EXAMPLES IN CATEGORY THEORY

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Abstract. Collection of more than 800 examples illustrating different ideas and applications of category theory. In particular, one can find among them various applications of fundamental category-theoretic concepts like Yoneda lemma, Grothendieck construction, limits and colimits, monads or coend calculus in different branches of mathematics such as topology, differential geometry, algebraic geometry, classical algebra, set theory or probability theory. Special chapters were dedicated also to more theoretical examples illustrating concepts from categorical sheaf theory, operads, enriched categories and higher category theory, were the lack of examples seems to be especially problematic.

Contents

1. Basic concepts

We recall the basic definition of a category mainly in order to clarify some conventions:

Definition 1.1. *A category* **Cat** *is a class of objects* ob(**Cat**) *together with a set of morphisms between each pair of objects* $\hom(X, Y)$ *such that:*

- *for any morphisms* $\varphi \in \hom(X, Y)$ *and* $\phi \in \hom(Y, Z)$ *there is a morphism* $\phi \circ \varphi \in \hom(X, Z)$
- *in every* $\hom(X, X)$ *there is an identity morphism* id*, such that for any other* $\varphi \in \hom(X, X)$ $\mathrm{id} \circ \varphi = \varphi \circ \mathrm{id} = \varphi$

Category defined in such a way, where we require the hom-sets to be sets, is sometimes called locally small. Since we won't consider here categories with hom-sets forming a proper class, it will be called a category in the rest of a text.
By an abuse of notation we will denote $X \in \mathrm{ob}(\mathbf{Cat})$ simply by $X \in \mathbf{Cat}$.

Definition 1.2. *A category is small if its objects form a set.*

Definition 1.3. *A functor* $F : \mathcal{C} \to \mathcal{D}$ *is a mapping associating to each object* X *of* $\mathcal{C}$ *an object* $F(X) \in \mathcal{D}$ *and to each* $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$ *a morphism* $F(f) \in \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ *in such a way that:*

- $F(\mathrm{Id}_{\mathbf{Cat}}) = \mathrm{Id}_{\mathcal{D}}$
- $F(g \circ f) = F(g) \circ F(g)$

Definition 1.4. *For any category* $\mathcal{C}$ *the opposite category* $\mathcal{C}^{op}$ *is a category with the same objects and reversed morphism, i.e. having domain and codomain interchanged.*

TODO

1.1. Categories.

Example 1 (Sets)**.** *Sets with functions form a category* **Set***.*
The category of sets is not small. It has unique initial object, and not unique final objects, singletons. It is complete and cocomplete. The identity functor make it a concrete category.

Example 2 (Algebraic structures)**.** *The most naturally appearing categories besides sets and topological spaces are encoding objects from algebra and their homomorphisms, providing a ton of examples such as*

- *groups* **Grp***, abelian groups* **Ab**
- *vector spaces* $\mathbf{Vect_K}$*, right, left or bimodules* R-**Mod**, **Mod**-R, R-**Mod**-S

- *monoids* **Mon**, *commutative monoids* **CMon**
- *semigroups, magmas*
- *rings* **Ring**, *commutative rings* **CRing**, *non-unital rings* **Rng**, *semirings* **Rig**
- *R-algebras, associative algebras R-***Mod**, *commutative algebras R-$CAlg$*
- *Lie groups* **LieGrp**, *topological groups* **TopGrp**
- *Lie algebras, Hopf algebras, coalgebras, bialgebras, C^*-algebras*
- *Fields*

Example 3 (Anti-homomorphisms)**.** *In algebra besides homomorphisms sometimes functions satisfying dual property $\varphi(ab) = \varphi(b)\varphi(a)$. In category of groups, as a consequence of looking at natural transformation inverting the elements we will see that anti-homomorphism do not bring any new structure there and can be canonically identified with ordinary homomorphism. The situation is drastically different in case of non-commutative rings, where inverting the order of multiplication can result in non-isomorphic objects. In case of such rings it means that no automorphism can be identified with anti-automorphism, so the distinction is significant.*

Rings and anti-homomorphisms do not form a category, as the composition of two anti-homomorphism is always an ordinary homomorphism, as it is nothing more than applying twice the same transposition. However, we can refine the category of rings to contain as arrows both homomorphisms and anti-homomorphism. It's easy to see that composition of them can be modelled by group of order two, with composing with homomorphism acts as identity on classes of maps, while anti-homomorphism flips the class. This construction is useful while working with some special kind of rings, where anti-homomorphisms appear naturally providing some additional structure, so that working in **Ring** *has the effect of lost information. It happens for example in case of group algebras KG, admitting canonical anti-automorphisms given of basis G by the inverse map $g \mapsto g^{-1}$. The same involution with even more significance in applications appears in case of Hecke algebras. The anti-involution is also a key feature of Hopf algebras, where it comes directly from the definition.*

Example 4 (Finite sets)**.** *Finite sets with functions* **Fin** *form a full subcategory category of* **Set**. *The inclusion make it a concrete category. It is not concrete nor cocomplete, but finitely complete and cocomplete.*

Example 5 (Paths)**.** *Given a graph G, we can form a category of its paths. Such a category takes all the vertices of G as objects, and morphisms $\mathrm{Hom}(a, b)$ are all the paths from a to b, where the composition* *of morphisms corresponds to concatenation of paths and the empty path is an identity element in each hom-set.*

Example 6 (Poset). *Any partially ordered set $(P, \geq)$ can be considered as a category, where objects are the elements of P, and hom-sets $\mathrm{Hom}(x, y)$ are singletons when $x \geq y$ and empty otherwise. Such a weird construction of morphisms obviously does not reflect any actual functions, but can be understood as the indicator of the relation $\geq$.*

Example 7 (Maximum). *In a poset $(P, \geq)$, a terminal object is the maximal element of P. Similarly, the initial object is a minimal element. It means that the existence of the initial and terminal objects depends on the considered poset, for example $[0, 1]$ has both initial and terminal objects, $\mathbb{R}$ has neither and $\mathbb{N}$ has initial, but not terminal.*

Example 8 (Wide subcategories). *A wide subcategory is dual to full, meaning that it contains all the objects of the parent, missing only some arrows. For example, for each subcategory there exists a wide subcategory containing only isomorphisms. A more common example is a standard category* **Met** *of metric spaces with short maps, containing only maps that do not increase the distance between points, or the category of smooth manifolds with smooth maps, which is the wide subcategory of the full subcategory of* **Top***, which objects are metric spaces. In general, every subcategory can be described as a wide subcategory of some full subcategory (as well as a full subcategory of a wide one).*

Example 9 (Delooping). *Every discrete group G naturally form a category $\mathbf{B}G$, called delooping of G, with a single object and arrows $\mathbf{B}G(\{\bullet\}, \{\bullet\}) = G$. The composition function $\mathbf{B}G(\{\bullet\}, \{\bullet\}) \times \mathbf{B}G(\{\bullet\}, \{\bullet\}) \to \mathbf{B}G(\{\bullet\}, \{\bullet\})$ is just the multiplication in G, and the identity arrow is identified with neutral element $e \in G$. Note that since every element of g is invertible, every morphism in $\mathbf{B}G$ is an isomorphism. In general, every category with single element with only invertible arrows corresponds to a group in the obvious way. This allows us not only to realise groups as subcategory of* **Cat***, but even to define a group as such category with these properties, since categorical definition as always the coolest.*

Similarly, when M is a monoid, the same construction yields its delooping $\mathbf{B}M$ with single object, and this time not necessarily invertible arrows $\mathbf{B}M(\{\bullet\}, \{\bullet\}) = M$. As each category with single object can be identified with monoid of endomorphisms $\mathcal{C}(\{\bullet\}, \{\bullet\})$, we can define monoids as categories with single object.

Not that, categorically, definition of monoid is more fundamental than the definition of a group. Later we will see further manifestation *of this motive, with monoidal categories serving as fundamental objects in modern category theory together with their monoidal products, extending the interplay between arrows and monoids to more sophisticated structures.*

1.2. **Functors.**

Example 10 (Powerset)**.** *Powerset of a set form a functor in several different ways. It can be seen as a covariant functor* $\mathbf{Set} \to \mathbf{Set}$*. A function* $X \to Y$ *induces a function* $P(X) \to P(Y)$*, where each subset* $A \subseteq X$ *is send to its image under* f*:* $P(f)(A) = f(A) \in P(Y)$*. However, powerset form also a contravariant functor* $\mathbf{Set}^{op} \to \mathbf{Set}$*, as taking a subset to a preimage under* f *is also functorial:* $P(f)(B) = f^{-1}(B) \in P(X)$*. The other way of making a functor out of the powerset is to add to it some additional structure. Note that a powerset naturally form a partial order under inclusion. Such partial ordering is also functorial, yielding another functor* $P : \mathbf{Set} \to \mathbf{Poset}$*.*

Example 11 (Counting elements)**.** *Consider a category of finite sets with inclusions* $\mathbf{IFin}$*. There is a cardinality functor* $\mathbf{IFin} \to \mathbf{B}\mathbb{N}$*, mapping each set* X *to its cardinality* $|X|$*. Since a domain of an inclusion has no more elements than the codomain, it induces a relation* $|X| \leq |Y|$*, thus is indeed functorial. Note that this result is not true in case of category* $\mathbf{Fin}$*, as any surjection which is not injective would induce a morphism from in the wrong direction, contradicting the total order on* $\mathbb{N}$*.*

We can also consider another variation of such a functor. Given any finite set X*, there is a functor* $|-| : P(X) \to \mathbf{B}\mathbb{N}$*, where* $P(X)$ *is a predered powerset of* X*. This map is clearly monotone, which is the same as a functor between preorders.*

Example 12 (Cardinality)**.** *We can form a preorder category* $\mathbf{ICard}$*, which objects are cardinalities of sets, to extend the previous example to any sets with injections. The cardinality functor can be constructed also without restricting to injective functions. Regarding* $\mathbf{Card}$ *as full subcategory of sets, denote its inclusion as* i*. Since there is a natural bijection* $c_X : X \to |X|$*, the identity of* $\mathbf{Card}$ *extends to a functor* $|-| : \mathbf{Set} \to \mathbf{Card}$*, sending* X *to* $|X|$ *and an arrow* $f : X \to Y$ *to the composition* $c_Y \circ f \circ c_X^{-1} : |X| \to |Y|$*.*

Example 13 (Uppersets)**.** *Consider any preorder* P*. An upper set in* P *is a upward closed subset, satisfying condition*

$$x \in U, y \geq x \Rightarrow y \in U$$

Uppersets of a preorder are also naturally preordered by inclusion, which provide an endofunctor **Pre** $\to$ **Pre**, *mapping* P *to its upperset preorder* $U(P)$.

Example 14 (Preorder of objects)**.** *From any small category* $\mathcal{C}$ *we can construct a preorder of objects* $P(\mathcal{C})$, *having* $\operatorname{ob}\mathcal{C}$ *as elements with relation* $X \leq Y$ *iff there is an arrow* $X \to Y$. *This construction forms a functor*

$$\mathbf{Cat} \to \mathbf{Pre}$$

Moreover, for any $\mathcal{C}$ *there is an unique functor* $\mathcal{C} \to P(\mathcal{C})$.

Example 15 (Chromatic functor)**.** *Given graph* G, *a regular colouring of* G *with a set* S *is a colouring in a usual sense - a labelling of vertices of* G *with elements from* S, *which is just a function* $V(G) \to S$, *such that if vertices are connected with an edge, their images are distinct. Colourings of a fixed graph assemble into an endofunctor on sets with injective maps*

$$\underline{\chi}(G, -) : \mathbf{Set}_{inj} \to \mathbf{Set}_{inj}$$

as any injection $S \hookrightarrow T$ *induces injection on the colouring* $\underline{\chi}(G, S) \hookrightarrow \underline{\chi}(G, T)$ *by composition.*

Example 16 (General linear groups)**.** *To any ring with unit we can associate it's linear group in a functorial way*

$$GL_n : \mathbf{Ring} \to \mathbf{Grp}$$

The morphism induced by any homomorphism $f_* : GL_n(R) \to GL_n(S)$ *is simply the composition with the coefficients of the matrix. Since* f, *as a homomorphism of rings, must map units to units, it preserves the unitality of all the determinants, which proves that the images of* f_* *are also invertible.*

Example 17 (Units)**.** *The other example of a commonly considered functor* $-^x\mathbf{Cring} \to \mathbf{Set}$ *maps a ring to it's group of units. In this case, the induced morphisms are practically unchanged, but theoretically the difference is huge - even though they acts the same as functions of underlying sets, now we consider them as homomorphisms of groups, not rings. In particular,* $f^* \notin H\operatorname{Hom}(R, S)$, *even though* $f \in \operatorname{Hom}(R, S)$ *and* f^* *was just taken to be* f.

Example 18 (Center is not a functor)**.** *However most algebraic operations or object are functorial and can be expressed categorically, there are some rare exceptions. Most notable non-functorial algebraic object is the center of a group. Even though the map* $G \mapsto Z(G)$ *is perfectly* *well-defined, construction of induced morphisms is impossible, as images of central elements need not to be central in general. There are lots of counterexamples: for example given any non-central element g of finite order n, consider the homomorphism*

$$\mathbb{Z}/n\mathbb{Z} \to G$$

mapping $1 \mapsto g$.

Example 19 (Permutations are not a functor). *Given any set X, we can form its group of automorphisms* $\mathrm{Aut}(X)$, *which is just a permutation group generated by X. However, such construction is not functorial, as there is no canonical way of constructing a homomorphism* $\mathrm{Aut(X)} \to \mathrm{Aut}(Y)$ *from a function $X \to Y$.*

Example 20 (Endomorphisms are not a functor). *By definition of a category, for any object $X \in \mathcal{C}$ its set* $\mathrm{Hom}(X,X)$ *form a monoid under composition. However, such map of objects $\mathcal{C} \to$* **Monoid** *do not naturally extend to a functor. Each morphism $X \to Y$ induces a morphism of sets* $\mathrm{Hom}(X,X) \to \mathrm{Hom}(X,Y)$, *however we have any natural arrow* $\mathrm{Hom}(X,X) \to \mathrm{Hom}(Y,Y)$.

Example 21 (Group representations). *Considering group G as a category with one object $\mathbf{B}G$, every functor*

$$\mathbf{B}G \to \mathbf{Vect}_{\Bbbk}$$

Can be identified with some $\Bbbk$-linear representation of G. The image of the single object $F(\bullet)$ specifies the vector space that is acted on by G, while the functoriality of F is equivalent to specifying the homomorphism of groups, corresponding to some representation.

$$G \to \mathrm{Aut}(V)$$

Example 22 (Permutation representations). *Let G be a finite group. Besides linear representations, we can consider also its permutation representation of G, where G acts by permutations on some finite set. The category of permutation representations of G can be identified with the functor category* $[\mathbf{B}G, \mathbf{Fin}]$.

Example 23 (Contravariant partitions). *A set of partitions of a set X, i.e. families of disjoint subsets $\{X_i \subseteq X\}$ satisfying $\bigcup X_i = X$, naturally form a preorder under refinement. Moreover, partitions of X admit also more convenient characterisations, as they can be uniquely identified with a surjection $X \to P$, where the sets X_i are in correspondence to preimages of points from P. Partitions of sets naturally assemble into a contravariant functor* $\mathbf{Set}^{op} \to \mathbf{Pre}$. *Given a function $X \to Y$ and a partition of Y associated to a surjection $Y \twoheadrightarrow P$, we* *obtain from composition a map $X \to P$. Such a map need not be surjective in general, but we can use the unique epi-mono decomposition to obtain a surjection $X \twoheadrightarrow Q \hookrightarrow P$ anyway. Since the surjective part makes Q a partition, while the injective part ensure the monotonicity, together it establish a functor*

$$\mathrm{Part}^{op} : \mathbf{Set}^{op} \to \mathbf{Pre}$$

Example 24 (Parallel transport)**.** *In differential geometry, two points from a manifold can be connected by a path. Given a tangent bundle of M, points from tangent spaces of x and y also can be moved by such a path by an operation called a parallel transport. Parallel transport has a nice categorical interpretation as a functor from path groupoid $P(M)$ to vector spaces $p : \mathbf{Vect}_{\mathbb{R}}$, which maps a point to its tangent space $p(x) = T_xM$ and a curve $\gamma : x \to y$ to its parallel transport $T_xM \to T_yM$.*

Example 25 (Towards Lebesgue integrals)**.** *Consider a category $\mathcal{C}_X$ with objects being continuous functions $X \to \mathbb{R}$. The homsets $\mathrm{Hom}(f, f)$ contain only the identity, and $\mathrm{Hom}(f, g)$ has a single element if $f(x) > g(x)$ for all x, otherwise being empty. Such a category is just a partial order on the set of real functions. Let $\mathbf{Top}(X)$ be the poset category of open subsets of X. In such a setting we can construct a functor $T : \mathcal{C}_X \to \mathbf{Top}(X)$, which maps a function f to the open set $f^{-1}(\mathbb{R}^+)$. If your familiar with measure theory, you can probably see that such a functor is indeed use in the process of integration (however, obviously, such a well-behaved setting of continuous functions is not really interesting; feel free to generalize this idea to all measurable functions and the category of σ-algebras)*

Example 26 (Alexandroff topology)**.** *There is a functorial way of constructing a topology on every poset $(P, \geq)$, called the Alexandroff topology. A subset $U \subseteq P$ in this topology is considered open if and only if is downward closed, meaning that for every $x \in U$, all elements smaller than x are also elements of U. It is easy to show that a continuous function between posets with Alexandroff topologies are exactly monotone functions, thus considered construction is functorial*

$$A : \mathbf{Poset} \to \mathbf{Top}$$

Example 27 (Comma category)**.** *Given two functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ we can consider the category $(F \downarrow G)$, where objects maps $\alpha : F(X) \to G(Y)$, and morphisms between α and α' are compatible pair of morphisms $f : X \to X'$ and $g : Y \to Y'$, such that the square*

(that can be identify the the morphism itself)

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(X') \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\alpha'} \\ G(Y) & \xrightarrow{G(g)} & G(Y') \end{array}$$

Example 28 (Rectangular bands)**.** *A band is a idempotent semigroup (i.e. with all elements idempotent). A specific class of bands, called rectangular, are also nowhere commutative, satisfying the relation* $xyx = x$*. The category of non-empty rectangular bands is equivalent to* $\mathbf{Set}_+ \times \mathbf{Set}_+$*, where* $\mathbf{Set}_+$ *is a category of non-empty sets. The equivalence is given by the universal presentation of a rectangular band as a product of two sets* $X \times Y$ *with multiplication* $(a, b)(c, d) = (a, d)$*.*

Example 29 (Group structures)**.** *A notion of a fiber from topology or algebra generalises gently to functors, producing some interesting constructions. Given a functor* $F : \mathcal{C} \to \mathcal{D}$*, its fiber* $F^{-1}(Y)$ *for some fixed object* $Y \in \mathcal{D}$ *is a maximal set of elements* $X \in \mathcal{C}$ *together with an induced isomorphism* $F(X) \to Y$*. Even from such an almost trivial functor as forgetful* $U : \mathbf{Grp} \to \mathbf{Set}$*, each fiber* $U^{-1}(S)$ *is quite an interesting object, classifying all the group structures on* S*.*

Example 30 (Classifying topologies)**.** *Similarly as group structure, fibers of a forgetful functor* $U : \mathbf{Top} \to \mathbf{Set}$ *classify all topological structures constructible on* U*. Similarly with other forgetful functors one can obtain all metric structures, measures, bases of vector spaces or even manifolds.*

1.3. **Category algebras.**

Example 31 (Convolution algebra)**.** *From any small category* $\mathcal{C}$ *and ring* R *we can construct the convolution (non-unital for infinite* $\mathcal{C}$*) algebra* $R[\mathcal{C}]$*. It has an underlying free* R *module generated by the arrows of* $\mathcal{C}$ *with multiplication defined on basis as*

$$f \cdot g = \begin{cases} f \circ g & f \text{ and } g \text{ are composable} \\ 0 & \text{otherwise} \end{cases}$$

Similarly as in case of group algebras, we can identify elements of $R[\mathcal{C}]$ *as functions* $\operatorname{arr}(\mathcal{C}) \to R$ *with finite supports. It this language the multiplication can be expressed as a convolution*

$$(\phi\varphi)(f) = \sum_{g \circ h = f} \phi(g)\varphi(h)$$

In case of discrete categories, $R[\mathcal{C}]$ is just the algebra of functions $\operatorname{ob}\mathcal{C} \to R$. As already hinted, the convolution algebra of the delooping $\mathbf{B}G$ is just a group algebra $R[G]$.

Note that the convolution algebra do not form a functor $\mathbf{Cat} \to R\text{-}\mathbf{Mod}$, as equivalent categories with different number of arrows have convolution algebras of different dimensions.

Example 32 (Matrix algebra)**.** *From a set we can construct a category in two canonical ways - discrete and codiscrete, the latter known also as the pair groupoid $\mathbf{Pair}(X)$. Its arrows can be identified with the set $X \times X$, with projections playing a role of source and target maps. The convolution algebra associated to the pair groupoid is the matrix algebra, with entries valued in R and size $|X| \times |X|$.*

Example 33 (Fine incidence algebra)**.** *A category $\mathcal{C}$ is called finely finite if any arrow $f : A \to C$ can be decomposed as $A \to B \to C$ in finitely many ways. Given finely fine category and a ring R, the fine incidence algebra of $R\mathcal{C}$ is constructed with the same convolution formula as convolution algebra $R[\mathcal{C}]$, but the functions $\operatorname{ar}(\mathcal{C}) \to R$ need not be compactly supported - it can be done, as the convolution formula is well defined due to finely finiteness of $\mathcal{C}$.*

Every fine incidence algebra has a special element called fine zeta function defined as a function with constant value 1 and a unit given by indicator of identity arrows . Its inverse, if exists, is called the fine Möbius function. In this case we say that $\mathcal{C}$ admits a fine Möbius inversion. A functor $F : \mathcal{C} \to \mathcal{D}$ between finely finite categories induce an algebra homomorphism $R\mathcal{C} \to R\mathcal{D}$ if and only if it is bijective on objects and has finite fibers. After denoting the category with such arrows as $\mathbf{Cat}_!$, we get a functor

$$R- : \mathbf{Cat}_! \to R\text{-}\mathbf{Mod}$$

In case of finite categories fine incidence algebras coincide with convolution algebras. In particular, for a finite group G, $R\mathbf{B}G \simeq R[\mathbf{B}G] \simeq R[G]$. Except trivial cases, it never admits a Möbius inversion.

Example 34 (Classical Möbius function)**.** *The best known and very important case of fine incidence algebra is associated to preorders. Preorder is finely finite if and only if is locally finite, i.e. each interval in P is finite. The objects of $R(P)$ are functions $\{(a, b) \in P^2 \mid a \leq b\} \to R$. Every locally finite poset admits a fine Möbius inversion, with Möbius function*

$$\mu_{\mathcal{C}}(a, b) = \sum_{n=1}^{\infty} (-1)^n |\{\text{chains } a = a_0 < a_1 < \cdots < a_n = b\}|$$

In case of $\mathcal{C} = \mathbb{Z}$, $\mu_{\mathbb{Z}}(n, m)$ *coincides with the classical Möbius function* $\mu(m/n)$ *known from number theory*

$$\mu(n) = \begin{cases} 1 & n \textit{ is a product of even number of distinct primes} \\ -1 & n \textit{ is a product of odd number of distinct primes} \\ 0 & \textit{otherwise} \end{cases}$$

Example 35 (Coarse Möbius inversion)**.** *Let* $\mathcal{C}$ *be a finite category. In this case we can construct its category in a third different way. Its course incidence algebra* $R_c(\mathcal{C})$ *has all the functions* $\operatorname{ob}\mathcal{C} \times \operatorname{ob}\mathcal{C} \to R$ *as generators and multiplication defined as*

$$(fg)(a, b) = \sum_{c \in \mathcal{C}} f(a, c) g(c, b)$$

Its unit is the Kronecker delta, indicator of the diagonal. The course zeta function is defined as

$$\zeta_{\mathcal{C}}(a, b) = |\operatorname{Hom}(a, b)|$$

and its inverse, if exists, is called the course Möbius function. Let's inspect some special cases

- *for finite posets* $R_c(P)$ *are functions* $P \times P \to R$*. It has fine incidence algebra as subalgebra, with fine and course Möbius functions coinciding*
- *in case of finite monoids* $R_c(\mathbf{B}M) \simeq R$ *and* $\zeta = |M|$
- *A category with course Möbius inversion must be skeletal*
- *All groupoids, categories with no non-trivial idempotents or admitting epi-mono factorisation have course Möbius inversion over* $\mathbb{Q}$*.*
- $R_c(\mathcal{C}) \simeq R(\mathbf{Pair}(\operatorname{ob}\mathcal{C}))$ *(thus it is functorial)*

Example 36 (Euler characteristic of categories)**.** *Coarse Möbius inversion provides a way of extending the Euler characteristic to some class of finite categories. If* $\mathcal{C}$ *is finite and admits the coarse Möbius inversion we define*

$$\chi(\mathcal{C}) = \sum_{a, b \in \mathcal{C}} \mu_{\mathcal{C}}(a, b)$$

Among two key features making this generalisation reasonable we have

- *relation with topological Euler characteristics* $\chi(\mathcal{C}) = \chi(|N\mathcal{C}|)$
- χ *depends only on underlying directed graph*

Example 37 (Patch incidence algebra)**.** *We can extend the construction of course incidence algebras to a larger class of categories, called* *patch finite. A patch* $[a,b]_{\mathcal{C}}$ *is the full subcategory of* $\mathcal{C}$ *containing objects* c *such that there exist a diagram* $a \to c \to b$*.* $\mathcal{C}$ *is called patch finite if all patches are finite, or equivalently, if is finely finite with finite homsets. The algebra* $R_p(\mathcal{C})$ *can then be defined equivalently as*

- $R(P(\mathcal{C}))$*, where* $P(\mathcal{C})$ *is the preorder of objects*
- *functions* $\operatorname{ob}\mathcal{C} \times \operatorname{ob}\mathcal{C} \to R$ *vanishing on pairs with empty patch with multiplication*
$$(fg)(a,b) = \sum_{c \in [a,b]_{\mathcal{C}}} f(a,c)g(c,b)$$
- *in case of finite* $\mathcal{C}$*, subalgebra of* $R_c(\mathcal{C})$ *defined as above*

The patch zeta function plays a role of generalised course zeta, thus is called just course zeta, as these two coincides on finite categories.

1.4. **Natural transformations.**

Example 38 (Functor category)**.** *For each small category* $\mathcal{C}$ *and any category* $\mathcal{D}$*, functors* $\mathcal{C} \to \mathcal{D}$ *form a category* $[\mathcal{C},\mathcal{D}]$ *with natural transformations as morphisms.*

Example 39 (Presheaves)**.** *For any category* $\mathcal{C}$ *(most commonly* **Ab**, **Ring***or* $R-$**Mod***) and topological space* X *the* $\mathcal{C}$*-presheaf over* X *are the contravariant functors from open subsets of* X *to* $\mathcal{C}$*, naturally forming the category* $[\mathbf{Top}(X)^{op},\mathcal{C}]$*, since* $\mathbf{Top}(X)$ *is always small.*

Example 40 (Gradation)**.** *Given any small discrete category* S *(which is nothing more than a set) and any category* $\mathcal{C}$*, we can construct the* S*-graded* $\mathcal{C}$ *as the category* $[S,\mathcal{C}]$*. The most common examples are graded rings, algebras or modules, with the set taken to be either* $\mathbb{N}$ *or* $\mathbb{Z}$*, however gradations by other sets such as* $\mathbb{Z}^2$ *are also in use reasonably common.*

Example 41 (Inverse of group)**.** *Note given any group* G *the function* $g \mapsto g^{-1}$ *is always a bijection, however not a homomorphism whenever* G *is not abelian. It is so called antihomomorphism, satisfying property dual to homomorphisms*
$$\varphi(ab) = \varphi(b)\varphi(a)$$
However, from the categorical perspective it is not less natural then ordinary homomorphism. Its construction is strictly dual - each group homomorphism is nothing more than a covariant functor between deloopings, while antihomomorphism are just contravariant functors
$$\mathbf{BG} \to \mathbf{BH}^{op}$$

Note also that the inverse map can in fact be seen as a homomorphism, but between G and its inverse group G^{op}. The construction of inverse groups is canonical and form the automorphism of op : **Grp** *and the inverse map forms a natural transformation $\mathbf{1}_{\mathbf{Grp}} \Rightarrow op$, which is also an isomorphism of functors. The conclusion we may take away is that every group in practice can be simply identified with its inverse group at no real cost, as this construction is an canonical isomorphism, natural in every aspect. The same reasoning can be repeated for every commutative algebra as well, however in case on non-commutative rings it fails on every level -*

Example 42 (Relations). *The category* **Rel***, which objects are sets, and morphisms are binary relations $R \subseteq X \times Y$ form a category. Note that a category of sets can be seen as a subcategory of* **Rel***, where a function $f : X \to Y$ is identified with a relation $\{(x, f(x))\} \subseteq X \times Y$. The category* **Rel** *has a rare feature of being isomorphic to its dual category, where the isomorphism is given as identity on objects, and on relations as a permutation $(x, y) \mapsto (y, x)$.*

Example 43 (Upper and power sets). *Recall that any set, and in particular any preorder, has a natural preordered powerset $P(X)$. Moreover, any preorder induces yet another preorder on subsets $U(X)$, constructed only from uppersets of X. There is a natural transformation between these functors $U \Rightarrow P$ araising from just an inclusion $U(X) \hookrightarrow P(X)$.*

Example 44 (Determinant). *Consider two already considered functors* **Ring** $\to$ **Grp** *- general linear group GL_n and group of units U. There is a natural transformation $GL_n \Rightarrow U$ given by determinant of a matrix. Since the matrices from general linear groups are invertible by assumption, its determinants are units of R, so $det_{n,R}(M) \in U(R)$ is a well defined function $GL_n(R) \to U(R)$. It is a functor due to the fundamental property of determinants $det(MN) = det(M)det(N)$. Moreover, its obviously also natural under change of rings.*

Example 45 (Not natural isomorphisms). *Recall that finitely generated abelian groups are all isomorphic to finite product of cyclic groups, as stated in their classification theorem. However, such isomorphism is not natural, in a sense that there is no corresponding natural isomorphism between identity and some endofunctor $F : \mathbf{Ab}_{FG} \to \mathbf{Ab}_{FG}$ with values $F(G) = \mathbb{Z}^k \oplus \bigoplus^m \mathbb{Z}_{n_i}$.*

The similar problem is encountered in classification of finite dimensional vector spaces as all isomorphic to K^n. Their lack of naturality *follows from the ambiguity of choosing a basis, not doable in a canonical way.*

Another not natural isomorphism of similar spirit arises from the Whitney embedding theorem of manifolds, saying that every smooth manifold is diffeomorphic to some subspace of $\mathbb{R}^n$, but does not construct such embedding

Example 46 (Cardinality)**.** *Consider two already considered functors* $\mathbf{Ring} \to \mathbf{Grp}$ *- general linear group GL_n and group of units U. There is a natural transformation $GL_n \Rightarrow U$ given by determinant of a matrix. Since the matrices from general linear groups are invertible by assumption, its determinants are units of R, so $det_{n,R}(M) \in U(R)$ is a well defined function $GL_n(R) \to U(R)$. It is a functor due to the fundamental property of determinants $det(MN) = det(M)det(N)$. Moreover, its obviously also natural under change of rings.*

Example 47 (Finite relations as boolean matrices)**.** *The category* **FinRel**, *a full subcategory of* **Rel** *with objects being finite sets, not only is isomorphic to its dual via the argument considered above, but also features additional equivalence with the category of boolean matrices, failing in the case of full category* **Rel**. *The category* **MatBool** *has objects indexed by natural numbers, and morphisms $n \to m$ are all the $n \times m$ matrices over 2-element Boolean semiring.*

Example 48 (Dagger relations)**.** *Given relation P, it has a natural opposite relation P^{op}. The identity function on underlying set of P is monotone as a morphism $P \to P^{op}$ if and only if the preorder is symmetric. Such preorders, called dagger preorders, are just equivalence relations. Symmetric preorders can be constructed from any preorder using the functor of symmetric closure, adding all opposite arrows to P. Thus the identity on sets induces a natural transformation of symmetric closure and its opposite, both considered as endofunctors* $\mathbf{Pre} \to \mathbf{Pre}$

$$id : \mathrm{Sym} \Rightarrow \mathrm{Sym}^{op}$$

Example 49 (Uppersets as functors)**.** *The upperset set-valued functor* $U : \mathbf{Pre}^{op} \to \mathbf{Set}$ *is representable by the preorder* $\mathbf{Bool} = \{0 < 1\}$, *meaning that there is a natural isomorphism of functors* $\mathrm{Hom}(-, \mathbf{Bool})$ *and U, as any upperset uniquely corresponds to its indicator function, and any indicator function is monotone if and only if the set is upper. But in this case we can get in fact more than representability, which later we'll generalised as enrichment. Any set of monotone functions between posets has naturally induced poset structure as well, with the relation $f \leq g$ checked pointwise, i.e. $\forall_x f(x) \leq g(x)$. This way the ordinary preorder-valued upperset functor can be constructed with this* *additional structure as well, as induced preorder on* $\mathrm{Hom}(X, P)$ *with the preorder on* $U(X)$ *induced by inclusion.*

Example 50 (Equivalent matrices)**.** *Recall that a matrix category* $\mathbf{Mat}_R$ *was a category having natural numbers as object, corresponding to matrix sizes, and matrices* $m \times n$ *as elements of* $\mathrm{Hom}(m, n)$. *Such category however does not provide sensible notion of a isomorphism between matrices, since it decodes them only as morphisms, not objects. Considering some functor categories fixes that problem. Objects of the category* $[\mathbf{2}, \mathbf{Mat}_R]$ *can be identified with matrices, as the values of* F *on two objects determines its shape, and the values on the non-identify arrow chooses the matrix. In this category two matrices are isomorphic if and only if they are equivalent, with the usual meaning of the equivalence from linear algebra:* $A \simeq B$ *iff* $A = PBQ^{-1}$. *Such an equivalence of matrices comes from the natural isomorphism of their corresponding functors*

$$\begin{tikzcd} F(0) \arrow[r, "A"] \arrow[d, dotted, "P"] & F(1) \arrow[d, dotted, "Q"] \\ G(0) \arrow[r, "B"] & G(1) \end{tikzcd}$$

Such a natural equivalence exists if $PB = AQ$, *and it is an isomorphism if matrices* P *and* Q *are invertible, which translates to the equivalence of matrices.*

Example 51 (Similar matrices)**.** *There is a different way of getting matrices as objects in a functor category. Regarding* $\mathbb{N}$ *as a categorified monoid, i.e. a category with a single object and arrows identifiable with non-negative integers, the objects of the functor category* $[\mathbb{N}, \mathbf{Mat}_R]$ *can be identified with square matrices via the morphism* $F(1) \in \mathbf{Mat}_R$. *In this case no non-square matrices can be obtained, and the morphisms between them also differ. For example two equivalent matrices need not be isomorphic anymore, since isomorphism now corresponds to stronger condition, requiring matrices to be similar. The natural transformation is now a single morphism* $F(\bullet) \to G(\bullet)$ *which commutes with* $A = F(1)$ *and* $B = G(1)$, *which can be described via the following diagram*

$$\begin{tikzcd} F(\bullet) \arrow[r, "F(1)"] \arrow[d, dotted, "P"] & F(\bullet) \arrow[d, dotted, "P"] \\ G(\bullet) \arrow[r, "G(1)"] & G(\bullet) \end{tikzcd}$$

Such a functor is an isomorphism if and only if it has an inverse, so where the matrix P is invertible, resulting in the similarity of matrices $A = PBP^{-1}$.

Example 52 (Arrows in vector spaces)**.** *Consider the arrow category* $\mathbf{aFinVect}_K$ *of finite dimensional K-vector spaces, i.e. having homomorphisms of vector spaces as objects and squares of a form*

$$\begin{tikzcd} V \arrow[r] \arrow[d, "f_1"] & W \arrow[d, "f_2"] \\ V' \arrow[r] & W' \end{tikzcd}$$

This category is naturally equivalent to the category of modules over the ring of lower triangular 2×2 matrices

$$A = \begin{pmatrix} K & 0 \\ K & K \end{pmatrix} \subseteq M_2(K)$$

Every such a module splits as a vector space as a direct sum $M = ME_{11} \oplus ME_{22}$, which we'll identify with the vector spaces W and V. Moreover, since $E_{21}E_{11} = E_{22}E_{21}$, multiplication by the element E_{21} determines the K-linear map between V and W by

$$\varphi(v) = \varphi(mE_{22}) = mE_{22}E_{21} = mE_{21}E_{11} \in W$$

Example 53 (Representations of bimodules)**.** *Consider an A−B−bimodule M. The category of representations of M generalizes the previous example to non-commutative case - it consists triples (X_A, Y_B, f), where X_A is an A-module, Y_B is a B-module, and f is a B-linear homomorphism*

$$X_A \otimes_A M \to Y_B$$

By similar argument as over a field, category $\mathbf{rep}(M)$ *is naturally equivalent to the category of modules over the ring*

$$\begin{pmatrix} B & 0 \\ M & A \end{pmatrix}$$

Moreover, using symmetrical arguments, it's also equivalent to modules over

$$\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$$

Example 54 (Opposite groups)**.** *In the non-commutative ring theory it's natural to consider opposite rings, sometimes providing not isomorphic objects, where multiplication is done in reversed order. The same can be done also in case of groups. The opposite group G^{op} has the same elements with multiplication $a \odot b = ba$. Unlike in case of rings, each* *group is canonically isomorphic to its dual via inverse map $g \mapsto g^{-1}$. It means that the dual group form an endofunctor $\mathbf{Grp} \to \mathbf{Grp}$, which is naturally isomorphic, but not equal, to the identity functor.*

Example 55 (Kronecker algebras)**.** *The equivalence of arrow category with modules over matrix algebras can be generalised to different shapes of diagrams. For example the double arrow category $[\downarrow\downarrow, \mathbf{FinVect}_K]$ can be identified over the modules over the Kronecker algebra*

$$\begin{pmatrix} K & 0 \\ K^2 & K \end{pmatrix}$$

In fact, by the Freyd-Mitchel embedding theorem, every functor category of a category of modules has such a presentation, however sometimes only as a full subcategory.

