-Planning_howie.pdf

`RRT-connect`—algorithmic description

Input: The C-space $\mathcal{X}$; start configuration x_{start} goal configuration x_{goal}; no. of iterations n; steering param η
Output: Path connecting x_{start} to x_{goal}

$\mathcal{T}_a$.`init`(x_{start}), $\mathcal{T}_b$.`init`(x_{goal})
for $i = 1$ to n **do**
  $x_{\text{rand}} \leftarrow$ `sample_random_state`($\mathcal{X}$)
  $x_{\text{near}} \leftarrow$ `nearest_neighbor`($\mathcal{T}_a, x_{\text{rand}}$)
  $x_{\text{new}} \leftarrow$ `extend`($x_{\text{rand}}, x_{\text{near}}, \eta$)
  if `collision_free`($x_{\text{near}}, x_{\text{new}}$) **then**
    $\mathcal{T}_a$.`add_vertex`(x_{new})
    $\mathcal{T}_a$.`add_edge`($x_{\text{near}}, x_{\text{new}}$)
    if `connect` ($\mathcal{T}_b, x_{\text{new}}$) **then**
      return `path`($\mathcal{T}_a, \mathcal{T}_b$)
  `swap`($\mathcal{T}_a, \mathcal{T}_b$)
return failure





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Input: The C-space $\mathcal{X}$; start configuration x_{start} goal configuration x_{goal}; no. of iterations n; steering param η
Output: Path connecting x_{start} to x_{goal}

$\mathcal{T}_a$.init(x_{start}), $\mathcal{T}_b$.init(x_{goal})
for $i = 1$ to n **do**
    $x_{\text{rand}} \leftarrow$ sample_random_state($\mathcal{X}$)
    $x_{\text{near}} \leftarrow$ nearest_neighbor($\mathcal{T}_a, x_{\text{rand}}$)
    $x_{\text{new}} \leftarrow$ extend($x_{\text{rand}}, x_{\text{near}}, \eta$)
    if collision_free($x_{\text{near}}, x_{\text{new}}$) **then**
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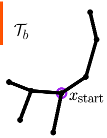
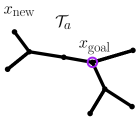


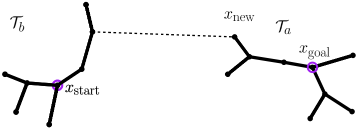


Figures adapted from https://www.cs.cmu.edu/~motionplanning/lecture/Chap7-Prob-Planning_howie.pdf

RRT-connect—algorithmic description

Input: The C-space $\mathcal{X}$; start configuration x_{start} goal configuration x_{goal}; no. of iterations n; steering param η
Output: Path connecting x_{start} to x_{goal}

```
𝒯a.init(x_start),    𝒯b.init(x_goal)
for i = 1 to n do
    x_rand ← sample_random_state(𝒳)
    x_near ← nearest_neighbor(𝒯a, x_rand)
    x_new ← extend(x_rand, x_near, η)
    if collision_free(x_near, x_new) then
        𝒯a.add_vertex(x_new)
        𝒯a.add_edge(x_near, x_new)
        if connect (𝒯b, x_new) then
            return path(𝒯a, 𝒯b)
    swap(𝒯a, 𝒯b)
return failure
```



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    swap(𝒯a, 𝒯b)
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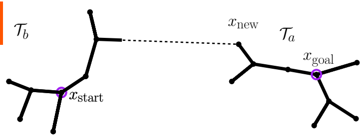

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        $\mathcal{T}_a$.add_edge($x_{\text{near}}, x_{\text{new}}$)
        if connect ($\mathcal{T}_b, x_{\text{new}}$) **then**
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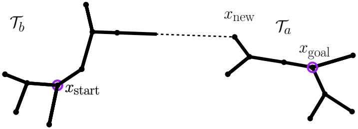

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    if connect ($\mathcal{T}_b, x_{\mathrm{new}}$) **then**
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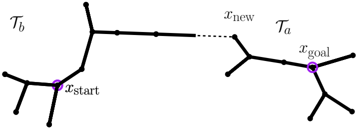
  swap($\mathcal{T}_a, \mathcal{T}_b$)
return failure

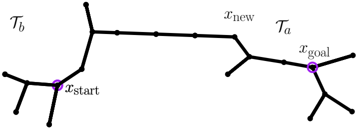


Figures adapted from https://www.cs.cmu.edu/~motionplanning/lecture/Chap7-Prob-Planning_howie.pdf

RRT-connect—algorithmic description

Input: The C-space $\mathcal{X}$; start configuration x_{start} goal configuration x_{goal}; no. of iterations n; steering param η
Output: Path connecting x_{start} to x_{goal}

```
T_a.init(x_start),    T_b.init(x_goal)
for i = 1 to n do
    x_rand ← sample_random_state(X)
    x_near ← nearest_neighbor(T_a, x_rand)
    x_new ← extend(x_rand, x_near, η)
    if collision_free(x_near, x_new) then
        T_a.add_vertex(x_new)
        T_a.add_edge(x_near, x_new)
        if connect (T_b, x_new) then
            return path(T_a, T_b)
    swap(T_a, T_b)
return failure
```

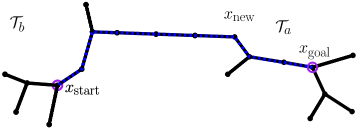


Figures adapted from https://www.cs.cmu.edu/~motionplanning/lecture/Chap7-Prob-Planning_howie.pdf

RRT-connect—algorithmic description

Input: The C-space $\mathcal{X}$; start configuration x_{start} goal configuration x_{goal}; no. of iterations n; steering param η
Output: Path connecting x_{start} to x_{goal}

```
Ta.init(x_start),    Tb.init(x_goal)
for i = 1 to n do
    x_rand ← sample_random_state(X)
    x_near ← nearest_neighbor(Ta, x_rand)
    x_new ← extend(x_rand, x_near, η)
    if collision_free(x_near, x_new) then
        Ta.add_vertex(x_new)
        Ta.add_edge(x_near, x_new)
        if connect (Tb, x_new) then
            return path(Ta, Tb)
    swap(Ta, Tb)
return failure
```



Figures adapted from https://www.cs.cmu.edu/~motionplanning/lecture/Chap7-Prob-Planning_howie.pdf

- Why do we swap the trees?
- How do we maintain the rapid exploration?
- What is the additional assumption taken and what are the implications?
  - Notice the exact connection made

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- How do we maintain the rapid exploration?
- What is the additional assumption taken and what are the implications?
  - Notice the exact connection made

RRT—Theoretical properties

- Rapid exploration (Voronoi bias)
- Prob. completeness
- (low) Quality of solutions

Voronoi Diagrams

Definition

Let $P = \{p_1 \ldots p_n\}$ be a set of n points (sites) in the plane. The Voronoi Diagram of P is the subdivision of the plane into n cells, one for each site, with the property that a point q lies in the cell corresponding to a site p_i if and only if $\text{dist}(q, p_i) < \text{dist}(q, p_j)$ for each $p_j \in P$ with $j \neq i$

- Can be extended to any metric space and any types of sites

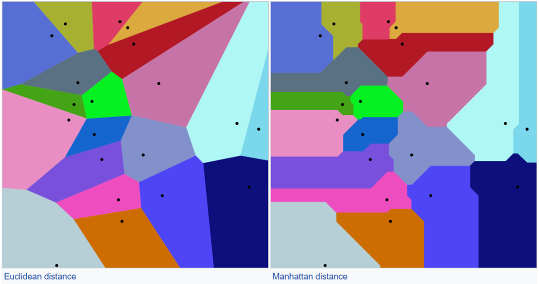


Figure adapted from https://en.wikipedia.org/wiki/Voronoi_diagram

Voronoi diagrams and RRTs

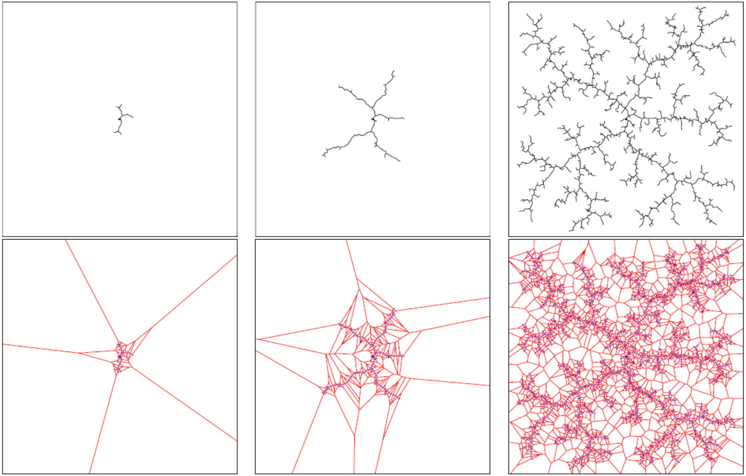


Figure adapted from [Kuffner, LaValle00]

Prob. completeness

- Let $\texttt{ALG}(n)$ be a sampling-based motion-planning algorithm that samples n configurations.
- Let $P_{\text{succ}}\left(x_{\text{start}}, \mathcal{X}, \mathcal{X}_{\text{goal}}\right)$ be the probability that $\texttt{ALG}(n)$ returns a collision-free path from x_{start} to $\mathcal{X}_{\text{goal}}$ in $\mathcal{X}$

Definition

An algorithm $\texttt{ALG}$ is said to be probabilistically complete if

$$\lim_{n\to\infty} P_{\text{succ}}\left(x_{\text{start}}, \mathcal{X}, \mathcal{X}_{\text{goal}}\right) = 1.$$

Prob. completeness

Thm [LaValle Kuffner01, Kleinbort et al.18]

The RRT algorithm is probabilistically complete.

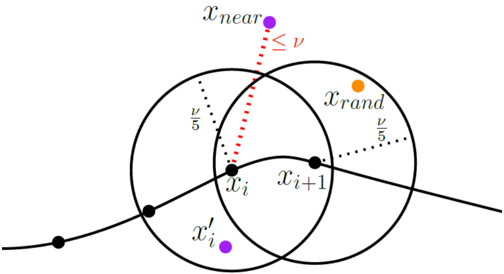


Figure adapted from [Kleinbort et al.18]

The quality of the solutions produced by RRT

- The probability for low-quality paths is bounded away from zero [Nechushtan, Raveh, Halperin10]
- This hold regardless if post-processing us applied
- Empirically, for certain scenarios the solution path is over 140 times worse than optimal in 5.9% of independent runs.