Example 56 (Δ)**.** *The category Δ is a category consisting of sequences $[n] = (1, 2, \cdots, n)$ together with order-preserving maps. It is very useful (especially while analysing the next example) to think about its objects as labeling of faces of a simplex. Every morphism $[n] \to [m]$ is in fact a composition of either "face inclusions" $[n] \to [n+1]$, where the map omit one vertex (this perpendicular to the face) or "face collapsing" $[n] \to [n-1]$, where we forget about one vertex (this one getting collapsed)*

Example 57 (Simplicial sets)**.** *In topology it is sometimes useful to present a space as some simplices glued together along faces, which is called a simplicial complex. Simplicial sets are a way of taking this combinatorial structure away from topology.*

Example 58 (Segre's Γ)**.** *The Segre's Γ category is a category with objects being finite sets and morphisms $X \to Y$ being functions $X \to \mathcal{P}(Y)$ (where $\mathcal{P}(-)$ is a powerset functor) such that all distinct points have disjoint images.*
Γ is naturally equivalent to ${\mathbf{Fin}_*}^{op}$ via identity on objects and

$$F(f)(y) = \begin{cases} x & y \in f(x) \\ * & \textit{otherwise} \end{cases}$$

Example 59 (Linear representations as modules over group algebra)**.** *If G is a finite group and field K of characteristic not dividing $|G|$, the category of finitely generated left (or right) $K[G]$-modules is isomorphic to the category of finite-dimensional linear representations of G over K, by identifying the module with n generators with action of G on K^n, by definition being its representation $G \to \mathrm{End}(K^n)$.*

Example 60 (Compact spaces and Banach algebras)**.** *Consider the category of commutative Banach algebras - unitary $\mathbb{C}$-algebras with the norm, making them complete Banach spaces. The involution x^* is an operation satisfying*

$$\|x\| = \|x^*\|, \quad \|xx^*\| = \|x\|^2$$

An example of an involution is a complex conjugation and the identity. One of the main results of the theory of commutative Banach algebras states that the category $\mathbf{Ban^{op}}$ *of commutative Banach algebras with involution is naturally equivalent to the category of compact (once and for all compact spaces are Hausdorff - otherwise we will call them quasi-compact) spaces. The equivalence is given by the functor* $C : \mathbf{Comp^{op}} \to \mathbf{Ban}$ *where $C(X)$ is the algebra of complex-valued functions on X with the norm $\|f\| = \sup_{x \in X} |f(x)|$ and involution $f^* = \bar{f}$.*

Example 61 (Gelfand-Kolmogorov theorem)**.** *Similar result can be obtain using the* Spec *functor, so is more algebraic in its spirit. In this setting we consider the $\mathbb{R}$-algebras of real-valued function on compact spaces. Gelfand and Kolmogorov have shown that the compact spaces are homeomorphic if and only if their rings $C(X)$ are isomorphic as $\mathbb{R}$-algebras. It can be reformulated (the categorical statement is in fact stronger, but follows directly from the proof of the original version) in the categorical fashion as showing that there is an inclusion (fully faithful, but not essentially surjective functor)*

$$\mathbf{Comp} \hookrightarrow \mathbb{R} - \mathbf{Alg}^{op}$$

Example 62 (Pontryagin duality)**.** *Consider the category of locally compact abelian groups* **LocCompAb** *- these are just abelian groups equipped with a topology, such that addition and subtraction are continuous (and obviously are locally compact). Main examples of such objects are $\mathbb{R}$ and $\mathbb{C}$ with euclidean topology, as well as their multiplicative versions $\mathbb{R}^\times$ and $\mathbb{C}^\times$ and all finite abelian groups with discrete topology. However, the group $\mathbb{Q}$ is not an object of* **LocCompAb***, as it is not locally compact, as well as $\mathbb{R}$ or $\mathbb{C}$ with discrete or anti-discrete topology.*

The Pontryagin duality is a result from the theory of characters. A character of a topological abelian group G is the map $G \to S^1$. The set of characters of G, denoted as $\hat{G}$, form a topological abelian group itself *with the uniform convergence topology on* $\mathrm{map}(G, S^1)$*. The duality theorem states that the* $\widehat{\bullet}$ *functor defines an equivalence of categories*

$$\mathbf{LocCompAb} \simeq \mathbf{LocCompAb}^{op}$$

In some sense the result is similar to the isomorphism of the dual vector space to itself in finite dimensional vector spaces, as well as to the Gelfand-Kolmogorov theorem considered previously.

Example 63 (Opposite of abelian groups)**.** *Similar equivalence to Pontryagin duality can be obtain for the category* $\mathbf{Ab}^{op}$*. Since every abelian group can be identified with topological abelian group with discrete topology, we can use the Pontryagin duality to inspect the category* $\mathbf{Ab}^{op}$ *itself. Given any discrete abelian group* A*, the dual group* $\hat{A} = \mathrm{Hom}_{\mathbf{TopAb}}(A, S^1)$ *with the product topology* $(S^1)^A$ *is always compact by the Tichonov theorem, which yields the equivalence*

$$\mathbf{Ab}^{op} \simeq \mathbf{CompTopAb}$$

Example 64 (Self duality of finite abelian groups)**.** *Considering yet another restriction of the previously considered equivalence, now considering only finite abelian groups, we again obtain the full duality, since every finite topological abelian group is necessarily discrete, so taking the essential image of the embedding* $\mathbf{FinAb^{op}} \hookrightarrow \mathbf{Ab^{op}} \simeq \mathbf{CompTopAb}$ *yields*

$$\mathbf{FinAb} \simeq \mathbf{FinAb}^{op}$$

Example 65 (Serre-Swan theorem)**.** *The theorem with two incarnations, algebraic, proven by Serre in 1955 and geometric, proven by Swan in 1962.*

The Swan's theorem claims that for any compact topological space X *there is a natural equivalence of the category of finitely generated projective modules over* $C(X)$ *(the ring of real-valued functions) and the category of finite-rank vector bundles over* X*.*

The Serre's version states, that for commutative, unital, Noetherian ring R *there is a natural equivalence of category of finitely generated* R*-modules with the category of algebraic vector bundles over* $\mathrm{Spec}R$ *(i.e. locally free sheaves of structure sheaf modules with constant, finite rank).*

Example 66 (Directed systems)**.** *Functors valued in the category $\mathcal{C}$ often decode some additionally structured objects of $\mathcal{C}$. For example functors $[\rightarrow, \mathcal{C}]$ can be identified with arrows of $\mathcal{C}$. More complicated preorders provide more interesting constructions with similar spirit, called directed systems in $\mathcal{C}$. For example functors from the preorder $\mathbb{N}$ of natural numbers we can identify with sequences*

$$X_0 \to X_1 \to X_2 \to \cdots$$

of objects in $\mathcal{C}$. It's not hard to construct countless examples of such sequences. Their morphisms - natural transformations - can be identified with commutative diagrams. The double covering defining real projective spaces provides for instance such an morphisms of sequences in **Top***, together with canonical equator inclusions of spheres*

$$\begin{tikzcd}
S^1 \arrow[r, hook] \arrow[d] & S^2 S^3 S^4 \arrow[r, hook] \arrow[d] & \cdots \\
\mathbb{R}P^1 \arrow[r, hook] & \mathbb{R}P^2 \arrow[r, hook] & \mathbb{R}P^3 \arrow[r, hook] & \mathbb{R}P^4 \arrow[r, hook] & \cdots
\end{tikzcd}$$

Choosing different preorders one obtains countless variations of such systems. For example the category $[\mathbb{R}, \mathcal{C}]$ categorifies continuous directed families of objects. Topological example of such family are for instance inclusions of intervals, defined for each real numbers $a \leq b$

$$[-a, a] \hookrightarrow [-b, b]$$

where the interval $[0, 0]$ we identify with the singleton $\{0\}$ and the interval $[a, -a]$ with the empty set whenever $a > 0$. Note that such sequences do not require the maps to be inclusions. The trivial example where every map is a constant $X \to X$ is a perfectly fine directed system for any indexing poset (but not preorder!). The continuous system with maps

$$f_t : \mathbb{R} \xrightarrow{\cdot t} \mathbb{R}$$

Is also perfectly fine with all the maps being isomorphisms except the single case of $t = 0$ (note that in make sense in several different categories, such as **Top***,* **Ab** *or* $\mathbf{Vect}_{\mathbb{R}}$*,* **Rng***, but not in* **Ring** *or* **Field***).*

Example 67 (Actions on categories)**.** *The monoid (or group) action on a set is a homomorphism $M \to \mathrm{End}(S)$. The similar construction can be made for small categories in even simpler way - since a functor transforms both objects and arrows, an action of M on a category $\mathcal{C}$ is nothing more than a functor $F : \mathbf{B}M \to \mathcal{C}$, which corresponds to a homomorphism $G \to \mathrm{End}(F(\bullet))$. Functors $[\mathbf{B}M, \mathcal{C}]$ are also called $\mathcal{C}$-representations of M (as they generalise the linear representations, which are just the special case of $\mathcal{C} = \mathbf{Vect}_{\mathbf{K}}$). Natural transformations* *of representations on the other hand directly generalise the notion of M-equivariant functions to M-equivariant functors (or interwining operators). Note that each sort of commonly considered actions arise as a special case of this construction*

- *Permutation representations of finite group:* $[\mathbf{B}G, \mathbf{Fin}]$
- *M-sets:* $[\mathbf{B}M, \mathbf{Set}]$
- *Linear representations:* $[\mathbf{B}M, \mathbf{Vect_K}]$
- *Transformation groups:* $[\mathbf{B}G, \mathbf{Top}]$

Note that this construction is functorial on two different levels - each homomorphism of monoids $M \to N$ and action $\mathbf{B}N \to \mathcal{C}$ induces by composition the action of M on $\mathcal{C}$. On the other hand, each functor $\mathcal{C} \to \mathcal{D}$ by composition extends the action to $\mathbf{B}N \to \mathcal{D}$.

Example 68 (Observations in dynamical systems)**.** *Consider some continuous time, autonomous dynamical system. Identifying $\mathbb{R}$ as a group of time-deletions, it can be modelled as some functor $\mathbf{B}\mathbb{R} \to \mathcal{C}$. The submonoid inclusion $\mathbb{N} \hookrightarrow \mathbb{R}$ induces a functor $[\mathbf{B}\mathbb{R}, \mathcal{C}] \to [\mathbb{N}, \mathcal{C}]$, which one can interpret as observations of the continuous systems in some sequence of discrete moments, transforming it into a discrete and future-oriented system.*

Example 69 (Homogeneous morphisms)**.** *Consider two actions of $\mathbb{R}$ on $\mathcal{C}$, $\rho, \eta : \mathbf{B}\mathbb{R} \to \mathcal{C}$, where $X = \rho(\{\bullet\}), Y = \eta(\{\bullet\})$. For any $p \neq 0$ we have an automorphism of $\mathbb{R}$*

$$f(x) = x^p$$

As the natural transformations $\rho \Rightarrow \eta$ we can identify with linear functions $X \to Y$, after composing η with f the $\mathbb{R}$-equivariant morphisms $\rho \Rightarrow f\eta$ represent homogeneous morphisms $X \to Y$ satisfying

$$\varphi(\lambda x) = \lambda^p \varphi(x)$$

Example 70 (Evolution of quantities)**.** *Physicist often distinguish two type of quantities of objects - described as functions, called intensive, or distributions - extensive. Intensive quantities can describe for example the temperature of the object as a real valued function on X, while extensive quantity is for example its density, described via differential form. This notion provides a new look at the function algebras - the functor*

$$\mathrm{Hom}(-, \mathbb{R})^{op} : \mathcal{C} \to \mathbb{R}\text{-}\mathbf{Alg}^{op}$$

if well defined (as for example when $\mathcal{C} = \mathbf{Top}$ or $\mathbf{Ring}$), can be thought of as describing real-valued intensive quantities of objects. Every monoid action of M on object $X \in \mathcal{C}$ induces the action on its intensive quantities, describing their evolution through states of X associated to the *action. For example, where $M = \mathbb{R}$ and the action describes the evolution through time of some object, $\mathbb{R}$ acts on he algebra $C^0(X)$ encoding the change of quantities of X, such as temperature, through time.*

Example 71 (Periodic systems)**.** *Consider the dynamical system encoding evolution through time of some object considered as the action of additive monoid $\mathbb{R}$ on category $\mathcal{C}$. The periodicity of such system can be formulated categorically as universal property using composition of functors. For every positive real number T we can consider a surjective homomorphism $\mathbb{R} \xrightarrow{\cdot T} S^1 = \mathbb{R}/\mathbb{Z}$. It can be used to express the T-periodic $\mathbb{R}$-actions as precisely functors $\mathbb{R} \to \mathcal{C}$ admitting the factorisation to a repeated period of length T*

$$\begin{tikzcd}
\mathbb{R} \arrow[rr] \arrow[dr, "T"'] & & \mathcal{C} \\
& S^1 \arrow[ur, dotted] &
\end{tikzcd}$$

Example 72 (Collapsing groupoid)**.** *The idea guiding us into the world of groupoids was to prevent losing information during such operations as taking quotients - for example in the action groupoid $X//G$ we simultaneously keep the information about group action and the points of X - in contrary to standard quotient X/G, where some of the points get collapsed. We can make sure that this process indeed do not lose information compared to the old lossy compression by functorial reconstruction of the quotient. It can be done by the functor $\pi_0 : \mathbf{Grpd} \to \mathbf{Set}$, that collapses the objects of a groupoid $\mathcal{G} \rightrightarrows M$ into the set $\pi_0(M)$, where all the isomorphic objects, i.e. connected by any arrow, so with non-empty set $s^{-1}(x) \cap t^{-1}(y)$, are glued together. In particular we have $\pi_0(X//G) = X/G$ as expected. This functor makes sense also for Lie groupoids or equipped with any other additional structure, where it compress $\mathcal{G} \rightrightarrows M$ into a topological space $\pi_0(M)$. Note that it is not a smooth manifold in general - for example if G acts on M in a dirty way, for example not freely, the action Lie groupoid has perfectly smooth structure, but the quotient may have nasty singularities. For instance in case of $\mathbb{Z}/2 \times \mathbb{Z}/2$ acting on $\mathbb{R}^2$ by reflections, the quotient groupoid will have a boundary and even a corner - it's homeomorphic to $[0, \infty)^2$.*

1.5. **Groupoids.**

Example 73 (Groupoid)**.** *Taking the delooping of a group to the opposite direction to monoids, relaxing the condition of having single object leads to the definition of a groupoid - a small category, which all arrows* *are isomorphism. Note that every groupoid is a disjoint union of isomorphic classes of objects. Identifying all isomorphic objects seeks to identifying groupoids with just collections of groups. This way the interpretations of groups as symmetries of some object extends (not very interestingly though) to correspondence of groupoids and symmetries of some collection of object, considered independently and one at a time.*

Example 74 (Action groupoid)**.** *Groupoids appear most frequently decoding some relations between points. Consider for example a group G acting on a set X. The orbit space X/G is a set constructed by identifying all the points lying in the same orbit. Using groupoids, similar construction can be done without loosing the information about the elements of X by considering the action groupoid $X//G$ instead. The set of objects of $X//G$ is X, and every two points of X are connected with a unique arrow if and only if they lie in the same orbit.*

Example 75 (Fundamental groupoid)**.** *Fundamental groupoid generalises the fundamental group of a topological space to non-pointed spaces. It is a groupoid which objects are in correspondence with points from X, and a morphism from x to y is a homotopy class of paths from x to y. We need to restrict ourselves to merely homotopy classes, as composition of paths is associative only up to reparametrisation, which is in particular homotopy invariant.*

Example 76 (Relative fundamental groupoid)**.** *The concept of relative homotopy groups, normally not applicable to the fundamental group, makes sense in case of the fundamental groupoid. The fundamental groupoid relative to $A \subset X$ is the wide subgroupoid*

$$\Pi_1(X, A) \hookrightarrow \Pi_1(X)$$

containing only paths with endpoints contained in A.

Example 77 (Path groupoid)**.** *The fundamental groupoid can be refined to remember information about stricter classes of paths than induced by ordinary homotopy. It uses the notion of a thin homotopy, intuitively corresponding to homotopies of paths $I^2 \to X$ with zero area. In case of smooth spaces and smooth homotopies, it has an intuitive definition of a homotopy with everywhere vanishing Jacobian. In general case however thin homotopies also made sense - this time we require the map to factor through some finite tree. The groupoid with objects X and morphisms $x \to y$ corresponding to thin homotopy classes of paths is called a path groupoid of X.*

Example 78 (Banal groupoid)**.** *Consider a surjective submersion of smooth manifolds $X \to Y$. Then the pullback inclusion $X \times_Y X \subset$*

$X \times X$ is an equivalence relation, thus canonically it forms a groupoid, called the banal groupoid G_F. For example, as each local diffeomorphism is a surjective submersion, any smooth covering space induces such a groupoid. In the classical case of the universal covering $\mathbb{R} \to S^1$, the pullback is the set $(x, x+n) \in \mathbb{R}^2 \subset \mathbb{R}^2$. Its banal groupoid is just the action groupoid obtained from $\mathbb{Z}$ acting on $\mathbb{R}$ by translation, one of the classical constructions of the manifold S^1. In case of n-covering $S^1 \to S^1$, the pullback is a circle n-times winded on the torus. In general, for any covering space, the banal groupoid is isomorphic to the action groupoid associated to associated action of fundamental groups.

Example 79 (Čech groupoid)**.** *Consider a poset of open coverings $\mathbf{Cov}(X)$ of a topological space X. There is a functor $\check{C} : \mathbf{Cov}(X) \to \mathbf{Grpd}$, assigning to each covering its Čech groupoid $\check{C}(\mathcal{U})$. Its objects are points from the disjoint union $\coprod_{\mathcal{U}} U_i$, and every point from an intersection $U_i \cap U_j$ establishes an arrow between its image in U_i and U_j.*

Example 80 (Groupoid of a puzzle)**.** *A Rubik cube can be solved by appropriate permutations of elements, which can be formalised in terms of a an action of a group of possible permutations. However, this approach does fail in general for different puzzles, where not all permutations of elements can be composed together. If on each state of the puzzle only some subset of moves is considered legal, but each action is invertible, instead of a group it can be formalised by an action of a groupoid, which points are possible states of the puzzle and arrows - possible permutations. A simple case of such a puzzle is the fifteen puzzle. —PIC —-*

Example 81 (Cancellative monoids)**.** *Unlike in case of groups, not all (even commutative) monoids satisfy the cancellation law*

$$ab = ac \Rightarrow b = c$$

The most notable examples of not cancellative monoids comes from differential topology. For any space X, the isomorphism classes of real vector bundles form a monoid $\mathrm{VB}(X)$, with the commutative operation $[B] + [B'] := [B \oplus B']$. This monoid is not cancelative, as there are non-trivial vector bundles that become trivial under the sum with a trivial bundle. The most famous example uses the hairy ball theorem stating that you can't comb a hairy ball, i.e. that there are no nowhere-vanishing vector fields on S^2. Since vector fields are global sections of the tangent bundle, it follows that the bundle TS^2 is not trivial. But its direct sum with normal bundle, which is its orthogonal complement, *is obviously trivial. Since NS^2 is trivial as well, the cancellation law in* $\mathrm{VB}(S^2)$ *would lead to combing a hairy ball.*

The cancellation law of monoids can be nicely described in categorical language - it can be reformulated as the property that every morphism in the delooping $\mathbf{B}M$ *is a monomorphism.*

Example 82 (Lie groupoids)**.** *In case of ordinary categories, groupoids can be thought of as a set of objects G_0 and a set of arrows G_1, together with two surjections encoding the source and target $s, t : G_0 \rightrightarrows G_1$ and injection $G_1 \hookrightarrow G_0$ corresponding to identity arrows. The composition of arrows is then a map from the pullback, containing only pairs of arrows with common target/source $G_1 \times_{s,t} G_1 \to G_1$. This definition can be extended to define Lie groupoids, where both G_0, G_1 are smooth manifolds and all maps are smooth. The only obstacle lies in the fact that not all pullbacks exist in* **Diff***. To ensure that it is also a manifold, we additionally require the maps s, t to be submersions (i.e. having surjective differential).*

Lie groupoid is usually denoted by $G_0 \rightrightarrows G_1$. Among the simplest examples of Lie groupoids we have

- $G \rightrightarrows \{\bullet\}$ *are just* ***Lie groups***
- $M \rightrightarrows M$ *where $s = t = \mathbb{1}$ are just* ***manifolds***
- *A* ***fundamental groupoid*** *of a manifold form a Lie groupoid*
- $M \times M \rightrightarrows M$ *is called a* ***pair groupoid*** *when s, t are projections*
- *the two trivial cases from above combines into a* ***trivial groupoid*** $M \times G \times M \rightrightarrows M$*, independently encoding a trivial action of G on M*
- *a Lie groupoid with $t = s$ is just a fiber bundle of Lie groups*
- *the Čech groupoid associated to some cover is a Lie groupoid* $\coprod_{\mathcal{U}} U_i \times_M U_j \rightrightarrows \coprod_{\mathcal{U}} U_i$
- *action groupoid of a Lie group on a manifold is a Lie groupoid*

Example 83 (Fundamental groupoid of a circle)**.** *The fact that homotopy classes of paths canonically admit smooth structure can be a bit surprising, as they seem to be discrete objects. As usual, instead of a formal proof we will illustrate such a structure on a simple example* $\Pi_1(S^1)$*.*

The homotopy classes of paths relative to endpoints between two points x, y from a circle are in correspondence with a any path connecting x, y composed with some loop rotating n times along S^1. All these paths can be realised via the exponential map xe^{it} defined on the interval $[0, 2\pi n + \arg(y)]$. It follows that the exponential map starting from X realises all the homotopy classes of paths starting from x

exactly once. It means that we may identify $\Pi_1(S^1)$ simply with

$$S^1 \times \mathbb{R} \rightrightarrows S^1$$

with source map being a projection $s(x,t) = x$ and the target map realising some path starting from x - $t(x,t) = xe^{it}$. The composition of maps we can identify with addition of periods

$$(xe^{it}, u) \circ (x,t) = (x, t+u)$$

Note how the component $\mathbb{R}$ plays two roles at the same time in this construction - its maximal integral multiplicity of 2π corresponds to homotopy class of a path, while the reminder $\mathrm{t mod} 2\pi$ specifies its end-point.

Example 84 (Local symmetry groupoid)**.** *Consider some space X with group of its automorphisms G (we'll stick for simplicity with $\mathbb{R}^n$ and $AO(n)$) and its subset A. Then the elements of G restricting to automorphisms of A form a subgroup $\Gamma := G^A \leq G$. Γ acts on X forming the action groupoid $X//\Gamma$. In this example we will form another groupoid of more local nature as an answer to two drawbacks of the global approach:*

- *Restricting (X, A) to some subset B in most cases breaks the symmetry of A, even though it looks the same inside B*
- *Different subsets such as $\mathbb{Z}\times\mathbb{Z} \subset \mathbb{R}^2$ and $\mathbb{Z}\times\mathbb{R}\cup\mathbb{R}\times\mathbb{Z} \subset \mathbb{R}^2$ has the same symmetry group, even though they can be distinguished by local symmetries*

Consider the example from above of $X = \mathbb{R}^2, A = \mathbb{Z} \times \mathbb{R} \cup \mathbb{R} \times \mathbb{Z}$ and $B = [-1,1]^2$. Restricting the global approach to B yields the symmetry group D_4 acting on B. The action groupoid $B//D_4$ has infinitely many orbits containing 4 points and one single point orbit with the origin.

The local groupoid $\mathcal{G}_{loc}$ is constructed to capture the local symmetries preserving distinguished regions - the region $P_1 = A \cap B$ of internal points from A, $P_2 = B \setminus A$ of points lying in the interior of squares and the outside set $P_3 = X \setminus B$. Points of $\mathcal{G}_{loc}$ are all the points from B, while the arrows $x \to y$ are such elements g of Γ that maps x to y and that for some neighbourhood U_x of x they preserve the regions, i.e. $g(U_x \cap P_i) \subset P_i$. This time we obtain much richer orbit structure, with finitely many orbits:

- *O_1 containing 4 corner points $\{(\pm1, \pm1)\}$*
- *O_2 containing 4 middle points of the edges $\{(0, \pm1), (\pm1, 0)\}$*
- *O_3 containing boundary edges without corners and centers*
- *O_4 containing the inner edges*
- *O_5 containing single point $(0,0)$*

- *O_6 containing all the points from P_2*

Stabilisers of points from each orbit are also interesting, isomorphic to

- *$\mathbb{Z}/2$ for points from O_1, O_2, O_3*
- *$\mathbb{Z}/2 \times \mathbb{Z}/2$ for points from O_4*
- *D_4 for point from O_5*
- *$O(2)$ for points from O_6*

1.6. **Size issues.**

Example 85 (Small category)**.** *A category is called small if its objects and morphisms are sets. Most categories associated to structures encountered outside category theory are not small, but a lot of fundamental category theoretic constructions are small. Moreover the Yoneda lemma is true only for small categories, so also most of the main results in category theory (as every proof is essentially just Yoneda lemma). Small categories form a category* **Cat**. *We'll address this annoying fact in a moment using Grothendieck universes. Among small categories we can find following examples*

- *The category of finite sets* **Fin**, *and more generally any subcategory of sets with bounded cardinality*
- *The delooping of a group or monoid $\mathbf{B}M$*
- *The simplicial category Δ of finite total orders*
- *The initial empty category and terminal category with one object and one arrow*
- *The discrete and full categories obtained from a set*
- *The arrow category $\bullet \to \bullet$*
- *The diagrams of (co-)equalisers and (pushouts) pullbacks*
- *The fundamental groupoid $\Pi_1(X)$ of a topological space*
- *The category of finite groups/modules/rings/fields*
- *The category of finitely generated groups/modules/rings*
- *The category of finite dimensional vector spaces*
- *The category of manifolds*
- *The category of finite dimensional vector bundles over fixed space*

Example 86 (Essentially small category)**.** *The notion of large and small categories has a fundamental flaw of breaking the equivalence principle. Some categories even though large, admit properties of a small category since are equivalent to some small category - we call them essentially small. In most cases such categories are rather artificial, as they must have admit a proper class of objects with only a set of isomorphism classes. Consider for example a category having sets as objects and exactly one element in each hom-set. This category is large,* *but since its objects are all isomorphic to each other, it is equivalent to the terminal category, which is its skeleton. In general, essentially small categories are exactly categories with small skeleton.*

Example 87 ((Very) large categories)**.** *There is consensus in the definition of a large category. Depending on the author, this term is used either to describe a category which is not small or, more generally, just any category (not necessarily small or locally small). Even though the latter is no different than saying merely "a category", it is used to emphasize that there is no assumption of all categories being locally small, in most cases made in the introduction chapter of a book.*

Throughout this chapter, we drop the conditions of local smallness and use the first definition of large categories, while referring to not necessarily small categories simply as just categories. Much wilder class of not locally small categories, which hom-sets are allowed to form a proper class, we'll call very large (however this nomenclature is not standard).

Among large, but not very large categories we have

- *The category of sets*
- *The category of groups, rings, modules, fields, monoids, etc.*
- *The category of topological spaces*
- *The homotopy category of topological spaces* **HTop**
- *The category of small categories*
- *The category of partial orders, preorders, total orders*
- *The category of relations*
- *The category of presheaves*
- *The category of functors valued in large category*

The flagship example of a category in this sense is a category **CAT** *of all categories and functors, playing the same role as sets in the theory of locally small categories.*

Example 88 (Ordinals)**.** *There is a large category* **ORD** *of ordinal numbers, an example of a very large monoid. Just as in case of classical monoids, it has only one object and a proper class of ordinal numbers as morphisms. Composition of morphisms is the multiplication of ordinals. Note that in case of very large categories the symmetry between delooping categories of monoids or groups and their algebraic counterparts breaks, as even though* **ORD** *is a monoid in categorical sense, it has no underlying algebraic monoid, as their construction demands well defined set of elements.*

Example 89 (Strict and weak categories)**.** *Note that in case of large categories usually there is no canonical definition of equality of objects,* *as equality is formally an equivalence relation, therefore a special kind of a subset. Even though the axiom of choice provides a canonical way of defining the equality of objects, it can be seen as some extra structure on a category. This subtle distinction leads to a notion of weak and strong categories. The equality of objects is well defined only in strong categories, while isomorphic objects in a weak category are basically indistinguishable.*

The classical definition of a category as a collection of objects and arrows with usual ZFC axioms makes all categories essentially strict. However, categories can be define in an alternative way, where exactly the same properties are achieved by considering only a family of morphisms with some additional axioms. This way we obtain weak categories. All the constructions on categories dualise as well - we may consider strict and weak functors, where strict functors are require to respect the equality relations, mapping equal objects to equal objects, as well as weak and strong natural transformations. As expected, these assemble into categories **WeakCat** *and* **StrCat***.*

The category of small strict categories is concrete, as we have a canonical functor $ob : \mathbf{StrCat} \to \mathbf{Set}$*. Such a functor does not exist in case of weak categories, however the axiom of choice provides a canonical way of its construction. Without the axiom of choice, this symmetry breaks. Even though each weak category always can be identified with strong category in a canonical way, it is not possible at the level of weak functor, called also anafunctors.*

Example 90 (Grothendieck universes)**.** *The most common approach of addressing size issues in category theory uses the so called Grothendieck universes. A universe is a set* U *which is an inaccessible cardinal, so their existence cannot be proven or disproven in ZFC. To fix that problem, usually we just add an additional axiom of their existence, having no practical effect on actual mathematics done after. The better understanding of their role is an equivalent direct definition of an universe as a set* U *satisfying following four axioms*

- *if* $a \in X \in U$*, then* $a \in U$
- *if* $X \in U$*, then* $\mathcal{P}(U) \in U$
- $\emptyset \in U$
- *for any function* $f : X \to U$*,* $\mathrm{im}(f) \in U$

The category of elements of U *form a full subcategory* $\mathbf{USet} \hookrightarrow \mathbf{Set}$*. The main trick solving size issues this way is that all the reasonable operations that can be done with any set can be done within some Grothendieck universe containing it, and each set is contained in some universe. It means that proving a result about the large category of sets* *can be effectively reduced to proving a result for all categories* **USet**, *which are locally small. In this spirit it in fact make sense to distinguish*

- *U-small categories, which set of objects and arrows belongs to U*
- *U-moderate categories, which set of objects belong to U*
- *U-large categories, which set of arrows do not belong to U*

Example 91 (Large limits). *In most cases when we talk about limits we require the indexing category to be small. Considering large limits, indexed by a large category, makes sense as well, however behaviour of such limits is sometimes odd. One of such counter-intuitive behaviour is reflected in the theorem already seen in case of small categories, stating that a bicomplete small category is necessarily a partial order. Surprisingly, that result holds also in the large case - any category with all large limits and colimits also must be a partial order.*

2. Representability and Yoneda lemma

2.1. Yoneda lemma.

Example 92 (Forgetful object). *Often it happens that the forgetful functor $\mathcal{C} \to$* **Set** *of some concrete category is representable. Such universal objects can be found for example in* **Top**, **Grp** *or* **Ring**:

- *in* **Top**, *the points of a space $|X|$ are represented by maps from terminal one-point set to X*
- *In the category of groups, forgetful functor is isomorphic to* $\mathrm{Hom}(\mathbb{Z}, -)$, *with elements $g \in G$ corresponding to unique homomorphism $\mathbb{Z} \xrightarrow{g} G$ sending* 1 *to* $\mathbb{Z}$.
- *For unital rings, similar role plays the ring* $\mathbb{Z}[t]$, *sending t to some element of R.*

Example 93 (Yoneda lemma on posets). *Yoneda lemma applied to a poset category states that given some elements x and y, if for all objects z $x \leq z$ iff $y \leq z$, then $x = y$. To see that, notice that the hom presheaf h_y is just a set of all elements less than or equal to y, and by the Yoneda lemma equality of such functors is equivalent to equality of underlying objects x and y.*

Example 94 (Zariski topology). *A very fundamental construction in commutative algebra and algebraic geometry is a spectrum of a ring* $\mathrm{Spec}A$. *Spectrum, as a set, consists of all the prime ideals of A. Even though it sounds like a weird thing to consider, it makes in fact a lot of sense, when we look at such ring as* $\Bbbk[x, y]$. *Its prime ideals are generated by irreducible polynomials, and are naturally identifiable with* *irreducible algebraic curves on the plane - exactly the subject of interest of algebraic geometers. Every such a spectrum has a canonical, however very weird topology, where closed sets are exactly algebraic ones, called Zariski topology. Formally, closed sets are defined as all possible vanishing sets of some ideals of* A

$$V(I) = \{\mathfrak{p} \mid I \subseteq \mathfrak{p}\}$$

Dually, we define the operation in the other way around, taking subset of the spectrum into the ideal vanishing at all these points (if it sounds as a complete nonsense to you, try to consider the case of a ring of continuous functions on a compact manifold - this model example probably provides the best intuition in understanding the spectrum; in particular, see that maximal ideals correspond to functions vanishing at a single point, and that construction of a vanishing set and the ideal of functions vanishing at some set can be translated into the same language as presented above, i.e. not referring to the elements of the ring at all, or at least any additional structure on them, for example that they are functions).

$$I(S) = \{f \in A \mid S \subseteq V(f)\}$$

After a bit long introduction, time for the centerpiece - a cool application of the Yoneda lemma, providing a quick proof that the vanishing sets of ideals $V(I)$ *indeed form a topology on* $\mathrm{Spec} A$ *- in particular, that closed sets are closed under arbitrary intersections. Recall that the Yoneda lemma in case of posets categories had a form*

$$a = b \textit{ iff } \forall c : c \leq a \Leftrightarrow c \leq b$$

Then we can use this fact to show that the intersection of vanishing sets is a vanishing set of the union:

$$\begin{gathered} T \subseteq V(\bigcup_{\alpha \in A} I_\alpha) \textit{ iff} \\ I(T) \subseteq \bigcup_{\alpha \in A} I_\alpha \textit{ iff} \\ \forall_{\alpha \in A} I(T) \subseteq I_\alpha \textit{ iff} \\ \forall_{\alpha \in A} T \subseteq V(I_\alpha) \textit{ iff} \\ T \subseteq \bigcap_{\alpha \in A} V(I_\alpha) \end{gathered}$$

In fact there is more to be explored in this example - for example the trick with interchanging V *and* I *is a special case of a very important concept of an adjoint, see chapter 4, where the entire construction* *will be generalised in a form of an Galois connection. From a slightly different angle we will go back to similar framework while analysing idempotent monads in chapter 5.3.*

—- NOT SURE; IT IS THE ENRICHED YONEDA EMBEDDING —–

Example 95 (Yoneda embedding of posets)**.** *In the poset category the Yoneda embedding also has a nice description. For any given x, the representable functor* $\mathrm{Hom}(-, x)$ *is just a set of all the elements less or equal to x. Such a functor is a known object called the principal downset $\downarrow x$ of x:*

$$\downarrow x = \{y \in P \mid y \leq x\}$$

The category of presheaves of a poset, on the other hand, can be understood as a poset category of downward closed sets. In this way, the Yoneda embedding identifies a poset P with a full subcategory of principal downsets in the poset category of all possible downsets.

Example 96 (Dynamical systems)**.** *Consider a category generated by a free monoid on one generator (which is just* $\mathbb{N}$*). The presheaves on* $\mathbb{N}$ *can be considered as dynamical systems - sets indexed by* $\mathbb{N}$ *with an endomorphism corresponding to evolution through time - a natural transformation $X(n) \to X(n+1)$. In this case the Yoneda lemma just states that such evolution of the system is just the same as multiplication of elements of the monoid corresponding to the states.*

Example 97 (Involution sets)**.** *Consider a setting similar to the previous example, but now taking the monoid M to have two elements* 1 *and σ, with $\sigma^2 = 1$. A presheaves* $[BM^{op}, \mathbf{Set}]$ *can be identified with sets together with a reflection, called involution sets, when the Yoneda lemma identifies reflecting the set with the multiplication $\sigma \cdot \sigma$ in M.*

Example 98 (Cayley graph of a group)**.** *The previous example can be generalized to monoids with any set of generators and presentations. The generators of M provide an additional parameter space to the set of underlying presheaf, while the relations shapes the graph of possible evolution of the system. When the monoid is a group, it is exactly its Cayley graph. Yoneda lemma identifies such M-set structures on X with right actions of M. Similarly, the left actions corresponds to presheaves of the opposite category, so with M^{op}-sets.*

Example 99 (Directed graphs)**.** *The category* **DirGrph** *can be constructed as a presheaf category, where the indexing category has two objects and two maps:*

$$V \underset{t}{\overset{s}{\rightrightarrows}} E$$

A functor $F \in [\mathcal{C}^{op}, \mathbf{Set}]$ associate to V some set of vertices, and to E a set of edges. The maps induced by s and t associate to each vertex its source and target.

Example 100 (Reflexive graphs)**.** *After considering the category from the previous example with additional arrow in the opposite direction*

$$\begin{tikzcd} V \arrow[r, shift left=2, "s"] & E \arrow[l] \arrow[l, shift left=2, "t"'] \end{tikzcd}$$

the presheaves $[\mathcal{C}^{op}, \mathbf{Set}]$ now correspond to directed graphs where each vertex has additional self-loop, prescribed by the additional morphism l. The category $\mathcal{C}$ is called a walking reflexive fork.

Example 101 (Cone functor)**.** *Recall that a diagram is any functor $F : \mathcal{I} \to \mathcal{C}$. A cone* $\mathrm{Cone}(F, c)$ *under F is a natural transformation $F \Rightarrow c$, where c is a constant diagram. We can use the Yoneda embedding to identify the constant diagrams as $\Delta : \mathcal{C} \hookrightarrow [\mathcal{I}, \mathcal{C}]$. This characterizes the cone functor* $\mathrm{Cone}(F, -) : \mathcal{C}^{op} \to \mathbf{Set}$ *as the hom-set in the full subcategory of $[\mathcal{I}, \mathcal{C}]$ being the image of Yoneda embedding:* $\mathrm{Cone}(F, -) \simeq \mathrm{Hom}(\Delta(-), F)$.

Given representable functor $F : \mathcal{C}^{op} \to \mathbf{Set}$, we can express its universal property in a fancy way, constructing such a category from objects of $\mathcal{C}$, that morphisms are "F-invariant". This will turn representants of F into terminal objects, allowing the universal property of the functor to be express in a very elegant way.

Definition 2.1 (Universal element)**.** *Given natural isomorphism $\nu : \mathrm{Hom}(A, -) \to F$, the universal element of $F(A)$ is the element defining the natural correspondence in Yoneda lemma, i.e. $\nu_A(\mathrm{Id}_A) \in F(A)$.*

Example 102 (Forgetting ring structure)**.** *The forgetful functor* $\mathrm{Fg} : \mathbf{Ring} \to \mathbf{Set}$ *is represented by the ring $\mathbb{Z}[x]$.*
The bijection $\varphi : Hom(\mathbb{Z}[x], R) \to \mathrm{Fg}(R)$ is given via evaluation $\varphi(f) = f(x)$, so $x \in \mathbb{Z}[x]$ is the universal element of F.

Example 103 (Representable G-sets)**.** *A functor $E : \mathbf{B}G \to \mathbf{Set}$ (that we can interpret as a G-set, since $\mathbf{G} - \mathbf{Set} \simeq \mathrm{Funct}(\mathbf{B}G, \mathbf{Set})$) if and only if $E \simeq Hom(*, -)$, so $E \simeq G$ in $\mathbf{G}$-$\mathbf{Set}$, where G is acted by itself by left multiplication. If $E \neq$ it implies that the action of G on E is free and transitive, and it happens to be also sufficient. The identity of $H(*, *)$ is the identity element $e \in G$, so the universal element of E is corresponding to choosing the image of the identity in E.*

Example 104 (Nilpotent elements)**.** *A functor* $\mathrm{Nil}_n : \mathbf{CRing} \to \mathbf{Set}$ *assigning to a ring* R *all nilpotent elements of degree* n*, i.e.* $\mathrm{Nil}_n(R) = \{r \in R \mid r^n = 0\}$ *is representable by the ring* $\mathbb{Z}[x]/(x^n)$*, as there is unique correspondence between such elements and homomorphism* $x \mapsto r$*. However, the functor* $\mathrm{Nil} : \mathbf{CRing} \to \mathbf{Set}$ *mapping* R *to its nilradical* $\mathrm{Nil}(R) = \{r \in R \mid \exists n \in \mathbb{N} : r^n = 0\}$ *is not representable. To show that, it is enough to notice that it does not preserve limits, in particular products. As a counterexample consider a ring* $R = \prod_{n=2}^{\infty} \frac{\mathbb{Z}[x]}{(x^n)}$*. The natural map*

$$\mathrm{Nil}(\prod_{\mathrm{n}=2}^{\infty} \frac{\mathbb{Z}[\mathrm{x}]}{(\mathrm{x}^{\mathrm{n}})}) \to \prod_{n=2}^{\infty} \mathrm{Nil}(\frac{\mathbb{Z}[\mathrm{x}]}{(\mathrm{x}^{\mathrm{n}})}$$

fails to be surjective, since the element $(x, x, x, \cdots)$ *is not surjective in* R*.*

Example 105 (Affine space as a G-torsor)**.** *A representable* G*-set without chosen universal element is called a* G*-torsor. An example of such an object is the affine space* A^n*, a representable* $\mathbb{R}^n$*-set. We can think of* A^n *as an euclidean space* $\mathbb{R}^n$ *acted by itself by addition of vectors, but without distinct origin. After choosing any element* $x \in A^n$ *as universal, we can reconstruct the group* $\mathbb{R}^n$ *by making from* x *an origin via identifying* $p \in A^n$ *with a vector* $v \in \mathbb{R}^n$ *such that* $x + p = v$*.*

Example 106 (Tensor product)**.** *Probably the most famous example of universal property is the tensor product. The definition already makes the tensor product representant of a functor, as* $\mathrm{Hom}(V \otimes W, Y) \simeq \mathrm{Bilin}(V, W; Y)$*. The natural bijection between bilinear map* $V \times W \to Y$ *and linear* $V \otimes W \to Y$ *is realised via the universal element* $V \times W \to V \otimes W$*, an element of* $\mathrm{Bilin}(V, W; V \otimes W)$*.*

Example 107 (Free completion)**.** *Suppose the category* $\mathcal{C}$ *is not complete (does not have all the limits). Then it is possible to artificially construct the limits by considering them in the Yoneda embedding, which is complete. More precisely, the category of presheaves over* $[\mathcal{C}^{op}, \mathbf{Set}]$ *is called also the free cocompletion of* $\mathcal{C}$*, and the object* $\mathrm{colim}(Y(X_i))$ *can be identified with the formal colimit of* X_i*, where* Y *is the Yoneda embedding. Dually, the opposite Yoneda embedding* $\mathcal{C} \to [\mathcal{C}, \mathbf{Set}]^{op}$*. The completion is free since it satisfies the universal property of free objects: every functor* $\mathcal{C} \to \mathcal{S}$*, where* § *is cocomplete, factor uniquely through the Yoneda embedding* $\mathcal{C} \to [\mathcal{C}^{op}, set] \to \mathcal{S}$*.*

Example 108 (Formal filtered colimits)**.** *Sometimes we are interested only to add some specific type of formal colimits or limits, in particular filtered colimits. The full subcategory of* $[\mathcal{C}^{op}, \mathbf{Set}]$ *consisting of the essential image of the Yoneda embedding and all the formal filtered colimits is called* $\mathrm{Ind}(\mathcal{C})$*, and the formal colimits of specific systems are called ind-objects.*

Example 109 (pro-manifolds)**.** *Consider the category of smooth manifolds* **Diff***. The pro-objects in* **Diff** *can be understood as (possibly) infinitely dimensional manifolds, but with only "mild" cardinality allowed, since only manifolds constructed as limits of finite-dimensional manifolds are allowed to exist. One of a main feature of pro-manifolds, where their "smallness" can be seen directly, is that they have properties of a paracompact spaces - in particular on every such a manifold there exist a good cover (a cover is called good if all its finite intersections are contractible; it is a crucial property for some topological constructions, such as nerves).*

Example 110 (Formal cofiltered limits)**.** *The opposite construction to formal filtered colimits leads to the category* $\mathrm{Pro}(\mathcal{C})$ *of formal cofiltered limits.*

Example 111 (Classifying object)**.** *There are two ways of describing subobjects in a category. Firstly, we can identify all the subobjects of* X *with monomorphisms* $S \hookrightarrow X$*. In some categories such as* **Set***, we can also say that the subobjects of* X *correspond uniquely to maps* $X \to \{0,1\}$*, which identify the elements of a subset with elements mapped to* 1*. However, such an description doesn't make sense in different categories, as* $\{0,1\}$ *is a specific object of the category of sets. In general, if such an objects exists, it is called the classifying object* Ω*.* Ω*, if exists, is an object representing the subobject functor* $\mathrm{Sub} : \mathcal{C} \to \mathbf{Set}$*. The classifying object together with a certain map makes a notion of a subobject classifier, which is described in more details in the section about limits.*

Suppose F is a covariant functor $\mathcal{C} \to \mathcal{D}$. The property making the representant X and its universal element $u \in F(X)$ special, is that for every Y and $y \in F(Y)$ there is a unique map $f : X \to Y$ such that $F(f)$ maps u to y. It follows instantly from the isomorphism $F(Y) \simeq Hom(X,Y)$ and Yoneda lemma. The other elements, on the other hand, may have more than one such a map or not have it at all. This information captures the category of elements:

Example 112 (Topology). *A Sierpiński space S is a non-Hausdorff space with 2 points $\{0,1\}$, with open sets being $\{\emptyset,\{0\},S\}$. Every map $f : X \to S$ can be identified with some open set $U \subseteq X$ by the preimage $f^{-1}(\{0\})$. Since such an identification is clearly functorial, it proves that S represents the functor*

$$Op : \mathbf{Top}^{op} \to \mathbf{Set}$$

Mapping space to its topology (set of open subsets).

Example 113 (Disconnections). *In a similar spirit every map $D \to X$, where D is a 2-point discrete space can be identified with a disconnection of X - a decomposition of a space into two disjoint open sets covering X. The identification is simply established by*

$$U_1 = f^{-1}(\{0\})$$
$$U_2 = f^{-1}(\{1\})$$

By the continuity of f both sets are open, and by the properties of preimage they must be disjoint and covering X. It follows that a 2-point discrete space represents the functor

$$D : \mathbf{Top}^{op} \to \mathbf{Set}$$

mapping X to the set of all its disconnections.

Example 114 (Characteristic classes). *A classical classification result from the theory of vector bundles states that isomorphism classes of n-dimensional complex vector bundles on paracompact space X are in natural bijection with homotopy classes of maps $X \to \mathrm{Gr}_n(\mathbb{C}^\infty)$, where $\mathrm{Gr}_n(\mathbb{C}^\infty)$ is the Grassmanian - the space parametrising n-dimensional linear subspaces in the countably-dimensional complex vector space. It means that the functor $\mathrm{HTop}^{op} \to \mathbf{Set}$ (where $\mathbf{HTop}$ is the homotopy category of some nice topological spaces that are paracompact, for instance CW complexes) is representable by the object $\mathrm{Gr}_n(\mathbb{C}^\infty)$.*

One of the most interesting construction in the theory of vector bundles are characteristic classes. Categorically, their a bit technical definitions can be reduced to simple characterisation as natural transformation of functors $\mathrm{Vect}_n \Rightarrow H^\bullet(-;\mathbb{Z})$. Thus, Yoneda lemma and representability of Vect_n reduces the general study of characteristic classes to calculation the cohomology of Grassmanians, which structure is quite well understood.

$$\mathrm{Nat}(\mathbf{Vect}_n, h^\bullet(-;\mathbb{Z})) \simeq H^\bullet(\mathrm{Gr}_n(\mathbb{C}^\infty);\mathbb{Z})$$

More concretely, cohomology classes of Grassmanians can be represented by Young diagrams, corresponding to Schubert cells of the Grassmanian. These objects are characterised also by several different constructions appearing often unexpectedly all across mathematics, such as symmetric functions, or representations of symmetric groups, all leading to some characteristic classes of vector bundles. For example the trivial representation of symmetric groups correspond to the Chern class, while the sign representation is associated with the Segre class.

Example 115 (Functor of points)**.** *This example requires a bit of theory from algebraic geometry or commutative algebra, but shows probably the most important application of Yoneda lemma in mathematics. It makes a functorial description of (R-)schemes - spaces glued from affine R-schemes, which are locally ringed spaces constructed as spectra of commutative R-algebras. The spectrum functor in fact identifies affine schemes with* R-$\mathbf{Mod}^{op}$*, being a contravariant equivalence. The Yoneda lemma embeds the category of R-schemes in presheaves functors*

$$\mathbf{Sch} \hookrightarrow [\mathbf{Sch}^{op}, \mathbf{Set}]$$

the representable functors $h_S(X) = \mathrm{Hom}(X, S)$ *are called the* S*-points of a scheme. These are most easy to describe in case of* $S = \operatorname{Spec} k$*, where they parameterise points from underlying topological space with residue field* k*. The key point is that since schemes are locally affine, a morphism from non-affine scheme is uniquely determined by its restrictions to affine open cover. It follows that from the set* $\mathrm{Hom}_{\mathrm{AffSch}}(-, X)$ *we can canonically recover the set of functors from all schemes* $\mathrm{Hom}_{\mathrm{Sch}}(-, X)$*, as the additional elements we need to add are uniquely determined by functors that are already there. This isomorphism leads to massive simplification of Yoneda embedding, as presheaves on affine schemes are just co-presheaves functors from* R*-algebras, meaning that the category of* R*-schemes can be embedded there.*

$$h : \mathbf{Sch}_R \hookrightarrow [R\text{-}\mathbf{Mod}, \mathbf{Set}]$$

This new embedding, called functor of points, reduces to restricted Yoneda embedding under composition with Spec.

2.2. Category of elements.

Definition 2.2 (Category of elements)**.** *Given covariant functor* $\mathcal{C} \to \mathcal{D}$*, its category of elements* $\int F$ *has objects* (X, y) *where* $X \in \mathcal{C}$ *and* $y \in F(X)$*. A morphism* $(A, y) \to (B, z)$ *is a morphism* $f : A \to B$ *such that* $F(f)(y) = z$*.*

As already mentioned, the category of elements can be seen as an extension of the category $\mathcal{C}$ to F-invariant one.

Example 116 (Colored graphs). *In the category of finite graphs* **FGraph** *we can consider the functor* $\mathrm{Col}_n : \mathbf{Graph}^{op} \to \mathbf{Set}$*, assigning to the graph all its possible n-colorings. Then the category $\int \mathrm{Col}_n$ is the category of n-colored finite graphs with coloring-preserving maps.*

Indeed the category $\int \mathrm{Col}_n$ is defined to consists graphs paired with some coloring, which is just a colored graph. The condition on morphisms between them forces them to be coloring-preserving, since for every map $f : G \to G'$ it is a morphism in $\int \mathrm{Col}_n$ only if $F(f)$ maps coloring of G to the coloring of G'.

Example 117 (Pointed categories). *The category of elements provide a tool for an universal construction of pointed categories from any concrete one as $\int U$, where U is a forgetful functor.*

Example 118 (Bundles). *A indexed family of sets can be constructed as a functor $B : \mathcal{I} \to$* **Set***, where $\mathcal{I}$ is a discrete category (having no morphisms). The objects of the category $\int \mathcal{I}$ can be identified with $\coprod_{i\in I} F_i$. The canonical projection $\int \mathcal{I} \to \mathcal{I}$ is the categorical analogue of a bundle, with the set $\coprod_{i\in I} F_i$ called dependent sum. The dependent product $\prod_{i\in I} F_i$ is the set of global sections of the projection.*

The distinguish element is naturally associated to the space via choosing some element of underlying set, and the morphism structure in $\int U$ force them to be basepoint-preserving.

Example 119 (Slice category). *Category of elements of the hom functor, $\int \mathrm{Hom}(A, -)$ are objects $X \in \mathcal{C}$ together with maps $A \to X$. The morphism is established by any map $f : X \to Y$ commuting with their maps from A, so we can identify them with f's from following commuting triangles:*

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X \arrow[rr, "f"] & & Y \end{tikzcd}$$

Such a category is called a slice category $X/\mathcal{C}$ under X. Equivalently in the contravariant case we get the slice category $\mathcal{C}/X$ over X with all the arrows reversed.

Using the Yoneda lemma we can also construct the category of elements as the comma category in yet another way - as the category

$(y/F)^{op}$, where y is the Yoneda embedding of $\mathcal{C}^{op}$ and F is the constant functor $1 \to [\mathcal{C}, \mathbf{Set}]$ choosing F. Since transformations $h_X \to F$ are by Yoneda with correspondence with elements of $F(X)$ for any $X \in \mathcal{C}^{op}$, the arrow in such slice after is nothing more than arrow $F(X) \to F(Y)$ preserving the elements chosen by $h_\bullet \to F(\bullet)$. Since we used the Yoneda embedding of $\mathcal{C}^{op}$, all the induced arrows goes in the wrong directions, which we fix by considering opposite category.

As already mentioned, category of elements is a tool allowing express universal property of representable functors as initial or terminal objects in a suitable category.

Lemma 2.1. *The covariant (contravariant) functor is representable if and only if its category of elements has initial (terminal) objects. Initial (terminal) objects of category of elements correspond to representants of a functor together with their universal elements.*

Example 120 (Pairs of sets)**.** *The category of elements corresponding to the contravariant powerset functor $\mathcal{P} : \mathbf{Set}^{op} \to \mathbf{Set}$ corresponds to the category of sets together with a distinguished subset, with maps mapping subsets to subsets. The terminal objects corresponding to representants of $\mathcal{P}$ and their universal elements are $(\{0,1\},\{0\})$ and $(\{0,1\},\{1\})$. Note that the terminal object is not used unique, even if we consider the functor in* sk$\mathbf{Set}$.

Example 121 (Construction as comma category)**.** *Using the notion of comma category, we obtain a simple alternative construction to the category of elements as $(\bullet \downarrow F)$ constructed from a constant functor $\bullet : 1 \to \mathbf{Set}$ choosing the terminal object $\{\bullet\}$ in $\mathbf{Set}$ and the functor F. Since objects of such a morphisms are arrows*

$$\{\bullet\} \to F(X)$$

we can easily identify them with a functorial choice of element from $F(X)$ corresponding to the image of the arrow.

Example 122 (Functorial comma construction)**.** *Using the Yoneda lemma we may construct $\int F$ from presheaf category in a similar way as above. For any functor $F : \mathcal{C}^{op} \to \mathbf{Set}$ via Yoneda lemma $F(X) \simeq \mathrm{Nat}(h_X, F)$, so the elements $(F(X), y)$ of $\int F$ are in bijection with natural transformations $\eta_X(y) : h_X \Rightarrow F$. Moreover, every morphism*

$\eta_X(y) \to \eta_{X'}(y')$ is a natural transformation $h_X \Rightarrow h_{X'}$, thus the category of elements can be constructed as a comma category $(Y \downarrow F)$, where F is a constant functor $1 \to [\mathcal{C}^{op}, \mathbf{Set}]$ and Y is a Yoneda embedding $\mathcal{C} \to [\mathcal{C}^{op}, \mathbf{Set}]$.

$$(X, y) \xrightarrow{f} (Y, f_*(y))$$

Example 123 (Discrete left fibration)**.** *The natural projection $\Pi : \int F \to \mathcal{C}$ is called a "projection" not without a reason, as it has similar properties to topological fibrations - functions locally behaving like projections, characterised by their lifting property. A functor with a categorical, discrete version of such a property is called a discrete fibration, left for a covariant case and right in contravariant. The lifting property of Π ensures that every map $f : X \to Y$ has a unique lift $(X, a) \to (Y, b)$ in $\int F$ compatible with Π, so making the diagram*

$$\begin{tikzcd}
(X, a) \arrow[r, dotted, "\bar{f}"] \arrow[d, "\Pi"] & (Y, b) \arrow[d, "\Pi"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

commute.

Example 124 (Grothendieck construction)**.** *Here we will consider an important example of a general phenomenon of representing some functors as functions and vice versa. Let's look for example at some discrete small category S, which we can identify with a set S. Then the functor category $[S, \mathbf{Set}]$ is equivalent to the slice category $\mathbf{Set}/S$ corresponding to functions over S. Similarly we can identify sheaves over X with Etale maps and bundles over X with maps from their total spaces. In similar fashion the category of presheaves is equivalent to the category of discrete left fibrations*

$$[\mathcal{C}^{op}, \mathbf{Set}] \simeq \mathbf{DFib}(\mathcal{C})$$

where the functor F can be associated with its fibration

$$F \mapsto \Sigma_F : \int F \to \mathcal{C}$$

Example 125 (Category of cones). *Recall that a cone over a diagram* $\mathcal{I} \to \mathcal{C}$ *sends* $c \in \mathcal{C}$ *to a set of natural transformations* $F \Rightarrow c$. *By Yoneda embedding we can express it as* $\mathrm{Hom}(\Delta(-), F)$, *where delta is the constant functor embedding. The notion of category of elements provide an elegant construction of category of cones as* $\int \mathrm{Cone}(F, -)$, *also naturally isomorphic to* $(\Delta \downarrow F)$.

Example 126 (Limits and colimits). *The universal property of limits and colimits can be expressed via representability: a limit of a diagram* $F : \mathcal{I} \to \mathcal{C}$ *exists if and only if* F *is representable and can be identified with its representant. Its universal element is a limit cone, defining the isomorphism*

$$\mathrm{Cone}(-, F) \simeq \mathrm{Hom}(-, \lim F)$$

In the language of category of elements, a limit is a terminal object in the category of cones over F, $\int \mathrm{Cone}(-, F)$, *being a tuple of a limit object in* $\mathcal{C}$ *together with a specified limit cone.*

Example 127 (Twisted arrow category). *The category of elements of a hom functor* $\mathrm{Hom} : \mathcal{C} \times \mathcal{C}^{op} \to \mathbf{Set}$ *is called the twisted arrow category* $\mathrm{Tw}(\mathcal{C})$. *Its elements can be identified with arrows* $A \to B$ *and morphism are diagrams of a form*

$$\begin{tikzcd} A_1 \arrow[d] & A_2 \arrow[l] \arrow[d] \\ B_1 \arrow[r] & B_2 \end{tikzcd}$$

The latter explains the phrase twisted, as its characterisation is identical to ordinary arrow category except that the direction of one arrow in the commutative square is flipped.

Example 128 (Intervals). *Let* P *be a preorder. Its twisted arrow category* $\mathrm{Tw}(P)$ *we can identified with non-empty intervals*

$$[a, b] = \{x \in P \mid a \leq x \leq b\}$$

Morphisms of intervals can be identified with inclusions, so $\mathrm{Tw}(P)$ *also naturally forms a preorder.*

—— TODO: I don't get it, but it will be covered in coends ———

Example 129 (Join). *Given two small categories* $\mathcal{C}$ *and* $\mathcal{D}$, *one can construct the join category* $\mathcal{C} * \mathcal{D}$ *by taking as object the disjoint union* ob$\mathcal{C} \sqcup$ ob$\mathcal{D}$ *and preserving all the morphisms, adding a single arrow between all the objects of* $\mathcal{C}$ *and* $\mathcal{D}$

$$\mathrm{Hom}_{\mathcal{C}*\mathcal{D}}(C_1, C_2) = \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$$
$$\mathrm{Hom}_{\mathcal{C}*\mathcal{D}}(D_1, D_2) = \mathrm{Hom}_{\mathcal{D}}(D_1, D_2)$$
$$\mathrm{Hom}_{\mathcal{C}*\mathcal{D}}(C, D) = \{\bullet\}$$

The join $\mathcal{C} * \mathcal{D}$ *has a canonical construction as a category of elements corresponding to the constant functor* $\mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$ *mapping all the elements to the singleton set.*

Example 130 (Pathed spaces)**.** *Consider the category of elements of the fundamental groupoid functor.*

2.3. **Slice and comma categories.** Slice and comma categories are very interesting on its own, providing a very flexible tool for some algebraic constructions such as modules. Until the rest of the book, if X is an object of $\mathcal{D}$ and $C_X : \mathcal{C} \to \mathcal{D}$ a constant functor pointing at X, the comma category $(C_X \downarrow F)$ we'll denote as $(X \downarrow F)$. Similarly, the comma involving the identity functor $(\mathbb{1}_{\mathcal{C}} \downarrow F)$ will be denoted simply by $(\mathcal{C} \downarrow F)$.

Example 131 (Slice category as special case of comma)**.** *Recall that the comma category* $F \downarrow G$ *for some functors* $F : \mathcal{A} \to \mathcal{C}$*,* $G : \mathcal{B} \to \mathcal{C}$ *decodes the pair of objects* A, B *together with a map between their images* $f : F(A) \to G(B)$*.*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(X') \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ G(Y) & \xrightarrow{G(g)} & G(Y') \end{array}$$

Taking the category $\mathcal{B} = \mathbb{1}$ *with a constant functor* $G(\bullet) = X_0$ *and* F *being identity functor on* $\mathcal{C}$*, we end up with morphisms of the form*

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & Y \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \alpha'} \\ X_0 & \xrightarrow{id} & X_0 \end{array}$$

which coincide with the slice category $(\mathcal{C}, X_0)$*.*

Example 132 (Pointed sets)**.** *Pointed sets (or any concrete category) can be constructed as* $\{\bullet\}/\mathbf{Set}$ *or* $(\{\bullet\} \downarrow \mathbf{Set})$*, as the distinguished element* x_0 *can be identified with the image of the morphism* $\bullet \mapsto x_0$*.*

Example 133 (R-algebras)**.** *For any commutative ring R, the category of R-algebras naturally arise as the slice category $R/\mathbf{CRing}$.*

Example 134 (Subtraction)**.** *Consider the monoid of natural numbers with addition $\mathbb{N}_+$. For every two of its elements n, m we have a tautological equivalence*

$$n \geq m \text{ iff } n - m \geq 0$$

This trivial expression however has no obvious place in categorical language - the left hand side is a statement between $\mathbb{N}$ as a poset, while the statement from the right regards it as a monoid, and these are completely different objects. The passage from on side to the other is given by the slice category. For any commutative monoid M, consider the slice $\bullet/M$. A morphism in this comma category is a commutative triangle

$$\begin{tikzcd} & \bullet \arrow[dl, "m"'] \arrow[dr, "n"] & \\ \bullet \arrow[rr, "x"] & & \bullet \end{tikzcd}$$

Such an arrow f can be identified with element satisfying $n + x = m$. Note that in general there can be a lot of such elements, which happens exactly when the monoid does not have the cancellation property. In case of cancellative monoids, the comma category form a poset and every such an arrow can be identified with unique element, which we can suggestively denote as $n - m := x$. This identification in fact is nothing more than a forgetful functor $0/M \to M$, categorifying the subtraction. The same way we may also make sense of division and divisibility - the second one corresponding to the relation in the poset $1/\mathbb{N}_\times$. Its functoriality can be expressed with another completely obvious fact about natural numbers

$$a|b \text{ and } b|c \Rightarrow \frac{c}{a} = \frac{c}{b}\frac{b}{a}$$

Note that this is yet another face of the category of elements, corresponding to the regular representation $\mathbf{BM} \to \mathbf{Set}$. Moreover, it shows some wrong intuitions one may have from analogies between full functors and surjective functions. The canonical projection $\int M \to \mathbf{B}M$, is surjective only on objects and do not form a full functor for any non-trivial monoid.

The poset obtain from this construction can be also identified with the action groupoid $M//M$ corresponding to the regular representation, i.e. M acting on itself by right multiplication.

Example 135 (Non-autonomous systems)**.** *The subtraction functor considered above, relating cancelative monoids with posets, together* *with Grothendieck construction provide a nice way of looking at non-autonomous dynamical systems. Recall that the category of elements of representable presheaf* $y(X)$ *was just the slice category* $\mathcal{C}/X$*. By Yoneda lemma follows that for general presheaves we have then*

$$\mathbf{Psh}(\int F) \simeq \mathbf{Psh}(\mathcal{C})/F$$

the identity in a spirit of the Grothendieck construction. Now consider some time monoid $\mathbf{B}T$*, such as* $\mathbb{R}$ *or* $\mathbb{N}$*. Since its category of elements we've identified with a poset* PT*, choosing the presheaf* $F : \mathbf{B}T^{op} \to \mathbf{Set}$ *to be its regular representations (which we'll just denote as* T *itself, as its just the Yoneda embedding) we get*

$$\mathbf{Psh}(PT) \simeq \mathbf{Psh}(\mathbf{B}T)/T$$

The objects from right hand side are dynamical systems equipped with T*-equivariant arrow* $X \to T$*. We can think about thing in terms of fibers - looking at each time unit* $t \in T$ *separately we can decompose* X *to a family of states* X_t*. The equivariance of such function means that each time translation of period* $s \in T$ *is respected by the fibers inducing for each* t *a function* $X_t \to X_{t+s}$*. With this in mind its easy the isomorphic is easy to understand, as a functor* $PT \to \mathbf{Set}$ *is just a family of sets* X_t *with maps* $X_t \to X_{t+s}$ *induced by each relation* $t \leq t+s$ *in* PT*.*

Such parameterised dynamical systems X_t *are models of non-autonomous systems - unlike in autonomous where there is some space of possible states* X *and time acting on* X *by changing one state to another, this time the space of states* X *is not globally defined, but point in time has its own custom state of possible states* X_t*, disjoint from all the other possibilities. In particular, such system cannot have any periodic point, as the fibers modelling possible states are all disjoint. Non-autonomous systems often appearing in nature comes from differential equations given by functions dependent on time. To see the difference in action without getting hands dirty imagine following equations: first describing the move along the circle with constant speed, and the second describing move around the circle, but with speed slowing down twice every hour. The first case is autonomous, as given your position on the circle you already know everything about your situation, even if you don't poses a watch. In the second case, the position doesn't tell the hole story - you know the direction you're moving, but without a watch have no when, or even if ever, you will visit the same place again.*

Example 136 (Covering spaces and fiber bundles)**.** *If you're familiar covering spaces, you know that we can consider morphisms between* *them being covering maps. In such a setting the category of covering spaces of some space X is a full subcategory of* $\mathbf{Top}/X$*. Obviously the construction naturally generalizes for fiber bundles. More about these objects you can find in the chapter about topological examples.*

Example 137 (Arrow category)**.** *From every category $\mathcal{C}$ we can construct the "category of morphisms"* $\mathrm{Arr}(\mathcal{C})$*. Not surprisingly its objects are all the morphisms in $\mathcal{C}$ and the morphism between $X_1 \to X_2$ and $Y_1 \to Y_2$ are commutative squares of the form*

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[d] \\ Y_1 \arrow[r] & Y_2 \end{tikzcd}$$

The arrow category can be naturally constructed as the comma category $(\mathcal{C} \downarrow \mathcal{C})$.

Example 138 (Rings with augmentation)**.** *The elements of* $(\mathbf{Ring} \downarrow \mathbb{Z})$ *are rings together with the augmentation map $\epsilon : R \to \mathbb{Z}$, called ring with augmentations. Example of a ring with natural augmentation is a group ring $\mathbb{Z}[G]$, where the augmentation is the mapping forgetting the group elements*

$$\epsilon(\sum n_g \cdot g) = \sum n_g$$

Augmentation map and its kernel is very important in the study of group rings, providing tools not available in vanilla **Ring** *category.*

Example 139 (Comma as functor)**.** *Fix small categories $\mathcal{C}, \mathcal{D}, \mathcal{E}$. The construction of a comma category $(F \downarrow G)$ out of functors $F : \mathcal{E} \to \mathcal{C}$ and $G : \mathcal{D} \to \mathcal{C}$ forms a functor with a wild domain*

$$[\mathcal{E}, \mathcal{C}]^{op} \times [\mathcal{D}, \mathcal{C}] \to \mathbf{Cat}$$

Example 140 (Natural transformations)**.** *Natural transformations between functors also can be described in the language of comma categories. Given functors $F, G : \mathcal{C} \to \mathcal{D}$, we can identify* $\mathrm{Nat}(F, G)$ *with functors $\mathcal{C} \to (F \downarrow G)$ which are sections of both forgetful functors $(F \downarrow G) \to \mathcal{C}$ and $(F \downarrow G) \to \mathcal{D}$. Writing that property down explicitly essentially leads to the standard definition of a natural transformation.*

3. Limits and colimits

3.1. Introduction.

Definition 3.1 (Diagram)**.** *A diagram is any functor from a small category $F : \mathcal{I} \to \mathcal{C}$. The category $\mathcal{I}$ is called a shape of F. A diagram*

c is constant if sends every object to some $c \in \mathcal{C}$ and every morphism to id_c.

Definition 3.2 (Cone)**.** *A cone over $F : \mathcal{I} \to \mathcal{C}$ with a summit $c \in \mathcal{C}$ is a natural transformation* $\mathrm{Cone}(c, F) : c \Rightarrow F$*, where c is a constant diagram. The cones with varying summit form a functor* $\mathrm{Cone}(-, F) : \mathcal{C}^{op} \to \mathbf{Set}$.
Similarly for in a dual category a cocone over F with a nadir $c \in \mathcal{C}$ is a natural transformation $\mathrm{Cocone}(F, c) : F \Rightarrow c$*. The cocones with varying nadir form a functor* $\mathrm{Cocone}(F, -) : \mathcal{C} \to \mathbf{Set}$.

Definition 3.3 (Legs)**.** *A leg is a single component of a natural transformation defining a cone (or cocone).*

Definition 3.4 (Limit)**.** *A limit (colimit) of F is an object representing a cone (cocone) functor over (under) F.*

Definition 3.5 (Product)**.** *A product is a limit of a diagram indexed by a discrete category.*

... TODO ...

3.2. Topological examples.

Example 141 (Making sense of topological product)**.** *Recall that the infinite product of topological spaces was constructed not in a most natural way. It seems arbitrary and the most commonly presented explanation of defining it this way rather than as the "box topology" to undergrads, which is "it behaves much better this way" is quite disappointing. Category theory provides much more inside - this is the one and only natural way of defining the product up to homeomorphism, as it satisfies the universal property of the product in* **Top**.

Example 142 (Coproduct)**.** *The coproduct in* **Top** *is always the disjoint union.*

Example 143 (Subspace)**.** *The equaliser of f and g*

$$\mathrm{Eq}(f, g) \longrightarrow X \underset{g}{\overset{f}{\rightrightarrows}} Y$$

Is the subset of X on which the maps coincide with the subspace topology, so it provides the canonical way of constructing subspaces.

Example 144 (Quotient space)**.** *The quotient space of Y identifying points $f(x) \sim g(x)$ for some maps $f, g : X \to Y$ is exactly their co-equaliser*

$$X \underset{g}{\overset{f}{\rightrightarrows}} Y \longrightarrow \mathrm{Coeq}(f, g) \cong Y/\sim$$

Example 145 (Unions and intersections)**.** *The union and intersection of two subsets can be constructed as a pushout and pullback with respect to inclusions:*

$$\begin{tikzcd} U \cap V \arrow[r] \arrow[d] & U \arrow[d, hook] \\ V \arrow[r, hook] & X \end{tikzcd} \qquad \begin{tikzcd} U \cup V & U \arrow[l, phantom] \\ V \arrow[u] & U \cap V \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}$$

Example 146 (Wild monomorphisms)**.** *Not all monomorphisms in* **Top** *comes from equalisers. Continuous function is a monomorphism if and only if it is injective on sets, which can be easily verified using some maps from discrete spaces, where the continuity becomes redundant condition. However all the equalisers are equipped with the subspace topology, monomorphism can be realised as an equaliser if and only if it is an embedding (homeomorphism onto the image).*

Example 147 (Gluing spaces)**.** *The pushout in* **Top** *corresponds to the attaching space $X_f \cup_g Y$, so the disjoint union of X and Y identified via images of the gluing maps $f(a) \sim g(a)$.*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d, dotted] \\ Y \arrow[r, dotted] & X_f \cup_g Y \end{tikzcd}$$

Example 148 (Mapping cylinder)**.** *Given a map $f : X \to Y$, its mapping cylinder $M(f)$ can be constructed as a pushout of*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & Y \arrow[d, dotted] \\ X \times I \arrow[r, dotted] & M(f) \end{tikzcd}$$

Example 149 (Cone)**.** *A cone over a space X is a following pushout:*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, hook] & X \times I \arrow[d, dotted] \\ * \times I \arrow[r, dotted] & C(X) \end{tikzcd}$$

Example 150 (Mapping cone). *The construction of mapping cylinder $M(f)$ and cone over X provide an easy construction of the mapping cone:*

$$\begin{tikzcd} X \times I \arrow[r, hook] \arrow[d, hook] & M(f) \arrow[d, dotted] \\ C(X) \arrow[r, dotted] & C(f) \end{tikzcd}$$

Example 151 (Wedge sum). *By considering some non-standard category of spaces we may get more interesting universal constructions. For example in the category of pointed spaces* $\mathbf{Top}_*$ *the colimit is not the disjoint union of spaces, but the wedge sum $X \vee Y$: disjoint union with the distinguish point glued together. Thus taking coproducts in* $\mathbf{Top}_*$ *leads to fundamentally different results than in* **Top** *- for example every coproduct of connected spaces is also connected.*

Example 152 (Fiber). *Given a map $f : X \to Y$ and point $y \in Y$, the fiber F_y of f (subspace being preimage of y) can be constructed as the pullback*

$$\begin{tikzcd} F_y \arrow[r, dotted] \arrow[d, dotted] & \{y\} \arrow[d, hook] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Example 153 (Fibered product). *We can generalize the previous example by replacing the one-point inclusion by arbitrary map. In such a case the pullback form a fibered product - the subspace of $X \times Y$ consisting of all the pairs of points with the common image.*

Example 154 (General case). *In fact there is a universal construction of all the limits and colimits in* **Top** *due to having well-behaved forgetful functor, preserving all the limits and colimits. Finding a limit in* **Top** *can be reduced to finding the limit of the underlying diagram in* **Set**, *which constitute a limit with the initial topology - the minimal with respect to inclusion making all the legs commute. Dually, the colimit is equipped with the final topology, the maximal one satisfying the same condition.*

Example 155 (Base change of bundles). *Let's some category of maps to a fixed space, for example all the maps $A \to X$ for some fixed X or some more interesting subcategory, as complex vector bundles over X,* $\mathbf{VectB}_{\mathbb{C}}(X)$ *(these are just some slice categories; we generalise this example in the abstract section). Then every map $B \to X$ induces a* *base change functor* $\mathbf{VectB}(X) \to \mathbf{VectB}(B)$ *via the pullback*

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, dotted] & E \arrow[d] \\ B \arrow[r, "f"] & X \end{tikzcd}$$

Example 156 (Submersions)**.** *Submersion is a map between smooth manifolds, which differential has maximal rank at every point. In the categorical language every map* $f : X \to Y$ *induces a map* $TX \to f^*TY$ *between the tangent space of* X *and the pullback of tangent space of* Y*.* f *being a submersion means exactly that this induced map is a surjection.*

Example 157 (Pullback of smooth manifolds)**.** *Pullbacks does not always exist in the category of smooth manifolds. Consider for example the pullback of two projection of a circle to an interval. The fiber product will be homeomorphic to* $S^1 \vee S^1$*, which indeed cannot be a smooth manifold. However, the pullbacks in* **Diff** *always exist along submersions.*

Example 158 (No coproducts in metric spaces)**.** *Consider a category of metric spaces with short maps (or weak contractions: functions not increasing distances between points)* **Met***. It can be easily proven that such a category does not have even binary coproducts. The argument is following: consider a 2-point metric space* $Z = \{-N, N\} \subset \mathbb{R}$*. For any two non-empty metric spaces* X *and* Y*, construct constant maps to* Z $f : x \mapsto N$ *and* $g : y \mapsto -N$*. Then choosing* $N > d(x, y)$*, existence of a coproduct is impossible, as the map*

$$X \coprod Y \xrightarrow{(f,g)} Z$$

would not be short.

Example 159 (Coequalizers in Hausdorff spaces)**.** *Recall that the quotient space of a Hausdorff space need not to be Hausdorff (with the standard example being the interval with double origin). It clearly makes the existence coequalisers on* **Haus** *problematic. The solution is bring by the it's not that bad theorem, stating that the quotient of Hausdorff space is Hausdorff whenever the equivalence relation of a quotient is closed as a subspace of* $X \times X$*. This allows the following construction of coequalisers in* **Haus***: given two maps*

$$\begin{tikzcd} A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & X \end{tikzcd}$$

the coequaliser in **Top** *would be constructed by factoring $X \times X$ by the relation $R \subset X \times X$, where $R = \{(f(a), g(a)) \mid a \in A\}$. In* **Haus**, *similar trick can be performed, but after replacing R with its closure $\bar{R}$ in the product topology of $X \times X$.*

Example 160 (Circle as cospan cotrace)**.** *A cospan cotrace is a construction generalising gluing two ends of an object together. In category with coproducts and pushouts consider a cospan of a form*

$$\begin{tikzcd} & X & \\ A \arrow[ur, "in"] & & A \arrow[ul, "out"'] \end{tikzcd}$$

From each such cospan one can construct another one, merging two legs together and adding a trivial one. We'll also need a trivial cospan with coproduct of identity arrows on the opposite leg

$$\begin{tikzcd} & X & & & X & \\ A \sqcup A \arrow[ur, "in \sqcup out"] & & 0 \arrow[ul] & 0 \arrow[ur] & & A \sqcup A \arrow[ul, "in \sqcup out"'] \end{tikzcd}$$

Concatenation of these cospans yields a cospan cotrace, define as its pushout

$$\begin{tikzcd} & & \mathrm{cotr} X & & \\ & A \arrow[ur] & & X \arrow[ul] & \\ 0 \arrow[ur] & & A \sqcup A \arrow[ul, "1 \sqcup 1"] \arrow[ur, "in \sqcup out"'] & & 0 \arrow[ul] \end{tikzcd}$$

The flagship example is the construction of a circle from unit interval. Choose $\epsilon < \frac{1}{2}$ and put $A = [0, \epsilon]$. Then the inclusion of A on the beginning and end of I yields a cotrace

$$\begin{tikzcd} & & \mathrm{cotr} I & & \\ & {[0,\epsilon]} \arrow[ur] & & I \arrow[ul] & \\ \emptyset \arrow[ur] & & {[0,\epsilon] \sqcup [0,\epsilon]} \arrow[ul, "1 \sqcup 1"] \arrow[ur, "in \sqcup out"'] & & \emptyset \arrow[ul] \end{tikzcd}$$

Such a pushout corresponds to gluing together the collars of the interval and is homeomorphic to S^1.

3.3. **Algebraic examples.**

Example 161 (Algebraic closures of finite fields)**.** *The algebraic closure of a finite field of characteristic p can be explicitly constructed as a colimit of all its finite extensions. It's a basic fact from field theory that given finite fields of order p^n and p^m, there is a unique inclusion $\mathbb{F}_{p^n} \hookrightarrow \mathbb{F}_{p^m}$ if and only if $m|n$. This observation reduces the construction to a single direct limit*

$$\overline{\mathbb{F}}_p = \varinjlim (\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2} \hookrightarrow \mathbb{F}_{p^6} \hookrightarrow \mathbb{F}_{p^{24}} \hookrightarrow \cdots)$$

Example 162 (p-adic integers)**.** *The ring of p-adic integers can be constructed as a limit of reduction morphisms of the cyclic groups*

$$\mathbb{Z}_p = \varprojlim \left(\cdots \hookrightarrow \mathbb{Z}/p^3\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{p} \mathbb{Z}/p\mathbb{Z}\right)$$

Example 163 (Prüfer groups)**.** *The Prüfer groups $\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$ can be thought of as "infinite finite cyclic groups" $\mathbb{Z}_{p^\infty}$, since are isomorphic to the following direct limit*

$$\mathbb{Z}_{p^\infty} = \varinjlim \left(\mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}/p^2\mathbb{Z} \hookrightarrow \mathbb{Z}/p^3\mathbb{Z} \hookrightarrow \mathbb{Z}/p^4\mathbb{Z} \hookrightarrow \cdots\right)$$

Example 164 (Formal power series)**.** *The ring $R[[x]]$ can be constructed as a limit in* **Ring***. It comes from the projective system*

$$\mathbb{R}[[x]] = \varprojlim \left(\cdots \to \frac{R[x]}{(x^3)} \to \frac{R[x]}{(x^2)} \to \frac{R[x]}{(x)}\right)$$

Example 165 ($\mathbb{Q}/\mathbb{Z}$)**.** *The group $\mathbb{Q}/\mathbb{Z}$ has a similar construction to Prüfer groups, but goes along all the primes in the denominators. It can be constructed in two different ways: as a colimit of the diagram of all possible inclusions of finite cyclic groups, or in a more comprehensive way as the direct limit of $\mathbb{Z}/n!\mathbb{Z}$:*

$$\mathbb{Q}/\mathbb{Z} = \varinjlim (\mathbb{Z}/2\mathbb{Z} \hookrightarrow \mathbb{Z}/6\mathbb{Z} \hookrightarrow \mathbb{Z}/24\mathbb{Z} \hookrightarrow \mathbb{Z}/120\mathbb{Z} \hookrightarrow \cdots)$$

Example 166 (Completion of $\mathbb{Z}$)**.** *When we use the diagram of all possible inclusions, but reverse all the arrow and replace inclusions with reduction morphisms considered in the case of p-adic integers, the limit yields the completion of $\mathbb{Z}$: $\prod_{p\in\mathbb{P}} \mathbb{Z}_p$.*

Example 167 (Localisation)**.** *The localisation of a module at some element x can be obtained as a colimit*

$$M_x = \varinjlim \left(M \xrightarrow{x} xM \xrightarrow{x} x^2M \xrightarrow{x} x^3M \hookrightarrow \cdots\right)$$

The localisation at a multiplicatively closed set $S^{-1}M$ is a colimit of the localisation on each element from S.