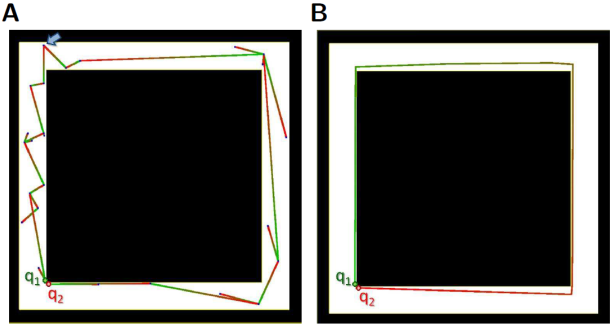


Figure adapted from [Nechushtan, Raveh, halperin10]

RRT * [Karaman Frazzoli11]

Input: The C-space $\mathcal{X}$; start configuration x_{start} goal configuration x_{goal}; no. of iterations n; steering param η
Output: Path connecting x_{start} to x_{goal}

```
T.init (x_start)
for i = 1 to n do
    x_rand ← sample_random_state(X)
    x_nearest ← nearest_neighbor(T, x_rand)
    x_new ← extend(x_nearest, x_rand)
    if collision_free(x_nearest, x_new) then
        T.add_vertex(x_new)
        T.add_edge(x_nearest, x_new)
        X_near ← nearest_neighbors(T, x_new, r_i)
        for all x_near ∈ X_near do
            rewire_RRT*(x_near, x_new )
        for all x_near ∈ X_near do
            rewire_RRT*(x_new, x_near )
```

RRT *—rewire operations

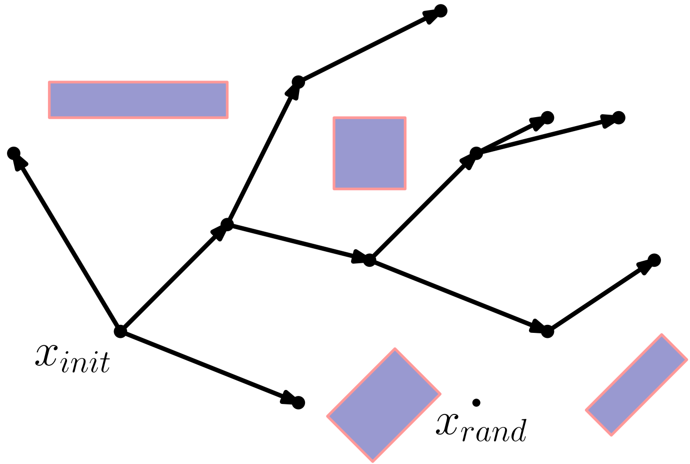
RRT* locally rewires nodes in $\mathcal{T}$ using a connection radius of

$$r(i) \approx \gamma(d) \cdot \left(\frac{\log i}{i}\right)^{1/d}$$

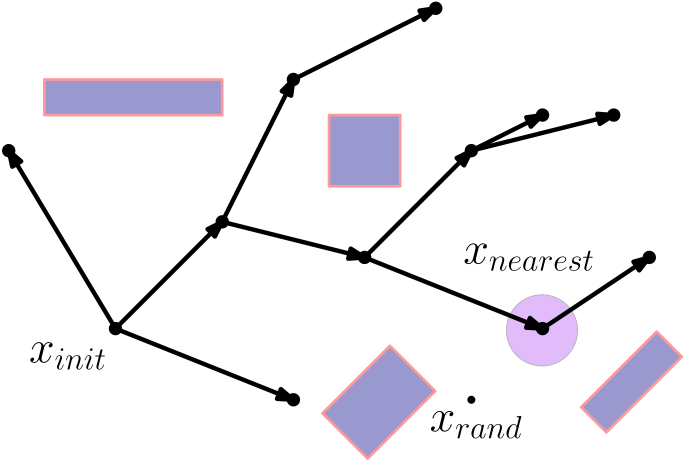
Input: A new potential parent $x_{\text{potential\_parent}}$ to childe node x_{child}
Output: Updated tree $\mathcal{T}$

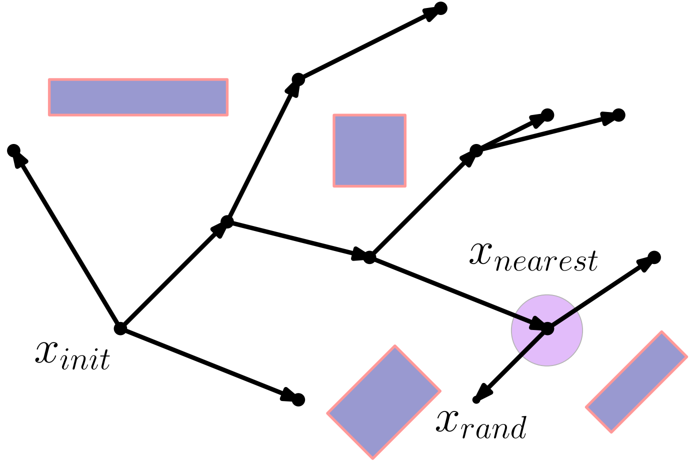
```
1: if (collision_free(x_potential_parent, x_child)) then
2:     c ← cost(x_potential_parent, x_child)
3:     if (cost_T(x_potential_parent) + c < cost_T(x_child)) then
4:         T.parent(x_child) ← x_potential_parent
```

RRT *—`rewire` operations

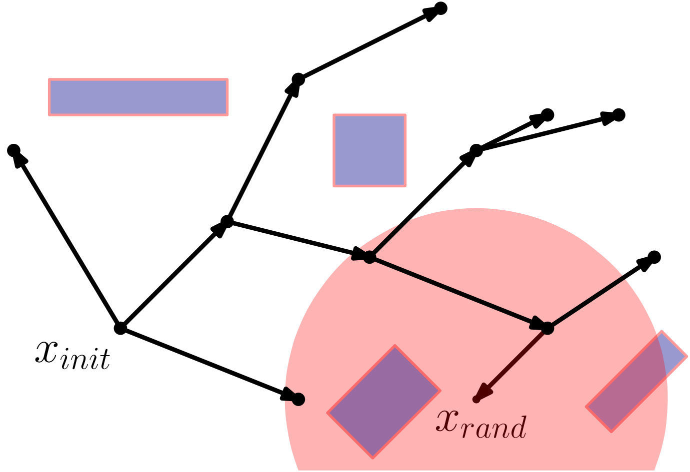


RRT*—rewire operations

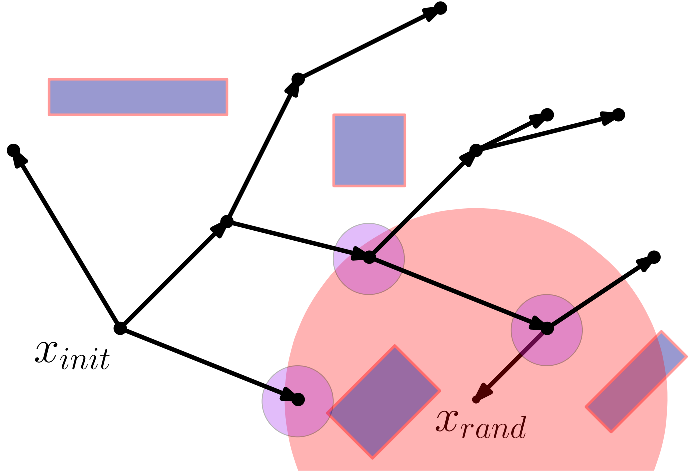




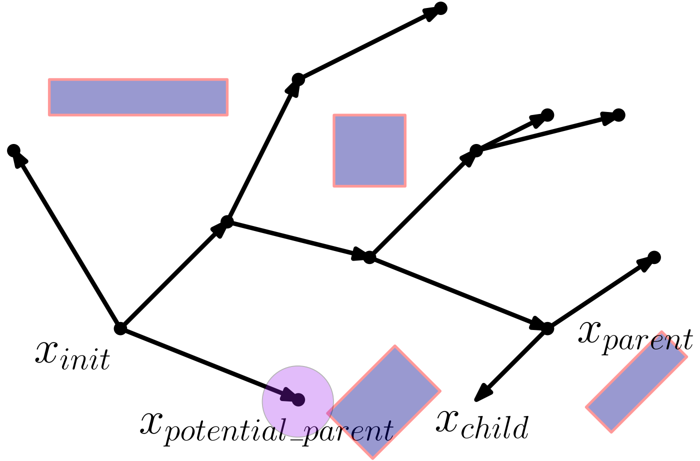
$x_{nearest}$
x_{init}
x_{rand}


x_{init}
x_{rand}

RRT*—`rewire` operations



RRT *—rewire operations



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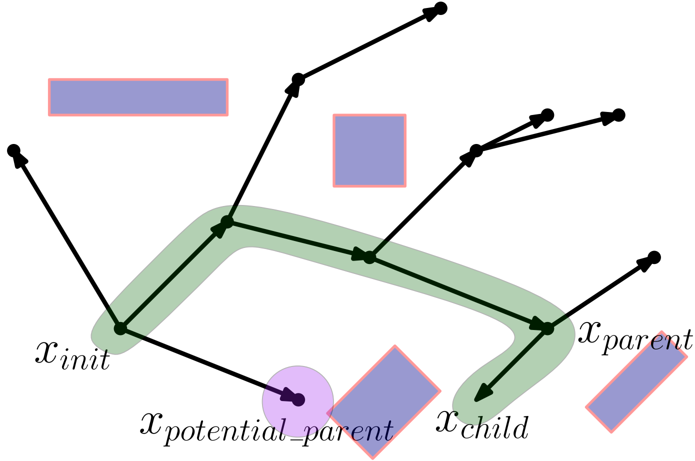
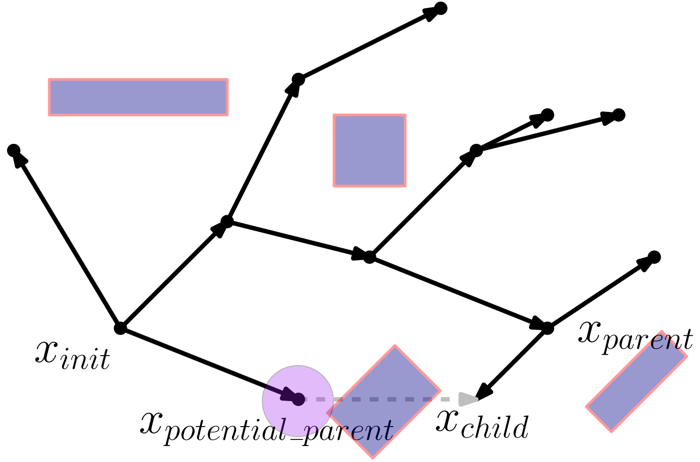
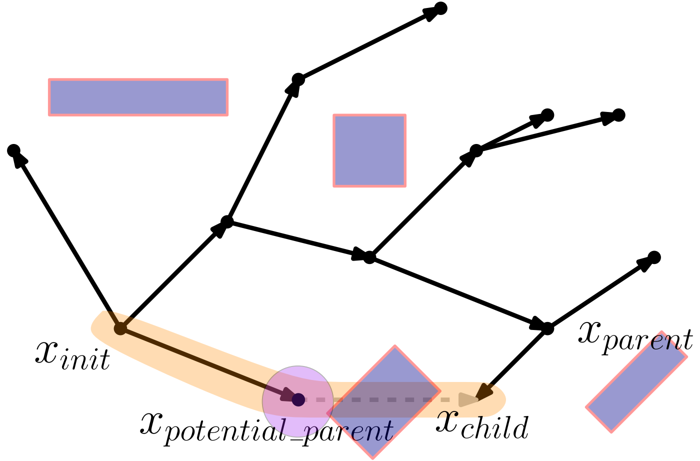
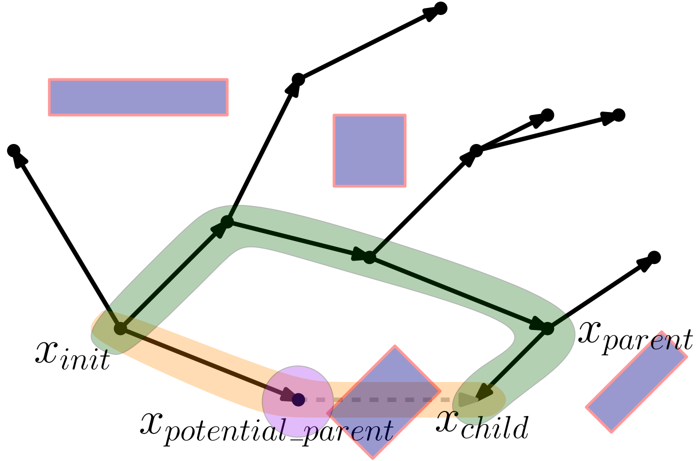


RRT*—`rewire` operations

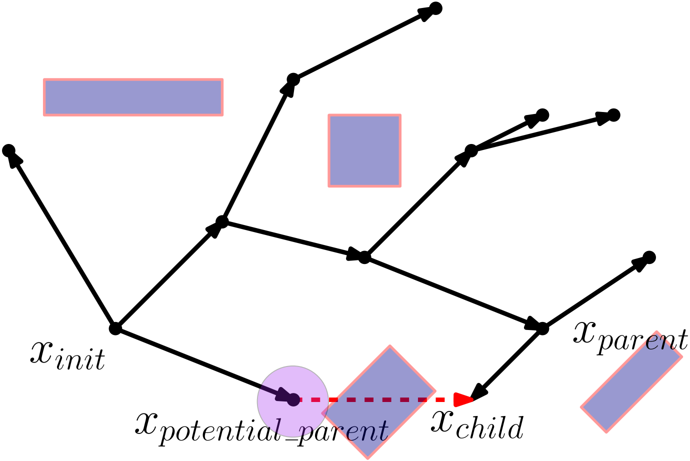


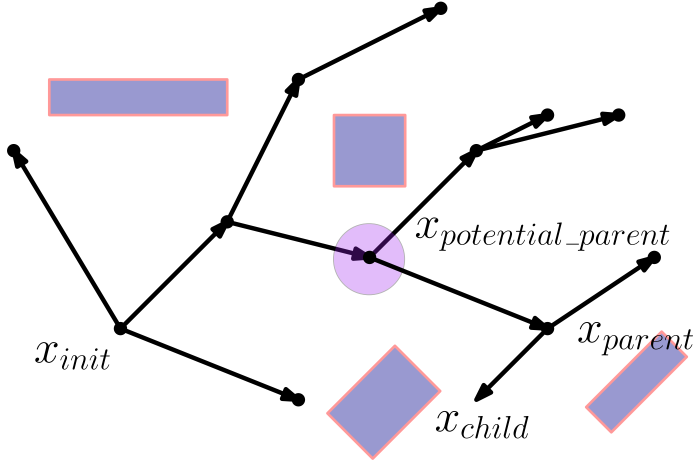

x_{init}
x_{parent}
$x_{potential\_parent}$
x_{child}

RRT *—rewire operations

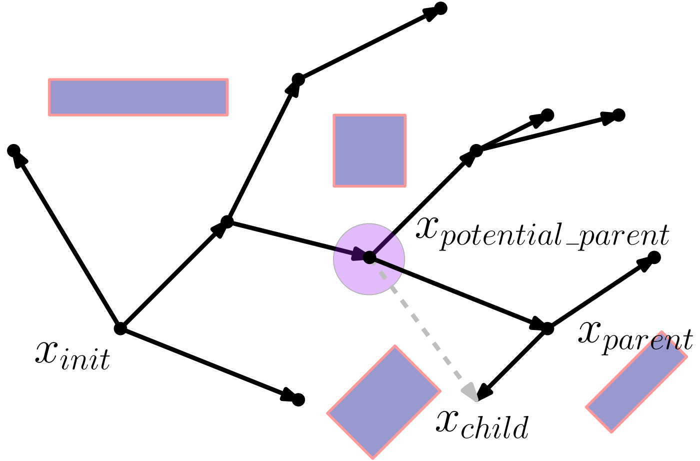


RRT *—`rewire` operations




$x_{potential\_parent}$
x_{init}
x_{parent}
x_{child}

RRT*—`rewire` operations



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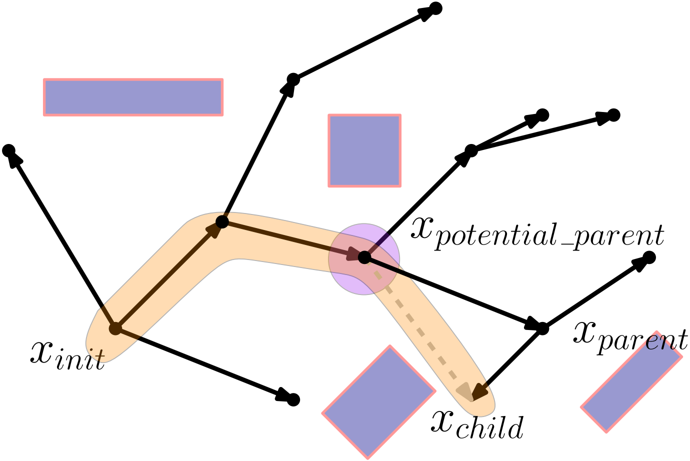
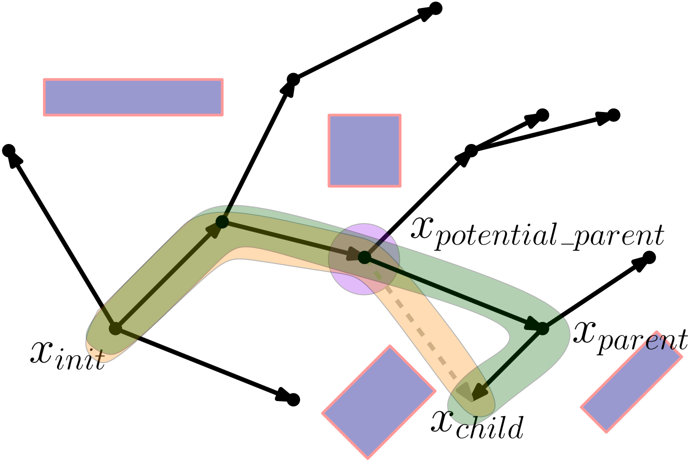
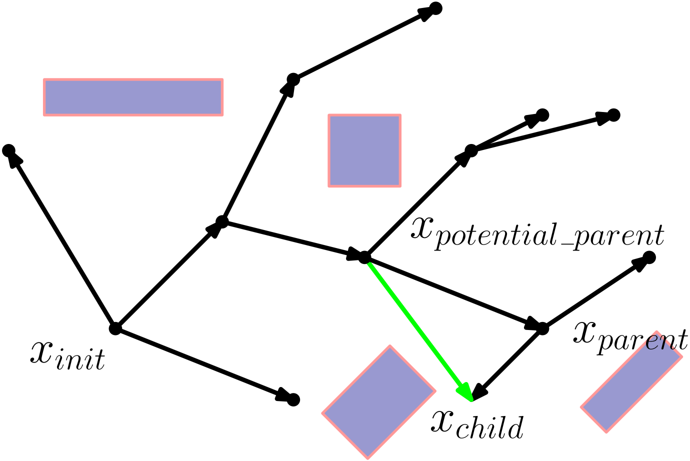


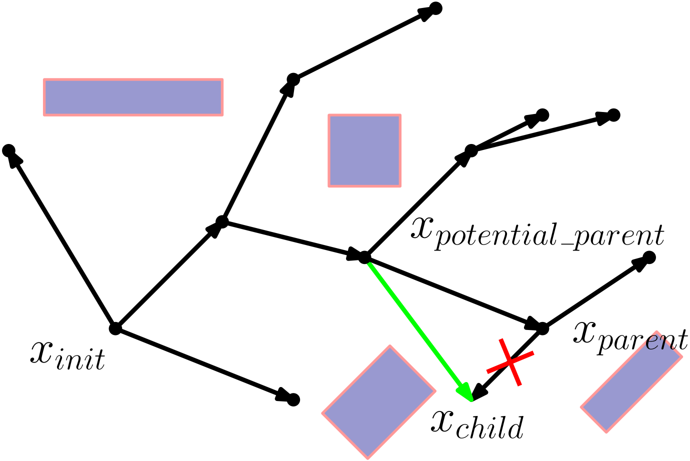


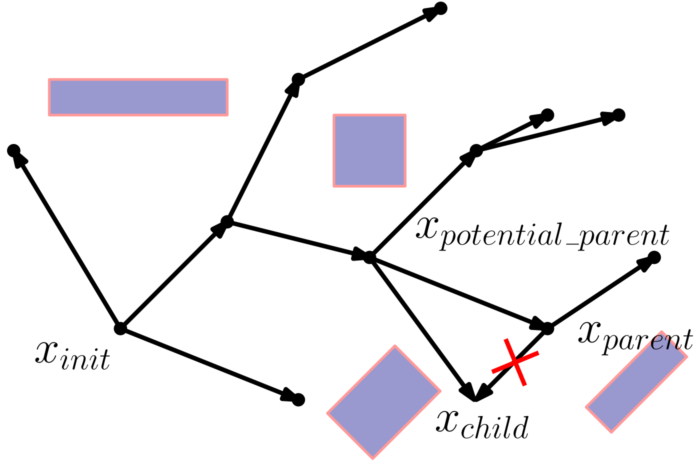
$x_{potential\_parent}$
x_{init}
x_{parent}
x_{child}

`RRT` *—`rewire` operations

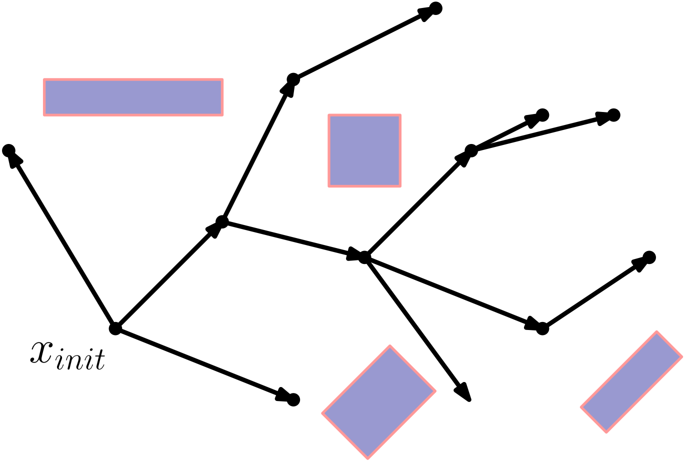


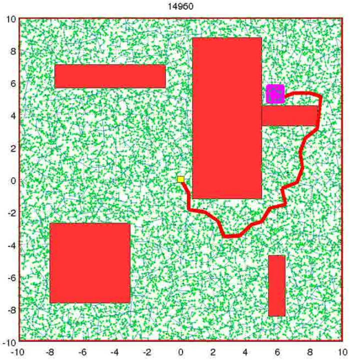
RRT*—`rewire` operations



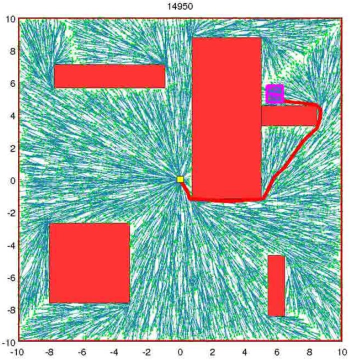

$x_{potential\_parent}$
x_{parent}
x_{init}
x_{child}

RRT *—`rewire` operations





RRT



RRT*

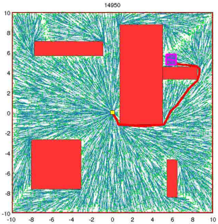
Videos created by Sertac Karaman, adapted from http://y2u.be/FAFw8DoKvik and http://y2u.be/YKiQTJpPFkA

RRT *—computational complexity

- The connection radius ensures that at the i^{th} iteration, we consider $O(\log i)$ nodes in $\mathcal{T}$ (in expectation) [Karaman, Frazzoli11]
- Nearest-neighbors computation takes $\Omega(n \log n)$ time [Karaman, Frazzoli11, Kleinbort, S., Halperin16]

Thm [Karaman, Frazzoli11, Kleinbort, S., Halperin16]

The complexity of the RRT* algorithm run with n samples is $\Omega(n \log n)$.