Example 168 (Lattice). *A lattice is a poset category with all binary products and coproduct. These two has a natural meaning in such a setting - a product $a \times b$ is a greatest lower bound of a and b, denoted also as $a \wedge b$. The coproduct, denoted as $a \vee b$, is a least upper bound of a and b. The initial and terminal terms $0, 1$, if exist, correspond to infimum and supremum of the poset.*

Example 169 (Finitely generated subgroups). *Every group/module/algebra or any algebraic structure in which such a statement makes sense, can be expressed as the colimit over its finitely generated subgroups (subobjects), indexed by their inclusion preorder.*

Example 170 (Presentation). *Consider the category of some algebraic structures, such as groups or modules. The presentation of an object is its realisation as a coequaliser of two free objects, generated by generators and relations*

$$F(R) \underset{\mathbb{1}}{\overset{r}{\rightrightarrows}} F(G) \longrightarrow X \cong Y/$$

Example 171 (Presentation of categories). *The presentation can be generalised also to small categories. Recall that we have a functor*

$$C : \mathbf{Grph} \to \mathbf{Cat}$$

assigning to each graph G a free category generated by G, where the morphisms are freely generated by the paths in G. A relation in a category can be considered as identities of a form

$$f_1 \circ \cdots \circ f_n = g_1 \circ \cdots \circ g_m$$

Given a graph G and such a single relation r, the category with presentation $\langle G \mid r \rangle$ can be realised as the coequaliser of the diagram

$$C(2) \underset{r'}{\overset{r}{\rightrightarrows}} C(G) \longrightarrow \mathcal{C}$$

where 2 is the graph with 2 nodes and a single vertex. Similarly, a finitely presented category generated by with presentation $\langle G \mid r_1, \cdots, r_n \rangle$ is the coequaliser

$$C(2 \times n) \underset{r_i'}{\overset{r_i}{\rightrightarrows}} C(G) \longrightarrow \mathcal{C}$$

Example 172 (The smallest non-free category). *Let's look at some concrete example of a presentation of a category. Consider a category with 2 objects and a unique isomorphism between them. Such a category* *is not free on any graph, as it has a loop, which corresponds to the relations*

$$f \circ f^{-1} = \mathbb{1}_X$$
$$f^{-1} \circ f = \mathbb{1}_Y$$

However, $\mathcal{C}$ is finitely presented. The simplest such a presentation is given by

$$C(2 \times 2) \underset{(\mathbb{1},\mathbb{1})}{\overset{(r_1,r_2)}{\rightrightarrows}} C(G) \longrightarrow \mathcal{C}$$

Where the graph G has a form

$$\bullet \rightleftarrows \bullet$$

Example 173 (Normal subgroups)**.** *Considering a group G as a category with single object BG, normal subgroups of G correspond bijectively to categorical congruence relations on BG, while the quotient groups of G/N to its quotient categories.*

Example 174 (Free product of groups)**.** *The coproduct of groups $G \star H$ is called a free product. A construction of a free product is similar to a free group - it involves making an alphabet of words from the disjoint union of elements of G and H constructing the free monoid from them, and then identify all the reduced words, producing the group structure.*

Example 175 (Amalgamated product)**.** *Consider groups G and H with common subgroup K. Then the amalgamated product $G \star_K H$ is the pushout*

$$\begin{tikzcd} K \arrow[r, hook] \arrow[d, hook] & G \arrow[d, dotted] \\ H \arrow[r, dotted] & G \star_K H \end{tikzcd}$$

Explicitly, $G \star_K H$ can be constructed by taking, similarly as in the case of a free group, all the formal letters from alphabets of G and H, and form a group by identifying the letters coming from elements of K (and, as always, all the reduced words).

Example 176 (Galois groups)**.** *The Galois group of a field extension*

$$K \hookrightarrow L$$

Can be expressed as a limit over Galois group of all intermediate fields. $K \subset E \subset L$. Thus taking as the indexing category $\mathcal{I}$ the poset of *inclusions of intermediate extensions, we can simply write down the Galois group as*

$$\mathrm{Gal}(L/K) = \lim_{E \in \mathcal{I}} \mathrm{Gal}(E/K)$$

Example 177 (Action groupoid)**.** *Recall that the action groupoid $X//G$ associated to an action of G on the set X has points of X as objects, and every two points of X are connected with a unique arrow if and only if they lie in the same orbit. The action groupoid can be constructed as a pullback in* **Cat** *under identifying the action of G with functor $\mathbf{B}G \to \mathbf{Set}$.*

$$\begin{tikzcd} X//G \arrow[r, dotted] \arrow[d, dotted] & \mathbf{Set}_* \arrow[d, "U"] \\ \mathbf{B}G \arrow[r] & \mathbf{Set} \end{tikzcd}$$

The similarity of this construction to the universal bundle, with the category of sets playing a role of universal bundle, classifying all possible group actions via pullback of the forgetful functor from pointed sets (playing a role of associated bundle). Fibers of the forgetful functor corresponds to points of chosen set. Since delooping category has only one object, the pullback $X//G$ is also the fiber of pullback map, so it has all choices of basepoint in X as the set of objects. Moreover, arrows g of $\mathbf{B}G$ induce changes of basepoints $x_0 \mapsto gx_0$, which determine the arrows in the pullback groupoid.

In extreme case where G acts trivially on X, the groupoid $X//G$ is just the coproduct of deloopings $X//G = \coprod_X \mathbf{B}G$. On the other hand, every free and transitive action produces the codiscrete groupoid with unique arrow between every pair of points. More generally the action is free if and only if its groupoid is contractible, and is transitive if and only if $X//G$ is a thin groupoid (or, equivalently, a symmetric preorder). Moreover, the group of endomorphisms of a point $X//G(x,x)$ is exactly its stabiliser.

Example 178 (Orbit-stabiliser theorem)**.** *The construction of action groupoid provides an elegant categorical proof of the orbit-stabiliser theorem, which states that for each point in a G-set X there is the equality*

$$|\mathrm{orb}(x)| = [G : \mathrm{stab}_G(x)]$$

To prove that, note that $|G|$ is exactly the number of arrows with codomain x in $X//G$ for any x_0. Moreover, there is a general fact true for any groupoid that the number of arrows from x to y is constant in the connected component of x. After fixing isomorphism φ between *y and z,* $\mathrm{Hom}(x, y)$ *is in bijective correspondence with* $\mathrm{Hom}(x, z)$ *by composition with* φ*.*

Now the theorem follows from simple counting of arrows within connected component of x*, which there are exactly* $\sum_{\pi_0(x)} |\mathrm{Hom}(x, y)| = |\mathrm{Hom}(x, x)||\pi_0(x)|$*. Since* $\mathrm{Hom}(x, x)$*, as we'd already established, is the stabiliser of* x*, it follows that*

$$|G| = |\mathrm{stab}_G(x)||\mathrm{orb}(x)|$$

Example 179 (Milnor square)**.** *Let* I *be an ideal of a ring* R *and* $\varphi : R \to S$ *a ring homomorphism mapping* I *injectively onto and ideal* J *in* S*. Then* R *can be realised as a pullback in the category of rings of a so called Milnor square*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & S \\ R/I \arrow[r] & S/J \end{tikzcd}$$

Example 180 (Conductor square)**.** *A particularly interesting example of a Milnor square is the conductor square. Suppose we have a finite extension of commutative rings* $R \to S$*. Let* I *be the largest ideal of* S *contained in* R*, i.e.* $I = \mathrm{Ann}_R(S/R)$*, called the conductor ideal. Then the ring* R *can be fully reconstructed from* I *and the quotient* R/I*, which we can interpret as decomposing* R *into the part forming an ideal in some bigger space and its reminder. The pullback associated to its Milnor square reconstructs* R *as pairs of elements from the ideal and from the reminder.*

Example 181 (Symmetric functions)**.** *Polynomials invariant under action of* S_n *on* $\mathbb{Z}[x_1, \cdots, x_n]$ *by permuting variables naturally form a graded ring* $\Lambda_n = \mathbb{Z}[x_1, \cdots, x_n]^{S_n}$*. By passing with* n *to infinity, we may naturally form a ring of symmetric functions in countably many variables - these are never polynomials if not constant, but all of finite degree. Categorically, this can be constructed both as a limit and colimit, with the first sequence setting last variable to zero, and the latter being natural inclusion. Interestingly, every ring* Γ_n *is generated by infinitely many complete homogeneous symmetric polynomials* h_i *- these are just sums of all monomials with degree* i*. Clearly these cannot be algebraically independent, as* Γ_n *is a subring of finitely generated ring of all polynomials. Surprisingly, these become indeed independent in the limit ring* $\Gamma = \varinjlim \Gamma_i \simeq \varprojlim \Gamma_i$*.*

3.4. **Different examples.**

Example 182 (Terminal and initial objects)**.** *The terminal object in any category is a trivial product, when the indexing category is empty. Dually, the colimit of such functor is always the initial object.*

Example 183 (GCD and LCM)**.** *Given any poset, any pullback of elements x and y is their least upper bound, while the pushout is the greatest lower bound. In particular, this perspective provides a construction of a greatest common divisor and least common multiple from number theory, as pullbacks and push-outs in the poset category of natural numbers with relation of divisibility.*

Example 184 (Generative effects)**.** *In data science a description of a system is often described in terms of preorders. An information about systems aka features is decoded by monotone maps, where $f : X \to A$ decodes some feature of X that is observed by A. Functors preserving products (meets) are the observations consistent on subsystems. The meet operation (product in preorders) $a \wedge b$ can be understood as restrictions to elements satisfying both conditions a and b, thus to some subsystem. The relation $f(a \wedge b) = f(a) \wedge f(b)$ means that the observation associated to f is still true in an intersection of two environments whenever it holds in each of them.*

Preservation of joins (coproducts) describes the opposite situation, where two systems are combined together. In particular, where a functor does not preserve joins, we say that it has a generative affect, so that after combining two systems together we can expect some additional information than encountered in two systems independently. In fact, whenever the joins exist, we always expect more stuff in combined systems, which is a slogan term for an always true in such cases relation

$$f(a) \vee f(b) \leq f(a \vee b)$$

In the next section we'll see that the class of functors consistent of subsystems but with generative effects are right adjoint, and are in bijective correspondence with description from the opposite direction, nicely gluing, but not consistent on subsystems. The capstone of this categorical approach to data science reader can find in the discussion of Kan extensions, where we'll develop some simple prediction and clustering techniques.

Example 185 (Covariant partitions)**.** *We've seen in chapter* 1 *that the preorder of partitions of a set can be seen as a contravariant functor* $\mathbf{Set}^{op} \to \mathbf{Pre}$*. Using pushouts, partitions can be seen also as a functor of covariant nature. Recall that a partition of X can be uniquely identified with some surjection $X \to P$. Given a function $X \to Y$, the* *pushout*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, two heads] & Y \arrow[d, dotted, two heads] \\ P \arrow[r, dotted] & P \sqcup_X Y \end{tikzcd}$$

yields a surjection $Y \to P \sqcup_X Y$ *in a functorial way, establishing a functor*

$$\mathrm{Part} : \mathbf{Set} \to \mathbf{Pre}$$

Example 186 (Terminal and initial categories)**.** *The category* $\mathbb{1}$ *with one object and one morphism is a terminal object in* **Cat***, so the trivial product. The initial category is the empty one.*

Example 187 (Fixed length paths in a graph)**.** *Every directed graph* G *can be presented with two arrows* $s, t : E \rightrightarrows V$*, specifying source and target vertices of each edge. The family of paths of length* 2*, pairs of two composable arrows, can be calculated as the pullback of 2 copies of associated span*

$$\begin{tikzcd} & & E \times_V E \arrow[dl] \arrow[dr] & & \\ & E \arrow[dl, "s"'] \arrow[dr, "t"] & & E \arrow[dl, "s"'] \arrow[dr, "t"] & \\ V & & V & & V \end{tikzcd}$$

as we see explicitly that $E \times_V E = \{(e_1, e_2) \mid t(e_1) = s(e_2)\}$*. Similarly taking the iterated pullback* n *times yields all the paths of length* n

$$E_n = E \times_V E \times_V \cdots \times_V E = \{(e_1, e_2, \cdots, e_n) \mid t(e_1) = s(e_2), t(e_2) = s(e_3), \cdots t(e_{n-1}) = s(e_n)\}$$

Finally, we can obtain all possible finite paths with the coproduct

$$\mathrm{Path}(G) = \coprod^{\infty} E_n$$

Example 188 (Euler method categorified)**.** *Note that the previously constructed set of all finite paths of directed graph, equipped with natural composition induced from pullbacks, establish in fact the arrows in the free category generated by* G*, regarded as image of free category functor adjoint to forgetting the composition rules. It can be used in quite an surprising construction of the categorified Euler method. Classically, given* C^1 *function* $f : \mathbb{R}^n \to \mathbb{R}^n$*, the Euler method approximates the solution of differential equation*

$$\frac{d\varphi}{dt} = f(\varphi(t))$$

Choosing some possibly small step $h > 0$, the discrete values of φ can be approximated by computing the iterated steps starting from chosen initial x_0

$$\varphi(t+h) = \varphi(t) + hf(\varphi(t))$$

Now consider the directed graph defined by the span

$$\begin{tikzcd} & \mathbb{R}^n \arrow[dl, "x"'] \arrow[dr, "x+hf(x)"] & \\ \mathbb{R}^n & & \mathbb{R}^n \end{tikzcd}$$

Its k-th pullback, classifying the paths of length k, is then the set

$$E_n = \{(x_1, \cdots x_1 + hf(x_1), x_1 + hf(x_1 + hf(x_1))), \cdots) \in \mathbb{R}n + k\}$$

containing the computed first k steps of the Euler method at each point. The free category generated from this graphs and arrows has the vectors from $\mathbb{R}^n$ as objects and the arrows with source $x_0 \in \mathbb{R}^n$ are all the approximate solutions to considered differential equations with finitely many steps, with target equal to the value computed in the last step. Thus the arrow $x \to y$ in $F(G)$ exists if and only if y is k-th Euler approximation with initial condition $x_0 = x$ for some finite number of steps $k \leq 0$, which can be identified with a tuple $(x = x_0, x_1, \cdots, x_k = y)$, where $x_i = x_{i-1} + hf(x_{i-1})$. In particular, for $k = 0$ we get the identity arrow corresponding to the tuple (x).

Example 189 (Kernel pair)**.** *A kernel pair of a morphism f is the pullback along 2 copies of f:*

$$\begin{tikzcd} K(f) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

In **Set***, the kernel pair of a function f has a form*

$$K(f) = \{(x_1, x_2) \mid f(x_1) = f(x_2)\}$$

So it stores all the pairs of points from X having the same image. The general idea behind kernel pair is that it represents the internal equivalence relation that was induced by f. In the case of **Set***, it just means that $K(f)$ stores all the points that are pairwise equivalent, where the equivalence is induced by their equality under the action of f. Finally, let's consider some concrete examples in* **Top***, such as the real absolute* *value function* $|x| : \mathbb{R} \to \mathbb{R}$. *The kernel pair*

$$\begin{tikzcd} K(f) \arrow[r] \arrow[d] & \mathbb{R} \arrow[d, "|x|"] \\ \mathbb{R} \arrow[r, "|x|"] & \mathbb{R} \end{tikzcd}$$

is the subspace of $\mathbb{R}^2$ *being an union of two perpendicular lines* $l_0(x) = x$ *and* $l_1(x) = -x$

$$K(f) = \{(x, \pm x) \mid x \in \mathbb{R}\} \simeq \mathbb{R} \vee \mathbb{R}$$

Example 190 (Coimage)**.** *While the kernel pair* $K(f)$ *stores all the information about the relation induced by* f*, which more or less dona by keeping track of all the equivalent pairs of elements, the quotient object encoding equivalence classes of such a relation, which in most cases is the most interesting to us, can be constructed as the coequalizer of the kernel pair*

$$\begin{tikzcd} K(f) \arrow[r, shift left] \arrow[r, shift right] & Y \arrow[r] & \mathrm{Coeq}K(f) \end{tikzcd}$$

Such coequaliser is called the coimage of f*. The idea behind its construction is dual to the one implemented via image, however with the same goal. The difference we will discuss in a moment, as firstly we must say a few words about image.*

Example 191 (Cokernel pair)**.** *The cokernel pair is the dual version of kernel pair, i.e. the pushout of* f *with itself*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d] \\ Y \arrow[r] & C(f) \end{tikzcd}$$

The space represented by this object is constructed from two copies of the codomain, with points lying in the image identified, while points from outside the image left doubled. The idea behind the cokernel pair is pretty clear as a special case of gluing constructions. Let's stick for a moment to topological spaces (or just sets). There are several ways of gluing them together. The most general one is via some equivalence relation, but in most practical example such a relation comes from some function. The pushout is a tool used for gluing two spaces, provided a function from them to common codomain, which specifies which pairs of points should be glued together by mapping them to the same value. In different scenario, when we want to only collapse some points inside *a space, such as orbits of some group action, we can do this via coequaliser by choosing two maps from X in order to specify which pairs of points should be glued together by mapping them to the same point. The cokernel pair provide a gluing mechanism simpler than both from above - given only a single map from a single space, it glues together points with common values, but not it collapse points in the space, but takes them from its distinct copy, such as the pushout did for two arbitrary spaces. A pretty famous example of the cokernel pair is the double ended interval. It comes from the inclusion $i : [0, 1) \hookrightarrow [0, 1]$. Its cokernel pair is a space locally isomorphic to $\mathbb{R}$, which fails to be a manifold by not being Hausdorff. The space $C(i) = I \sqcup_{[0,1)} I$ comes from two unit intervals, in which all equal points where glued together except the pair with value 1. Since every neighbourhood of both of these points consists only themselves and points that were glued, separating them is impossible.*

Example 192 (Image)**.** *The image of $f : X \to Y$ is defined, dually to the coimage, as the equaliser of cokernel pair. The best understanding of this construction in my opinion comes from understanding how it differs from the coimage, despite trying to achieve the same goal. Relation image/coimage is similar to relation between left and right Kan extensions - both are approximating the same object, but from different sides of the wall. Again will analyse some examples from* **Top***, since every considered construction has an intuitive meaning then, which is pretty simple to write down, while also they do not equal in general - which is in fact not so common among categories that we like considering, for example they coincide in all abelian categories (by definition), as well as in* **Set***. The construction of image can be summarised in following 3 steps:*

(1) *Take two copies of Y*
(2) *Glue together points lying in the image*
(3) *Keep only the points that were glued*

Similarly we can describe the construction of coimage:

(1) *Take all the pairs of points from X*
(2) *Keep only the pairs mapped by f to the same value*
(3) *Glue together pairs mapped to common value*

Both these approaches try to model the space of values of f, but in the image the construction starts from the codomain Y, while coimage is built by manipulating the domain X.

Example 193 (Connected components of a graph). *Recall that we associated the category of directed graph with the presheaves of the category $\mathcal{C}$ of a form*

$$V \underset{t}{\overset{s}{\rightrightarrows}} E$$

The functor $\Pi_0 : \mathbf{DirGrph} \to \mathbf{Set}$ *mapping a graph to the set of its connected components is exactly the coequaliser of the diagram from above*

$$V \underset{t}{\overset{s}{\rightrightarrows}} E \longrightarrow \mathrm{Coeq}(s, t)$$

Such a coequaliser identifies each pair of arrows, in which the source of one arrow is the target of the second, which naturally collapses each connected component to a single element of $\Pi_0(G)$*.*

Example 194 (Colimits as connected components). *The functor* Π_0 *of connected components together with the category of elements provide a neat way of constructing the colimit in any small category with a simple formula*

$$\operatorname{colim} F = \Pi_0 \int F$$

Example 195 (Limits as sections). *The dual theorem to the previous one also exists, but instead of the connected component functor uses the sections. More concretely, in any small category the limit of* F *can be identified with the set of sections of the natural projection* $\int F \xrightarrow{\pi} \mathcal{C}$*. Note that such a section is just a terminal object in* $\int \mathrm{Cone}(-, F)$*, so the idea isn't particularly new, nevertheless elegant*

Example 196 (Category of elements as pullback). *In this example we will consider a very confusing construction - taking a limit in* $\mathbf{Cat}$*. Given a functor* $\mathcal{C} \to \mathbf{Set}$*, its category of elements can be considered as "freely adjoining to* X *some element from* $F(X)$*", which can be formally expressed as the pullback along forgetting the point from the category* $\mathbf{Set}_*$*:*

$$\begin{tikzcd} \int F \arrow[r] \arrow[d] & \mathbf{Set}_* \arrow[d] \\ \mathcal{C} \arrow[r, "F"] & \mathbf{Set} \end{tikzcd}$$

Example 197 (Sheaves). *Recall that the presheaf is a functor* $\mathbf{Top}(X)^{op} \to \mathcal{C}$*. A sheaf is a presheaf satisfying additional two axioms:*

- ***(Locality)****: If two sections* $s, t \in F(U)$ *agrees (have equal restrictions) on any open cover of* U*, they must be equal.*

- ***(Gluing)***: *If $\{U_i\}$ is an open cover of U and there are sections $s_i \in F(U_i)$ that pairwise agree on intersections, then they are a restriction of a unique section $s \in F(U)$*

The condition can be reformulated to a more categorical language by requiring $F(U)$ to be a equaliser of a diagram in the category $\mathcal{C}$:

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \overset{f}{\underset{g}{\rightrightarrows}} \prod_{i,j\in I} F(U_i \cap U_j)$$

for any open cover $\{U_i\}_{i\in I}$ of any open set $U \in \mathbf{Top}(X)$. The equalised maps are product of all maps induced by inclusions of first and second component. Even more elegant interpretations can be stated in category $\mathbf{Top}(X)$. For any its element U, the I-indexed family of its subsets $U_i \subset U$ covers U if the diagram of inclusions their pairwise intersections has colimit U. Then a presheaf is a sheaf if and only if it preserves these colimits. Since it is contravariant, it means that F maps them to limits in $\mathcal{C}$, and the previous condition is just a classical presentation of this limit using products and equalisers.

Example 198 (Base change)**.** *We can generalise the example of base change of bundles to a more general case. Whenever the category $\mathcal{C}$ has all pullbacks, every morphism $X \to Y$ induces the base change functor on the slice categories $\mathcal{C}/Y \to \mathcal{C}/X$*

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d, dotted] & E \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The most notable example are sheaves and presheaves - the map between base spaces can be "lifted" in this way to transform the entire sheaf in some compatible (we can say even "canonically the most compatible") way.

Example 199 (Subobject classifier)**.** *Consider a category $\mathcal{C}$ with finite limits. The subobjects of a given object X can be describes as all the monomorphisms $S \to X$, or in some cases, where the functor* $\mathrm{Sub} : \mathcal{C} \to \mathbf{Set}$ *is representable, by the morphisms* $\mathrm{Hom}(X, \Omega)$. *Where the classifying object Ω exists, there is a very interesting correspondence between these two representations of subobjects: there is an unique morphism $1 \to \Omega$, such that given a representation of a subobject via* *the morphism $X \to \Omega$, the monic representation $S \to X$ can be constructed canonically via a pullback*

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook, dotted] & 1 \arrow[d] \\ X \arrow[r, "f"] & \Omega \end{tikzcd}$$

A basic example of a subobject classifier is a function $\{0\} \to \{0,1\}$ mapping $0 \mapsto 0$.

Example 200 (Classifying vector bundle). *Consider the category of n-dimensional complex vector bundles over a paracompact space X. It is a classical result in a theory of vector bundles that the subobject classifier is a map from the Stiefel manifold V_k, whose points are tuples of k orthonormal vectors in $\mathbb{C}^\infty$, to the Grassmanian, which is the classifying object called $BU(n)$ - the space of all n-dimensional subspaces of $\mathbb{C}^\infty$.*

Example 201 (Classifier in G-sets). *Consider the category $G - \mathbf{Set}$. Given any object $X \times G \to G$, its subobjects are all the subsets of X closed under the action of G. Notice that the complement of any subobject S also is G-invariant, as if $xg \in S$, then $(xg)g^{-1} = x \in S$. Thus the G-invariant maps $X \times G \to \{0,1\} \times G$ determines the subobjects, so the subobject classifier is similar as same as in $\mathbf{Set}$: $\{0\} \times G \to \{0,1\} \times G$. In the next example we will see that such a reasoning fails in the case of action of monoids, and the subobject classifier takes a much more interesting form.*

Example 202 (Classifying action of monoids). *As already mentioned, the subobject classifier of a form $1 \to 2$ as in $\mathbf{Set}$ or $G - \mathbf{Set}$ fails to classify subobjects in $M - \mathbf{Set}$, the category of spaces acted by some monoid M. Before fixing the problem let's see what actually goes wrong. Consider the real line acted by $M = (0, \infty)$ by multiplication. Clearly the subset $[2, \infty) \subset \mathbb{R}$ is M-invariant, thus a subobject of $\mathbb{R}$. So the classifying map $\mathbb{R} \to \{0,1\}$ should take $(2, \infty)$ to 1 and $(-\infty, 2]$ to 0. But such a map clearly fails to be M-invariant! For example then we would have $\varphi(1) = 0$, thus also $0 = 2 \cdot 0 = \varphi(1+2) = \varphi(2) = 1$. In the case of $\{0,1\} \times M$ serving as classifying object, we would have an additional on a subobject*

$$xg \in S \Rightarrow x \in S$$

Which certainly holds if M is a group, but not monoid in general. To fix this problem we must enlarge the classifying object Ω to contain all the *subsets of M closed under multiplication, called (for obvious reasons) ideals. Then the M-invariant map $\varphi : X \to \Omega$ defined as*

$$\varphi(x) = \{m \in M \mid xm \in S\}$$

indeed determines the subobject S uniquely. Let's again go back to our example. The ideals of $[0, \infty)$ are the subsets $[a, \infty)$ or (a, ∞). Thus the map $X \to \Omega$ corresponding to $S = [2, \infty)$ is

$$\varphi(x) = \{c \geq 0 \mid c \geq 2 - x\} = [\max(0, 2 - x), \infty)$$

The subobject classifier is then a map $\{\bullet\} \to \Omega_M$, where Ω_M is a set of all the right ideals of M with an action of M by right multiplication. Given such a morphism, the subobject can be identified as the preimage of the maximal ideal $M \lhd M$.

Example 203 (Half-truth)**.** *Similar obstruction we encounter in the case of the arrow category $[2, \mathbf{Set}]$. The subobject of the arrow $X_1 \xrightarrow{f} X_2$ are the subsets S_1, S_2 fitting into the commutative square*

$$\begin{tikzcd} S_1 \arrow[r, hook] \arrow[d] & X_1 \arrow[d, "f"] \\ S_2 \arrow[r, hook] & X_2 \end{tikzcd}$$

Now the elements of X_2 can be of two types - belonging to S_2 or not, as usual. However, the elements of X_1 have 3 possible types: they can either belong to S_1, and if they don't their image $f(x)$ can either belong to S_2 or not. Then the mapping

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[d] & X_2 \arrow[d] \\ \{0,1,2\} \arrow[r] & \{0,1\} \end{tikzcd}$$

Uniquely determines the sets S_1 and S_2. Without adding the additional truth value to the first set, the subsets of arrows, where image of S_1 is strictly smaller than S_2 cannot possibly be realised as a morphism to the universal object. In general, in the category of sequences of n-maps, the subobject classifier will have a form

$$\begin{tikzcd} \{\bullet\} \arrow[r] \arrow[d] & \{\bullet\} \arrow[r] \arrow[d] & \cdots \arrow[r] & \{\bullet\} \arrow[d] \\ \{0, \cdots, n+1\} \arrow[r] & \{0, \cdots, n\} \arrow[r] & \cdots \arrow[r] & \{0,1\} \end{tikzcd}$$

Such a classifier has a nice logical interpretation - the evaluation $X \to \{0,1\}$ can be thought of as assigning the truth value - $f(x) = 1$ iff x is true, and otherwise x is false. In this setting, we got an additional value in the first step - if we change the syntax a little bit, $X_1 \to \{0, \frac{1}{2}, 1\}$ corresponds to evaluation

$$f(x) = \begin{cases} 0 & x \text{ is true} \\ \frac{1}{2} & x \text{ is false, but will become true in the future} \\ 1 & x \text{ is false} \end{cases}$$

In the case of sequences of length n, the place in the sequence can be identified with time. Then, again changing a little bit the syntax to $X_n \to \{0, 1, \cdots, n, \infty\}$, we can interpret $f(x) = n$ as "x will become true in n time steps". In particular when $f(x) = 0$ x is already true, and if $f(x) = \infty$ x will always stay false.

Example 204 (Fuzzy logic)**.** *The last example can be taken to the limit, which will result in some variation of a fuzzy logic. Considering the category of infinite chain of functions $[\mathbb{N}, \mathbf{Set}]$, the subobject classifier will have a form of a limit over the finite cases, which is*

$$\begin{tikzcd} \{\bullet\} \arrow[r] \arrow[d] & \{\bullet\} \arrow[r] \arrow[d] & \cdots \\ \mathbb{N} \cup \{\infty\} \arrow[r] & \mathbb{N} \cup \{\infty\} \arrow[r] & \cdots \end{tikzcd}$$

Which using the previous interpretation as a truth value, allows us to considering values from some continuous spectrum. Obviously, if you refuse to consider a countable set of values continuous, you can freely change $\mathbb{N}$ to any cardinal number you like.

Example 205 (Final functor)**.** *This motivation is inspired by the fact, that the direct limit of any system do not change after ignoring all but infinitely many terms, for example considering only even indices all ignoring first 100 ones. A subcategory $\mathcal{I} \hookrightarrow \mathcal{J}$ is called final, or more generally a functor $F : \mathcal{I} \to \mathcal{J}$ is final, if every colimit can be computed after composing with F (which in case of a subcategory is the inclusion). This property has a very easy to check condition: F is final if and only if for every $j \in \mathcal{J}$ the slice category j/F is non-empty and connected. Considering again the case of N, we can identified its final subcategories with all unbounded, so having infinitely many objects. Some other simple examples of final subcategories or functors are*

- *inclusion of final object*

- *right adjoint functors*
- **Cart** $\hookrightarrow$ **SmoothManifolds**, *where* **Cart** *is full subcategory containing open subsets of* $\mathcal{R}^n$
- *good covers (with all finite intersections contractible) in* **Top**(X) *when* X *is paracompact*
- *free modules in projective modules*
- *trivial bundles into finite dimensional vector bundle over paracompact space*
- $[0] \rightrightarrows [1] \hookrightarrow \Delta^{op}$
- *non-empty subsets of total orders with the same maximum*

Example 206 (Cofinal diagrams)**.** *Similar situation as before can be considered with respect to diagrams. Two diagrams are called cofinal if they have equivalent colimits. This provide a slight generalisation of the previous example, as there can be no functor between their indexing category. For example any two final subcategory are cofinal.*

Example 207 (Hemicompact spaces)**.** *Consider the poset of compact subsets of topological spaces. In some cases there exist a handy cofinal diagram in the shape of a direct system* $(K_i)_{\mathbb{N}}$*. Such a diagram is cofinal if the subspaces* K_i *satisfy*

- $K_i \subseteq \operatorname{int}(K_{i+1})$
- $\bigcup K_i = X$

Spaces admitting such a cofinal diagram are called hemicompact. As an example $\mathbb{R}$ *is hemicompact with the sequences* $[-n, n]$*, however the long line is not hemicompact.*

Example 208 (Dense subcategory)**.** *A dense subcategory* $\mathcal{D} \subset \mathcal{C}$ *generalises the notion of a dense subspace in a sense, that all objects from* $\mathcal{C}$ *are constructible as a colimit of objects from* $\mathcal{D}$*. The analogy comes from the fact that given any function* $f : A \to X$ *where* X *is compact and image of* f *is dense, than any map* $X \to Y$ *is determined uniquely by its composition with* f*, so inclusion of any dense subspace functionally determines the entire space. Among examples of dense subcategories are (where a set of objects are understood as a full subcategory):*

- $\{\bullet\} \hookrightarrow$ **Set**
- $\{R \oplus R\} \subseteq R$-**Mod**
- $\{\mathbb{Z}\}$ *is not dense in* **Ab**
- *every Yoneda embedding (as we shall see later, as a result of coYoneda lemma)*

Example 209 (Dense inclusions do not compose)**.** *Surprisingly, given a chain of dense subcategories* $\mathcal{C} \subset \mathcal{D} \subset \mathcal{E}$*, does not imply that* $\mathcal{C}$ *is* *dense in $\mathcal{E}$. A counterexample comes from simplicial category theory and involves the inclusions*

$$\Delta_{\leq[n]} \hookrightarrow \Delta \hookrightarrow \mathbf{Cat}$$

The first inclusion is easy to see elementary, while the second inclusion follows from the fact, that **Cat** *can be embedded in simplicial sets, and any simplicial set as a presheaf is a colimit of representable objects, which Yoneda lemma identifies with Δ. However, not every category can be constructed as a colimit of only terminals and arrows.*

Example 210 (**Top** is huge)**.** *Basic categories such as* **Set** *and* **Ab***, despite being big have dense subcategories consisting only a single object. However, this is not the case with* **Top***, as it has no small subcategory at all. A proof of that is quite pretty: if a space is homeomorphic to colimit of some diagram made of spaces X_i, then its topology is uniquely determined by topologies on all possible images of maps $X_i \to X$. But taking a space Y of cardinality bigger than any object from alleged subcategory $\mathcal{C}$ with Zariski-like topology, when open sets are exactly having complement of smaller cardinality than the space itself, gives a contradiction - this way every image of map with domain in $\mathcal{C}$ is discrete, while the space itself is not. A category with a small dense subcategory is called* ***bounded****.*

Example 211 (It cannot be proven if integers are dense)**.** *A weird thing happens when ones try to determine dense subcategories in $\mathbf{Set}^{op}$. Even though can find a condition of their existence, we never know if they exist or not. A cardinal number κ is called accessible if there exist a two-valued measure which is κ-additive. It can be shown that a dense subcategory of $\mathbf{Set}^{op}$ exists if and only if measurable cardinals form a proper class, which on the other hand have been proven by Isbell to be unprovable in ZFC. Moreover, the full subcategory of $\mathbb{Z}$ is dense if and only if there are no measurable cardinals at all, which is also unprovable. Interestingly, equivalently it implies the existence of an endofunctor* $\mathbf{Set} \to \mathbf{Set}$ *which is exact, but not naturally isomorphic to the identity.*

Example 212 (Codense square)**.** *Dually to dense subcategories, $\mathcal{D} \subset \mathcal{C}$ is codense if every object in $\mathcal{C}$ is a limit of objects in $\mathcal{D}$. A quite unexpected example is a one-object subcategory of compact spaces with object I^2, which can be proven to be codense.*

Example 213 (Natural numbers as directed circle)**.** *In the topological examples we've constructed the topological circle as cotrace cospan,* *gluing the endpoints of unit interval. This time we'll repeat that construction in* **Cat**. *The interval object in small categories is the arrow category, which we'll in this example denote as I. The cotrace we're interesting in is identical as in topological case, so we'll jump straight to its final form, which is*

$$\begin{tikzcd}
\mathrm{cotr}(I) & I \arrow[l] \\
1 \arrow[u] & 1 \sqcup 1 \arrow[l] \arrow[u, "0 \sqcup 1"']
\end{tikzcd}$$

Following the algorithm of calculating pushouts of categories, we can notice that it is equivalent to the fundamental category of the directed circle (it's in fact its skeleton)

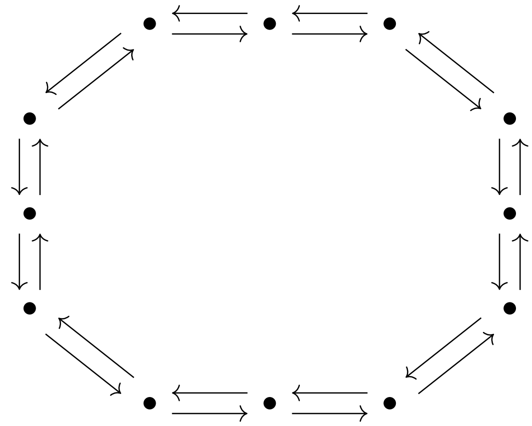
$$\begin{tikzcd}
 & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[dr] & \\
\bullet \arrow[ur] & & & & \bullet \arrow[d] \\
\bullet \arrow[u] & & & & \bullet \arrow[d] \\
\bullet \arrow[u] & & & & \bullet \arrow[dl] \\
 & \bullet \arrow[ul] & \bullet \arrow[l] & \bullet \arrow[l] &
\end{tikzcd}$$

Since such skeleton has one object and countably many arrows, each one of them corresponding to n full circumnavigation around the circle, we can clearly identify it with $\mathbf{B}\mathbb{N}$ *with the "winding number functor". Note that this innocently looking result suggests that the monoid of natural numbers plays the same role in category theory as circle in topology, however as we'll see in the next example, the categorical universe admits here more elaborate structure, distinguishing between directed and not directed circles.*

Example 214 (Integers as categorical circle)**.** *Let's once again construct the circle object, this time restricting ourselves to the category of groupoids. The interval object in* **Grpd** *differs with categorical interval, as it must admit only invertible arrows. This time the interval* I_{iso} *is the category with two isomorphic objects. Similar argument as* *in* **Cat** *shows that the pushout*

$$\begin{tikzcd}
\mathrm{cotr}(I_{iso}) & I_{iso} \arrow[l] \\
1 \arrow[u] & 1 \sqcup 1 \arrow[l] \arrow[u, "0 \sqcup 1"]
\end{tikzcd}$$

is now the fundamental category of undirected circle



Its skeleton has one object and arrows generated freely by clockwise or counterclockwise loops, so again the winding number functor provides its equivalence with $\mathbf{B}\mathbb{Z}$. *In particular note that* $\mathbf{B}\mathbb{Z}$ *is a group, while* $\mathbf{B}\mathbb{N}$ *only a monoid - thus the categorical models of circle differs from the directed circle (however, all the reasonable models of circles inspired by topology have weakly homotopic nerves).*

Example 215 (Weak limit)**.** *The weak limit is an object satisfying the weaker property of limits, that require the universal arrow to exists, but not necessarily uniquely. Weak limits are not in general unique up to isomorphism - in fact they in most cases form a quite a large class of different objects. Consider the simplest example of weakly terminal sets. The non-uniqueness in this case means that from any set there must exist an arrow to* X *- thus all non-empty sets are weakly terminal.*

Example 216 (Split epimorphisms)**.** *Consider the general case of weak pullbacks in any category* $\mathcal{C}$. *These are all such objects* P, *that any* X

filling the square has some factorisation through P.

$$\begin{tikzcd}
X \arrow[dr, dotted] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& P \arrow[r] \arrow[d] & A \arrow[d] \\
& B \arrow[r] & C
\end{tikzcd}$$

In particular if the pullback of such square exists, P factors through pullback, while pullback factors through P in some way. These conditions are exactly expressing the existence of split epimorphism $P \to A \times_C B$.

4. Adjoint functors

Example 217 (Hom-product adjunction)**.** *In some categories, called Cartesian monoidal categories, the covariant* Hom *functor is adjoint to the product. The canonical example is the category* **Set***, where we have indeed*

$$\mathrm{Hom}(A \times B, C) \simeq \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

More details about this fancy name you can find in the chapter about monoids.