Definitions

Definition

The δ-interior (of $\mathcal{X}_{\text{free}}$), denoted by $\text{int}_\delta(\mathcal{X}_{\text{free}})$, is the set of all points $x \in \mathcal{X}_{\text{free}}$ that are within distance δ from $\mathcal{X}_{\text{obs}}$.

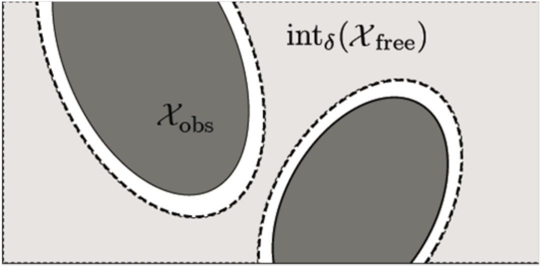


Figure adapted from [Karaman Frazzoli11]

Definitions (for quality of paths produced by RRT *)

Definition

A collision-free path σ is said to have strong δ-clearance if $\forall \tau \in [0,1]\ \sigma(\tau) \in \mathrm{int}_\delta(\mathcal{X}_{\mathrm{free}})$

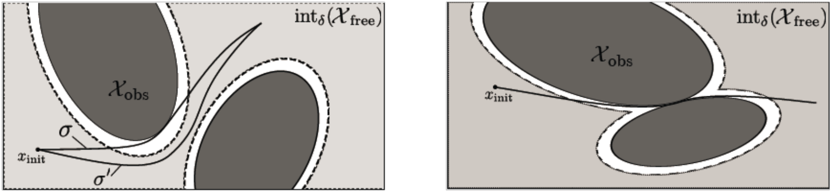
Definition

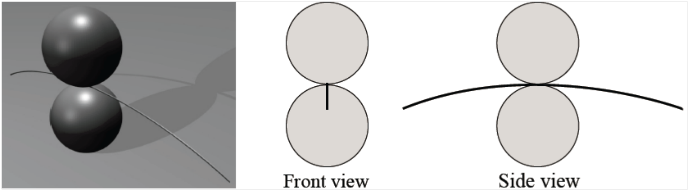
Two collision-free paths are said to be in the same homotopy class if there exists a continuous deformation between the paths that is in $\mathcal{X}_{\mathrm{free}}$

Definition

A collision-free path σ is said to have weak δ-clearance if there exists a path in its homotopy class with strong δ-clearance

Path type - strong or weak δ-clearance





Figures adapted from [Karaman Frazzoli11]

Definitions (for quality of paths produced by RRT *)

Definition

A feasible path $\sigma^* \in \mathcal{X}_{\text{free}}$ is said to be a robustly optimal solution if:

- It is optimal (i.e. $c(\sigma^*) = \min\{c(\sigma),\ \sigma \text{ is feasible}\}$ for a cost c
- It has weak δ-clearance
- For any sequence of collision free paths $\{\sigma_n\}$ s.t. $\lim_{n\to\infty} \sigma_n = \sigma^*$, $\lim_{n\to\infty} c(\sigma_n) = c(\sigma^*)$

Definitions (for quality of paths produced by RRT *)

Definition

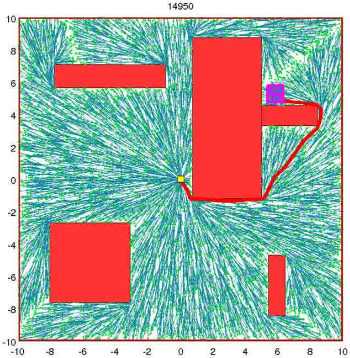
An algorithm ALG is asymptotically optimal if, for any path planning problem $(\mathcal{X}_{\mathrm{free}}, x_{init}, \mathcal{X}_{goal})$ and cost function $c : \Sigma \to \mathbb{R}_{\geq 0}$ that admit a robustly optimal solution with finite cost c^*

$$\Pr\left(lim\ sup_{n\to\infty} Y_n^{ALG} = c^*\right) = 1.$$

Where Y_n^{ALG} is the random variable corresponding to the cost of the minimum-cost solution included in the graph returned by ALG at the end of iteration n

Thm [Karaman, Frazzoli11]

The RRT* algorithm is asymptotically optimal.



Proof Outline - High level

Let σ^* denote a robustly optimal path

- Define a sequence $\{\delta_n\}_{n\in\mathbb{N}}$ where $\delta_n > 0$ and $\lim_{n\to\infty}(\delta_n) = 0$ and construct a sequence of paths $\{\sigma_n\}_{n\in\mathbb{N}}$ s.t.
  - σ_n has strong δ_n-clearance
  - σ_n converges to σ^*
- Define a sequence $\{q_n\}_{n\in\mathbb{N}}$ and construct a sequence $\{B_n\}_{n\in\mathbb{N}}$ of overlapping balls, each with radius q_n that collectively cover σ_n
- Show that for large n the probability that each ball in B_n has at least one vertex in V is one
- Show that for every two points in consecutive balls
  - They are no more than $r(n)$ apart
  - The straight line connecting them is in $\mathcal{X}_{\text{free}}$
- $\Rightarrow$ PRM* will generate a path passing through the balls
- Show that a sequence of such paths converges to σ^* and the cost of the best path in the roadmap converges to c(σ^*)

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Lemma

Let σ^* be a path with weak δ-clearance, let $\{\delta_n\}_{n \in \mathbb{N}}$ be a sequence of real numbers s.t. $lim_{n \to \infty}(\delta_n) = 0$, and $0 \leq \delta_n \leq \delta$.

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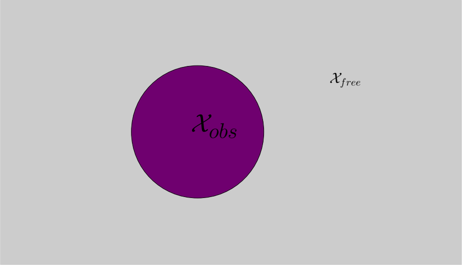

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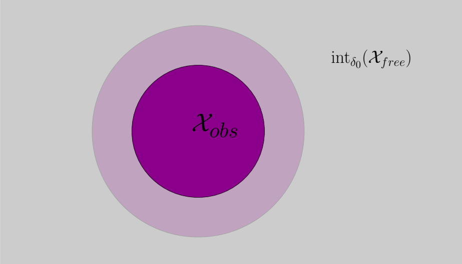

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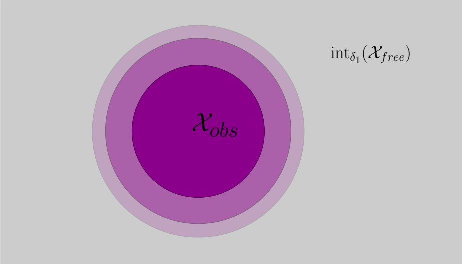

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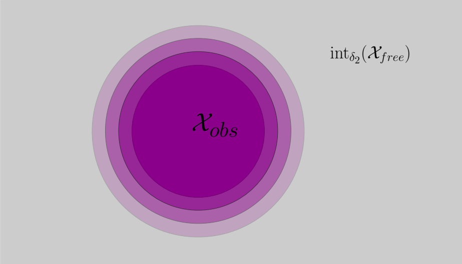

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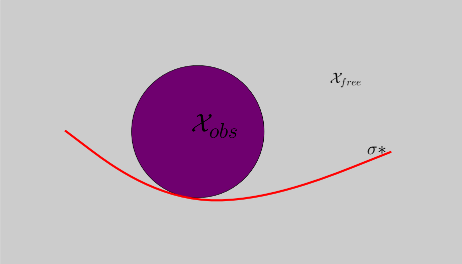

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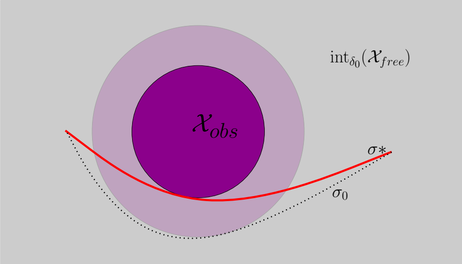

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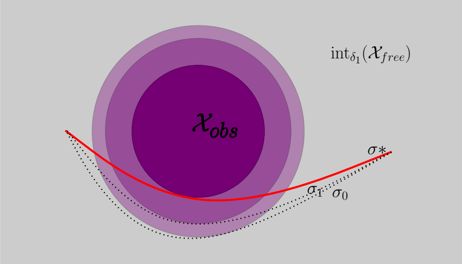

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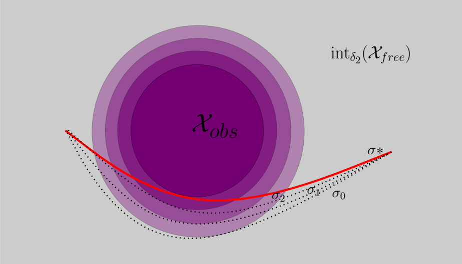

RRTs: quality vs. computational complexity

Setting: Single-query motion planning
Common approach: Sampling-based (RRTs)
Optimize: Path-length

Scenario taken from OMPL

- RRT [LaValle Kuffner01] — Fast, not optimal
- RRG, RRT* [Karaman Frazzoli 11] — Slower, asymptotically optimal

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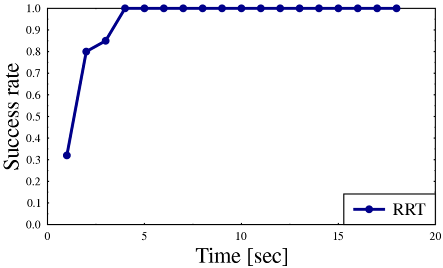