Example 218 (Internal Hom)**.** *In some categories, in particular monoidal (again, unsurprisingly more details can be found in the section about monoids), there is a natural construction of Hom as an object of $\mathcal{C}$, and Hom can be expressed as the bifunctor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{C}$. Its left adjoint is called the tensor product (or exponential object), which is consistent with its standard definition in categories of modules. However, this notion can be misleading - for example the tensor product in* **Set***, due to the previous example, is just a categorical binary product, the tensor product in $R-\mathbf{Mod}$ is the standard algebraic biproduct, while the algebraic tensor product in $R-\mathbf{Alg}$ is not adjoint to* Hom *at all, but it is just a coproduct.*

Example 219 (Complexification)**.** *Since $\mathbb{C}$ is naturally an $\mathbb{R}$-algebra, we have a natural forgetful functor from complex to real vector spaces/algebras/Lie algebras. As a special case of restriction of scalars, this functor is right adjoint to extension of scalars, in this notable case called the complexification $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathcal{C}$*

Example 220 (Field of fractions)**.** *The field of fraction $Q(R)$ of some integral domain R is left adjoint to the forgetful functor*

$$\mathbf{Field} \to \mathbf{ID}_m$$

Where the category $\mathbf{ID}_m$ *is a category of integral domains with only injective morphisms. It is a rare case, where the field of fields is somehow useful, however it obviously came with a price of considered rings only with some strange collection of morphisms, making the existence of any functor to fields possible.*

Example 221 (Reflective subcategory). *The subcategory* $\mathcal{C} \hookrightarrow \mathcal{D}$ *is called reflective if the inclusion functor has a left adjoint, called a localisation.*

Example 222 (Abelianisation). *The forgetful functor* $\mathbf{Ab} \to \mathbf{Grp}$ *has a left adjoint, which maps a group to its abelianisation, making it a reflective subcategory. The fist thought about how can such a thing be constructed can be misleading, since as we've seen at the beginning, the function* $G \mapsto Z(G)$ *corresponding to most obvious choice fails to form a functor. However, there is a different way of making a group abelian, which is not only functorial, but also optimally free (adjoint to forgetful functor). It can be computed as the quotient group of the commutator subgroup*

$$G^{ab} = G/[G,G]$$

Where the commutator subgroups is just a subgroup decoding every pair of elements of G *that fails to commute:*

$$[G,G] = \{ghg^{-1}h^{-1} \mid g,h \in G\}$$

This is pretty non-standard adjunction to a forgetful functors, as almost all the other are repeating similar free object construction, so to get better understanding let's take a look at its unit and counit. For each one of these there only one reasonable possibility: the unit $G \to G/[G,G]$ *is just a projection on the quotient, which get rid of non-commutative elements within the realm of category of groups, while the counit just do nothing, as once the group is made abelian, it is not bothered anymore.*

Example 223 (Commutativity and anti-commutativity). *Both commutative and anti-commutative algebras are reflective subcategories of associative algebras. The reflector quotient out the algebra by its commutator or anti-commutator ideal*

$$J = \begin{cases} (xy - yx \mid x,y \in A) & \textit{commutator} \\ (xy + yx \mid x,y \in A) & \textit{anti-commutator} \end{cases}$$

Example 224 (p-groups). *The category of* p*-groups form a reflective subcategory of* $\mathbf{Grp}$*. The reflector maps a group* G *to a quotient* $G/\mathbf{O}^p(G)$*, where* $\mathbf{O}^p(G)$*, called the* p*-residual subgroup of* G*, is the intersection of all normal subgroups* $H \lhd G$ *such that* G/H *is a* p*-group.*

Similarly, the category of abelian p-groups is also reflective in **Grp** *via similar argument, only now considering the quotient by $A^p(G)$, constructed as intersection of normal subgroups, which quotients are abelian p-groups.*

Example 225 (Elementary abelian groups)**.** *The other reflective subcategory of* **Grp** *are the elementary abelian p-groups - abelian groups with all non-zero elements having order p. A reflector takes a quotient of G by the intersection of all normal subgroups of index p, which turns out to be the largest elementary abelian p-group surjecting onto G.*

Example 226 (Global sections)**.** *The global sections functor of $\mathcal{C}$-valued sheaves $\Gamma : \mathbf{Shv}_{\mathcal{C}}(X) \to \mathcal{C}$ is left adjoint to the constant sheaf functor.*

Example 227 (Adjunctions in monoid and groups)**.** *Analysing the adjunctions between single-pointed categories representing monoids or groups $\mathbf{B}G \to \mathbf{B}H$ turns out to be not interesting - every such a functor has an adjoint if and only if it is an isomorphism. Since every such a functor is just an algebraic homomorphism of underlying monoids, adjointness condition translates exactly to the morphisms being inverses of each other.*

Example 228 (Dual vector space)**.** *The functor ${\mathbf{Vect}_{\mathrm{k}}}^{op} \to \mathbf{Vect}_{\mathrm{k}}$ is adjoint to itself, since*

$$\mathrm{Hom}_{\mathbf{Vect^{op}}}(V^*, W) = \mathrm{Hom}_{\mathbf{Vect}}(W, V^*) =$$
$$\mathrm{Hom}_{\mathbf{Vect}}(W, \mathrm{Hom}(V, \mathrm{k})) \simeq \mathrm{Hom}_{\mathbf{Vect}}(V \otimes W, \mathrm{k} \simeq$$
$$\mathrm{Hom}_{\mathbf{Vect}}(V, W^*)$$

Example 229 (Loopspace-suspension)**.** *In the category $\mathbf{Top}_*$ we can consider the loopspace functor. Loop in pointed space (X, x_0) is just a pointed map $S^1 \to X$, and the loopspace is the space of all such maps with the compact-open topology, with trivial loop distinguished.*

$$\Omega(X, x_0) = map_*(S^1, X)$$

Loopspace form an endofunctor $\mathbf{Top}_* \to \mathbf{Top}_*$, having a famous left adjoint Σ - the reduced suspension functor. More on suspensions and loopspace you can find in the section about groups and examples from topology.

Example 230 (Limits and colimits)**.** *The most classical definition of limits and colimits is defining them as adjoint to the constant diagram functor. More precisely, for any small index category $\mathcal{I}$, the constant diagram (sometimes called also diagonal) functor $\Delta : \mathcal{C} \to [\mathcal{I}, \mathcal{C}]$ maps* *object of $X \in \mathcal{C}$ constantly to all available spots in the diagram, and all the arrows fills with identity morphism of X. A colimit of a diagram of shape $\mathcal{I}$ is defined as the left adjoint functor to Δ, while the limit - as right adjoint.*

Example 231 (Diagonal arrow). *One of the most important constructions that the adjoint definition of limits provide is the diagonal arrow. Since the product $- \times -$ is the limit of a diagram of a shape of two points with only their identity arrows, it's codomain is the functor category $[1 \sqcup 1, \mathcal{C}]$, which via the classic identity* $\mathrm{Hom}(X \sqcup Y, Z) \simeq \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z)$ *can be identified with the product $\mathcal{C} \times \mathcal{C}$. Thus the product functor $- \times -$ is right adjoint to the "paste two identical copies" functor*

$$\begin{tikzcd} \mathcal{C} \times \mathcal{C} \arrow[r, bend left, "- \times -"] & \mathcal{C} \arrow[l, bend left, "\Delta"] \end{tikzcd}$$

The unit of this adjunction is the anticipated diagonal arrow δ_X, which is a canonical map to the product in the wrong direction

$$\begin{tikzcd} & X \arrow[dl, "\mathbb{1}"'] \arrow[d, dotted, "\delta_X"'] \arrow[dr, "\mathbb{1}"] & \\ X & X \times X \arrow[l, "p"'] \arrow[r, "p"] & X \end{tikzcd}$$

The diagonal arrow is extremely important in algebraic topology, as it plays a crucial role in the ring structure of cohomology theories.

Example 232 (Tensor algebra). *The forgetful functor from category of commutative R-algebras to modules is right adjoint to the tensor algebra. Tensor R-algebra is a structure providing free multiplication, but requiring the underlying abelian group to be at least acted on by R, which makes tensoring over R possible. More concretely, given R-module M, its tensor algebra is constructed by freely adjoining all final formal products of its elements:*

$$TM = \bigoplus_{n=0} M^{\otimes n}$$

Example 233 (Exterior algebra). *The exterior algebra ΣM is a twin construction of the tensor algebra, but in the case of anti-commutative graded algebras (where the multiplication has reversed sign whenever the sum of gradation of its arguments is odd). The most noticeable application of the exterior algebras is its key role in the definition of differential forms.*

Example 234 (Collapsing a subspace). *An operation of collapsing a subspace of a topological space is common especially in homotopy theory. Such an operation has in fact arises from an adjoint pair, when considered in a suitable category* $\mathbf{Top}^{(2)}$*, containing topological spaces together with distinct subspaces. A morphism in* $\mathbf{Top}^{(2)}$ $f : (X, A) \to (Y, B)$ *is a continuous function* $X \to Y$*, such that* $f(A) \subseteq B$*. Collapsing a subspace forms a functor* $\mathbf{Top}^{(2)} \to \mathbf{Top}_*$*, mapping* (X, A) *to* $(X/A, A)$*, and such a functor is left adjoint to the functor* $R : (X, x_0) \mapsto (X, \{x_0\})$ *changing the basepoint to one-point subspace.*

Example 235 (Exponential and universal cover). *Recall that the exponential map takes any connected Lie group to its Lie algebra, establishing a functor from connected (real/complex) Lie groups to finite dimensional (real/complex) Lie algebras* $\mathbf{ConnectedLieGrp} \to \mathbf{LieAlg}_{fd}$*. This functor has a left adjoint* Γ*, taking every finite dimensional Lie algebra to its universal covering, which is a simply connected Lie group. Moreover, it can be shown that* Γ *is fully faithful, providing an equivalence of categories*

$$\mathbf{LieAlg}_{fd} \simeq \mathbf{SimplyConnectedLieGrp}$$

And realising finite dimensional Lie algebras as coreflective subcategory of Lie groups.

Example 236 (Universal enveloping algebra). *From any* R*-algebra, not necessarily associative, one can form a canonical derivation via the Lie bracket* $[x, y] = xy - yx$*. Such a construction provide a functor*

$$R - \mathbf{Alg} \xrightarrow{Lie} \mathbf{Lie}(R)$$

which takes an algebra to the Lie algebra over R *with the same underlying set with multiplication given by the Lie bracket constructed above. A left adjoint to the* Lie *functor is the universal enveloping algebra* $U(\mathfrak{g})$*. In particular, with this construction every* $\mathfrak{g}$*-module, coming from some linear representation* $\mathfrak{g} \to \mathfrak{gl}(V)$ *in fact has every sensible property of a standard module over an associative algebra, as it is canonically also a* $U(\mathfrak{g})$*-module.*

Example 237 (Galois connection). *Consider two poset categories,* $\mathcal{P}$ *and* $\mathcal{Q}$*. A functor* $\mathcal{P} \to \mathcal{Q}$ *is obviously just some preorder preserving map, i.e. morphism* $P \to Q$ *in the category* **Poset***. The pair of functors between posets are adjoint if they satisfy an interesting relation, called the Galois connection: identifying functors with order-preserving maps, it can be formulated as*

- $f \circ g \circ f = f$
- $g \circ f \circ g = g$

The other way of looking at Galois connection is via explicit formulas on posets:

- $a \leq g(f(a))$ *(unit)*
- $b \leq f(g(b))$ *(counit)*

Every Galois connection form an idempotent adjunction, as substituting the first formulas to the latter, expressing units and counits, immediately imply that these are idempotent.

Example 238 (Galois correspondence)**.** *A special case of a Galois connection is the Galois correspondence - the case where f and g form an adjoint equivalence, so satisfy the simpler conditions*

- $a = g(f(a))$
- $b = f(g(b))$

Surprisingly (and trivially), every Galois connection induces naturally a Galois correspondence, by simply considering the images $f(A)$ and $g(B)$. It means that again we see the pattern of adjunction restricting to equivalence (encountered in case of every idempotent adjuction)

$$\begin{tikzcd}
A \arrow[r, bend left, "f"] \arrow[d, "f"] & B \arrow[l, bend left, "g"] \arrow[d, "g"] \\
f(A) \arrow[r, bend left, "g"] & g(B) \arrow[l, bend left, "f"]
\end{tikzcd}$$

The sets $f(A)$ and $g(B)$, on which the restriction happens, can be naturally be understood as fixed point of the Galois connection between A and B.

Example 239 (Adjoints in Δ)**.** *Consider a map $f : [n] \to [m]$ in Δ ($= \mathbb{N}$). This poset category is particularly simple, and its Galois correspondences can be explicitly classified. Identifying f as a functor $\mathbf{n} \to \mathbf{m}$, it has a left adjoint if and only if $f(n) = m$. Moreover, it is determined uniquely by the formula*

$$g_l(k) = \min\{i \in [m] \mid f(i) \geq k\}$$

which follows directly from the Galois connection condition. Similarly we get the formula for the right adjoint, with existence condition $f(0) = 0$

$$g_r(k) = \max\{i \in [m] \mid f(i) \leq k\}$$

Example 240 (Partitions)**.** *Recall that we've constructed preorder of partitions as both covariant and contravariant functors from sets to preorders. Any function $f : X \to Y$ induced a monotone function $\mathrm{Part}(X) \to \mathrm{Part}(Y)$ constructed from pushout along surjection associated to a partition, as well as $\mathrm{Part}(Y) \to \mathrm{Part}(X)$ associated to epi-mono factorisation of composition. These two maps, regarded as functors between preorders, are in fact adjoint to each other, forming a (covariant) Galois connection between $\mathrm{Part}(X)$ and $\mathrm{Part}(Y)$.*

Example 241 (Galois correspondence in Galois theory)**.** *Galois correspondence usually are conspicuous due to their characteristic feature of reversing the inclusion. It owns its name to the most famous example, appearing in the main theorem of the classical Galois theory. For every separable and normal field extension $K \subset L$, it manifests itself as the bijective correspondence*

$$\{\textit{Itermediate field extensions } K \subset E \subset L\} \simeq \{\textit{Subgroups of } \mathrm{Gal}(L/K)\}$$

The adjoint pair defining the correspondence is given by

$$\begin{tikzcd} \mathbf{Subfields}(L/K)^{op} \arrow[rr, bend left, "\phi"] & & \mathbf{Subgroups}(\mathrm{Gal}(L/K)) \arrow[ll, bend left, "\varphi"] \end{tikzcd}$$

Where $\phi(E) = \mathrm{Aut}_E(L)$ maps the field to a subgroup of automorphisms fixing E, while $\varphi(H) = L^H$ - a subfield fixed by automorphisms contained in the subgroup H.

Example 242 (Strong Nullstellensatz)**.** *The other famous example of Galois correspondence can be found in Hilbert's Nullstellensatz. Consider the polynomial ring $A = \Bbbk[x_1, \cdots, x_n]$. Given any set of polynomials $S \subseteq A$, we can consider the set of common zeros of polynomials from S, called the vanishing set of S:*

$$V(S) = \{\mathbf{x} \in \Bbbk^n \mid \forall f \in S f(\mathbf{x}) = 0\}$$

Also going in the other direction, with each subset X of $\Bbbk^n$ we can associate ideal of polynomials vanishing at X

$$I(X) = \{f \in A \mid f(X) = 0\} \lhd A$$

The magic behind the strong Nullstellensatz is provided by the weak Nullstellensatz, which for any algebraically closed field $\Bbbk$ identifies maximal ideals of A with points from $\Bbbk^n$. More precisely, every such an ideal must have a form

$$\mathrm{m} = (x_1 - a_1, \cdots, x_n - a_n)$$

which uniquely corresponds the the point $(a_i) \in \Bbbk^n$. The set of all prime ideals of a ring A is called its spectrum $\mathrm{Spec}A$*, and usually considered together with a weird (since rarely even T_1!) topology named after Zariski, in which closed sets are defined exactly to be the sets of a form $V(I)$. Such sets, being common zeros of some polynomial equations, are called algebraic subsets of $\Bbbk^n$. This way we obtain the Galois connection.*

$$\begin{tikzcd} \mathbf{Ideals}(A)^{op} \arrow[r, bend left, "V"] & \mathbf{Top}(\mathrm{Spec}A) \arrow[l, bend left, "I"] \end{tikzcd}$$

More about properties of the Zariski topology you'll find in the section about Yoneda lemma and monads. Meanwhile, as the most important feature of this connection is that it provides a description of algebraic sets, let's use the weak Nullstellensatz and restrict our attention only to maximal ideals on A, regarded as a subspace $\mathrm{Spec}_{max}A \subset \mathrm{Spec}A$*, which allows us to identify it with $\Bbbk^n$ (still with weird topology though). The most interesting part of the theorem is the induced Galois correspondence. The strong Nullstellensatz is basically a theorem providing its unit and counit as*

$$V(I(E)) = \overline{E}$$
$$I(V(J)) = \sqrt{J}$$

Since the closed sets in Zariski topology were defined as image of V, it is not surprising that taking a set of common zeros of the ideal generated by polynomials vanishing on E is just its topological closure $\overline{E}$. More surprising is the second part. By the square root symbol we denote the radical ideal of J, defined as

$$\sqrt{J} = \{f \in A \mid f^n \in J\}$$

The surprise can be significantly reduced by considering some example, such as the ideal $J = (x^2) \lhd \mathbb{C}[x]$. Its set of zeros is just a single point $\{0\} \in \mathbb{C}$, which after again making it into ideal of vanishing polynomials, turns into the radical ideal $(x) = \sqrt{J}$. So, finally, we can summarize this massive example with the square of Galois connection-correspondence:

$$\begin{tikzcd}
\mathbf{RadicalIdeals}(A)^{op} \arrow[r, bend left, "V_{max}"] \arrow[d, hook] & \mathbf{AlgebraicSubsets}(\Bbbk^n) \arrow[l, bend left, "I"] \arrow[d, hook] \\
\mathbf{Ideals}(A)^{op} \arrow[r, bend left, "V_{max}"] & \mathbf{Subsets}(\Bbbk^n) \arrow[l, bend left, "I"]
\end{tikzcd}$$

And that's not even the end, as one more interesting restriction can be found, also forming a Galois correspondence on even more interesting subsets. Every algebraic subset can be constructed from sum of smaller algebraic subsets. As it happens, each of them can be decomposed into finitely many minimal pieces, called irreducible. Consider for example the set generated by $x(x^2+y^2-1)$. It's not hard to see that its vanishing set is the unit circle together with a line $x = 0$. Each of them forms a strictly smaller algebraic set of its own, however, it's impossible to press the reduction further - both the line and the circle do not form an union of finitely many smaller pieces that are algebraic. Using the machinery from above, it's not hard to prove that irreducible algebraic sets correspond bijectively to prime ideals of A, also via the Galois correspondence. Irreducible algebraic subsets are also called affine varieties.

$$\begin{tikzcd}
\mathbf{PrimeIdeals}(A)^{op} \arrow[r, bend left, "V_{max}"] \arrow[d, hook] & \mathbf{AffineVarieties}(\Bbbk^n) \arrow[l, bend left, "I"] \arrow[d, hook] \\
\mathbf{RadicalIdeals}(A)^{op} \arrow[r, bend left, "V_{max}"] & \mathbf{AlgebraicSubsets}(\Bbbk^n) \arrow[l, bend left, "I"]
\end{tikzcd}$$

Example 243 (Galois connection coming from a relation)**.** *The construction used to define the Zariski topology has a more general form, constructed via relation. It is interesting mostly because it in fact classifies all the Galois connections defined on powersets.*

Fix two sets X, Y and a relation $R \subset X \times Y$. For convenience we'll abbreviate $(x, y) \in R$ by $R(x, y)$. In each such a case, there are *canonical adjoint functors between their poset of subsets:*

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "V"] & \mathcal{P}(Y)^{op} \arrow[l, bend left, "I"] \end{tikzcd}$$

while again, for bigger clarity we write V and I instead of more precise V_E and I_E, as we've fixed our relation once and for all. Our functors can be explicitly described with formulas

$$V(S) = \{y \in Y \mid \forall x \in S R(x, y)\}$$
$$I(T) = \{x \in X \mid \forall y \in T R(x, y)\}$$

The adjunction $I \dashv V$, equivalent to the condition

$$T \subset V(S) \Rightarrow S \subset I(T)$$

follows simply from equivalent, more symmetric condition

$$S \times T \subset R$$

The rest of the construction and features are very similar to these already seen in case of Zariski topology. However, it's worth to at least sketch a proof, that such a connection classifies all connections defined on powersets. Consider such an adjunction

$$\begin{tikzcd} \mathcal{P}(X) \arrow[r, bend left, "f"] & \mathcal{P}(Y)^{op} \arrow[l, bend left, "g"] \end{tikzcd}$$

Now the relation can be constructed surprisingly simply, by taking

$$R(x, y) \textit{ iff } y \in f(\{x\})$$

f, as left adjoint functor, takes colimits in $\mathcal{P}(X)$ (unions) to colimits in $\mathcal{P}(Y)^{op}$, which are intersections in $\mathcal{P}(Y)$. It then follows easily that

$$f(S) = \bigcap_{x \in S} f(\{x\}) = \{y : \forall_{x \in S} R(x, y)\}$$
$$T \subseteq f(S) \textit{ iff } T \subseteq \{y : \forall_{x \in S} R(x, y)\} \textit{ iff}$$
$$S \times T \subseteq R$$

Meaning that every powerset Galois connection is also constructible via the relation R. Of course, such a feature fails whenever we consider any proper subset of powersets - in particular, the Zariski topology cannot be considered as its special case, as underlying poset of ideals was clearly not a full powerset of the ring.

In fact, there is much more powerful tool, that does classify all the possible Galois connections

Example 244 (Annihilator-span adjunction)**.** *Consider some bilinear map of modules $B : M \times N \to P$ over commutative ring R. The annihilator of some subset $S \subseteq M$ is a vanishing set of M with respect to evaluation by B:*

$$\mathrm{Ann} S = \{n \mid \forall_{s \in S} B(s, n) = 0\}$$

Such an annihilator form a Galois connection between subsets of M and N together with the Span *functor, being the counit of adjunction (called the closure operator in case of Galois connections)*

$$\mathrm{Span} S \subseteq \mathrm{Ann}\mathrm{Ann} S$$
$$\mathrm{Ann}\mathrm{Ann}\mathrm{Ann} S \subseteq \mathrm{Ann}$$

Example 245 (Annihilator of a module)**.** *The special case of a Galois connection constructed from annihilator is the annihilator of a module, where the bilinear map is taken to be the multiplication in a module $B : M \times R \to M$. The annihilator takes here a more classical form*

$$\mathrm{Ann}_R(M) = \{r \mid rM = 0\}$$

It can be also constructed in a different way, also involving adjoint functor - $\mathrm{Ann}_R(M)$ *is the kernel of the action $R \to \mathrm{End}(M)$, which correspond to the identity map $M \to M$ in the isomorphism established from the hom-tensor adjunction*

$$\mathrm{Hom}(M \otimes R, M) \simeq \mathrm{Hom}(R, \mathrm{Hom}(M, M))$$

Example 246 (Orthogonal subspace)**.** *The other noticeable special case of Galois connection of annihilators is a fundamental construction of the orthogonal complement in vector spaces. Given any non-degenerate bilinear form on a vector space $\langle -, - \rangle : V \times V \to K$, the annihilator of $\langle -, - \rangle$ is just the orthogonal subspace, so the annihilator-span adjunction can be stated as*

$$\mathrm{Span} S \subseteq (S^{\perp})^{\perp}$$
$$((S^{\perp})^{\perp})^{\perp} \subseteq S^{\perp}$$

Example 247 (Commuatative monoid completion)**.** *The most important functor in the K-theory, the Grothendieck group, is constructed from a commutative monoid via the abelian monoid completion $M \mapsto M^{-1}M$, left adjoint to forgetful functor.*

$$M^{-1}M : \mathbf{CMonoid} \to \mathbf{Ab}$$

Given a monoid, its completion group $M^{-1}M$ is a group constructed by equivalence classes of sums, which is in some sense artificial introduction of inverse elements. For example when $M = \mathbb{N}$, its completion *has elements of a form $[(m,n)]$, where (m_1,n_1) and $(m_2.n_2)$ are considered equivalent if $m_1+m_2=n_1+n_2$, which leads just to the group isomorphic to $\mathbb{Z}$, generated by elements from the class $[(1,0)]$, which inverse, corresponding to -1, is the class $[(0,1)]$. In general, when the monoid fails to be cancelative, we may need to additionally add some multiple of 1 to both sides. Such an example we can find in the topological K-theory, when the monoid M is taken to be the semiring of vector bundles over some space X. Vector bundles form a commutative monoid under direct sums, which fails to be cancelative - there is no such thing as direct difference. In this case the elements of a group are classes of tuples (E,F), and the pairs (E_1,F_1), (E_2,F_2) are equivalent iff $E_1 \oplus F_1 \oplus \epsilon^k \simeq E_2 \oplus F_2 \oplus \epsilon^k$ for some trivial bundle ϵ^k*

Example 248 (Monoid completion)**.** *Non-commutative monoids admit completion functor as well, left adjoint to forgetful*

$$\hat{-} : \mathbf{Monoid} \to \mathbf{Grp}$$

In case of abelian monoids completion coincides with the Grothendieck group. In general it has much worse properties though - even for cancelative monoids the canonical map $M \to \hat{M}$ sending m to its class $[m]$ is not injective in general.

Example 249 (Semiring completion)**.** *The monoid completion does properly extends to semi-ring completion, a functor*

$$\hat{-} : \mathbf{CSemiring} \to \mathbf{CRing}$$

Which is also left adjoint to the inclusion and with forgetting the multiplicative structure, i.e. the following diagram of functors is commutative

$$\begin{tikzcd}
\mathbf{CSemiring} \arrow[r, "\hat{-}"] \arrow[d, "U"] & \mathbf{CRing} \arrow[d, "U"] \\
\mathbf{CMonoid} \arrow[r, "\hat{-}"] & \mathbf{Ab}
\end{tikzcd}$$

Example 250 (Reflection of a preorder)**.** *The category of partial orders* $\mathbf{PartOrd} \hookrightarrow \mathbf{PreOrd}$ *is a reflexive subcategory of preorders (recall that the difference lies in the additional assumption, that a partial order is anti-symmetric). Its left adjoint is called the reflection - a partial order made from a preorder by identifying elements satisfying both $x \leq y$ and $y \leq x$.*

Example 251 (Even and odd numbers)**.** *Consider $\mathbb{Z}$ as a poset category. The subcategory consisting of even numbers is a reflective subcategory of $\mathbb{Z}$, as the inclusion $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$ has a right adjoint $R(x) = \left\lfloor \frac{x}{2} \right\rfloor$.*

Interestingly, such a functor is itself left adjoint to the inclusion of odd numbers, which makes them the coreflective subcategory.

$$Even \dashv \left\lfloor \frac{x}{2} \right\rfloor \dashv Odd$$

where the adjunctions come from the inequalities

$$2\lfloor n \rfloor \leq 2n \leq 2\lfloor n \rfloor + 1$$

Example 252 (Floor and ceil)**.** *Triple adjunction of quite opposite character to the one above can be constructed from the inclusion of posets* $\mathbb{Z} \hookrightarrow \mathbb{R}$*. This time the inclusion makes* $\mathbb{Z}$ *simultaneously reflective and coreflective subcategory of* $\mathbb{R}$*, as it is right adjoint to the floor function, while right adjoint to ceil. I called this example opposite, since the subcategories in the previous examples, even though isomorphic, weren't both reflective and coreflective, but there were a single functor being both reflector and coreflector. This case is dual - there is a single reflective and coreflective subcategory, but the reflectors and coreflector differ. Abviously the adjunction is established by the inequalities*

$$\lfloor x \rfloor \leq x \leq \lceil x \rceil$$

Example 253 (Reflective subcategories of **Set**)**.** *It has been shown that there exist exactly* 3 *reflective subcategories of* **Set** *assuiming classical logic - the category of singleton sets, the category of empty or singleton sets, and the* **Set** *itself. In the first two cases the reflector is just a constant functor, varying only on the value of* $\emptyset$*, which in the first case is mapped to* $\{\emptyset\}$*, and to itself in the second one. Working with different logic however may result in more such subcategories (see https://ncatlab.org/nlab/show/Lawvere-Tierney+topology)*

Example 254 (Polynomials)**.** *Consider a category of pointed rings* $\mathbf{Ring}_*$ *- rings with distinguished point* (R, x)*, with distinguished-point-preserving morphisms. The forgetful functor* $\mathbf{Ring} \to \mathbf{Ring}_*$ *has a left adjoint functor, which is just a polynomial ring with variable* x *distinguished:*

$$P(R) = (R[x], x)$$

Example 255 (Sheafification)**.** *The inclusion* $\mathbf{Pshv} \hookrightarrow \mathbf{Shv}(X)$ *has a left adjoint, called sheafification, which makes sheaves a reflective subcategory of presheaves. This pretty funny name reflects quite well what's actually going on. We can think about sheaves as presheaves "storing only local data". The additional conditions make sure that every information can be restored from local information, so there is no* *"emergence". The sheafification makes something exactly in this fashion - it takes a presheaf, take the local data from every point, throwing away all the data that cannot be stored in such a way, and then glue them together again, creating all the global data gluing the pieces freely together.*

Example 256 (Stalks and skyscrappers)**.** *A fundamental concept used in the sheaf theory is a stalk. For every $x \in X$, it forms a functor $\mathbf{Shv}(X) \xrightarrow{\mathrm{Stalk_x}} \mathbf{Set}$. A stalk is expressed by the limit taken over neighbourhoods of x, $\mathcal{F}_x = \lim \mathcal{F}(U_x)$. It can be taught of as a sheaf evaluate at x + some infinitesimally small neighbourhoods, or as the common information contained in all its neighbourhoods. A left adjoint functor to the stalk is a skyscrapper sheaf, which is a sheaf theoretic version of indicator*

$$\mathcal{S}_x(U) = \begin{cases} \{1\} & x \in U \\ \emptyset & x \notin U \end{cases}$$

Example 257 (Germs and sections)**.** *Fix a topological space X and consider the category of its bundles (just maps to X) $\mathbf{Top}/X$. Then to every bundle $E \to X$ one can assign its sheaf of sections $\Gamma(-, E) \in \mathbf{Psh}(X)$. Such a functor has a left adjoint, assigning to every presheaf a bundle of sections ΛP. A space ΛP is a disjoint union of all the germs of sections of P. In such a setting every section s of P becomes a function $\bar{s} : X \to \Lambda P$, mapping x to its germ at x. We can use this trick to topologize ΛP by taking a basis made out of all the images of $\bar{s}$. Note that every ΛP is an Etalè space, while every sheaf of sections is a sheaf, so our adjunction has a nice additional feature - it restricts to isomorphism of subcategories $\mathbf{Etal\acute{e}(X)} \to \mathbf{Shv(X)}$.*

Example 258 (Skyscrapper diagrams)**.** *Somewhat similar constructions to skyscrapper sheaves can be done in any functor category (where $\mathcal{I}$ is small) $[\mathcal{I}, \mathcal{C}]$. Such a category is a category of diagrams, with nodes indexed by some set I. For any $k \in I$, we have a natural projection from the diagram to its k-th node, which is a functor*

$$\pi_k : [\mathcal{I}, \mathcal{C}] \to \mathcal{C}$$

If $\mathcal{C}$ has enough products, such a functor has a right adjoint $k_* : \mathcal{C} \to [\mathcal{I}, \mathcal{C}]$, such that $k_*(A)$ maps a copy of A to arrow ending in k. Explicitly, it is given by the formula

$$k_*(A)(i) = \prod_{i \to k} A$$

where all the induced morphisms are projections to the coordinate index by the map.

Example 259 (Completion of a metric space). *Consider a forgetful functor* **CompleteMetricSpace** $\to$ **MatricSpace**. *Its left adjoint* Comp *free functor is the completion of a metric, so for example we have* $\mathrm{Comp}(\mathbb{Q}) = \mathbb{R}$ *with euclidean topology, or if you're into that,* $\mathrm{Comp}(\mathbb{Q}) = \mathbb{Q}_p$ *with p-adic metric.*

Example 260 (Cocompletion of a poset). *Recall that the colimits in posets are just suprema of some sets of elements, thus a poset is cocomplete when every subset has a supremum in P. Cocomplete posets form a subcategory* **CPoset** $\hookrightarrow$ **Poset** *which is not full, as we require the morphisms to preserve the supremum, so that a morphisms are monotone functions $P \to P$ additionally satisfying the condition*

$$f(\mathrm{sup} a_i) = \mathrm{sup} f(a_i)$$

For example the function $f : I \to I$ given by the formula

$$\{(x) = \begin{cases} 0 & x \in [0, 0.5) \\ 1 & x \in [0.5, 1] \end{cases}$$

is the morphism in **Poset**, *but not in* **CPoset**, *as* $\mathrm{sup}_{x\in[0,0.5)} f(x) = 0$, *but* $f(\mathrm{sup}_{x\in[0,0.5)}) = f(0.5) = 1$. *The category* **CPoset** *form a reflexive subcategory of* **Poset**, *as the forgetful functor is right adjoint to the free completion of P, which can be construction as a poset of higher sets in P. Similarly one can construct the free completion of P, which is just the same as considering the cocompletion of P^{op}.*

Example 261 (Unitalisation). *A forgetful functor from unital rings to non-unital rings (or just rngs) has a left adjoint called unitalisation. It "freely adjoints" a unit to the ring in a most economical way, which is adding a direct summand $\mathbb{Z}$ to the ring. Similar functor exists also in any kind of non-unital algebras. However, since the inclusion* **Rng** $\hookrightarrow$ **Ring** *is not full, as the morphisms in unital rings maps* 1 *to* 1, **Rng** *is not considered to be reflective.*

Example 262 (Torsion abelian groups). *Dually to reflective subcategories, where the inclusion functor has left adjoint, we can consider the opposite case of coreflective subcategories, where is has a right adjoint. Such examples are much less common, one of them being the category of torsion abelian groups* **TAb**. *The inclusion* **TAb** $\hookrightarrow$ **Ab** *has right adjoint, which unsurprisingly is the torsion subgroup functor* $T(A) = \{a \in A \mid an = 0\}$

Example 263 (Kelleyfication). *A topological space is compactly generated if its open sets have following property: U is open if and only if for any map $C \to X$, where C is some compact space, its preimage* *is open. The category of compactly generated spaces (called also Kelley spaces or k-spaces)* **kTop** *is a coreflective subcategory of* **Top**. *The inclusion has a right adjoint called the kelleyfication, which changes the topology of X by adding to its closed sets all subsets, which intersection with all compact subsets is closed.*

Example 264 (Stone-Čech compactification)**.** *The full subcategory of compact Hausdorff spaces* **CompHaus** $\hookrightarrow$ **Top** *is reflective. It's left adjoint β is called the Stone-Čech compactification. The universal property of βX characterizes it as a compact Hausdorff space, such that every map to a compact Hausdorff space $X \to C$ factors uniquely through βX. The explicit description of βX are rarely known and the proof of its existence does not hint how to construct even a single point. However the structure of βX for a discrete space is better understood through ultrafilters and it will appear in more details in chapter* 5.

Example 265 (Hausdorffification)**.** *The subcategory*

$$\mathbf{Haus} \hookrightarrow \mathbf{Top}$$

is reflective. To construct its adjoint, recall condition equivalent to being Hausdorff, is that the diagonal

$$\{(x, x) \mid x \in X\} = \Delta_X \subset X \times X$$

is a closed subspace of $X \times X$. Moreover, any space X can be constructed as the quotient space $X \times X/\Delta_X$. This trick leads to the localisation $\mathbf{Top} \xrightarrow{L} \mathbf{Haus}$, *constructed as the qoutient of the closure of diagonal*

$$\mathrm{L}(X) = (X \times X)/\bar{\Delta_X}$$

Example 266 (Groupoids are bireflective)**.** *The subcategory of groupoids* **Grpd** $\hookrightarrow$ **Cat** *is both reflective and coreflective. The reflector takes $\mathcal{C}$ to its maximal localisation $\mathcal{C}[\mathcal{C}^{-1}]$, adding free inverses to all the arrows. The coreflector on the other hand is the core groupoid* $\mathrm{Core}(\mathcal{C})$, *which removes all non-invertible arrows.*

Example 267 (Kolmogorov quotient)**.** *The spaces distinguished by separation axioms often form reflexive subcategories. Starting from the wildest, recall that space is T_0 (or Kolmogorov) if any there are no two points lying in exactly the same open sets. Notice that it naturally form an equivalence relation on points of X - points x, y are equivalent iff the sets of their neighbourhoods are equal. Taking a quotient space $X/\sim$, called the Kolmogorov quotient, form a functor left adjoint to the inclusion, so the reflector of a subcategory*

$$\mathbf{T}_0 - \mathbf{Top} \hookrightarrow \mathbf{Top}$$

Example 268 (T_1 reflection). *Frèchet T_1 spaces are also reflective. Recall that a space is T_1 if all of its one-point subsets are closed. The reflector takes X to a quotient by an equivalence relation constructed as the intersection of all equivalence relations on X having closed equivalence classes.*

Example 269 (Complete regularisation). *Completely regular spaces, satisfying the $T_{3\frac{1}{2}}$ axiom, are also reflexive. Recall that these spaces are characterized by a property, that each pair of points x and y can be separated by a real valued function by taking there different values. Such a space is completely determined by its family of real valued functions $C(X)$, and can be characterised as spaces, which topologies coincides with finest topologies generated by $C(X)$, which makes such an operation the reflector associated to the embedding*

$$\mathbf{CReg} \hookrightarrow \mathbf{Top}$$

Example 270 (Adjoints of pullback). *As mentioned in the example of a pullback bundle, the pullback can be generalised to any slice category. When $\mathcal{C}$ is a category with pullbacks, every morphism $X \to Y$ induces a functor $\mathcal{C}/Y \to \mathcal{C}/X$ defined via pullback diagram (see: base change of bundles). The adjoint functors to the pullback (if exist) are the dependent product and dependent sum, functors described in Example 16.*

Example 271 (Group invariants). *Given any group G, a (left) G-module is an abelian group acted on by G, so with additional multiplication $(g, a) \mapsto ga$. The universal construction of something acting on something has a natural categorical construction as $[BX, \mathcal{C}]$, in our example $[BG, \mathbb{Z} - \mathbf{Mod}]$, which is equivalent to the category of modules over the integral group ring $\mathbb{Z}G - \mathbf{Mod}$. It means that every such a module consists of formal sums $\sum_{g \in G} n_g g$ with multiplication taken from the multiplication in G. From every module A we can extract the invariant subgroup $A^G = \{a \in A \mid \forall g \in G : ga = a\}$. The canonical construction of invariant elements in case of finite group is through the normalising element $N = \sum_{g \in G} g$ - for example in case of a cyclic group of order n with a generator ϵ, $N = 1 + \epsilon + \cdots \epsilon^{n-1}$. A^G is the maximal submodule of A with a trivial G-action, which make it a right adjoint functor to the trivial module functor $\mathbf{Ab} \to \mathbb{Z}G - \mathbf{Mod}$. It follows then that we can express invariant subgroup as*

$$A^G \simeq \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, A^G) \simeq \mathrm{Hom}_{\mathbb{Z}G}(\mathbb{Z}, A)$$

Example 272 (Group coinvariants). *The left adjoint of a trivial G-module functor also has a pleasant form. It is a largest trivial quotient* *module, called coinvariants $A_G = A/\langle a - ga\rangle$. It can be thought of as a orbit space of a module, as we just collapse all the orbits to a single point, or choose their representants. Since the trivial module functor is just a restriction of scalar, we know that it's left adjoint is the extension of scalars $- \otimes_{\mathbb{Z}G} \mathbb{Z}$, thus*

$$A_G \simeq \mathbb{Z} \otimes_{\mathbb{Z}G} \mathbb{Z}$$

More about these functors you can find in the section about group homology.

Example 273 (Polynomial functor)**.** *Given any morphisms $W \leftarrow X \to Y \to Z$ in a category $\mathcal{C}$ with pullbacks, and where dependent products and sums exist, we can form a functor from their composition*

$$\mathcal{C}/W \xrightarrow{f^*} \mathcal{C}/X \xrightarrow{\prod_g} \mathcal{C}/Y \xrightarrow{\sum_h} \mathcal{C}/Z$$

It's name is not coincidental: in **Set** *the polynomial functor P induces by $\{*\} \leftarrow A \to I \to \{*\}$ sends a set A to a function*

$$P_A(X) = \coprod_{i \in I} X^{A_i}$$

But what exactly the 4 underlying correspond to in this analogy? This example shows us that the the map $X \to Y$ captures how the monomials are ensemble into a sum, since Y is the indexing space, and the map attach to every monomial its place in the sum. The next example shows that the spaces W and Z correspond to the domain and codomain.

Example 274 (Linear functor)**.** *The linear function is a special case of a polynomial, where the exponents are the same as indices in a sum. It can be modelled as a polynomial functor corresponding to*

$$A \longleftarrow M \xrightarrow{\mathbb{1}} M \longrightarrow B$$

Now, when the spaces W, Z are non-trivial, it categorifies the linear transformations. The functor has a familiar form looking as matrix multiplication by a vector, where the elements of A indexes the input vector, while B is indexing the image, and the matrix M is behaving like $A \times B$ matrix familiar from linear algebra:

$$P(X)_a = \left(\coprod_{a \in A} X_a \times M_{ab} \right)_{b \in B}$$

Example 275 (Discrete category)**.** *Consider the functor* ob : **Cat** $\to$ **Set***, mapping a small category to its set of objects. The left adjoint functor to* ob *is the functor D :* **Set** $\to$ **Cat***, associating to each set* *a discrete category $D(S)$, which set of objects is S, and the only morphisms are identities.*

Example 276 (Full category)**.** *Previously considered functor* ob : $\mathbf{Cat} \to \mathbf{Set}$ *has also the right adjoint functor $F : \mathbf{Set} \to \mathbf{Cat}$, which again takes S to the category with objects formed out of S, but now every hom(x, y) has exactly one element, so the category can be visualised as a full graph on $|S|$ vertices.*

Example 277 (Connected components)**.** *The discrete category functor $D : \mathbf{Set} \to \mathbf{Cat}$, right adjoint to* ob *has a left adjoint itself - the functor $\Pi_0 : \mathbf{Cat} \to \mathbf{Set}$, mapping the small category to the set of its connected components, defined as the quotient $S/(x \sim y$ iff hom$(x, y) \neq)$*

Example 278 (Equalisers-cokernel pairs)**.** *Recall that the image of a morphism can be defined as the equaliser of the cokernel pair, which is a pushout along itself. To avoid confusions, let's denote the category with 2 object and one arrow by* $\mathbf{2}$ *and the category with 2 objects and 2 arrows by $\downarrow\downarrow$. If cokernel pairs exist in $\mathcal{C}$, assigning to morphism its cokernel pair forms a functor $[\mathbf{2}, \mathcal{C}] \to [\downarrow\downarrow, \mathcal{C}]$. If $\mathcal{C}$ has equalisers, not only they can be constructed dually as a functor $[\downarrow\downarrow, \mathcal{C}] \to [\mathbf{2}, \mathcal{C}]$, which maps a pair of morphisms to its equalising arrow, but it is also right adjoint to the cokernel pair.*

Example 279 (Powers and copowers)**.** *By the analogy to arithmetic, the $n - th$ power of an object is the n-times iterated product $X^n = X \times X \times \cdots \times X$. Similarly the copower $n \times X$ is the similar construction with respect to coproduct written additively. In any category $\mathcal{C}$ with powers and copowers, copower is left adjoint to power functor for every n.*

Example 280 (Projection from a slice)**.** *In a category $\mathcal{C}$ with finite coproduct, the projection from a slice category $\Pi : A/\mathcal{C} \to \mathcal{C}$*

$$\Pi(A \to X) = X$$

has a left adjoint, freely adjoining A to X via coproduct, resulting in the arrow

$$A \to X \sqcup A \in A/\mathcal{C}$$

Example 281 (Boolean algebra)**.** *As we've seen in the example of lattices, binary products and coproducts in a poset correspond to greatest lower bound and least upper bound, and they can imitate the language of logical operators. However, such a language is too poor to capture all the axioms, as we do not have any notion of negation. To fix that, we can consider a special kind of a lattice - firstly, we want it to be* *distributive, so such that the d'Morgan laws applies. The lattice L is distributive if the following condition holds:*

$$a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$$

If a lattice has initial term 0 *and terminal term* 1*, the element a has complement of an element* $\sim a$ *if it satisfies*

- $\sim a \wedge a = 0$
- $\sim a \vee a = 1$

In a distributive lattice if a complement exists, it is unique. A boolean algebra is a distributive lattice with 0 *and* 1 *where all the complements exist. Such an additional structure is completely equivalent to the standard language of propositional calculus.*

Example 282 (σ-algebras)**.** *Every σ-algebra on a space form a poset which is a Boolean algebra, since, we require the complement of an element to also belong to the algebra. Similar statement does not hold in general for a topology on a space, however it does in some specific cases, for example a discrete and indiscrete topologies are Boolean.*

Example 283 (Heyting algebra)**.** *Now we'll construct a category corresponding to different logical system. A Heyting algebra is a finitely complete and cocomplete, Cartesian closed poset category. What does it mean? It is just a lattice with* 0 *and* 1*, additionally having an exponential object $a \Rightarrow b$, naturally corresponding to the implication. By the definition, the implication $a \Rightarrow b$ is a right adjoint functor to the binary product, thus we have*

$$z \leq (x \Rightarrow y) \text{ iff } z \wedge x \leq y$$

We can understand it as a biggest possible element, such which product with x still lies below y. Every boolean algebra is a Heyting algebra, but the converse does not hold. An interesting example of such a case is a poset of open sets of some topological space X. In general, not every such a poset has a complement, since a complement of an open set does not need to be open. However, the exponential object does indeed exists - $U \Rightarrow V$ is a biggest open set, which intersected in U is contained in V. More generally, given any category, the poset of subobjects of any object X form a Heyting algebra, but not a boolean algebra in general. It can be nicely seen on the examples of G-sets, where the subobject classifier is $1 \to \{0, 1\}$, so we do get a Boolean algebra, while in case of sets acted by a monoid, the subobject classifier takes more values, which is a consequence of a fact that the complement of a subobject needs not to be a subobjects - a fact equivalent to saying that subobjects of X do not form a Boolean algebra. It also shows us *some correspondence between Boolean algebras structures and subobject classifiers - the classifier has a simple form, similar to the case of sets, if and only it the poset* $\mathrm{Sub}_{\mathcal{C}}(X)$ *has a structure of a Boolean algebra, and it is more complex when it fails to be Boolean.*

Example 284 (Quantifiers as adjoints)**.** *Now, as we've already constructed the categorification of a propositional calculus, we would like to go a step further to obtain some model of a first order logic. Consider some predicate* $S(x, y)$ *- a sentence with 2 variables, which can be true or false depending on the values of* x *and* y*. For example the sentence* $S(x, y) = x > y$ *is a predicate, which just evaluate expressions like* $S(1, 1) = 0$ *as it is not true that* $1 > 1$ *and* $S(2, 1) = 1$*, since* $2 > 1$*. Supposing that* x *belongs to some Boolean algebra* X *and* y *to* Y *(since we obviously need to built the first order logic on some already constructed model of propositional logic, and nothing stops us to use 2 different such a models on different variables, for example we may consider only natural* x *and real* y*, or we may even let* y *to be some real valued function; in each of these cases our toy sentence* S *makes perfect sense). Every predicate can be seen just as an object in a category* $\mathcal{P}(X \times Y)$*, corresponding to the Boolean algebra of all subsets of* $X \times Y$*. Considering an usual projection* $X \times Y \to Y$*, consider its pullback with respect to some subset of* Y

$$\begin{tikzcd}
p^*A \arrow[r] \arrow[d] & A \arrow[d, hook] \\
X \times Y \arrow[r, "p"] & Y
\end{tikzcd}$$

It is really an overly complicated way of saying that $p^*A = X \times A \subset X \times Y$*, but seeing it as a pullback helps to see the similarity with the general pattern and philosophy behind this type of examples (moreover, the main idea behind this book is to present even most trivial constructions as some aspect of abstract nonsense, nay, the trivial the better). Such a pullback is in fact a functor mapping subsets of* Y *to subsets of* $X \times Y$

$$p^* : \mathcal{P}(Y) \to \mathcal{P}(X \times Y)$$

Now the magic happens: the quantifiers $\forall_{x \in X} S(x, -)$ *and* $\exists_{x \in X} S(x, -)$ *are respectively the right and left adjoint to the pullback! To see that, notice that for any set* $T \in \mathcal{P}(Y)$ $\mathrm{Hom}(S, p^*T)$ *is non-empty iff* $S \subset X \times T$*, so when for any* $t \in T$ *there exists such an* x*, that* $S(x, t) = 1$*, which means exactly that* $\mathrm{Hom}(S, p^*T) = \mathrm{Hom}(\exists_x S, T)$*. Similarly,* $\mathrm{Hom}(p^*T, S)$ *is non-empty when* $X \times T \subset S$*, meaning that for all* t *and all* x *we have* $S(x, y) = 1$*, yielding the second adjoint formula*

$\mathrm{Hom}(p^*T, S) = \mathrm{Hom}(T, \forall_x S)$. *Obviously this example is a part of a much more general picture, which was already described earlier. If you like that kind if stuff, try to make an explicit connection between this example with the adjunctions of the base change functor, slice categories and discrete fibrations.*

Example 285 (Lawvere's Hyperdoctrine Diagram)**.** *For every set X, its subobjects are in bijective correspondence with inclusions $A \hookrightarrow X$. This naturally forms an embedding of its poset of subsets $P(X)$ in the slice category $\mathbf{Set}/X$. Such an inclusion is reflective, as it is adjoint to he functor $\sigma_X : \mathbf{Set}/X \to P(X)$, mapping function to the poset of its subobjects in the slice category (all its restrictions). Putting it together with some previously considered adjunctions, it forms an impressive diagram of adjunctions called the Lawvere's Hyperdoctrine Diagram*

$$\begin{tikzcd}
\mathbf{Set}/X \arrow[r, bend left, "\Pi"] \arrow[r, bend right, "\Sigma"'] \arrow[d, bend left, "\sigma"] & \mathbf{Set}/Y \arrow[l, "f^*" description] \arrow[d, bend left, "\sigma"] \\
P(X) \arrow[u, hook, bend left] \arrow[r, bend left, "\forall"] \arrow[r, bend right, "\exists"'] & P(Y) \arrow[l, "f^{-1}" description] \arrow[u, hook, bend left]
\end{tikzcd}$$

Example 286 (Localisation of a ring)**.** *Localisation of a ring can be described as adjunction, but in quite complicated fashion. Consider a category $\mathcal{C}$, where the objects are pairs (R, M), with R being a commutative ring and M being a submonoid or R (since every commutative ring is an abelian group additively + commutative monoid multiplicatively; now we use just a second structure, forgetting about the additive one). In the category $\mathcal{D}$ we consider less structured objects, considering the second object from a pair (R, M) only as a submonoid of the group of units $U(R)$ of R. In such a setting given some multiplicatevely closed subset S (in other words a multiplicative submonoid of R), the localisation $S^{-1}R$ comes from an adjoint functor to the forgetful $F : \mathcal{C} \to \mathcal{D}$. The universal property of a localisation is indeed exactly the adjunction isomorphism on homs:*

$$\mathrm{Hom}_{\mathcal{C}}((A, S), F(B, T)) \simeq \mathrm{Hom}_{\mathcal{D}}((S^{-1}A, j(S)), (B, T))$$

where the map j is the canonical map $j : A \to S^{-1}A$. Since the morphism in $\mathcal{C}$ correspond exactly to maps $f : A \to B$ mapping S to units in B, it is just reformulation of the universal property of localisation, *providing a unique factorisation of such a map by the localisation*

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "j"'] & & B \\ & S^{-1}A \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

Example 287 (Fix points of group action)**.** *Let G be a topological group and N its normal subgroup. Then the quotient homomorphism $G \to G/N$ induces by precomposition a functor*

$$G/N - \mathbf{Top} \to G - \mathbf{Top}$$

Left adjoint functor of such precomposition is the fix locus X^N - a G/N-space of fixed points.

Example 288 (Residual Weyl group action)**.** *The situation considered previously is more interesting when the subgroup $H < G$ is not normal. In such a case, the projection onto the quotient does not make sense anymore, however the space of fix points still can be considered. It is well defined as a topological space, so the only missing piece is the induced group action. The proper way of constructing such one is to consider the Weyl group of H $W_H = N(H)/H$, where $N(H)$ is the normalizer of H in G. This functor again has right adjoint functor, but now instead of simple precomposition it is a pullback*

$$\begin{tikzcd} R(X) \arrow[r, dotted] \arrow[d, dotted] & X \arrow[d] \\ N(H)/H \arrow[r] & G/H \end{tikzcd}$$

Where G/H is a coset space, not a quotient group. Such an adjunction in fact can be factored as a composition of two intermediate adjoint pairs:

$$\begin{tikzcd} G\text{-}\mathbf{Top} \arrow[r, shift left, "G\times_{N(H)}-"] \arrow[rr, bend left, "(-)^H"] & N(H)\text{-}\mathbf{Top} \arrow[l, shift left, "i^*"] \arrow[r, shift left, "p^*"] & N(H)/H\text{-}\mathbf{Top} \arrow[l, shift left, "\mathrm{map}(N(H)/H,-)^{N(H)}"] \arrow[ll, bend left, "R"] \end{tikzcd}$$

Example 289 (Specialisation preorder)**.** *From a topological space X one can construct the specialisation preorder with the relation $x \leq y$ iff x lies in the closure of $\{y\}$. Note that interesting preorders, which* *are not discrete, can be obtained only from wild spaces, which are not* T_1*, such as Zariski topologies on spectra. On the other hand every preorder can be given the Alexandrov topology* $T(P)$*, where open sets are the upper closed sets, satisfying the condition*

$$x \in U, x \leq y \Rightarrow y \in U$$

Functors

$$\mathbf{Pre} \underset{W}{\overset{T}{\rightleftarrows}} \mathbf{Top}$$

form an adjoint pair.

5. Internal objects

5.1. **Monoidal categories.**

Definition 5.1 (Monoidal category)**.** *A monoidal category is a category with a tensor product (bifunctor* $-\otimes- : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ *and identity element* I*. We require the tensor to be associative and identity to behave like identity, which formal statement is surprisingly really messy. It's so ridiculous that I will treat it as a fun fact and really present it here.*

The monoidal category is a category $\mathcal{C}$ *with a bifunctor* $-\otimes- :$ $\mathcal{C}\times\mathcal{C}\to\mathcal{C}$ *and object* I *together with three natural isomorphisms:*

- *associator (natural in each of the three arguments)* $\alpha_{A,B,C} : A\otimes(B\otimes C)\to(A\otimes B)\otimes C$
- *left unitor* $\lambda_A : I\otimes A\to A$
- *right unitor* $\rho_A : A\otimes I\to A$

Moreover these natural isomorphisms satisfy following coherence condition, expressed by commutativity of following diagrams

$$\begin{tikzcd}
 & (A\otimes B)\otimes(C\otimes D) \arrow[dr, "\alpha_{A\otimes B,C,D}"] & \\
A\otimes(B\otimes(C\otimes D)) \arrow[ur, "\alpha_{A,B,C\otimes D}"] \arrow[d, "\alpha_{\mathbb{1}_A\otimes\alpha_{B,C,D}}"] & & ((A\otimes B)\otimes C)\otimes D \\
A\otimes((B\otimes C)\otimes D) \arrow[rr, "\alpha_{A,B\otimes C,D}"] & & (A\otimes(B\otimes C))\otimes D \arrow[u, "\alpha_{A,B,C}\otimes\mathbb{1}_D"']
\end{tikzcd}$$

$$\begin{tikzcd}
A\otimes(I\otimes B) \arrow[rr, "\alpha_{A,I,B}"] \arrow[dr, "\mathbb{1}_A\otimes\lambda_B"] & & (A\otimes I)\otimes B \arrow[dl, "\rho_A\otimes\mathbb{1}_B"] \\
 & A\otimes B &
\end{tikzcd}$$

Example 290 (Cartesian monoidal category)**.** *Every category with finite limits has a monoidal structure with a product being a tensor functor and terminal object (existing since it's a trivial product) serving as the identity. In this way we can form a monoidal structure on* **Set***,* **Cat***,* **Top***, in fact in almost all known categories.*

Example 291 (Cocartesian monoidal category)**.** *We can dualise that construction and use the coproduct as a tensor functor and initial object as the identity. This way we obtain the monoidal structure for example* *in* **CRing** *or more generally* $R-$**Alg**, *where the tensor product* $-\otimes_R-$ *is a binary coproduct.*

Example 292 (Wedge product of sets)**.** *Besides coproducts and products we know one more valid symmetric monoidal product on sets. The wedge sum combines these two with the formula* $A \vee B = A \sqcup A \times B \sqcup B$, *assembling the non-standard symmetric monoidal category* $(\mathbf{Set}, \emptyset, \vee)$.

Example 293 (Smash product)**.** *The canonical monoidal structure on* $\mathbf{Set}_*$ *is non-Cartesian, but using the smash product and the 2-point set instead of Cartesian product and 1-point set. The smash product is a product with the points having the basepoint on at least one coordinate "smashed" into a single point (the basepoint, of course), so a space*

$$X \wedge Y = (X \times Y)/(X \vee Y)$$

where $\vee$ *is a wedge sum (disjoint union with base-points identified), a coproduct in* $\mathbf{Set}_*$.

The reason behind the 2*-point set is, obviously, preventing the colonisation of the entire smash to the point, as it happens in the case of a singleton. The* 2*-point set under the smash product behaves exactly as the singleton under the Cartesian product, which in some sense reduces the pointed case to classical, where the intuition behind hom-product is more clear.*

Example 294 (Join of categories)**.** *In section* 2 *we've seen the operation of join between small categories. Recall that a join* $\mathcal{C} * \mathcal{D}$ *is a category with objects* $\mathrm{ob}\mathcal{C} \sqcup \mathrm{ob}\mathcal{D}$ *and morphisms*

$$\mathrm{Hom}_{\mathcal{C}*\mathcal{D}}(C_1, C_2) = \mathrm{Hom}_{\mathcal{C}}(C_1, C_2)$$
$$\mathrm{Hom}_{\mathcal{C}*\mathcal{D}}(D_1, D_2) = \mathrm{Hom}_{\mathcal{D}}(D_1, D_2)$$
$$\mathrm{Hom}_{\mathcal{C}*\mathcal{D}}(C, D) = \{\bullet\}$$

The join operation provides a monoidal structure on **Cat**, *with the empty category taken as a unit.*

It is a good place to slow down and contemplate. Recall that "the old, good tensor product" was characterized as an adjoint of Hom, so by a natural isomorphism

$$\mathrm{Hom}(A \otimes B, C) \simeq \mathrm{Hom}(A, \mathrm{Hom}(B, C))$$

However, the "new, fancy tensor" does not need to satisfy such an axiom. In fact, you can have plenty of completely different tensor bifunctors, such as product and coproduct in almost any category, and much more. This coincidence of names is on one hand slightly confusing, but not accidental. When such an option is available, the good old tensor is always preferable as a new fancy tensor. This is exactly the case of a smash product, since smashing all the points touching any basepoint in the product it's exactly the thing to do in order to get the good old tensor property

$$\mathrm{Hom}_*(X \wedge Y, Z) \simeq \mathrm{Hom}_*(X, \mathrm{Hom}_*(Y, Z))$$

The smash product is a great example showing the need of a "convenient category of topological spaces". In full generality topological spaces can be extremely obscure and do not satisfy some "completely obvious" properties. Since algebraic topologists care only about CW-complexes and general topologists despise category theory, there is a natural need to find the subcategory of **Top** that has the nicest possible properties, while still big enough to contain all CW-complexes. The category of CW-complexes is not nice, as it's not complete. On the other hand **Top** is also not nice, and the trouble can be seen in this example - the smash product in $\mathbf{Top}_*$ fails to be associative, thus cannot be taken as a tensor product. Most troubles are caused by non-Hausdorff spaces, but the category of Hausdorff spaces is again not complete (with the classical counter-example of double ended interval, non-Hausdorff colimit of unit intervals). The first "really nice" category was found by Steenrod.

Example 295 (Symmetric monoidal preorders)**.** *Consider some preorder category with a monoidal structure. Moreover, it is convenient to add some "commutativity" to the monoidal structure - such structures are called symmetric. A symmetric monoidal preorder is a preorder* $(X, \geq)$ *together with a monoidal tensor product* $\otimes : X \times X \to X$*, with following additional features:*

- ***identity element**:* $1 \in X$
- ***monotonicity**:* $a \leq b$ *and* $x \leq y$ *implies* $a \otimes x \leq b \otimes y$
- ***unitality**:* $1 \otimes x = x \otimes 1 = x$
- ***associativity**:* $(x \otimes y) \otimes z = x \otimes (y \otimes z)$
- ***symmetry**:* $x \otimes y = y \otimes x$

If you're familiar with the notions of monoidal categories introduced in the section about monoids, you'll notice that it is just a monoidal structure on a preorder with one additional symmetry axiom, which is just making the monoid commutative. The simplest example of a symmetric monoidal preorder is a category 2*, that can be considered as a category of truth values with ordering* $true \geq false$*. There are two different symmetric monoidal structures on* 2 *- equipping it either with the product* $\wedge$ *or coproduct* $\vee$*. Formally, these categories can be denoted as* $(2, \geq, true, \wedge)$ *and* $(2, \geq, false, \vee)$*.*

Example 296 (Symmetric structure on powerset)**.** *For any set S, the poset of subsets of S, i.e. the poset $(\mathcal{P}(S), \subseteq)$, has a symmetric monoidal structure with S as the identity and the intersection $\cap$ as a tensor product $\otimes$*

$$(\mathcal{P}(S), \subseteq, S, \cap)$$

Example 297 (Chemical catalyst)**.** *The chemical reactions can be expressed in a monoidal category, which in an elegant (but obviously oversimplified) way expresses the role of catalysts. Recall that a catalyst of a reaction is an additional substance required for the reaction to take place, but not used as a substrate. For example some reactions take place only in acidic environment, at the same time fixing the overall amount of acid. We can consider chemical reactions as a symmetric monoidal preorder of matter $(\mathbf{Matter}, \to, 0, +)$, which is just the free abelian group generated by particles, which relation $A \to B$ if there is a reaction taking particles A to particles B.*

Now consider a following relations $A + K \to C + L$, $B + C \to D$, $D + L \to Y + K$. Monoidal symmetric structure assemble them together into a reaction

$$A + B + k \to Y + K$$

The particle K can be seen as a catalyst of such reaction, as even though staying fixed, the reaction $A + B \to Y$ need not to be valid.

Example 298 (Divisibility)**.** *The natural (positive) numbers are naturally ordered by divisibility, with $m \leq n$ iff $m|n$. Such preorder is strict monoidal with respect to multiplication and unit 1, as we have $a \cdot c | b \cdot c$ iff $a|b$. Note that in case of natural numbers defined as non-negative integers, this category no longer has sense, and even if we somehow manage to make sense out of divisibility by 0, the implication $a|b \Leftrightarrow 0|0$ fails no matter which value $0|0$ takes.*

Example 299 (Convenient category of topological spaces)**.** *The most basic convenient category of topological spaces is a category of compactly generated spaces* $\mathbf{CG}$*, characterized by the property that open sets are exactly all subsets having open preimage from some map from compact space. In use are also compactly generated Hausdorff spaces* $\mathbf{CGH}$*, or nowadays most common compactly generated weakly Hausdorff spaces* $\mathbf{CGWH}$*, with the additional separation axiom lying between T_1 and T_2, satisfied when all images of maps from compact spaces are closed.*

Example 300 (Topological smash)**.** *Taking any pointed convenient category of topological spaces, the smash product in $\mathbf{Top}_*$ is associative and define a monoidal structure together with 2-element set.*

Example 301 (Rings)**.** *Now for a change let's look at a particularly simple but elegant algebraic example. The category* **Ab** *has at least* 3 *different possible functors making it monoidal. In such a case, since the tensor product of abelian groups is well define and form an exponential object, its structure is highly preferred. Now it's not hard to see, that monoids in* $(\mathbf{Ab}, \otimes, \mathbb{Z})$ *are just unitary rings!*

Example 302 (Unit interval as closed commutative monoidal preorder)**.** *An unit interval* $[0, 1]$ *can be seen as a commutative monoidal preorder* $(I, \otimes, \leq, 1)$*. Its monoidal product is the standard multiplication. Additionally, its monoidal structure is compatible with the preorder structure, meaning that we have*

$$a \leq b, x \leq y \Rightarrow a \otimes x \leq b \otimes y$$

Moreover, this monoidal category is closed, so there exists the internal hom functor $[a, b]$*, which is right adjoint to the monoidal product*

$$x \otimes y \leq z \text{ iff } x \leq [y, z]$$

it is given by the formula

$$[x, y] = \begin{cases} \frac{y}{x} & x > 0 \\ 1 & \text{otherwise} \end{cases}$$

Also, note that the monoidal product differs from the categorical product, as $x \otimes y = xy$*, while* $x \times y = \min(x, y)$

Example 303 (Braided category)**.** *In algebra, sometimes objects have property similar to commutativity, but somehow twisted, for example anti-commutative algebras with commutative relations with flipped sign. A monoidal category of similar property is called braided, with braiding transformation*

$$A \otimes B \to B \otimes A$$

Satisfying the same axioms as the trivial braiding appearing in case of symmetric monoidal categories. A canonical example of a braided are graded modules over commutative ring R*. Fixing any non-zero element* $u \in R$*, we may consider a braiding*

$$(a, b) \mapsto u^{|a||b|} b \otimes_R a$$

Note that whenever $u^2 = 1$*, the category is symmetric. Most common example involves* $u = -1$*, which leads to the anti-commutative structure, such as in the algebras of differential forms.*

Example 304 (Braid category)**.** *Another central example of a braided category is a category of braids itself. Recall that braids with* n *strands* *form a group B_n under concatenation. The braid category $\mathcal{B}$ is a groupoid constructed as a coproduct of all the braid groups*

$$\mathcal{B} = \coprod_n = 1^\infty B_n$$

Given two braids with n and m strands, putting them next to each other defines a natural product $B_m \times B_n \to B_{n+m}$. Clearly such a product fails to be associative, but with a little additional twist, putting the top side of the second braid at the left of the second one and the bottom on the right, while keeping all strands at the top, it can be made braided, as it satisfies all the hexagon relations, while failing to be symmetric.

Example 305 (Crossed G-sets)**.** *A wilder example of a braided category are crossed G-sets, where G is a discrete group. A set acted on by G is crossed, if it is equipped with an additional map*

$$|\cdot| : X \to G$$

satisfying the twisted relation $|gx| = g|x|g^{-1}$. Crossed G-sets has a monoidal product taking underlying sets to the Cartesian product with the cross map

$$(x, y) \mapsto |x||y| : X \times Y \to G$$

Such a category has a natural braiding

$$(x, y) \to (|x|y, x) : X \ timesY \to Y \times X$$

Example 306 (Braided coherence)**.** *A braid category is central the theory due to astonishing theorem of braided coherence, where it relates the braiding of $\mathcal{C}$ to the underlying category $\mathcal{C}_0$ with forgotten braiding. For any braided monoidal category $\mathcal{C}$ there is a natural isomorphism*

$$[\mathcal{B}, \mathcal{C}]_{BMC} \simeq \mathcal{C}_0$$

where $[\mathcal{B}, \mathcal{C}]_{BMC}$ is the set of braided functors.

Example 307 (One-point compactification)**.** *One point compactification does not extend to a functor on* **Top**, *however it is well defined as a functor from locally compact Hausdorff spaces, and its image is naturally pointed*

$$C : \mathbf{Top}_{LCH} \to \mathbf{Top}_*$$

Moreover, C is a monoidal functor, interchanging the Cartesian product with the smash product in $\mathbf{Top}_*$, which the most natural choice of monoidal structure on pointed spaces.

$$C(X \times Y) \simeq C(X) \wedge C(Y)$$

Example 308 (Three faces of forgetful functor). *Consider a forgetful functor $U : \mathbf{Ab} \to \mathbf{Set}$. Such a functor, depending on the choice of monoidal structures, can be both strict, lax or not monoidal at all. Firstly, consider Cartesian structures on both categories. Since the product of abelian groups is, by definition, Cartesian product of underlying sets with induced group structure, U is naturally strict monoidal, as we have*

$$U(A \times B) = U(A) \times U(B)$$

Considering $\mathbf{Ab}$ with its most natural structure $(\mathbf{Ab}, \otimes, \mathbb{Z})$, U can be still made monoidal, however only lax, as there is a natural transformation

$$U(A) \times U(B) \to U(A \otimes B)$$

but it is not an isomorphism in general, with $\mathbb{Z}/2 \otimes 0 \simeq 0$ serving as the smallest counterexample. Changing the monoidal structure on $\mathbf{Set}$ to cocartesian takes the situation to the other extreme, as the natural transformation

$$U(A) \sqcup U(B) \to U(A \otimes B)$$

is this time just impossible to construct.

Example 309 (Monoidal functors on preorders). *A lot of examples of lax, strong and colax functors can be easily constructed on some simple monoidal preorders. Consider for example the inclusion of strict monoidal preorders under addition $\mathbb{N} \hookrightarrow \mathbb{R}$. This functor is obviously strict monoidal, as the product and relations do not change after inclusion.*

The floor functor $\mathbb{R} \hookrightarrow \mathbb{N}$ however is only lax monoidal. As a monotone function it is a functor, and it is lax due to $\lfloor x + y \rfloor \geq floor x + \lfloor y \rfloor$. However, the case of $1 = \lfloor 0.5 + 0.5 \rfloor > \lfloor 0.5 \rfloor + \lfloor 0.5 \rfloor = 0$ shows that it is not strict. On the hand a variant of the same example shows that the ceil functor is also not strict, but colax.

Example 310 (Hom is lax, and sometimes weak). *Let $\mathcal{C}$ be symmetric monoidal and $\mathbf{Set}$ symmetric Cartesian monoidal. Then the hom profunctor $\mathrm{Hom}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$ is a lax monoidal functor, as there is a canonical transformation*

$$\mathrm{Hom}(A, C) \times \mathrm{Hom}(B, D) \to \mathrm{Hom}(A \otimes B, C \otimes D)$$

sending (f, g) to $f \otimes g$, which is not an equivalence in general. In the same way in $\mathbf{Vect}$-enriched categories, $\mathbf{Vect}$-valued hom functor is also lax monoidal. In some most common examples, however, Hom *is* *often weak monoidal, for example in case of finite dimensional vector spaces, where there is a canonical isomorphism*

$$\mathrm{Hom}(A, C) \otimes \mathrm{Hom}(B, D) \simeq \mathrm{Hom}(A \otimes B, C \otimes D)$$

Example 311 (Vector spaces and algebras as monoidal functors)**.** *Let* $\mathbb{N}$ *be a discrete category on natural numbers with strict monoidal product given by addition*

$$n \otimes m := n + m$$

Consider any functor (not necesserily monoidal) $F : \mathbb{N} \to \mathbf{Vect}_K$*. Each such a functor naturally form a graded vector space* $|F|$ *with obvious gradation* $x \in F(n) \Rightarrow |x| = n$*.*

$$|F| = \bigoplus^{\infty} F(n)$$

Now suppose the functor is lax monoidal, so equipped with natural transformation

$$\varphi_{n,m} : F(n) \otimes F(m) \to F(n + m)$$

It provides $|F|$ *with additional multiplication* $xy := \varphi_{|x|,|y|}(x \otimes y)$*, making it a graded* K*-algebra. Similar trick can be used if the functor is colax, i.e. equipped with transformation in the other direction*

$$\phi_{m,n} : F(n + m) \to F(n) \otimes F(m)$$

This time, dually, $|F|$ *can be equipped with the* K*-coalgebra structure with comultiplication*

$$\Delta(x) = \sum_{i+j=|x|} \phi_{i,j}(x)$$

Example 312 (Dual objects)**.** *A symmetric category is called compact closed if every object* c *has a dual* c^* *together with*

- *unit* $1 \to c \otimes c^*$
- *counit* $c \otimes c^* \to 1$

satisfying coherence relations, called sometimes snake equations. It can be easily shown that each compact closed category is closed, and that all its objects has duals unique up to isomorphisms and are themselves isomorphic to their double duals.

Here there are a few familiar examples of compact closed categories

- *Discrete compact closed categories are just abelian groups, with* $x^* = -x$
- *The delooping of a monoid* $\mathbf{B}M$ *is compact closed with trivial dual* $\bullet^* = \bullet$

- *Finite dimensional vector spaces* $\mathbf{FinVect}_K$ *are compact closed with standard dual vector spaces* $V^* = \mathrm{Hom}(V, K)$
- *Relations* **Rel** *with Cartesian structure are compact closed with* $R^* = R$

Example 313 (Correlations)**.** *A correlation between sets* $c : A \to B$ *is an equivalence relation on* $A \sqcup B$*. Correlations can be composed - given correlations* $c : A \to B$ *and* $c' : B \to C$*,* $(c' \circ c) : A \to C$ *is an equivalence relation, with points equivalent iff related by* c, c' *or are both equivalent to some element of* B*. The category of correlations with cocartesian structure is moreover compact closed, with each set being self dual. Since the monoidal unit is the empty set, the axioms of compact closed require the existence of canonical correlation between* $A \sqcup A$ *and the empty set, which can be obtained by identifying pairs of cloned points.*

Example 314 (Grothendieck group of category)**.** *Note that each small symmetric monoidal category forms naturally an abelian monoid under isomorphism classes of objects acted on by monoidal product. From such abelian monoid we can form an abelian group via monoid completion and obtain the Grothendieck group of category* $K_0(\mathcal{C})$*. We've seen already a few examples of those - the representation ring, constructed from classes of representations of a finite group, the Picard group, constructed from category of vector bundles and the Burnside ring, associated to category of finite G-sets or the classical Grothendieck group of a ring, classifying finitely generated projective modules. Note that all these examples are in fact rings, not merely abelian groups. This structure is provided by the additivity of all considered categories, but for now we will ignore these additive properties and consider vanilla symmetric monoidal categories. Most basic examples representing addition and multiplication operations can be constructed from the category of finite sets. We have two natural choices of monoidal product on* **Fin** *- coproduct or product. In the first case, the abelian monoid associated to classes of objects is just the free additive monoid* $\mathbb{N}$ *with completion* $\mathbb{Z}$*. In case of Cartesian structure, this time we obtain natural numbers under multiplication. Its monoid completion is the multiplicative group of positive rational numbers*

$$K_0(\mathbf{Fin}_\times) \simeq \mathbb{Q}_+$$

Example 315 (Faithfully projective modules)**.** *Recall that rank of finitely generated projective module is a function from the spectrum of a ring* R *to natural numbers, naturally locally constant. Under the correspondence between projective modules and vector bundles, rank is just* *the local dimension of fibers. Previously we restricted vector bundles, algebraic or geometric, to have constant rank. In fact, such assumption can be replaced by milder condition, allowing bundles to have varying rank on different connected component. A faithfully projective module is finitely generated and with nowhere vanishing rank, not necessarily constant. Note that this class of modules is closed under tensor product, thus together with R as a unit form a symmetric abelian category. The category $K_0(\mathbf{FP}(R))$ turns out to be pretty interesting, as it fits into a short exact sequence connecting Grothendieck group of R with zeroth cohomology of its spectrum, both with rational coefficients*

$$0 \to \tilde{K}_0(R) \otimes \mathbb{Q} \to K_0(\mathbf{FP}(R)) \to H^0(\operatorname{Spec} R, \mathbb{Q}_+) \to 0$$

Example 316 (Brauer groups)**.** *A central simple algebra is finite dimensional K-algebra admitting no non-trivial ideals and a center K. Wadderburn-Artin theorem classifies all such algebras as isomorphic to matrix groups $M_n(D)$ for some division ring D. Under fixed field K, central simple K-algebras are closed under tensor product, thus form an symmetric monoidal category $\mathbf{Az}(K)$. The matrix rings $M_n(K)$ are not interesting class of such objects, as they form identical submonoid for any field, with multiplication*

$$M_n(K) \otimes M_m(K) \simeq M_{nm}(K)$$

The Brauer group of a field K restricts attention only to non-trivial objects, using this as a sneaky way of developing the group structure on the classes of division algebras. Note that, as algebraically closed fields admit no non-trivial division algebras, their Brauer group always vanish. The first interesting, and maybe a bit surprising example is $\mathbb{R}$. It admits 2 non-trivial division rings, complex numbers and quaternions. However, only the latter can be used to construct central algebras - centers of matrix rings $M_n(\mathcal{C})$ is always $\mathcal{C}$, not $\mathbb{R}$. This does not happen in case of quaternions, forming central simple algebras for all matric rings. Since we quotient out the information about matrix sizes, we may identify this Brauer group as having only two classes of objects due to the isomorphism $[\mathbb{H} \otimes \mathbb{H}] = [M_4(\mathbb{R})] = 0$

$$\operatorname{Br}(\mathbb{R}) \simeq \mathbb{Z}/2$$

Brauer groups, despite messy definition, have a very simple presentation with only two relations

$$[A \otimes B] = [A] + [B]$$

$$[M_n(K) = 0]$$

Example 317 (Grothendieck-Witt ring). *In similar fashion to division algebras, using the Grothendieck group for any fixed field K we can encode all possible geometries that it encodes with bilinear forms in an abelian group, and even a ring. Consider the category* $\mathbf{Sbil}(K)$ *which objects are K-vector spaces equipped with symmetric inner products. Morphisms in this category are isometries, and the direct sum applied both to spaces and forms make it into symmetric monoidal category. Its Grothendieck group* $K_0(\mathbf{Sbil}(K))$ *encodes all the isometry classes of inner products, where dot products form a trivial class. A crucial role in this ring plays the hyperbolic plane, 2-dimensional space with bilinear form*

$$B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

More generally, spaces isometric to some direct sum of hyperbolic planes is called simple hyperbolic, and form a non-trivial class. Moreover, we can actually equip this group with a ring structure under tensor product, called the Grothendieck-Witt ring.

To restrict attention only to classes different than hyperbolic, sometimes it's useful to consider also the Witt ring $W(K)$, constructed as quotient over ideal generated by hyperbolic spaces. Unlike in case of $\mathbb{R}$ or $\mathcal{C}$, for some fields classification of classes of bilinear forms can be rather messy. For instance, the forms

$$q(x, y) = (x + y)^2$$
$$p(x, y) = xy$$

are isomorphic if and only if K contains a subfield with 4 elements.

5.2. **Groups.** Group objects in a category $\mathcal{C}$ can be defined in a similar fashion as monoids. In the category having terminal object G is a group object if it has all binary products (notice that in this case we don't allow different kinds of products, such as tensors or coproducts, but on the other hand we don't need all binary products to exists - existence of the functor $G \times -$ is enough to make everything work), and satisfy coherence conditions corresponding to axioms of a group.