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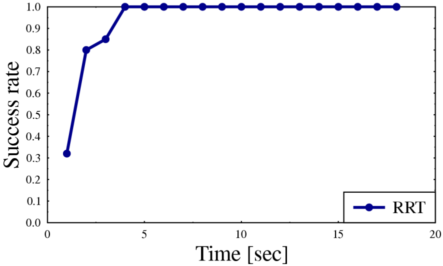

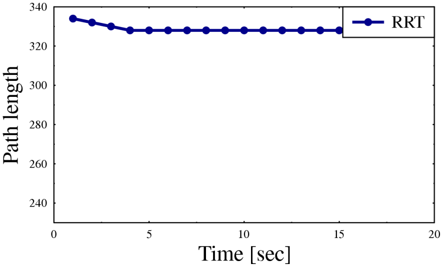

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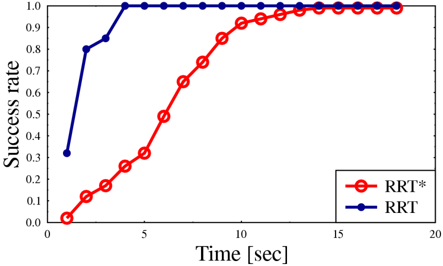

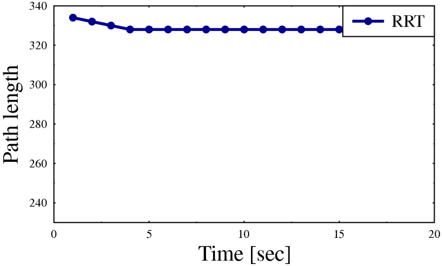

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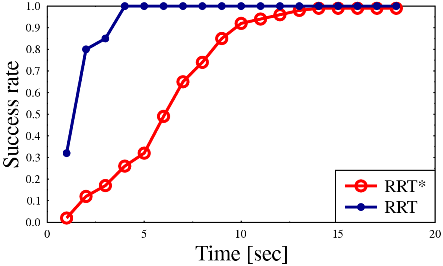

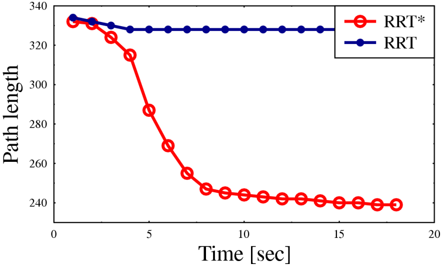

Related work - high-quality RRTs (partial list)

- Improving the quality of RRT
  - Post-processing existing paths [GO07]
  - Path hybridization [REH11]
  - Changing the sampling strategy [LTA03, US03, SWT09]
  - Changing connection scheme to a new milestone [US03, SLN00]
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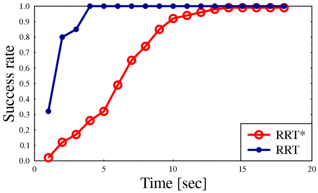

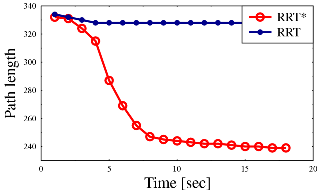

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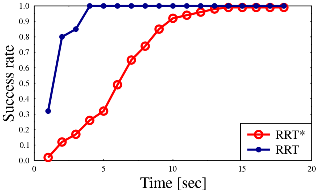

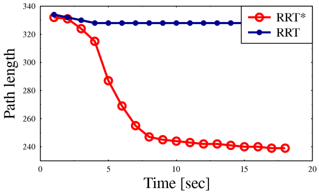

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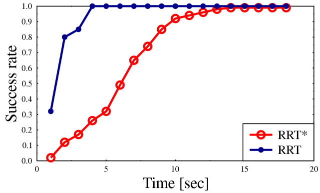
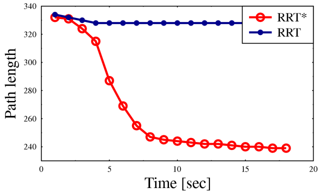




Lower Bound Tree-RRT (LBT−RRT) [S. Halperin16]

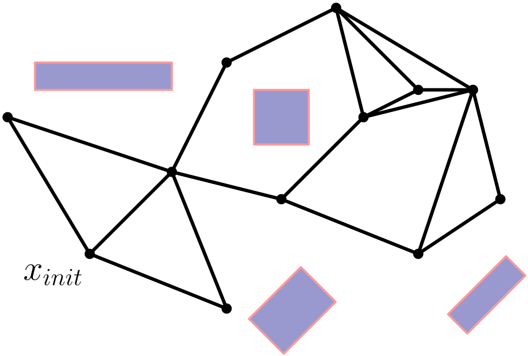
- Lower Bound Tree-RRT (LBT−RRT) is an asymptotically near-optimal planner
- LBT−RRT continuously interpolates between the fast RRT and the asymptotically optimal RRT*

Approximation factor ($1+\varepsilon$)	**Behavior**
No approximation ($\varepsilon = 0$)	like RRT* (asymptotically optimal)
Unbounded approximation ($\varepsilon = \infty$)	like RRT (fast)
In between ($0 < \varepsilon < \infty$)	higher-quality paths than RRT faster than RRG, RRT*

Algorithmic background - RRG

- Explores the configuration-space by constructing a graph
- Uses connection radius $r(n) \approx \gamma_{\text{RRG}}(d) \left(\frac{\log n}{n}\right)^{1/d}$

  n - number of nodes, d - dimension of configuration space



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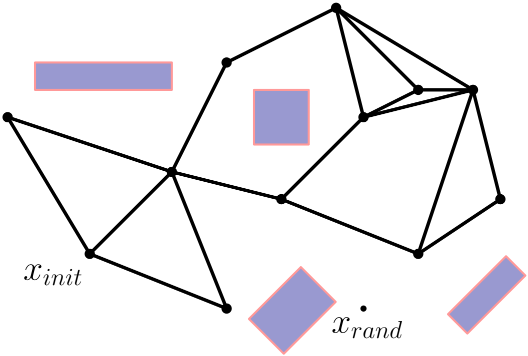

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- Uses connection radius $r(n) \approx \gamma_{\text{RRG}}(d)\left(\frac{\log n}{n}\right)^{1/d}$
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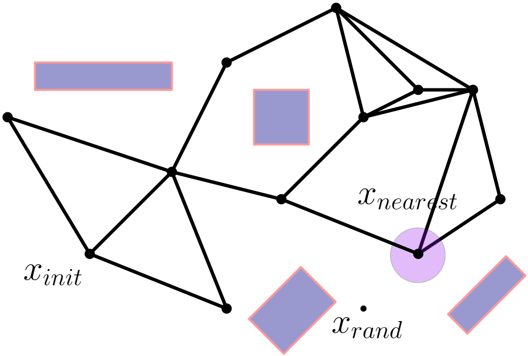

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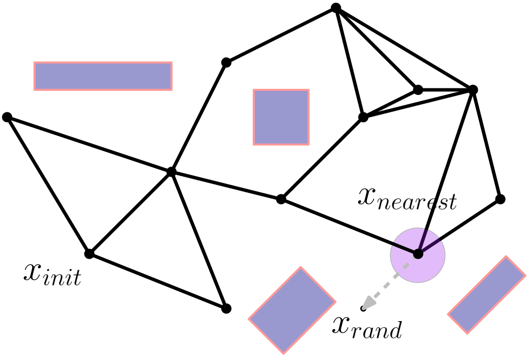

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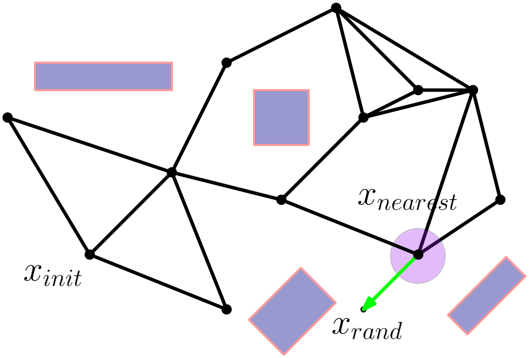

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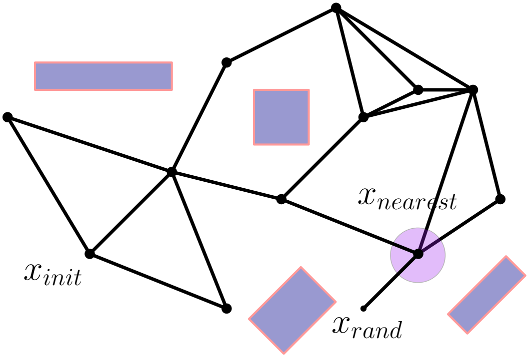

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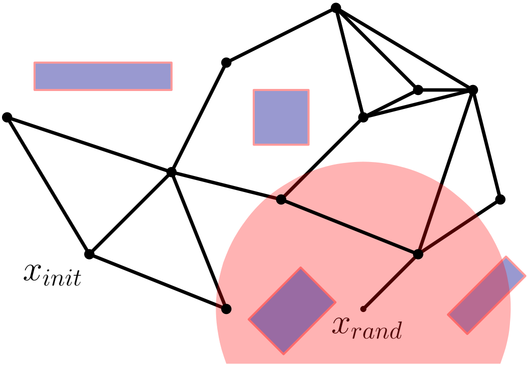

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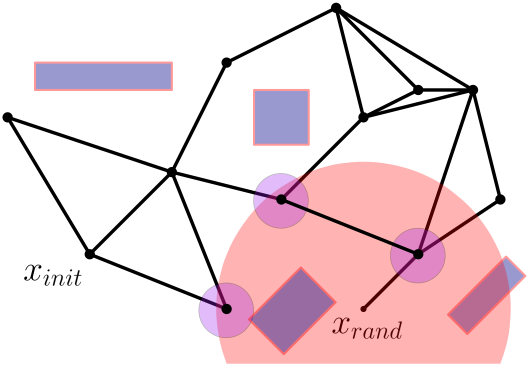

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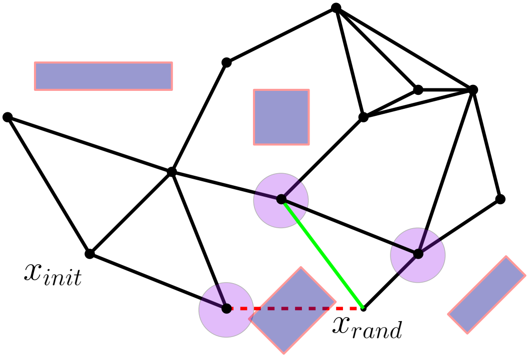

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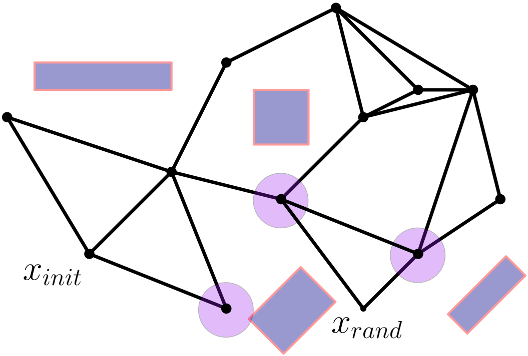