Example 318 (Topological groups). *A topological group is a group object in* **Top**. *It is a topological space with a group structure, such that the multiplication and inverse functions are continuous. Most common examples of topological groups are Lie groups, having additional structure of a smooth manifold. Among topological groups not forming a manifold are for example all the groups with a discrete topology, called discrete groups, as well as $\mathbb{R}^n$ and $\mathbb{Q}^n$ with euclidean topology. More exotic example is a Cantor set, which also form a group, in fact it* *equivalent to the standard topology of p-adic integers $\mathbb{Z}_p$. Its quotient field, p-adic numbers $\mathbb{Q}_p$, also form a very interesting topological group, which is not a Lie group.*

Example 319 (Lie groups)**.** *A Lie group is a group object in the category of smooth manifolds. The most common among them are matrix groups, with natural subspace topology of $M_n(\mathbb{R}) \simeq \mathbb{R}^{n^2}$. Among matrix groups one can find*

- *complex or real general linear groups* $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$*, formed by all the invertible $n \times n$ matrices*
- *complex or real special linear groups* $\mathrm{SL}_n(\mathbb{R})$, $\mathrm{SL}_n(\mathbb{C})$ *of volume and orientation preserving linear map (matrices with determinant* 1*)*
- *orthogonal group* $\mathrm{O}(n)$*, group of isometries of* $\mathbb{R}^n$ *(orthogonal matrices of determinant* 1 *or* -1*)*
- *special orthogonal group* $\mathrm{SO}(n)$*, group of rotations in* $\mathbb{R}^n$ *(orthogonal matrices of determinant* 1*)*
- *projective special linear group* PSL(n) *- linear automorphisms of* $\mathbb{R}P^n$*, isomorphic to* $\mathrm{GL}_n(\mathbb{R})/\mathbb{R}$.
- *unitary groups* $\mathrm{U}(n)$ *- isometries of* $\mathbb{C}^n$ *(unitary matrices)*
- *special unitary groups* $\mathrm{SU}(n)$ *- isometries preserving volume form (which is a complex analogue of preserving orientation, but the "sign" in this case has a value from* S^1 *instead of* S^0*)*
- *compact symplectic groups* $\mathrm{Sp}(n)$ *- unitary group over quaternion algebra* $\mathbb{H}^n$

There are a few examples of Lie group that are not isomorphic to any matrix group - an example is a metaplectic group $\mathrm{Mp}(n)$ *- a double cover of a symplectic group.*

Example 320 (Suspension)**.** *Dually to group objects we can consider cogroup objects, which are just groups in $\mathcal{C}^{op}$. A classical example of a cogroup object is a reduced suspension on* **HTop**$_*$*. It is constructed as a pushout*

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & * \arrow[d] \\
* \arrow[r] & \Sigma X
\end{tikzcd}$$

More geometrically, the reduced suspension is a double cone, with additionally all the points consisting x_0 identified:

$$\Sigma X = X \times I/(X \times \{0,1\} \cup \{x_0\} \times I)$$

It has following cogroup structure: two suspensions can be duplicated by a wedge sum, which gives the comultiplication $\Sigma X \to \Sigma X \vee \Sigma X$.

Example 321 (Cogroup). *Dually to group objects, we can consider cogroup objects, which are just groups in $\mathcal{C}^{op}$. To define the cogroup structure we must define a comultiplication $X \to X \otimes X$, which is rather some kind of self duplication than multiplication. Alternatively we can characterise cogroups as objects which $Hom(X, -)$ functor can be lifted to the category of groups. The multiplication of maps $X \xrightarrow{f} Y$, $X \xrightarrow{g} Y$ is defined exactly through the duplication, where on one copy we apply the map f, and on the second copy we apply g and obtain the map $f * g : X \to X \otimes X \xrightarrow{f \otimes g} Y$.*

Example 322 (Reduced suspension). *A classical example of a cogroup object is a reduced suspension on $\mathbf{HTop}_*$. It is constructed as a pushout*

$$\begin{array}{ccc} X & \longrightarrow & * \\ \big\downarrow & & \big\downarrow \\ * & \longrightarrow & \Sigma X \end{array}$$

More geometrically, the reduced suspension is a double cone, with additionally all the points consisting x_0 identified:

$$\Sigma X = X \times I/(X \times \{0, 1\} \cup \{x_0\} \times I)$$

It has following cogroup structure: two suspensions can be duplicated by a wedge sum, which gives the comultiplication $\Sigma X \to \Sigma X \vee \Sigma X$.

*The cogroups in $\mathcal{C}$ form a category consisting of all representable functors $\mathcal{C} \to$ **Grp**.*

Example 323 (Free groups as cogroups). *Consider the free group $F(S)$ on some set S. The adjoint relation $Hom(F(S), G) \simeq G^S$, define the group structure on hom-sets of $F(S)$, so they have a natural cogroup stricture. There is a theorem due to Kan stating that free groups are the only cogroups in the category of groups.*

Example 324 (Groups and cogroups in abelian categories). *Since in abelian categories every hom-set has the structure of an abelian group, each object is both an abelian group object and abelian cogroup object. Moreover, in the category **Ab** such a structure is unique, while it does not need to be the case in other categories. For example, since every R-module is also an $\mathbb{Z}$-module (as the underlying abelian group), the functors $Hom_{\mathbb{Z}}(-, -)$ and $\mathrm{Hom}_R(-, -)$ are both valid lifts of Hom to **Ab** and are not equivalent in general (however they can be even when $R \not\simeq \mathbb{Z}$; see: Morita equivalence in chapter 1).*

Example 325 (Modules). *Classically modules are associated to rings, but there are also several other types of modules, such as modules over a* *group, being abelian groups with acted on by G or modules over Lie algebras, corresponding to their linear representation. As you can imagine, every such a construction can be retrieved from an underlying abstract nonsense notion of a Beck module. Given category with $\mathcal{C}$, a module over an object A is just an abelian group object in the slice category $\mathcal{C}/A$. This way an R-module is such an object in* **CRing**$/R$ *with operations, where the multiplication by elements from R are decoded by the addition operation. We need pullbacks in $\mathcal{C}$ to obtain the monoidal structure on the slice category, taking the tensor operator as a pullback along itself*

$$\begin{tikzcd} M \otimes M \arrow[r] \arrow[d] & M \arrow[d] \\ M \arrow[r] & R \end{tikzcd}$$

Similarly the module over Lie algebra $\mathfrak{g}$ is such an object in $\mathbf{Lie}(R)/\mathfrak{g}$ *and the module over a group G is abelian group object in* $\mathbf{Grp}/G$. *This construction connects also modules with their sheaf-theoretic siblings - Modules over ringed spaces. We can also consider modules over monoids or other kind of algebraic objects, for example modules over differential graded algebras, constructed over the tensor product of chain complexes in some abelian category.*

Example 326 (Eckmann-Hilton argument)**.** *Eckmann-Hilton argument is a simple, but extremely powerful observation, that given two binary operations $\bullet, \circ : X \times X \to X$ having the same identity element and satisfying the interchange identity $(a \circ b) \bullet (x \circ y) = (a \bullet x) \circ (b \bullet y)$, the two operations must be equal and commutative. A classical consequence of this result is the classification of the group objects in the category of groups as just abelian groups. Similarly a commutative monoid is a monoid object in* **Monoid**. *All this combined provides a funny chain construction identifying commutative ring as monoids in monoids in groups in groups in sets.*

Example 327 (Higher homotopy groups are commutative)**.** *The second important consequence of the Eckmann-Hilton argument is that The group* $\mathrm{Hom}(A, B)$ *where A is a cogroup object and B is a group object is an abelian group, as can be chased down to a group object in* **Grp**. *This provides a one-line categorical proof that for every $n > 1$ the homotopy group $\pi_n(X) = [S^n, X]_* := \mathrm{Hom}_{\mathbf{HTop}_*}(S^n, X)$ is abelian, since using the fact that $S^n = \Sigma S^{n-1}$ and the loop suspension adjunction we obtain*

$$[S^n, X]_* = [\Sigma S^{n-2}, \Omega X]_*$$

Since suspension is a cogroup object in $\mathbf{HTop}_*$*, while the loop space is a group object.*

Example 328 (Endomorphisms of identity functor)**.** *Consider the identity functor* $\mathbb{1}$ *on some small category* $\mathcal{C}$*. A natural transformations* $\nu, \mu : \mathbb{1} \Rightarrow \mathbb{1}$ *can be composed in two different ways, horizontally and vertically. Since the identity transformation acts as identity on both of them, by Eckmann-Hilton argument* $\mathrm{End}(\mathbb{1}) := \mathrm{Nat}(\mathbb{1}, \mathbb{1})$ *has a natural structure of a commutative monoid. This monoid can be drastically different across different categories (if the category is large it also works, if we use the categorical definition of a monoid as a category with one object, without requirement to be small). For example in the category of sets and groups such a monoid is trivial, but in* **Ab** *it has a form* $\{n_X : n_X(x) = nx\}$ *and is isomorphic to* $\mathbb{Z}$*. More generally, for a commutative ring* R*,* $\mathrm{End}(\mathbb{1}_{R-\mathbf{Mod}})$ *is isomorphic to* R *and containing all the transformations induced by scalar multiplication.*

Example 329 (Group functor)**.** *A group object in the functor category is called a group functor. It is particularly useful to define algebraic groups or group schemes. TODO GRP*

5.3. **Monads.** As everyone already knows, monad is just a monoid in the category of endofunctors. This joke becomes funny when you realise how simpler the definition of a monad compared to monoids.

Definition 5.2 (Monad)**.** *The monad is an endofunctor* $T : \mathcal{C} \to \mathcal{C}$ *together with a unit* $\eta : \mathbb{1} \Rightarrow T$ *and multiplication* $T^2 \Rightarrow T$ *satisfying similar relations as monads, formalising that multiplication is associative and the unit behaves like a unit. The diagrams decoding them are much simpler, so are no longer a fun fact and I'm happy to skip them.*

Example 330 (Adjoint functors form a monad)**.** *Given the adjoint pair of functors with unit* $\eta : \mathbb{1} \Rightarrow UF$ *and counit* $\epsilon : FU \Rightarrow \mathbb{1}$*, they naturally assemble into a monad* $T = UF$*, with unit being identity and multiplication constructed from the counit as*

$$\mu = U\epsilon F : FUFU \Rightarrow FU$$

The triangle identities of the adjunctions translate perfectly into the coherence considitions of a monad, that I was too lazy to include, but it's on Wikipedia so you can fulfill your curiosity with not too big of an effort.

Example 331 (Maybe monad)**.** *The forgetful functor* U *from pointed sets has an adjoint* F *that adds to a set a new disjoint point*

$$\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Set}_*$$

The corresponding monad T maps a space to the one with additional point $T(X) = UF(X) = X \cup \{*\}$. The unit $\eta_X : X \hookrightarrow X \cup \{*\}$ is the standard inclusion, while the multiplication $\mu : X \cup \{*\} \cup \{\bullet\} \to X \cup \{*\}$ merges two additional points, restricting to the idenditity on X. Similar monad can be constructed in category with coproducts and for any freely adjoint object, in particular in $\mathbf{Top}_*$.

Example 332 (List monad)**.** *The free monoid monad (or a list monad) is induced by the free-forgetful adjunction from monoids to sets*

$$\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Monoid}$$

The endofunctor T sends the set A to all finite lists of elements in A

$$T(A) = \coprod_{n=0} A^n$$

The unit $\eta : A \hookrightarrow T(A)$ is the coproduct inclusion, while the multiplication $\mu : T^2(A) \to T(A)$ concatenates the list of lists into a single list

$$[[a_1, \cdots, a_n], ..., [z_1, \cdots z_k]] \mapsto [a_1, \cdots, a_n, ..., z_1, \cdots z_k]$$

Example 333 (Free module monad)**.** *Similarly as in the previous case, the free-forgetful functors in $R - \mathbf{Mod}$ induces the monad taking the set A to $R[A]$, the set of formal linear combinations with coefficients in R. The unit $\eta : A \to R[A]$ sends the element to its one-element formal sum, and the multiplication $\mu : R[R[A]] \to R[A]$ evaluates formal sums of formal sums to formal sums via multiplication of the coefficients.*

Example 334 (Free group monad)**.** *The free group functor $\mathbf{Set} \to \mathbf{Grp}$ together with its forgetful adjoint induces a free group monad, taking the set A to the set of finite words on the letters $\{a, a^{-1} \mid a \in A\}$.*

Example 335 (Ultrafilters)**.** *An ultrafilter $\mathcal{F}$ of a set S is a family of subsets of S that is upward closed (so for every subset U it contains, it must contain also all the bigger subsets containing U), closed under finite intersections and not containing the complement of any of its elements.*

We divide ultrafilters into two types: principal ultrafilters $\mathcal{F}_a$, containing singleton $\{a\}$ together with all the subsets containing a, and free ultrafilters, not containing any singletons. The principal ultrafilters are just all the neighbourhoods of some point a in a discrete space and are pretty easy to understand. Free ultrafilters, on the other hand, are wild, since their construction is impossibly without some version of the axiom of choice (however proving their existence does not). Even *a free ultrafilter, despite the lack of minimal element, always converges to a unique point.*

From the categorical perspective ultrafilters can be understood as a unit of a double powerset monad, induced by the self-adjoint contravariant powerset functor. The monad $T(X) = \mathcal{P}^2(X)$ *sends set* X *to its double powerset, while the unit* η_X *associate a point* $x \in X$ *to its principal ultrafilter* $\mathcal{F}_x \in P^2(X)$*. The multiplication is pretty wild to write down, but the idea is similar as in previous examples, in particular the list monad*

Example 336 (Stone representability theorem)**.** *Ultrafilters provide an interesting embedding of categories* $\mathbf{BA}^{op} \hookrightarrow \mathbf{Set}$*, where* $\mathbf{BA}$ *is the categories of Boolean algebras. Indeed, any ultrafilter* U *is a natural object of a Boolean algebra, satisfying following axioms:*

- $1 \in U$
- *closed under* $\wedge$
- *downward closed*
- *maximal (for any* x *either* $x \in U$ *or* $\sim x \in U$*)*

The functor $Ult : \mathbf{BA}^{op}\mathbf{Set}$ *is representable, since it can be expressed as*

$$Ult(A) = \mathrm{Hom}_{\mathbf{BA}}(A, 2)$$

Moreover, this functor is fully faithful, thus is an embedding of categories, where the inverse functor is the contravariant powerset. The sets from the essential image of Ult *have a natural topology induced by the ultrafilters, and are called Stone spaces, thus the Stone representability theorem states that every Boolean algebra can be represented as some Stone space.*

—- TODO —-

Example 337 (Stone-Čech compactification of discrete space)**.** *As it has been announced in chapter 4, there is a constructive description of the Stone-Čech compactification of a discrete space, moreover the construction can presented as a monad! Consider the forgetful functor from compact Hausdorff spaces to sets. Since it is a reflective subcategory of* **Top***, it factors through the forgetful functors to* **Top***, since both components have left adjoints.*

$$\mathbf{CompHaus} \underset{\beta}{\overset{U'}{\rightleftarrows}} \mathbf{Top} \underset{D}{\overset{U}{\rightleftarrows}} \mathbf{Set}$$

The induced monad sends a set X *to the underlying set of the Stone-Čech compactification of* X *with a discrete topology. On the other hand,* *it has an elegant construction as a functor mapping every set to its set of ultrafilters.*

There are also plenty of monads araising naturally, not derived from any adjoint pair (which a priori doesn't mean they cannot be constructed in such way, as we will see in a moment that they can)

Example 338 (Powerset monad)**.** *The covariant powerset is itself a monad, with structure similar to the list monad: the unit sets an element of X to the singleton subset $\{x\} \subseteq X$ and the multiplication takes the set of subsets to their union.*

Example 339 (Vector space)**.** *The ⅂-vector spaces are monads in* **Set** *- the functor $V(S)$ defines a set of linear combinations of elements of S with coefficient in ⅂. The functoriality constitutes linear maps, the unit $\eta : S \to V(S)$ constitute S as its base and the addition is encoded in $\mu : V(V(S)) \to V(S)$, which takes some formal linear combination of elements to their evaluation, so it takes some "abstract" linear combination, being linear combination of linear combinations, and evaluate one of them into the "actual" linear combination.*

Example 340 (Trees)**.** *The other monad in* **Set** *is a tree monad T, associating set with all the trees having element of x as its leaves. The unit of T constitute the single point as a trivial tree, while multiplication make sure that a tree having some other trees as nodes naturally assembles into a single big tree via concatenation.*

Example 341 (Counting monad)**.** *The indexing of sets has a natural monad structure via the functor $- \times \mathbb{N} : \mathbf{Set} \to \mathbf{Set}$, with the multiplication and identity inherited from $\mathbb{N}$. In the obvious way this example can be generalized by replacing $\mathbb{N}$ by any monoid and* **Set** *with any monoidal category (not complete; the product here is used in a role of a tensor bifunctor, not a limit)*

Example 342 (Closure of a set)**.** *Consider any poset category $(P, \leq)$. Endofunctors of such a category correspond to order preserving functions $T : P \to P$, and such an endofunctor is a monad if and only if it has a unit (thus for all $x \in P : x \leq Tx$) and multiplication ($T^2x \leq Tx$, which merge together into the conditions*

$$\forall_{x \in P} x \leq Tx \quad \textit{and} \quad T^2x = Tx$$

Such a monad is called a closure operator, since its central example is the topological closure functor, where the poset is formed from all subsets of some space X and their inclusions.

Example 343 (Interior)**.** *Dually to monads, we can consider comonads, as monads in the opposite category. The comonoad in a poset category satisfies unsurprisingly the opposite conditions to closure*

$$\forall_{x \in P} x \geq Tx \quad and \quad T^2 x = Tx$$

And their central example is the interior functor. However, such monads are confusingly called kernel operators, not interior operators as one could suppose.

5.4. **Algebras.** Let's once again take a look at a monad. With your favourite monad in mind, we can always find an analogy of it as encoding some new way of interaction between elements in any object, where all the possible interactions within X are stored in $T(X)$. For example it allows elements to group into formal linear combinations (vector space monad), connect with each other via edges of some tree etc.) The encoding has 3 key parts: functoriality, which provides some suitable morphisms respecting the interaction that were made between objects, providing a map $T(X) \to T(X)$, unity, which justifies treating $T(X)$ as object of X with some extra stuff by identifying an element with the element not interacting with others at all by a map $X \to T(X)$. Finally the multiplication justifies the notion of "interactions", which encodes its structure by defining how the composition works, making the addition of formal sums possible (and such a possibility obviously is equivalent to it being a formal sum, so the key description of the structure is stored there). We can visualise possible maps as:

$$\begin{tikzcd} X \arrow[r, "\eta"] & T(X) \arrow[r, shift left, "\eta"] & T^2(X) \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\eta"] & \cdots \arrow[l, shift left, "\mu"] \end{tikzcd}$$

The only map that we're missing is $T(X) \to X$, making possible to evaluate the structure back into. We can make such an evaluation already, but only provided that the interaction is already established. On some object, however, such an evaluation does indeed exist, and there we can make object interacting and then evaluate how good it went - the trivial example is obviously $T(X)$.

Definition 5.3. *Given a monad T, an object A is called a T-algebra if it has an evaluation maps $T(A) \to A$ interacting well with T, so* *making commuting following diagrams:*

$$
\begin{tikzcd}
A \arrow[r, "\eta_A"] \arrow[dr, "\mathbb{1}_A"'] & T(A) \arrow[d, "f"] \\
& A
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2(A) \arrow[r, "\mu_A"] \arrow[d, "Tf"] & T(A) \arrow[d, "f"] \\
T(A) \arrow[r, "f"] & A
\end{tikzcd}
$$

Example 344 (Eilenberg-Moore category)**.** *Given any monad T in $\mathcal{C}$, all T-algebras form a Eilenberg-Moore category of T, $\mathcal{C}^T$, with morphisms being a maps commuting with evaluation.*

Example 345 (Recovering the adjoint)**.** *Let's take some functor having and adjoint, for example let it be the hom-tensor. Given it's monad $-\otimes_{\mathbb{Z}} R : \mathbf{Ab} \to \mathbf{Ab}$, the Eilenberg-Moore category fully reconstruct the codomain of primary functor, as well as its adjunct functor (forgetting the algebra structure). So the algebras of the $-\otimes_{\mathbb{Z}} R$ monad are exactly R-modules. In exactly the same the free point monad constructs $\mathbf{Set}_*$, the free monoid monad constructs $\mathbf{Monoid}$, the closure and kernel operators on $P(X)$ reconstruct the topology of X and so on. The trick works also in the other direction - the induced inclusion of $\mathcal{C}$ in $\mathcal{C}^T$ is adjoint to the forgetful functor, so every comes from some adjoint pair, as well as all adjoint functors comes from some monad.*

You can think that it implies that there is some sort of equivalence of monads with adjoint pairs, but it's wrong. Given an adjoint pair we can indeed find its monad in a unique (up to natural isomorphism) way, but a monad can come from many functors with different codomains. The Eilenberg-Morre category is not even a unique canonical construction if such, however it is universal in a sens that it is the biggest possible such one, which makes the adjunct functor the forgetful one (that intuition will be formalised later). The other canonical construction, this time "the smallest" one is the Kleisli category.

Example 346 (Kleisli category)**.** *Given a monad T in $\mathcal{C}$, the Kleisli category $\mathcal{C}_T$ is a category with the same object as $\mathcal{C}$, and the morphisms $X \rightsquigarrow Y$ are exactly morphisms $X \to T(Y)$ in $\mathcal{C}$, composing via multiplication μ and with identities given by the unit η.*

Example 347 (Partially defined functions)**.** *The Kleisli category of a maybe monad have objects being sets, and morphisms being functions $X \to Y \cup \{*\}$. Each such a function can be identified with partially defined functions, indentifying points on which is undefined with preimage of the extra point.*

Example 348 (Multivalued functions)**.** *The Kleisli category of a free monoid monad are sets, this time with morphisms $X \rightsquigarrow Y$ being functions $X \to \coprod_{n=0} Y^n$, which can be understood as multivalued functions, in a sense that each point is mapped to some finite sequence of elements of Y, with possibly varying size.*

Example 349 (Functions with side effects)**.** *The hom-product adjunction in* **Set** *for a fixed set S induces a monad $\mathrm{Hom}(S, S \times -)$. The morphisms $A \rightsquigarrow B$ are the functions $A \to \mathrm{Hom}(S, S \times B)$, which are naturally equivalent to $A \times S \to B \times S$. If we take the set S as storing some possible states, we can think about morphisms in this category as functions that additionally depend on some current state, and together with standard output return also the updated state. It's not hard to believe that such a concept is useful in computer science, especially in functional programming.*

Example 350 (Markov kernels)**.** *Reading this probabilistic example, keep in mind the basic principle of the quantuum mechanics, that the particle does not have any specific position is space, but rather its position forms some form of a function decoding the probability of it being at any given point (and in fact decoding more than that, so we cannot think about particles as being naturally isomorphic to probability distributions, sadly). Similar idea lies behind the Markov kernel. For any morphism of measure spaces $(X, \mathcal{A}) \to (Y, \mathcal{B})$ a Markov kernel is a function $X \times \mathcal{B} \to I$, nicely behaving in both variables (restricting to a measurable function for a fixed event and to a probability measure for a fixed point). Such a strange object just assigns to each point $x \in A$ and measurable subset $U \subseteq B$ the probability of x ending up in U. The category of Markov kernels is exactly the Kleisli category of the Giry monad.*

Example 351 (Discrete time Markov chains)**.** *The previous example provides an elegant categorical construction of Markov chains. Consider an object $(X, \mathcal{A})$ of* **Meas** *being a finite set with discrete σ-algebra. Then in the Kleisli category of the Giry monad, endomorphism of $(X, \mathcal{A})$ is the function $X \to \mathcal{A}$, which is the probability distribution of transition to other states, making it a discrete time Markov chain.*

Example 352 (Category of adjoints)**.** *As we've seen, for every monad T there is possibly plenty of different adjoint pairs inducing T. All these pairs form a category $\mathbf{Adj}_T$, with object being fully specified (with already chosen specific unit and counit, or the specific isomorphism of Homs) adjoint pair inducing T. The morphisms are, similarly as in the slice category, functors commuting with both functors from the* *pair, which can be depicted with the diagram*

$$\begin{tikzcd} \mathcal{D} \arrow[rr, "F"] \arrow[dr, shift left] & & \mathcal{D}' \arrow[dl, shift left] \\ & \mathcal{C} \arrow[ul, shift left] \arrow[ur, shift left] & \end{tikzcd}$$

All the other commutativity relations that we'd expect follows already from adjunctions, such as that the compositions are also adjunct, composition commute with counits and so on.

Example 353 (Universality of Kleisli and Eilenberg-Moore categories)**.** *The category of adjoint pairs inducing T allows to formalize previous claims about universal properties of Eilenberg-Moore and Kleisli categories as being respectively the initial and terminal objects in $\mathbf{Adj}_T$.*

Example 354 (Free algebras)**.** *Recall that free objects are canonically defined as left adjoints to the forgetful functors. Since the adjoint pair inducing T via the forgetful functor is precisely the Eilenberg-Moore category, the functor $F_T : \mathcal{C} \to \mathcal{C}^T$ carries an object a to its free T-algebra $F^T(A)$.*

Example 355 (Category of free algebras)**.** *The problem with previous construction of the free algebra is that the functor $F^T : \mathcal{C} \to \mathcal{C}^T$ is not fully faithful, so does not provide the suitable construction of category of free T-algebras as it's not an embedding (as two different object can induce the same free algebra). The better way to do that is to use the functor (taken from the unique morphism $F : \mathcal{C}_T \to \mathcal{C}^T$ in $\mathbf{Adj}_T$) from the Kleisli category. Since the Kleisli category is an initial object in $\mathbf{Adj}_T$, it implies that F is faithful, while the terminality of $\mathcal{C}^T$ assures it is full. It allows us to identify the Kleisli category with the essential image of F, which is precisely the full subcategory of $\mathcal{C}^T$ consisting of free algebras. Such a characterisation immediately implies that the Kleisli category is equivalent to the Eilengerg-Moore category if and only if the embedding is essentially surjective, so when all T-algebras are free.*

tl;dr the facts that might got lost in the wall of text, but are the most important to have in mind at the end of the day: we can identify the Eilenberg-Moore category $\mathcal{C}^T$ as the category of T-algebras, while the Kleisli category $\mathcal{C}_T$ is equivalent to free T-algebras.

Example 356 (Pointed sets are partial functions)**.** *A simple example of equivalence following from the last corollary of previous example* *comes from the maybe monad. Since the algebras of T are sets capable of deleting some distinguished point, they can be identify as $T(X)$, which is a functor adjoining that point, so all of them are free. Such a fact implies the (rather obvious anyway, but who cares) equivalence of its Kleisli and Eilengerg-Moore categories, which are, as mentioned in previous examples, pointed sets and partially defined functions*

$$\mathbf{Set}_* \simeq \mathbf{Set}^\partial$$

Example 357 (Monadic functors)**.** *Consider any functor $F : \mathcal{C} \to \mathcal{D}$ that has a left adjoint. Such a pair induces a monad T and is an element of $\mathbf{Adj}_T$. In general the category $\mathcal{D}$ need not be neither the Kleisi category of free T-algebras nor the Eilenberg-Moore category of all T-algebras, but can be somewhere in the middle, so informally defining some class of T-algebras, consisting some not free algebras, but not all of them. It means that not always it's possible to use such a functor to define the T-algebra structure on $\mathcal{D}$, but it can be done in some cases - when $\mathcal{D}$ is equivalent to Eilenberg-Moore. Such an equivalence, if exists, has a canonical construction, as both pairs are then terminal in category of adjoints over T. This special class of functors are called monadic, and it's adjoint pair monadic - adjunctions, and the category $\mathcal{D}$ is said to be monadic over $\mathcal{C}$. Ale the free-forgetful adjunctions are*

monadic. It allows us to identify for example abelian groups as algebras for the free abelian group monad, and **Ab** *as monadic category over* **Set***.*

Example 358 (Idempotent monad)**.** *Every inclusion of a reflective category also is monadic. It's easy to see that all of them come from idempotent monads, where the multiplication $T^2 \Rightarrow T$ is an isomorphism, so the itself is an equivalence of categories. Moreover, the opposite implication is also true - every idempotent monad defines a reflective subcategory.*

Example 359 (Almost all algebraic structures are just algebras)**.** *Using the free-forgetful adjoint pairs we can construct almost all (with fields being the main exception) algebraic structures that algebraists care about:* $\mathbf{Grp}$, $\mathbf{Ab}$, $\mathbf{Ring}, \mathbf{CRing}, \mathbf{Monoid}, \mathbf{CMonoid}, R-\mathbf{Mod}$, $\mathbf{Vect}_\neg, \mathbf{Aff}_\neg, G-\mathbf{Set}, \mathbf{Set}_*$*. Among counterexample are fields, division algebras or any topological categories (the fact that topological spaces are not considered as algebraic structures isn't very surprising).*

Example 360 (Generators and relations)**.** *In the algebraic case, one of the most important features of algebras (as well as other algebraic structures from above) are their presentations, allowing construct every* *algebra as quotient of a free algebra and some relations. The categorical algebra also has this feature, moreover it really nicely formalise the common motive that we already seen, that the multiplication is almost some kind of a nested evaluation. Algebras, having in addition the "real" evaluation, can presented as the difference between formal and real evaluations. Let's start with some concrete example, for which intuition you already have - abelian groups. Every abelian group can be presented in planty different ways, but only one functorial - the only way to avoid the need for choosing elements is to pick all of them. This way every abelian group can be expressed as the following coequaliser:*

$$\mathbb{Z}[\mathbb{Z}[A]] \underset{\mu_A}{\overset{\mathbb{Z}[p]}{\rightrightarrows}} \mathbb{Z}[A] \xrightarrow{p} A$$

Going from the left, we get the free group of a free group - which consists all formal sums of formal sums of A. Now we ask when the formal sum in $\mathbb{Z}[A]$ corresponds to a the real sum, evaluated as an element of A. The diagram from above says: the group A are formal sums of its elements, but all the sums with the same evaluation are considered equal. The same reasoning works for any algebras, which means that they can be constructed via analogous equaliser.

$$T^2(A) \underset{\mu_A}{\overset{T(p)}{\rightrightarrows}} T(A) \xrightarrow{p} A$$

Example 361 (Group structures)**.** *The group structure on a set G can be boiled down to the following diagram of operations*

$$\begin{tikzcd} G \times G \arrow[r, "mul"] & G & G \arrow[l, "inv"'] \\ & 1 \arrow[u, "u"] & \end{tikzcd}$$

Such a structure can be reduced to a a single natural map

$$1 \coprod G \coprod G^2 \to G$$

From this point it's easy to realise that the group structures are naturally equivalent to evaluations $F(X) \to X$ of the endofunctor

$$F(X) = 1 \coprod X \coprod X \times X$$

*which provides a realisation of a category of group structures on sets as the category $F -$ **Alg**.*

Example 362 (Involutions)**.** *Recall that an involution is an automorphism satisfying $f^2 = \mathbb{1}$. In any monoidal category $\mathcal{C}$ with distributive* *finite coproducts, the involutions are in $\mathcal{C}$ are captured by the object $2 := 1 \sqcup 1$ with the universal involution* NOT : $2 \to 2$ *swapping the order of objects. 2 has also a natural structure of monoid, which taking the analogy with logic gates further we can identify with* FALSE : $1 \to 2$ *and* XOR : $2 \otimes 2 \to 2$. *This way the mapping $I(X) = X \otimes 2$ form a monad in $\mathcal{C}$, which algebras are exactly the involutions in $\mathcal{C}$.*

Example 363 (Finitary structures)**.** *Similarly to group structures, one can construct the category of ring structures by considering the endofunctor $P(X) = 2 + X + 2X^2$ where 2 corresponds to assigning multiplicative and additive units, X defines the inverse of additive structure, and $2X^2$ are parts constructing operations of addition and multiplication. Similarly, polynomial $P(X) = 1+X$ constructs of Peano algebras (basically natural numbers), $P(X) = 1 + X^2$ stands behind monoids, $P(X) = X^2$ behind semigroups, $P(X) = 2+2X^2$ - semirings and so on. In general, given any such a polynomial functor, we can consider its algebras as some algebraic structures, axiomatically defined similarly to groups or rings, called finitary structures.*

Example 364 (Lambek's theorem)**.** *The initial algebras, which are just initial objects in $P - \mathbf{Alg}$, always have isomorphic evaluation. In fact, the problem of finding such an initial algebra can be just reduced to finding the "minimal least point" of the endofunctor, since it's determined by solving the equation*

$$P(X) \simeq X$$

Moreover, clearly every category of monad algebras has such a initial object. This simple observations provide a simple argument to an very annoying fact, that the category of fields cannot be constructed as algebras in any reasonable way, even though it doesn't seem to be very different from rings, groups, monoids and similar structures on the level of axioms, and on the level of properties is clearly the best behaved.

Lambek's theorem has also another interesting interpretation - thinking about natural isomorphisms as equalities, this result states that every initial algebra is a fixed point, as it satisfies the equation

$$F(X) \simeq X$$

Thus we can think about initial algebras as smallest fixed point of a functor. Similar result one can get by dualising the theorem to coalgebras. In this case, the terminal coealgebra is the largest fixed point of a functor.

Example 365 (Natural numbers)**.** *Consider a maybe monad, induced by the endofunctor $F : \mathbf{Set} \to \mathbf{Set}$ adjoining a point*

$$F(X) = X \cup \{*\}$$

The initial algebra of F, by Lambek's theorem, is the smallest fixed point of F. One can easily see that it is just a set $\mathbb{N}$ of natural numbers. In general, one can replace the category **Set** *with any topos in such a construction, for example $\mathbf{Psh}(X)$, $\mathbf{Shv}(X)$ or $G - \mathbf{Set}$. The resulting initial algebra is called a natural number object in $\mathcal{C}$.*

Example 366 (Extended natural numbers)**.** *Now let's take a look at coalgebras of the same endofunctor of a form $F(X) = X + 1$. A final F-coalegebra is a set X with a structure map $X \to F(X)$, which is a final object in the category of F-coalgebras. Such an object can be identified with $\bar{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$, with the structure map $f : \bar{\mathbb{N}} \to \bar{\mathbb{N}} \cup 0$*

$$f(n) = n - 1$$

with the obvious identification $\infty - 1 = \infty$

Example 367 (Free monoid)**.** *Let A be a set and consider a linear endofunctor*

$$F(X) = \{\bullet\} \cup A \times X$$

To come up with its initial algebra, we can try to informally solve its fixed point equation

$$F(X) \simeq X$$
$$1 + A \times X \simeq X$$
$$X \simeq \frac{1}{1 - A}$$
$$X \simeq 1 + A + A^2 + \cdots$$

Even though such operations doesn't make sense at all, they help to make a (correct) educated guess, that the initial algebra has a structure

$$A^* = \{\bullet\} \coprod A \coprod A \times A \coprod \cdots$$

Now it's easy to identify such an algebra with a free monoid generated on A

Example 368 (Finite binary trees)**.** *Consider an endofunctor of* **Set**

$$F(X) = \{\bullet\} \cup X \times X$$

Its initial algebra is a set of equivalence classes of binary trees, with the evaluation $X \times X \cup \{\bullet\}$ maps the singleton component into the root of *a new tree, and two trees from a tuple to right and left children of a root*

$$(T_1, T_2, \bullet) \mapsto T_1 \leftarrow \bullet \rightarrow T_2$$

Since a set of binary trees is a fixed point, which is also a minimal one, it form an initial F-algebra due to the Lambek's theorem.

Example 369 (Binary trees). *By considering final coalgebras assosiated to the same endofunctor $F(X) = X^2 + 1$ in* **Set**, *we end up with a situation already seen in the case of natural numbers - the final F-coalgebra can be also identified with binary trees, but this time infinite (countable) trees are also allowed. The structure map $f : X \to X^2 \cup \{\bullet\}$ this time splits a tree into two branches and a a root,*

Example 370 (Binary trees with metric). *This time we will again analyse the same endofunctor $F(X) = X^2 + 1$ applied to different categories. Replacing* **Set** *with metric spaces* **Met** *results in a very similar result - its initial algebra can be identified with binary trees with natural metric, where $d(x, y)$ is just a minimal path length between nodes x and y. Surprising change appears in the category of complete metric spaces - this time the final algebra contains also infinite trees!*

Example 371 (Seven trees in one). *The fact that classes of finite binary trees T form an initial algebra of the endofunctor $F(X) = 1 + X^2$ has a funny consequence. Such a construction provide an natural isomorphism*

$$T \simeq T^2 + 1$$

By purely algebraic calculation, the equation $x = x^2 + 1$ correspond to the equation of 6-th roots of unity, so every its solution satisfies also the equation

$$x = x^7$$

After translating such this result into the language of endofunctors, it implies that there is a natural equivalence between T and T^7, meaning that there is a canonical way of representing every tuple of seven binary tree in a single binary tree, and every binary tree has a unique decomposition into tuple of seven such trees. This equivalence has an explicit formula involving 5 different cases depending of structures of trees $T_1, \cdots, T_7$ up to depth 4. For more details, see https://arxiv.org/pdf/math/9405205.pdf

Example 372 (Planar trees). *The construction of binary trees as initial algebra of an endofunctor generalizes to all planar trees. Consider a functor mapping X to its free monoid X^*, which is, as pointed out in earlier example,*

$$F(X) = 1 + X + X^2 + X^3 + \cdots$$

Similar argument as in the case of binary tree identifies its initial algebra with equivalence classes of all finite planar trees.

Example 373 (Prüfer group)**.** *Consider a slice category $\mathcal{C} = \mathbb{Z} \downarrow \mathbf{Ab}$. There is an endofunctor $F_p : \mathcal{C} \to \mathcal{C}$, mapping the arrow $\mathbb{Z} \to A$ to the pushout*

$$\begin{tikzcd} \mathbb{Z} \arrow[r, dotted, "\cdot p"] \arrow[d] & \mathbb{Z} \arrow[d] \\ A \arrow[r, dotted] & A' \end{tikzcd}$$

The initial algebra of such a functor is the Prüfer group $\mathbb{Z}_{p^\infty}$. For another construction of the Prüfer group, see Limits and colimits: Algebraic examples.

Example 374 (L^1 functions)**.** *Consider an example similar to the one above, involving slice category. This time we take the category $\mathbb{C} \downarrow$ **Ban**, arrows in the category of Banach algebras with maps of norm bounded by* 1*. The endofunctor F acts on $\mathcal{C}$ by squaring:*

$$\begin{tikzcd} \mathbb{C} \arrow[rr] & {} \arrow[d, Rightarrow, "F"] & X \\ \mathbb{C} \arrow[rr] & {} & X \times X \end{tikzcd}$$

It is a theorem due to Tom Leinster that the initial algebra of this endofunctor is the algebra of L^1 functions on the unit interval.

Example 375 (Algebraic coalgebras)**.** *In algebra, the R-coalgebra of a ring R is a structure dual to algebras - an R-module C together with a R-linear maps*

$$\Delta : C \to C \otimes C$$
$$\epsilon : C \to R$$

which are associative with identity on C. Categorically, coalgebras can be constructed as coalgebras of the endofunctor $F : R - \mathbf{Mod} \to R - \mathbf{Mod}$, mapping $X \mapsto X \otimes_R X$.

Example 376 (Countable product)**.** *A product of countably many copies of some set S can be expressed as the terminal coalgebra of the product endofunctor $P : \mathbf{Set} \to \mathbf{Set}$*

$$P(X) = X \times S$$

Clearly we have a natural isomorphism

$$P(\prod^{\infty} X) \simeq \prod^{\infty} X$$

Moreover, it is the largest set with such property.

Example 377 (Divided power coalgebra)**.** *The polynomial ring $R[x]$ has a natural (commutative) coalgebra structure in* **CRing** *called divided power coalgebra. Δ operation of this coalgebra is defined as*

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k}$$

And $\epsilon : R[x] \to R$ maps are identity on scalars, and 0 otherwise. Identifying $R[x] \otimes_R R[x]$ with $R[x, y]$, the example of the diagonal Δ in action is

$$\Delta(x^3) = x^3 + 3x^2y + 3xy^2 + y^3 \in R[x, y]$$

Example 378 (Cantor set)**.** *Consider an endofunctor F :* **Top** $\to$ **Top**

$$F(X) = X \coprod X$$

and let C be a Cantor set. It is clear that the Cantor set is a fixed point of F. Moreover, it satisfies the universal property that for any continuous function $f : X \to X \coprod X$ there is an unique map η making the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\eta"] & X \coprod X \arrow[d, "(\eta,\eta)"] \\ C \arrow[r] & C \coprod C \end{tikzcd}$$

It makes the Cantor set the terminal coalgebra of F.

Example 379 (Freyd's construction of unit interval)**.** *Let $\mathcal{C}$ be a category of diagrams in* **Top** *of a form*

$$X \rightrightarrows Y$$

where the two arrows are continuous closed injections with disjoint image. The endofunctor $F : \mathcal{C} \to \mathcal{C}$ is informally gluing the space (X, Y, f, g) end to end, which formally can be expressed as a pushout *(with $F(X) = (F_0, F_1, f', g')$)*

$$\begin{tikzcd}
 & & F_1 & & \\
 & Y \arrow[ur, dashed] & & Y \arrow[ul, dashed] & \\
F_0 = X \arrow[ur, "f"] & & X \arrow[ul, "g"'] \arrow[ur, "f"] & & X \arrow[ul, "g"']
\end{tikzcd}$$

Then the unit interval I is the terminal F-coalgebra. When we drop the condition of closedness of the injections, we also get the unit interval as the terminal coalgebra, but this time with the anti-discrete topology.