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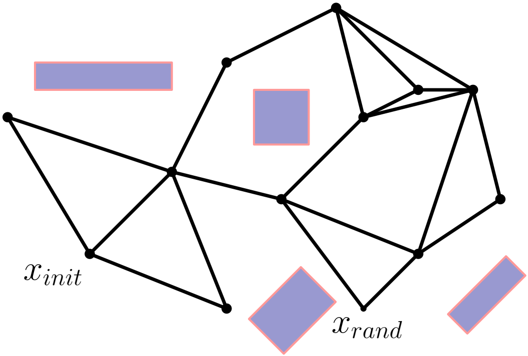

Lower Bound Tree-RRT (LBT−RRT) - motivation

Problem:

- Rewiring may call the expensive local planner $O(\log n)$ times per sample
- This is one of the most time-consuming parts of RRT*

Solution:

- LBT−RRT maintains two roadmaps: $\mathcal{G}_{lb}$ and $\mathcal{T}_{apx}$ (over the same set of vertices as the RRG roadmap)
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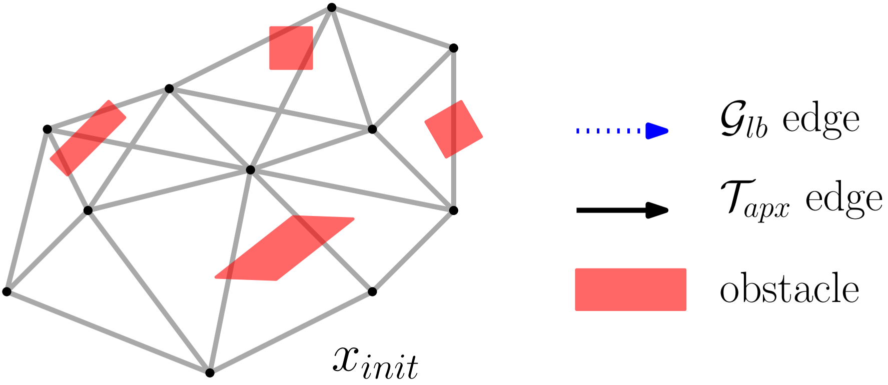

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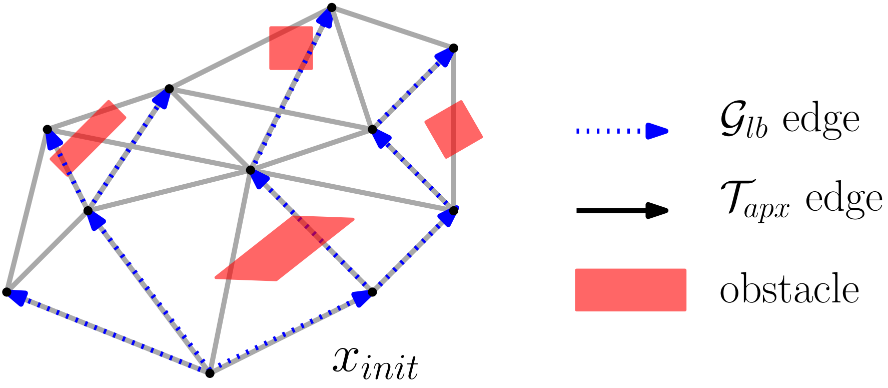

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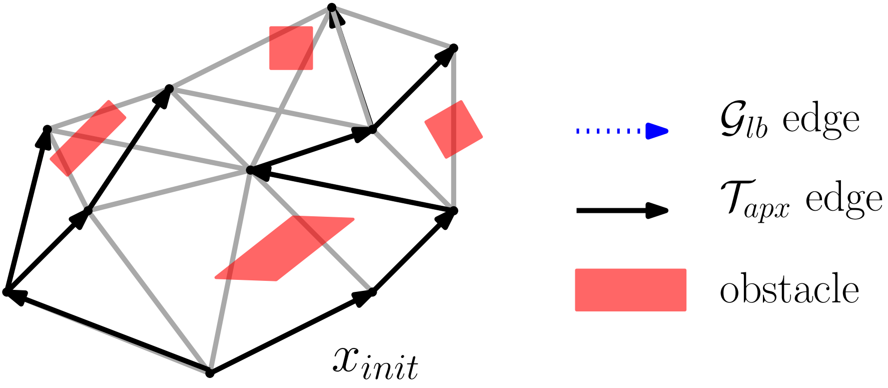

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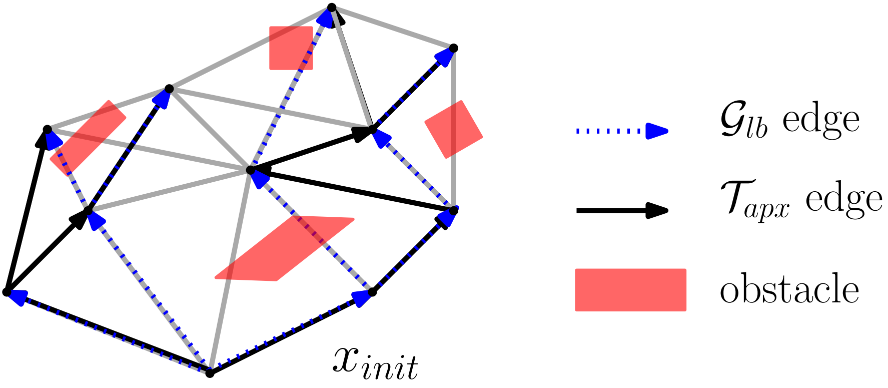

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Given a parameter ε, the following invariants are maintained:

Bounded approximation invariant

For every node $x \in \mathcal{G}_{lb}, \mathcal{T}_{apx}$, $\texttt{cost}_{\mathcal{T}_{apx}}(x) \leq (1+\varepsilon) \cdot \texttt{cost}_{\mathcal{G}_{lb}}(x)$.

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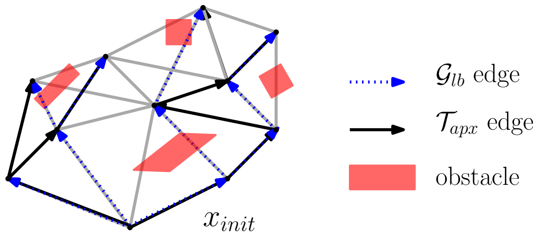

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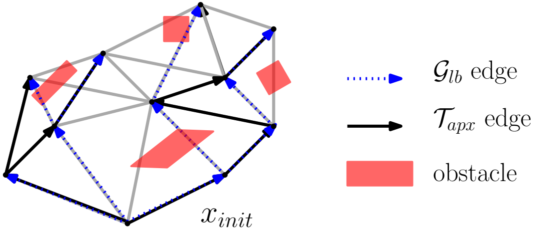

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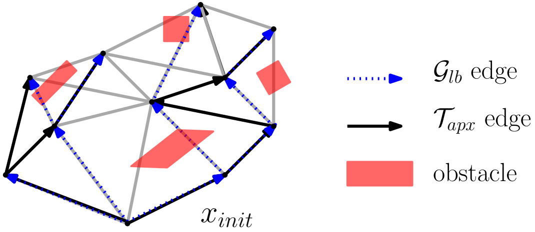

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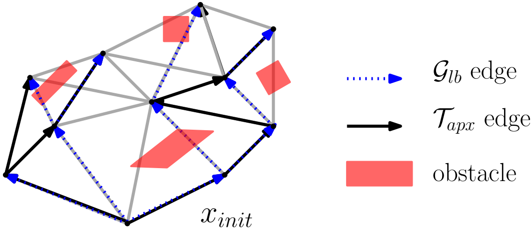
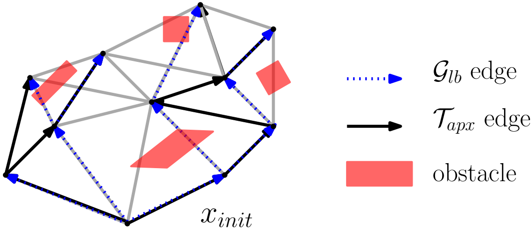
The combination of the two invariants ensure that LBT-RRT is asymptotically near-optimal with an approximation factor of $1+\varepsilon$



Maintaining the trees

LBT-RRT follows the same structure as RRT, RRG RRT* with respect to adding a new milestone



It differs with respect to the additional edges considered

In contrast to RRT* where rewiring is always performed, and RRG where the local planner is called for all all neighbors, LBT-RRT only calls the local planner for edges necessary to maintain the lower bound invariant

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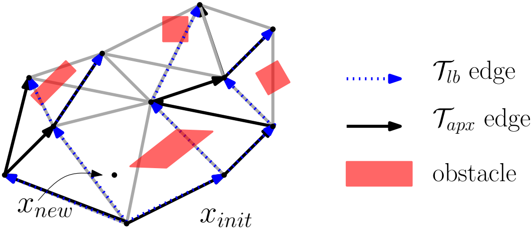

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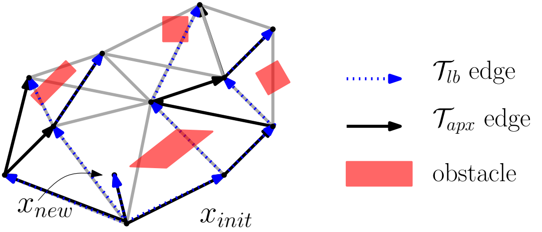

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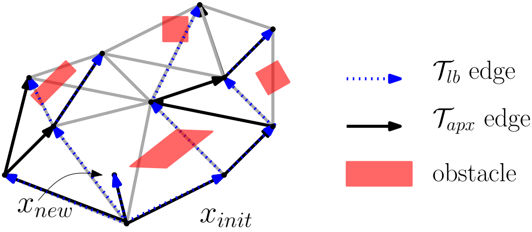

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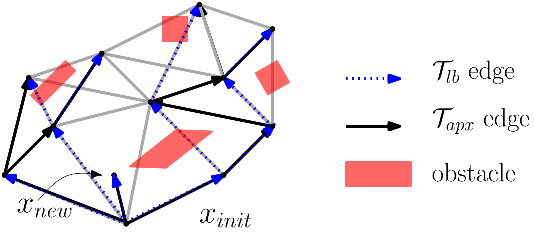

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Dynamic single-sink shortest-path problem (SSSP)

Let $G = (V, E)$ be a graph that undergoes a series of edge insertions and edge deletions.

Maintaining the shortest-path from a given node to every other node in V is referred to as the Dynamic single-sink shortest-path problem

Efficient algorithms for Dynamic SSSP exist (e.g. [RR96, FSN00])

consider_edge(x_1, x_2)

Given two nodes x_1 and x_2 such that:

- $x_{new} \in \{x_1, x_2\}$
- $\|x_1, x_2\| \leq r(n)$

`consider_edge` adds (lazily) the edge (x_1, x_2) to $\mathcal{G}_{lb}$
$\Rightarrow$ $\texttt{cost}_{\mathcal{G}_{lb}}$ possibly decreases

It then ensures that the bounded approximation invariant is maintained for all nodes by:

- Removing (in-collision) edges from $\mathcal{G}_{lb}$
  $\Rightarrow$ $\texttt{cost}_{\mathcal{G}_{lb}}$ possibly increases
- Adding (collision-free) edges to $\mathcal{T}_{apx}$
  $\Rightarrow$ $\texttt{cost}_{\mathcal{T}_{apx}}$ possibly decreases

Updates are performed using the procedures: $\texttt{insert\_edge}_{\texttt{SSSP}}$, $\texttt{remove\_edge}_{\texttt{SSSP}}$

consider_edge(x_1, x_2) - cont.

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while $Q \neq \emptyset$ **do**
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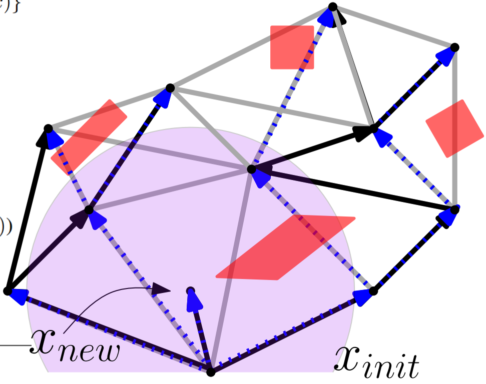

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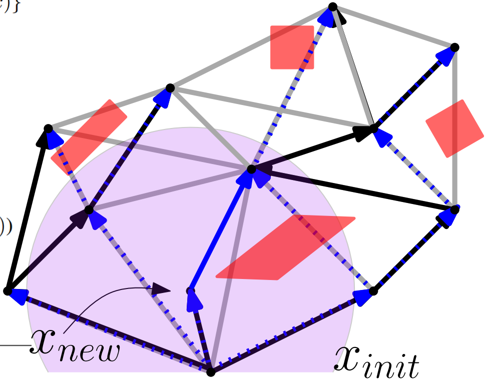

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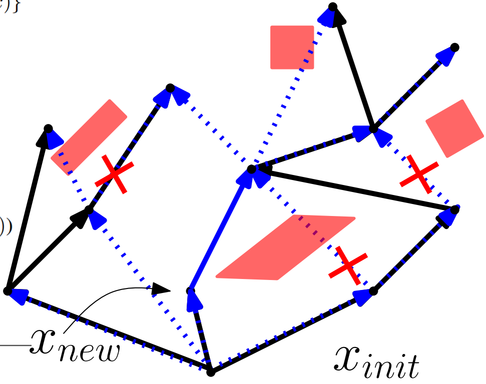

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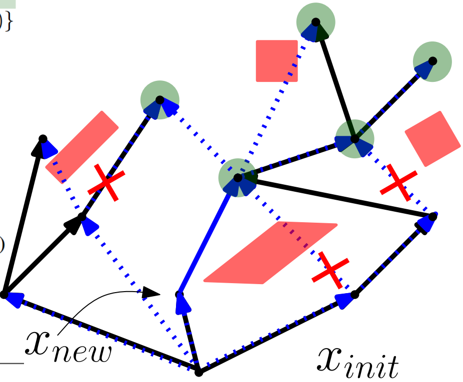

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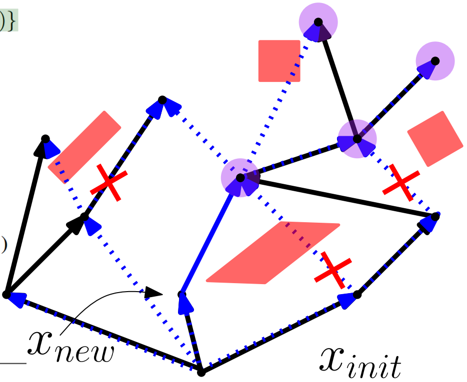

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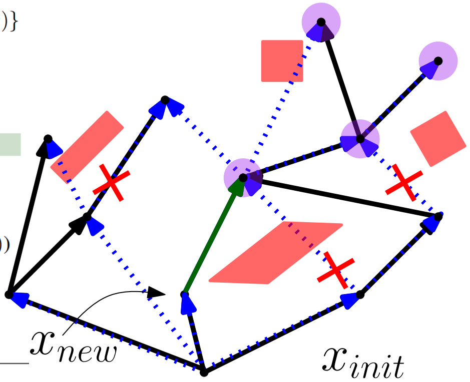

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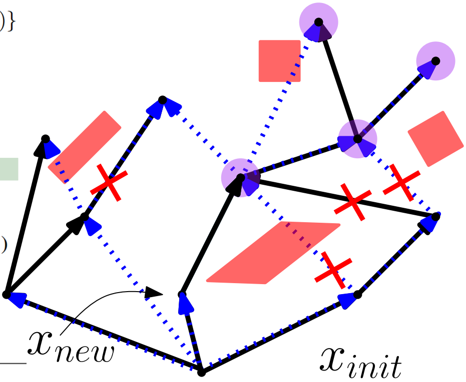

Algorithm 6 `consider_edge`(x_1, x_2)

$I \leftarrow$ `insert_edge`$_{\texttt{SSSP}}(\mathcal{G}_{lb}, (x_1, x_2))$
$Q \leftarrow \{x \in I \mid \text{cost}_{\mathcal{T}_{apx}}(x) > (1+\varepsilon) \cdot \text{cost}_{\mathcal{G}_{lb}}(x)\}$
while $Q \neq \emptyset$ **do**
  $x \leftarrow Q$.`top`();
  if $\text{cost}_{\mathcal{T}_{apx}}(x) > (1+\varepsilon) \cdot \text{cost}_{\mathcal{G}_{lb}}(x)$ **then**
    $x_{parent} \leftarrow$ `parent`$_{\texttt{SSSP}}(\mathcal{G}_{lb}, x)$
    if (`collision_free` (x_{parent}, x)) **then**
      $\mathcal{T}_{apx}$.`parent`$(x) \leftarrow x_{\text{parent}}$
      Q.`pop`()
    else
      $D \leftarrow$ `delete_edge`$_{\texttt{SSSP}}(\mathcal{G}_{lb}, (x_{\text{parent}}, x))$
      for all $y \in D \cap Q$ **do**
        Q.`update_cost`(y)
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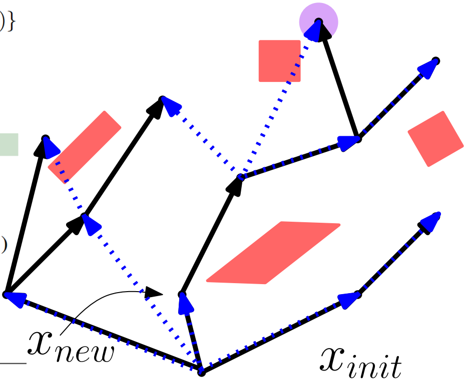

consider_edge(x_1, x_2) - cont.

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    if (collision_free (x_{parent}, x)) **then**
      $\mathcal{T}_{apx}$.parent$(x) \leftarrow x_{\text{parent}}$
      Q.pop()
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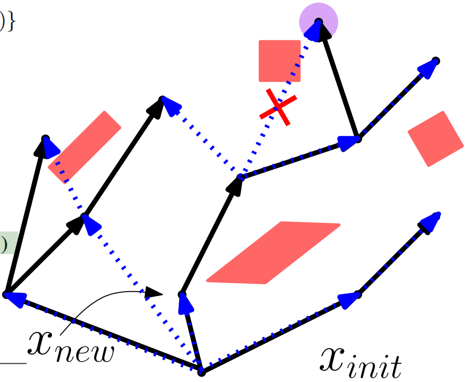

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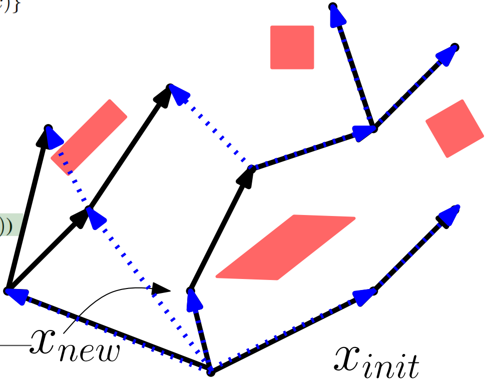

Analysis - lower bound invariant

Observations

- A node x is added to $\mathcal{G}_{lb}$ and to $\mathcal{T}_{apx}$ if and only if x is added to $\mathcal{G}_{RRG}$
- Both LBT-RRT and RRG consider the same set of $k_{RRG} \log(|V|)$ nearest neighbors of x_{new}
- Every edge added to the RRG roadmap is added to $\mathcal{G}_{lb}$
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Corollary

After every iteration of LBT-RRT the lower bound invariant is maintained

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If the bounded approximation invariant holds prior to a call to the procedure $\texttt{consider\_edge}_{\texttt{SSSP}}(x_1, x_2)$ then the procedure will terminate with the invariant maintained

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Maze scenario - 3 degrees of freedom (DOFs)

Scenario taken from OMPL

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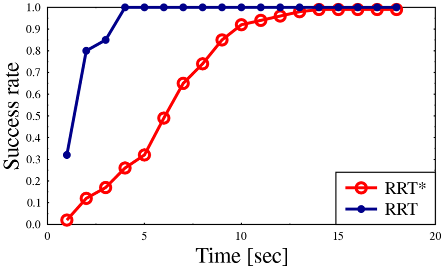

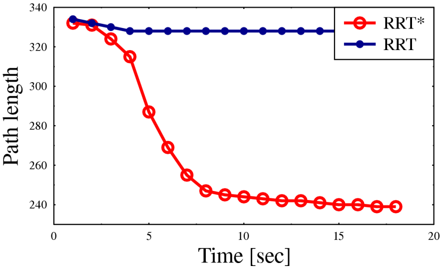

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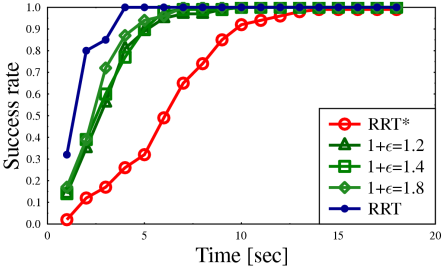

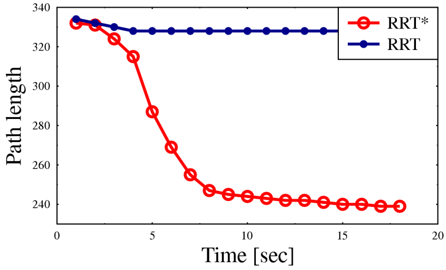

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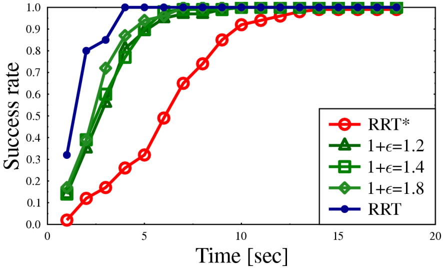

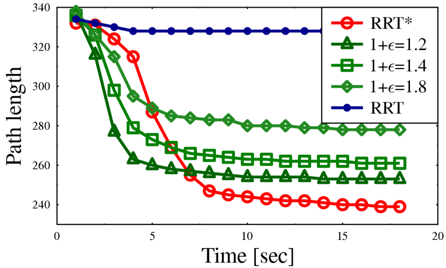

Simulations (cont.)

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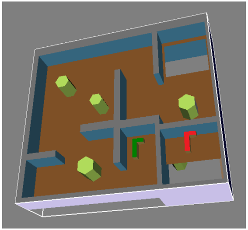
Cubicles scenario (2 robots) - 12 DOFs



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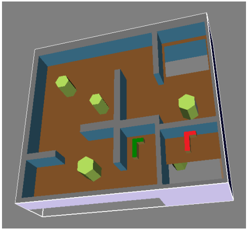

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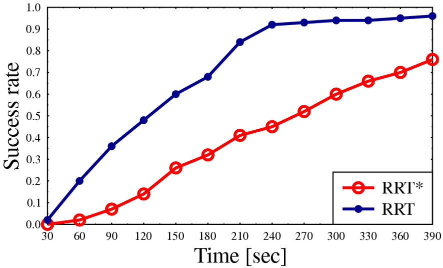

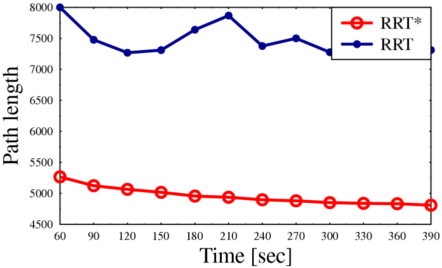

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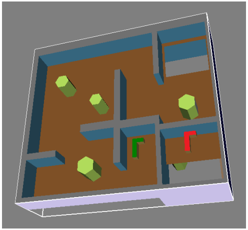

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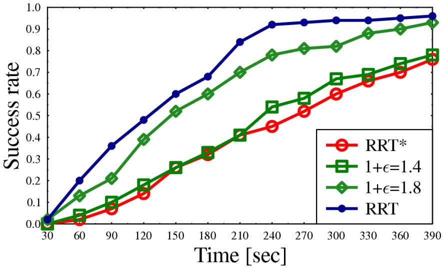

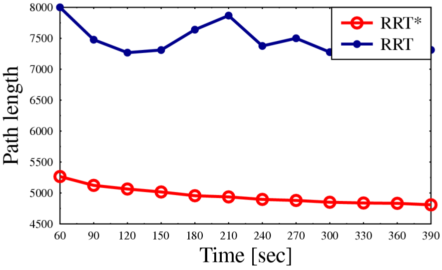

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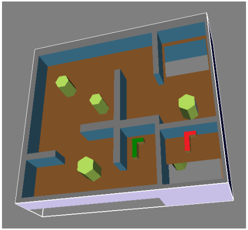

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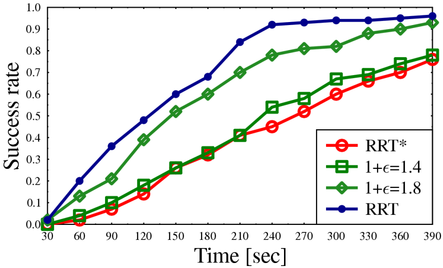

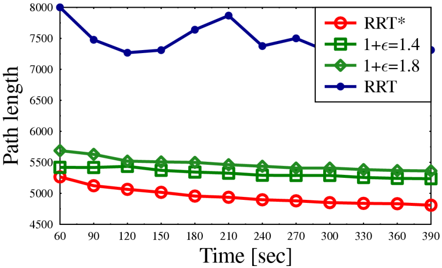

Summary (LBT-RRT)

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Appendix—Prob. completeness of RRT

Thm [LaValle Kuffner01, Kleinbort et al.18]

The RRT algorithm is probabilistically complete.

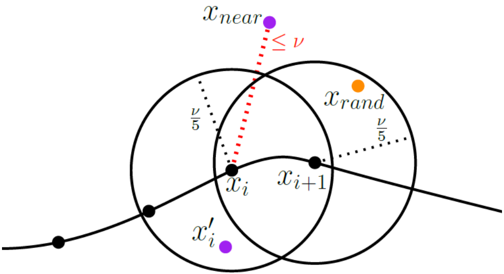


Figure adapted from [Kleinbort et al.18]

Assumptions & notation

- $\exists \pi \subset \mathcal{X}_{\text{free}}$ s.t. $\pi[0] = x_{\text{start}}$ and $\pi[1] = x_{\text{goal}}$
- $\mathcal{X} = [0,1]^d$ (proof is much more complex for the general case)
- $B_r(x)$—ball of radius r centered at x
- δ—clearance of π
- L—length of π
- η—extend parameter
- $\nu := \min(\delta, \eta)$
- $m := \frac{5L}{\nu}$

Sequence of points—X

Define a sequence of $m+1$ points $X = x_0, \ldots x_m$ such that

- $x_i \in \pi$, $x_0 = \pi[0]$, $x_m = \pi[1]$
- The length of the sub-path between every two consecutive points is $\nu/5$
- Thus, $\forall i, ||x_{i+1} - x_i|| \leq \nu/5$
- Define $T = \{B_{\frac{\nu}{5}}(x_i) | 0 < i < m\}$

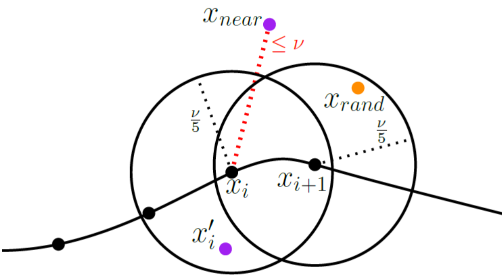
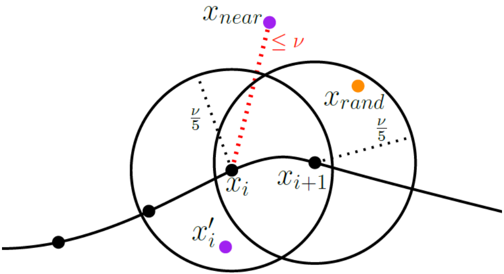


Figure adapted from [Kleinbort et al.18]

Reaching a specific ball

Lemma [Kleinbort et al.18]

Suppose that RRT has reached $B_{\nu/5}(x_i)$, that is, $\exists x'_i \in T \cap B_{\nu/5}(x_i)$. If $x_{\text{rand}} \in B_{\nu/5}(x_{i+1})$, then $\overline{x_{\text{rand}} x_{\text{near}}} \in \mathcal{X}_{\text{free}}$



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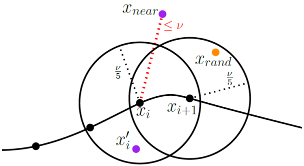
Reaching a specific ball—cont.

$$\begin{aligned}
\|x_{\text{near}} - x_i\| &\leq \|x_{\text{near}} - x_{\text{rand}}\| + \|x_{\text{rand}} - x_i\| && \text{/* triangle inequality */} \\
&\leq \underbrace{\|x_i' - x_{\text{rand}}\|}_{(1)} + \underbrace{\|x_{\text{rand}} - x_i\|}_{(2)} && \text{/*} x_{\text{near}} \text{ is NN */} \\
&\leq \nu && \text{/* 2 + 3 = 5*/}
\end{aligned}$$



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(1) $\|x_i' - x_{\text{rand}}\| \leq \|x_i' - x_i\| + \|x_i - x_{i+1}\| + \|x_{i+1} - x_{\text{rand}}\| \leq 3 \cdot \frac{\nu}{5}$

(2) $\|x_{\text{rand}} - x_i\| \leq \|x_{\text{rand}} - x_{i+1}\| + \|x_{i+1} - x_i\| \leq 2 \cdot \frac{\nu}{5}$

Reaching a specific ball—cont.

$$||x_{\text{near}} - x_i|| \leq ||x_{\text{near}} - x_{\text{rand}}|| + ||x_{\text{rand}} - x_i||$$ /* triangle inequality */

$$\leq \underbrace{||x_i' - x_{\text{rand}}||}_{(1)} + \underbrace{||x_{\text{rand}} - x_i||}_{(2)}$$ /* x_{near} is NN */

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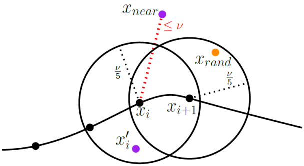

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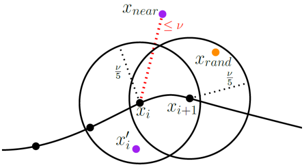

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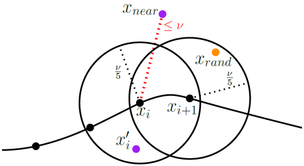

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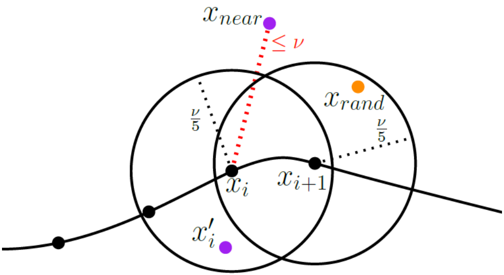
Reaching a specific ball—cont.

- We showed that $||x_{\text{near}} - x_i|| \leq \nu$
- Thus $x_{\text{near}}, x_{\text{rand}} \in B_\nu(x_i)$
- As $\nu \leq \delta$, we have that $\overline{x_{\text{rand}} x_{\text{near}}} \in \mathcal{X}_{\text{free}}$



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Prob. completeness proof

- Assume that $B_{\nu/5}(x_i)$ contains an RRT vertex
- The prob. p of sampling in $B_{\nu/5}(x_{i+1})$ is $|B_{\nu/5}|/|[0,1]^d| = |B_{\nu/5}|$
- If RRT successfully moves from one ball to the next m times, it will find a path

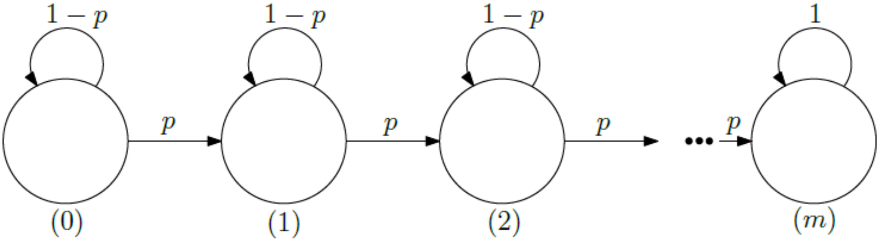

Figure adapted from [Kleinbort et al.18]

Prob. completeness proof (cont.)

- X_n—the number of successes after n samples
- We want to show that $\lim_{n\to\infty} \Pr[X_n < m] = 0$

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\begin{aligned}
\Pr[X_n < m] &= \sum_{i=0}^{m-1} \binom{n}{i} p^i (1-p)^{n-i} \\
&\leq \sum_{i=0}^{m-1} \binom{n}{m-1} p^i (1-p)^{n-i} && /^* m \ll n^* / \\
&\leq \binom{n}{m-1} \sum_{i=0}^{m-1} (1-p)^n && /^* p \leq 1/2^* / \\
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- We just showed that $\Pr[X_n < m] \leq \frac{m}{(m-1)!} n^m e^{-pn}$

As p, m are fixed

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