Example 380 (Dyadic rationals and unit interval)**.** *The construction of I as final coalgebra can be also done in a different way, showing along the way also a different interesting phenomenon. Consider a category of bipointed metric spaces $\mathbf{Met}_*^*$ and the endofunctor $F((X, x_0, x_1)) = ((X \coprod X)/(i_0(x_1) \sim i_1(x_0)), i_0(x_0), i_1(x_1), \frac{1}{2}d)$ which maps X to its two copies, where the second point of first copy is glued with the first point of the second copy, and with the quotient metric shrinked by 2. The final coalgebra of such a functor again can be identified with the unit interval, while the initial algebra can be identified with dyadic rationals lying in the interval*

$$X = \{\frac{n}{2^m} \in I\}$$

with subspace topology. One can notice that the initial algebra and final coalgebra of F are not unrelated, as the final coalgebra I is the Cauchy completion of the initial algebra X. Indeed, such a theorem can be proven for any endofunctor in $\mathbf{Met}_*^*$.

Example 381 (Space of walks)**.** *Let the set of natural numbers $\mathbb{N}$ contain 0. Choosing any natural number n. A walk starting at n is a sequence of steps, when in each time period one can either go right or left except for the element 0 - here we can either go right or do nothing. The space of all walks starting at n is a subset of $\mathbb{N}^{\mathbb{N}}$. We will consider it as a topological space W_n with pro-finite topology, for any finite sequence $(a_1, \cdots, a_n)$, the walks starting with such a sequence form a closed set. The object (W_n) has a coalgebra structure in the category of sequences of spaces $\mathbf{Top}^{\mathbb{N}}$. It can be identified with a terminal coalgebra of the endofuntor*

$$F(X_\bullet)_n = \begin{cases} X_{n-1} \coprod X_{n+1} & n > 0 \\ X_0 & n = 0 \end{cases}$$

Example 382 (Lexicographic order)**.** *As we've already seen, the countable product $L^{\mathbb{N}}$ is the terminal coalgebra of the functor $X \mapsto X \times L$. Suppose that L is linearly ordered and $L^{\mathbb{N}}$ has a coalgebra structure given by*

$$L^{\mathbb{N}} \xrightarrow{[} (h,t)]L \times L^{\mathbb{N}}$$

Then the lexicographic order on L can be defined by the corecursion, i.e. using the coalgebra structure of $L^{\mathbb{N}}$:

$$x < y \text{ iff } h(x) < h(y) \text{ or } h(x) = h(y) \text{ and } t(x) < t(y)$$

In the next example we will see that the case of $L = \mathbb{N}$ provides a construction of non-negative real numbers.

Example 383 (Real numbers as terminal coalgebra)**.** *Categorifying the idea of continued fractions, it is possible to construct real numbers as a terminal coalgebra. Consider a category of* **Loset** *of linearly ordered sets and the endofunctor $G(X) = \mathbb{N} \times X$ with lexicographic order. Then the set $[1, \infty)$ has a structure of terminal coalgebra of G. To see that, denote the "strict floor function" taking a rational number x to the smallest strictly bigger integer as* sf(x). *Then the coalgebra structure on $[1, \infty)$ is given by the funny function*

$$f(x) = (\mathrm{sf}(x), \frac{1}{x - \mathrm{sf}(x)}) \in \mathbb{N} \times [0, \infty)$$

It was shown by Pavlovic and Pratt that such a coalgebra is indeed terminal.

Example 384 (Rational numbers as initial algebra)**.** *Similarly as in the previous example, let's consider a category* **Loset***, now with the endofunctor $G(X) = \mathbb{N} \times X \cup \{\bullet\}$ with the lexicographic order, with the point $\{\bullet\}$ being maximal. The initial H-algebra is lexicographicly ordered free monoid of positive integers $M(\mathbb{Z}_+)$. On the other hand, $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ also is an H-algebra, with the evaluation $\epsilon : \mathbb{N} \times \widehat{\mathbb{Q}} \cup \{\bullet\} \to \widehat{\mathbb{Q}}$*

$$\epsilon(n, x) = n + 1 - \frac{1}{x}$$
$$\epsilon(\bullet) = \infty$$

Moreover, $\widehat{\mathbb{Q}}$ is a fixed point of G, as it has an inverse

$$\epsilon^{-1}(x) = \begin{cases} \left(\lceil x \rceil - 1, \frac{-1}{1 - \lceil x \rceil}\right) & x \in \mathbb{Q} \\ \bullet & x = \infty \end{cases}$$

The isomorphism of linear orders between the initial algebra $M(\mathbb{Z}_+)$ and $\widehat{\mathbb{Q}}$ can be explicitly expressed using continued fractions: if $a_1 \cdots a_n$ is a word in the monoid $M(\mathbb{Z}_+)$, it can be expressed as an element of $\widehat{\mathbb{Q}}$ via

$$a_1 \cdots a_n \mapsto 1 + a_0 - \frac{1}{1 + a_1 - \frac{1}{1+a_2-\cdots}}$$

6. Examples from topology

Example 385 (Initial and final topologies)**.** *Consider some family of functions $\{f_i : X \to Y_i\}$. The initial topology associated to such a family is a coarsest topology on X making all the maps f_i continuous. Let F be a functor indexing the family with some discrete category $\mathcal{I}$ and U be a forgetful functor $\mathbf{Top} \to \mathbf{Set}$. Then topologies making f_i continuous can be seen as cones under F, and the universality characterising the coarsest structure can be captured by adjunction relation with the functor*

$$\mathrm{Cone}(F) \to \mathrm{Cone}(U \circ F)$$

Dual construction obviously captures the final topology, which is the opposite, finest solution to a problem with arrows reversed.

Example 386 (Covering spaces)**.** *The covering space is a surjection $E \to B$, such that it is trivially a projection from a disjoint union. It means that every point $b \in B$ has some open neighbourhood U_b, such that its preimage is homeomorphic to the disjoint union of U_b. As already mentioned, covering spaces can be categorified as a subcategory of the slice category $\mathbf{Top}/B$, so morphisms between them are following commutative triangles*

$$\begin{tikzcd}
E_1 \arrow[rr, "f"] \arrow[dr, "p_1"] & & E_2 \arrow[dl, "p_2"] \\
& B &
\end{tikzcd}$$

Example 387 (Fiber bundles)**.** *The fiber bundles are a direct generalisation of covering spaces, where instead of the discrete space we choose any fixed fiber $F \in \mathbf{Top}$ and demand every point b to have a neighbourhood U_b, which preimage is canonically isomorphic to $U_b \times F$ (where by canonically we mean that it composed with p is a product projection). It is categorified exactly the same way.*

Example 388 (Vector bundles)**.** *Vector bundles are fiber bundles, which fiber has a vector space structure. It is very tempting to claim that the category of vector bundles is abelian, as we can make there fiber-wise operations associated to their vector spaces, so the biproduct and tensor product is well defined, we have the zero object, as well as kernels and cokernels can be defined fiberwise. The kernels and cokernels unfortunately fail to be vector bundles themselves when the morphism has not locally constant rank. Consider for example the map* $\mathbb{R}^2 \to \mathbb{R}^2$ *(cosidered as trivial bundles over* $\mathbb{R}$*) given by* $(x,t) \mapsto (t, tx)$*. Then the kernel would be* $(-\infty, 0) \times \mathbb{R} \cup \{0\} \cup (0, \infty) \times \mathbb{R}$*, which is not even a fiber bundle (but makes sense in the category of sheaves, which is abelian).*

Example 389 (The problem with maps of constant rank)**.** *The solution to problem stated above is considering only morphisms between vector bundles of locally constant rank. Then both kernel and cokernel are indeed well-defined and are vector bundles. The only (but huge) problem with this approach is that vector bundles with morphisms of constant rank do not even form a category! Consider following maps between trivial* $\mathbb{R}$*-bundles:*

$$f : \mathbb{R}^2 \to \mathbb{R}^3 \quad f(t,x) = (t, xt, x) g : \mathbb{R}^3 \to \mathbb{R}^2 \quad g(t,x,y) = (t,x)$$

The composition $(gf)(t,x) = (t, xt)$*, which as we have already seen, is not a map of constant rank, even though both* f *and* g *have constant rank* 1*.*

Example 390 (Principal bundle)**.** *—-nLab thinks its principal bundle in the wide sense, so TODO —— Consider some topological group* G *acting on a topological space* X*. Such an action naturally forms a bundle* $E \to B = E/G$*, as there is a natural surjection from* E *to its orbit space with the quotient topology. A principal* G*-bundle is a bundle corresponding to the free and transitive action. There is a nice way of a categorical condition for a bundle being principal:* $E \to B$ *is principal iff the map* φ *is an isomorphism*

$$\begin{tikzcd}
E \times G \arrow[r, "\simeq"] & p^*E \arrow[r] & E \arrow[d, "p"] \\
& E \arrow[r, "p"] & B
\end{tikzcd}$$

Where the $\varphi : E \times G \to E \times_B E$ *is given by* $\varphi(x,g) = (x, xg)$*. The reason behind such a condition is that we can identify free actions with inducing injective* φ *and transitive with inducing surjective ones, thus bundles are principal if and only if* φ *is an isomorphism*

Example 391 (Fundamental group). *Consider some pointed compactly generated Hausdorff space (X, x_0). Its fundamental group $\pi_1(X, x_0)$ is a group of homotopy classes of pointed maps $S^1 \to X$. In more categorical language we can say that π is a functor $Hom((S^1, *), -)$: $\mathbf{HTop}_* \to \mathbf{Grp}$, where the values can be considered in the category of groups since S^1 is a reduced suspension of S^0, and reduced suspensions are cogroup objects in $\mathbf{HTop}_*$ (see the section about group objects for more detailed explanation).*

Example 392 (Universal cover). *The main result about covering spaces states that for every path-connected pointed Hausdorff space (X, x_0) the category of $\pi_1(X, x_0)$-sets is equivalent to the category of covering spaces over X $\mathbf{Cov(X)}$. It means that every set acted on by $\pi_1(X, x_0)$ naturally forms a covering space of X, that this correspondence is $1-1$. The universal covering space of X is the covering corresponding to the trivial principal $\pi_1(X, x_0)$-bundle over X.*

Example 393 (Covering space as a monomorphism). *Consider the category of pointed, connected, locally connected and locally path-connected spaces. These conditions are sufficient to prove that every covering projection has an unique lifting, which translates categorically to a weird fact, that every non-trivial covering space is a monomorphism in this category. Obviously it is also an epimorphism, so it is one of these weird examples of maps, which are both epi and mono, but are not isomorphisms (unless the covering is taken to be the identity map).*

Example 394 (Epimorphisms of Hausdorff spaces). *While the previous example has shown that monomorphisms in category of some topological spaces different than* **Top** *(when everything works as expected!) need not be injective as maps of sets, now we'll see that also epimorphisms need not to be surjective. Our category of interest is* **Haus***, a full subcategory of* **Top** *of Hausdorff spaces. Epimorphisms in* **Haus** *are exactly continuous functions with dense image, not necessarily surjective. In this example again we can easily construct a morphism which is both epi and mono, but is not an isomorphism, by considering any embedding of dense subspace, for example*

$$\mathbb{Q} \hookrightarrow \mathbb{R}$$

Example 395 (Homotopy groups). *Consider S^n for $n \geq 2$. We know that $S^n \simeq \Sigma S^{n-1} \simeq \Sigma^2 S^{n-2}$. Due to the loopspace-suspension adjunction in $\mathbf{HTop}_*$, we have $\mathrm{Hom}(S^n, Y) \simeq \mathrm{Hom}(\Sigma S^{n-2}, \Omega Y)$. Since ΣX is an H-cogroup and ΩY is an H-group, $\mathrm{Hom}(S^n, -)$ lifts to the category of abelian groups. The lifting $\pi_n(X, x_0) = \mathrm{Hom}((S^n, *), (X, x_0))$ is called the nth homotopy group of X.*

Example 396 (Open sets that don't exist?). *An idea behind a locale is both elegant and crazy. Consider some stupid condition defining a set, like surjective functions from* $\mathbb{N}to\mathbb{R}$*. Such a set is obviously empty, so it's boring, but when we ignore this fact it happens to be less boring, as we can talk for example about it's open subspaces with some suitable topology, for example we may consider a subspace of functions with only one zero. Such a system of subspaces form some poset, just like in the case of topological spaces that have some actual points inside its open subsets, and have similar properties to them, especially categorical, when the system of open subsets is a main object of study.*

Example 397 (Hurewicz homomorphism). *In this example we consider all spaces to be pointed, but since basepoints are not relevant, we ignore writing them down. Every disk* D^n *has a canonical homeomorphism with the standard simplex* Δ^n*. Such an homeomorphism* h *generates the* n*-th homology group of the sphere* $H_n(S^n) \simeq \mathbb{Z}[h]$*. Due to homotopy invariance of homology groups, every homotopy class of maps* $[f] \in \pi_n(X)$ *induces a map* $f_* : H_n(S^n) \to H_n(X)$*. Evaluating such a map on the canonical generator constructed above, we get the canonical natural transformation called the Hurewicz homomorphism* $\varphi : f \mapsto f_*(h)$

$$\begin{array}{ccc} \pi_n(X) & \xrightarrow{\varphi} & H_n(X) \\ \downarrow{\scriptstyle g_*} & & \downarrow{\scriptstyle g_*} \\ \pi_n(Y) & \xrightarrow{\varphi} & H_n(Y) \end{array}$$

Example 398 (Directed spaces). *On a topological space we can naturally consider all paths, but there is also a construction allowing to decode paths going only in specific direction. For a space* X *and its path space* $P(X)$ *(the usual one, constructed as* $\mathrm{map}(I, X)$ *with compact-open topology). Then the d-structure on* X *is a subset* $\mathfrak{D} \subseteq P(X)$*, such that*

- *contains all the constant paths*
- *is closed under concatenation*
- *is closed under non-decreasing reparametrisations*

d-space is called a pair $(X, \mathfrak{D})$*. d-spaces naturally form a category* $d - \mathbf{Top}$*, with d-structure preserving continuous functions, which is both complete and cocomplete.*

Example 399 (Directed real line). *A standard example of a directed space is a directed real line* $\vec{\mathbb{R}}$*, with* $\mathfrak{D}$ *consists of all non-decreasing* *paths. Similarly we can construct directed cube* $\vec{I_n}$ *with coordinate-wise non-decreasing paths and directed simplex* $\vec{\Delta^n}$ *with paths non-decreasing in baricentric coordinates. Obviously none of these structures are unique - for every space always we can choose only the constant paths, which will be the coarser structure (similar to indiscrete topology) or all possible paths, the richer (discrete) option.*

Example 400 (Completion of directed space)**.** *Some directed structures on a space are better behaved than others. The situation is similar to completeness of a metric space, from which comes the naming convention. For a d-space* $(X, \mathfrak{D})$ *a path* $\gamma : I \to X$ *is almost directed if on every open subset* $U \subseteq X$ *it can be "straighten" to directed one, meaning that for every* $[s,t] \subseteq I$ *such that* $\gamma([s,t]) \subseteq U$ *there exists a directed path* $\alpha \in \mathfrak{D}$ *with common endpoints* $\alpha(0) = \gamma(s)$, $\alpha(1) = \gamma(t)$. *As an example take a plane with all smooth maps (or more precisely, having some smooth parametrisation). Clearly not every non-decreasing map is smooth, but every can be locally smoothen. Thus all paths are almost directed in such a d-space. But considering non-decreasing smooth paths, all non-decreasing paths are almost directed, while decreasing maps are not. A set of almost directed paths forms itself a d-structure on* X*, called the completion* $\overline{\mathfrak{D}}$. *In fact, it forms an idempotent monad*

$$\overline{(-)} : d(-)\mathbf{Top} \to d(-)\mathbf{Top}$$

All the previously considered d-spaces are complete, with both canonical, discrete and anti-discrete d-structures. For interesting properties of d-spaces you can find in the section about simplicial methods.

Example 401 (Alexandrov-discrete spaces)**.** *Recall that every preordered set can be given a natural Alexandrov topology* $T(P)$ *constructed from all the upper closed sets, and from every topological spaces one can construct the specialisation preorder* $W(X)$ *from relation* $x \leq y$ *iff* $x \in \bar{\{y\}}$*, which form a right adjoint functor to* T. *Topological spaces araising from this constructed can be uniquely identified with Alexandrov-discrete spaces, characterised by following equivalent conditions*

- *both open and closed sets are closed under arbitrary unions and intersections*
- *all the points have minimal neighbourhoods*
- *the topology is finest with respect to inclusion of finite subspaces*

Functors T *and* W *restricted to the full subcategory of Alexandrov-discrete spaces* $\mathbf{Alex} \subset \mathbf{Top}$ *form an isomorphism of categories cnAlex* $\simeq$ $\mathbf{Pre}$. *Moreover, this subcategory turns out to be both reflective and coreflective, where both reflector and coreflector is the composition* $T \circ W$.

Let's stand this example with a few properties of any Alexandrov discrete space X and the category **Alex**

- **Alex** *has finite products*
- *subspace of X is again Alexandrov-discrete*
- *X is locally compact*
- *X is locally path-connected*

Example 402 (Infinite-dimensional manifolds)**.** *Classically manifolds are defined as spaces locally homeomorphic/diffeomorphic to Euclidean spaces with some additional axioms of being second-countable and Hausdorff. Similarly we can treat infinite-dimensional manifolds, modelled on some broader class of spaces. This way we obtain for example*

- *Hilbert manifolds* **HilbMan***, modelled on Hilbert spaces*
- *Banach manifolds* **BanMan***, modelled on Banach spaces*
- *ILH manifolds* **ILHMan***, modelled on inverse limits of Hilbert spaces*
- *Fréchet manifolds* **FrMan***, modelled on Fréchet spaces (complete, metrisable, locally convex Hausdorff vector spaces*

With inclusions

$$\begin{tikzcd} & \mathbf{ILHMan} \arrow[dr, hook] & \\ \mathbf{HilbMan} \arrow[ur, hook] \arrow[dr, hook] & & \mathbf{FrMan} \\ & \mathbf{BanMan} \arrow[ur, hook] & \end{tikzcd}$$

Most interesting non-trivial examples are smooth mapping spaces between smooth manifolds, all having a structure of a Fréchet manifolds. Among Banach manifolds we have unitary groups of infinite dimensional spaces. The direct and inverse limits of manifolds are also often Fréchet manifolds, as in particular in case of infinite sphere S^∞ and projective spaces $\mathbb{R}P^\infty, \mathbb{C}P^\infty$.

Example 403 (Associated bundle)**.** *Given fixed principal G-bundle $P \to M$, from any G-space X we can form the associated bundle. Since G acts on P on the right and on X on the left, there are two maps*

$$P \times G \times X \rightrightarrows P \times X$$

Coequaliser of such pair $P \times_G X$ is the associated bundle, encoding the twisted product of two actions, identifying the points (pg, x) with (p, gx). Associated bundle forms a functor

$$G\text{-}\mathbf{Top} \to \mathbf{Bundle}(M)$$

Example 404 (Mapping torus)**.** *A special case of the associated bundle construction is the mapping torus, constructed with the principal $\mathbb{Z}$ bundle $\mathbb{R} \to S^1$ and diffeomorphism $\Phi : X \to X$, with $\mathbb{Z}$ acting on X by powers of Φ. The mapping torus $\mathbb{R} \times_{\mathbb{Z}} X$ is then the quotient space of $\mathbb{R} \times X$ identifying points $(0, x) \sim (1, \Phi(x))$ and form an bundle over S^1 with fiber X. This construction provides a lot of interesting topological spaces, for example*

- *when $\Phi = 1_X$, $\mathbb{R} \times_{\mathbb{Z}} X \simeq S^1 \times X \to S^1$ is the product bundle*
- *in case of $X = \mathbb{R}$ and $\Phi(x) = -x$, $\mathbb{R} \times_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ is the Möbius strip*
- *every compact connected manifold with fundamental group isomorphic to finite extension of $\mathbb{Z}$ can be constructed as a mapping torus*

Example 405 (Cohomotopy groups)**.** *The dual version of homotopy groups, i.e. the contravariant functor* **HTop** $\to$ **Set** *representable by spheres do not in general form a group. However, restricting our categories only to* **Man**(n) *- n-dimensional manifolds with boundary, the homotopy classes of maps $[M, S^n]$ can be classified by the degree of a map. It can be computed by approximation the map by some homotopic smooth function, and then after choosing some regular value, by counting the (finitely many) points from the fiber taking into account the orientation of its tangent. It follows that the cohomotopy functor π^n has an unique lift to the category of groups, such that*

$$\pi^n(M) \simeq \begin{cases} \mathbb{Z} & M \text{ is orientable}, \partial M = \emptyset \\ \mathbb{Z}/2 & M \text{ is non-orientable}, \partial M = \emptyset \\ 0 & \partial M \neq \emptyset \end{cases}$$

However, these isomorphisms are not natural!

7. Examples from differential geometry

Example 406 (Atiyah Lie groupoid)**.** *Consider some G-principal bundle $P \to X$. The Atiyah Lie groupoid* $\mathrm{At}(P)$ *(also known as gauge groupoid or transport groupoid) encodes all the isomorphisms between different fibers of P respecting the G-structure, so being G-invariant. Thus categorically it is a groupoid with objects X and arrows*

$$\mathrm{At}(P)(x, y) = \mathrm{Hom}_G(P_x, P_y)$$

To get the smooth structures we need a more careful look. Firstly, note that since P is principal, G-invariant maps are always isomorphisms. Moreover, since fibers are isomorphic to G, they correspond to group multiplication, thus are determined by the image of identity element of

P_x. It means that any pair of points $(p,q) \in P \times P$ determines unique isomorphism of their fibers, and moreover only pairs of a form (gp, gq) determine the same isomorphisms as (p,q). Thus all arrows of the groupoid are in correspondence with points of the manifold $(P \times P)/G$, the quotient under diagonal action, so the groupoid can be identified as

$$(P \times P)/G \rightrightarrows X$$

Note that there is an unique functor $\mathrm{At}(P) \to \mathrm{Pair}(X)$ that is an identity on objects, since $\mathrm{At}(P)$ is transitive. Any section of this functor $\mathrm{Pair}(X) \to \mathrm{At}(P)$ is just a trivialisation of P

Example 407 (Parallel transport)**.** *The notion of connection, central notion of differential geometry, however admitting a definition quite obscure, has surprisingly a very elegant and simple abstract nonsense presentation. Let's start with the intuitive introduction to what we're trying to achieve by considering connections. Consider some fiber bundle $E \to B$ and a path $I \to B$. It's easy to see "at the picture" on some simple examples, such as Möbius band, that it's reasonable to expect that such a path in B moving x_0 to x_1 can also linearly move the fibers E_{x_0} to E_{x_1}, locally just extending the path linearly to all the other points of a fiber. Now to express that categorically, for a moment restrict ourselves to principal bundles. Recall that linear isomorphisms of fibers were encoded in the Atiah groupoid $\mathrm{At}(E)$, while paths in B in the fundamental groupoid $\Pi_1(B)$. Since we want to move points of B identically by a path and induced transport of fibers, consider the pullback diagram*

$$\begin{tikzcd} \mathrm{At}'(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_1(B) \arrow[d] \\ \mathrm{At}(E) \arrow[r] & \mathrm{Pair}(B) \end{tikzcd}$$

Note that this is just the change of base of the Atiyah groupoid, as we want it to have the base in paths, not points. This way the connection can be expressed just as the section of such projection!

$$\Pi_1(B) \to \mathrm{At}'(E)$$

However we have cheated a little bit in the process - recall that categorically it was pretty hard to represent paths in a space, as they in general refuse to compose associatively, and the groupoid we've used encodes them only up to homotopy. Thus we do not now if all connections can be constructed this way, as every such transformation by construction must be homotopy invariant. Even though most interesting constructions in topology are homotopy invariant, this subject lies *closer to differential geometry than topology and the analogy cracks, but not breaks. The class of connections that are homotopy invariant can be intrinsically characterised as the flat connections, having zero curvature. We can interpret such section as a map from a path between points (up to homotopy) to the G-invariant isomorphism between fibers attached to its endpoints. Any such section is called the parallel transport of the connection.*

Example 408 (Connections)**.** *We can refine the previous construction of flat connection to all connections by refining our classes of paths. Note that expect fundamental groupoids we've considered also a weaker, however more obscure category encoding classes of paths - the path groupoid $P_1(B)$. Its morphisms were paths only up to so called thin homotopy. A homotopy between paths is just a map $I^2 \to B$. Intuitively, such homotopy is called thin if its image has zero area, which corresponds to the condition that its differential has nowhere full rank. Moreover, we insist that not only such homotopy fixed endpoints of γ, but also some of its neighbourhood. Under such conditions it the sections*

$$\begin{tikzcd} \mathrm{At}'(E) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \Pi_1(B) \arrow[l, dashed, bend right, "tra"'] \arrow[d] \\ \mathrm{At}(E) \arrow[r] & \mathrm{Pair}(B) \end{tikzcd}$$

do indeed classify parallel transports of all possible connections, as it can be shown that all of them even if not homotopy invariant, are invariant under thin homotopies. The lack of homotopy invariance is visualised in the picture below

Example 409 (Differential equation)**.** *The equivalence between flat connections and representations of path groupoid can be also considered in the context of ordinary differential equations. Firstly note that we have a correspondence between ordinary differential equations on a manifold with flat connections. Every such equation, for simplicity of notation having order* 2

$$u'' + \alpha u' + \beta t = 0$$

We may reduce to system of order 1 *equations by substitution $u' = v$*

$$\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\beta & -\alpha \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

Then using the derivative explicitely as differential operator we get the equation

$$\left(d + \begin{pmatrix} 0 & -1 \\ \beta & \alpha \end{pmatrix} dt\right) \begin{pmatrix} u \\ v \end{pmatrix} = 0$$

It's easy to check that the operator from the left hand side satisfies the properties of a connection. Moreover, since solutions of ODE depends only of the homotopy class of integral lines from initial condition to solution, such a connection must be flat.

On the other hand, the integral lines of a solution form a linear map between fibers of some bundle E, having possible solutions as sections. This map, coinciding with parallel transport of ∇, can be seen as a functor $\Pi_1(M) \to \mathrm{GL}(E)$.

Example 410 (Integration)**.** *Flat connections can be seen in yet another context. Formally, the flatness is defined by vanishing of the curvature tensor*

$$\nabla_{[X,Y]} - \nabla_X \nabla_Y - \nabla_Y \nabla_X = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y] = 0$$

This property is nothing more than the condition satisfied by representations of Lie algebra $\mathfrak{X}(M)$. This correspondence is in fact bijective as well and we classify flat connections with a category in two different ways

$$\begin{gathered} \mathbf{Rep}(\mathfrak{X}(M)) \simeq \mathbf{Rep}(\Pi_1(M)) \simeq \\ \{\textit{vector bundles with flat connections}\}/\textit{iso} \simeq \\ \{\textit{general solutions of ODEs}\} \end{gathered}$$

The correspondence of parallel transport of γ along section $s \in E_{t_0}$ looks like

$$\begin{gathered} \{\textit{vector field along } \gamma \textit{ acting on } s\} \simeq \\ \{F(\gamma) \in \mathrm{GL}(E) \textit{ evaluated at } s\} \simeq \\ \{\textit{parallel transport along } \nabla - P_{t,\gamma} : E_{t_0} \to E_t\} \simeq \\ \{\textit{integral lines under initial condition given by } s\} \end{gathered}$$

This construction is very important in the theory of groupoids and very instructive. Reader may have noticed the analogy with ordinary Lie groups and algebras - starting from representation of paths, by applying differential operator we get representation of Lie algebra of vector fields, that in some sense formalise infinitesimal paths. We can also go the other way - starting from Lie algebras of vector fields, solving differential equation leads us to representation of actual paths (modulo *homotopy). We've seen that the latter are just integral lines, so calling this "integration" won't be a huge abuse of notation.*

Example 411 (Classifying principal bundles)**.** *Let's M be a manifold, $\mathcal{U} = \{U_i\}$ its open cover and G a Lie group. Recall that each Lie group could have been constructed as the Lie groupoid $\mathbf{B}G : G \rightrightarrows \{\bullet\}$. Moreover, from any open cover we can also construct the Čech groupoid, which also is smooth and equivalent to the pair groupoid.*

$$\breve{C}(\mathcal{U}) : \coprod_{\mathcal{I}\times\mathcal{I}} U_i \cap U_j \rightrightarrows \coprod_{\mathcal{I}} U_i$$

The morphisms of groupoids

$$\breve{C}(\mathcal{U} \to \mathbf{B}G$$

are in correspondence with principal G-bundles having trivialisation on $\mathcal{U}$. In particular, if we take $\mathcal{U}$ as the entire atlas of M, its Čech groupoid $\breve{C}(M)$ classifies all principal G-bundles on M as elements of $\mathrm{Hom}_{\mathbf{Grpd}}(\breve{C}(M), \mathbf{B}G)$. *This behaviour of the Čech groupoid is not coincidental and reflects its deeper universal property - $\breve{C}(M)$ co-represents all the matching families of covers that are used in the definition of sheaves.*

Example 412 (Groupoid action)**.** *Group actions naturally generalise also to groupoids and have there a lot of extra structure. Consider Lie groupoid $\mathcal{G} \rightrightarrows M$ and any morphism $p : A \to M$, called the moment map. Lie groupoids, unlike groups, act not just on spaces, but on such moment maps. The action*

$$\phi : (\mathcal{G} \rightrightarrows M) \curvearrowright (A \xrightarrow{p} M)$$

Is the map from so called good pullback (i.e. pullback with good differential properties - inducing the pullback also on tangent spaces)

$$\begin{tikzcd}
\mathcal{G} \times_M A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "p"] \\
\mathcal{G} \arrow[r, "s"] & M
\end{tikzcd}$$

to A

$$\phi : \mathcal{G} \times_M A \to A$$

which we denote as $\phi(g, a) = \phi_g(a)$, such that

- $p(\phi_g(a)) = t(g)$
- $\phi_{e_x}(a) = a$
- $\phi_g \phi_h = \phi_{gh}$

We can think that ϕ realises arrows of $\mathcal{G}$ as symmetries of fibers of p, as every arrow $x \xrightarrow{g} y$ induces some diffeomorphism $A_x \to A_y$ in similar fashion as every element of a group induces automorphism of, this time unique, object via group action. Thus groupoid action works pretty much the same way, but this time act on points of A spread across several different clusters, which are encoded by the moment map. I think this is a key to finally see how groupoids are encoding local symmetries, unlike groups encoding only global ones - a group action permutes all the points of A at once, while the action of groupoid also permute points, but between two local clusters. As always, let's at first take a look at some trivial examples

- *Actions of manifolds $(M \rightrightarrows M) \curvearrowright (A \to M)$ are all trivial*
- *A group action is equivalent to groupoid action of a form $(G \rightrightarrows \{\bullet\}) \curvearrowright (A \to \{\bullet\})$*

The next quite simple type of groupoids - corresponding to a submersion - have a bit more interesting actions. A given submersion $M \to N$ and moment map $A \to N$, the actions $(M \times_N M \rightrightarrows M) \curvearrowright (A \to M)$ correspond to solutions to certain universal problem - such equivalence relations R on A, that the square

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A/R \arrow[d, "p"] \\ M \arrow[r, two heads] & N \end{tikzcd}$$

are pullbacks.

Example 413 (Equivariant vector bundles)**.** *The natural next target to copy from group theory once we've constructed actions are linear representations. In case of groups, these were just actions on vector spaces. Since groupoids are just groups with parameter, they should act on vector spaces with parameters - vector bundles. Groupoid via a linear representation $(\mathcal{G} \rightrightarrows B) \curvearrowright (E \to B)$ acts by arrows $g : x \to y$ on fibers, inducing linear isomorphism $E_x \to E_y$. There are a few interesting constructions appearing this way - for example the linear representations of the action groupoid $G \times M \rightrightarrows M$ correspond to G-equivariant vector bundles - such that the action on M extends to the action on fibers, inducing additionally linear isomorphism $E_x \to E_{gx}$. For example for each Lie group G, its tangent bundle is naturally G-equivariant.*

Example 414 (Frame groupoid)**.** *In group theory we had a few equivalent definitions of the linear representation - as action on vector space,* *as homomorphism $G \to GL(V)$ or as KG-module. The second one can be quite directly translated to general linear groupoids. It arises from more general construction of a frame groupoid, generalising also the Atiyah groupoids to non-principal bundles. It can be associated to any fiber bundle E, or even any surjective submersion, and its arrows are just (G-equivariant) isomorphism between fibers*

$$\{E_x \xrightarrow{\simeq} E_y \mid x, y \in B\} = \mathrm{Fr}(E) \rightrightarrows B$$

In case of vector bundles, it is called general linear groupoid $\mathrm{GL}(E)$*. We can also considered other variants - for example for manifolds equipped with metric more natural is the orthogonal linear groupoid $O(E) \rightrightarrows B$, which contains only isometric isomorphisms with respect to intristic metric of M. General linear groupoids, generalising the general linear group, classifies the linear representations of $\mathcal{G}$, as we have a natural bijection*

$$\mathrm{Rep}_E(\mathcal{G}) \simeq \mathrm{Hom}_{\mathbf{Grpd}}(\mathcal{G}, \mathrm{GL}(E))$$

Example 415 (Lie algebroid as core)**.** *Lie algebroids pop up in several different ways during the study of Lie groupoids. One of their constructions can be seen through linear representations. Similarly as group actions can be presented as maps $G \times X \to X$, we can describe in similar fashion groupoid representations as morphisms of groupoids $\mathcal{G} \times_M A \to \mathcal{G}$. These can be unwrapped into a following commutative square*

$$\begin{tikzcd}
\mathcal{G} \times_M A \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & \mathcal{G} \arrow[d, shift left] \arrow[d, shift right] \\
A \arrow[r, "p"'] & M
\end{tikzcd}$$

Note that in case of linear representation, the horizontal arrows are both vector bundles, and both groupoid arrows between total spaces are also vector bundle morphisms. Such squares we'll call VB-groupoids, as they merge vector bundles and groupoids into one object. Having two bundle morphisms inside any VB-groupoid, we can use one (source) to get another vector bundle as its kernel. Using the unit inclusion $M \hookrightarrow G$, such a kernel restricted to M we call a core $C \to M$.

$$\begin{tikzcd}
0 \arrow[r] \arrow[dr] & C \arrow[r] \arrow[d] & \mathcal{H} \arrow[r, shift left, "s"] \arrow[r, shift right, dashed] \arrow[d] & A \arrow[d, "p"'] \\
& M \arrow[r] & \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M
\end{tikzcd}$$

Finally taking the derivative of both arrows, we can form a tangent VB-groupoid, which core is the associated Lie algebroid.

$$\begin{tikzcd} 0 \arrow[r] & A_{\mathcal{G}} \arrow[r] \arrow[d] & T\mathcal{G} \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & TM \arrow[d] \\ & M \arrow[r, hook] & \mathcal{G} \arrow[r, shift left] \arrow[r, shift right] & M \end{tikzcd}$$

Example 416 (General Lie algebroids)**.** *Lie algebroids, similarly to Lie algebras, can be studied independently to Lie groupoids. Recall that a Lie algebra is just a vector space equipped with the Lie bracket, and under such multiplication it forms a not-associative K-algebra. A natural generalisation of this concept just adds parameter to a vector space by replacing it with vector bundle. A Lie algebroid over M is thus a vector bundle $E \to M$ together with the Lie bracket $[-,-]$ on its space of sections and the so called anchor map - morphism of bundles preserving the bracket $a : E \to TM$, providing the bracket with a derivation structure*

$$[s, f \cdot t] = f \cdot [s,t](m) + a(s)(f) \cdot t$$

Note that each such anchor map induces a natural Lie algebra homomorphism $\Gamma(E) \to \mathfrak{X}(M)$, as the space of vector fields comes with natural structure of $C^\infty(M)$-Lie algebra with Lie bracket $[X,Y] = XY - YX$.

Example 417 (Morita equivalence)**.** *Note that in case of categories we always prefer to look at them up to equivalence, not isomorphism, as the notion of isomorphism is just way too strong. Similar situation happens in case of Lie groupoids - their isomorphisms are even stronger than isomorphisms of underlying categories, as additional smooth structure also play a role there. As in the case of categories, this is sometimes too restrictive. Consider for example the trivial groupoid of a manifold $M \rightrightarrows M$ and its Čech groupoid $\check{C}(M)$. Clearly their realisations π_0 are diffeomorphic, as correspond to gluing together the cover. Also all the fibers are diffeomorphic, but we do not have such isomorphism in* **Grpd**. *It can be fixed by taking more category theoretic perspective and considering weaker class of equivalences, similar to equivalence of categories, but taking into account the smooth structure as well. The morphism $f : (\mathcal{G} \rightrightarrows M) \to (\mathcal{H} \rightrightarrows N)$ is called essentially surjective if the map*

$$\mathcal{H} \times_N M \to N$$

is a surjective submersion, what expresses the fact that each element $n \in N$ can be reached from some $m \in M$ by an arrow $f(m) \to n$. The *analogue of being fully faithful in a smooth way is expressed on the level of sets by existence of a pullback*

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & H \arrow[d] \\ M \times M \arrow[r, "f\times f"] & N \times N \end{tikzcd}$$

And to get the proper smooth structure we must add the condition that this pullback is good, i.e. the diagram of differentials between tangent spaces is also a pullback.

Using the weaker notion of Morita equivalence instead of isomorphisms, we get exactly the properties we've wanted - the Čech groupoid is equivalent to its manifold, but all the group actions are still distinguishable - for example no groupoids $\{\bullet\}//G$ are equivalent if the groups differ, and even such subtle examples as $\mathbb{R}^n//O(n)$ and $\mathbb{R}^n//SO(n)$ are still distinguishable from each other, as they have different stabilisers of origins, even though for example diffeological spaces, also making a pretty decent job of preserving information about the action in the quotient, do not distinguish these two. Every Morita equivalent groupoids have

- *Homeomorphic realisations* $\pi_0(\mathcal{G})$
- *Diffeomorphic fibers* $\mathcal{G}(x,y) \simeq \mathcal{H}(f(x),f(y))$ *for each* x,y

A particular example of Morita equivalent groupoids identifies

$$\mathbf{Rep}(\Pi_1(M)) \simeq \mathbf{Rep}(\pi_1(M,x_0))$$

Adding another element to our long sequence of isomorphic objects derived from flat connections.

Example 418 (Features of Lie groupoids)**.** *A lot of topological or differential features of underlying maps of Lie groupoids have some consequences in its general properties. Among most interesting features we have*

- $\mathcal{G}$ *is called* ***étale*** *if maps* s,t *are local diffeomorphisms. Étale groupoids have discrete stabilisers*
- $\mathcal{G}$ *is called* ***proper*** *if* $\mathcal{G} \xrightarrow{s\times t} M\times M$ *is a proper map, i.e. preimages of compact sets are compact. Proper groupoids have compact stabilisers.*
- $\mathcal{G}$ *is called* ***transitive*** *if it has single orbit. Transitive groupoids are always isomorphic to some Atiyah groupoids for some principal bundle.*

Example 419 (Projective line)**.** *A nice toy example of Lie groupoid araises from projective geometry. Consider the real projective plane* *without two points $M = \mathbb{R}P^2 \setminus \{S, T\}$ and choose some distinguished projective line $l : S^1 \hookrightarrow \mathbb{R}P^2$. As in a projective plane there is always unique line connecting two points, for each point we obtain two unique points from l, obtained from intersection of line connecting p with S and T. These will be our source and target map*

$$\mathbb{R}P^2 \setminus \{S, T\} \rightrightarrows S^1$$

any two points with common source and target, has associated four lines, intersecting 3 times on the line l, on distinguished removed points S, T and in exactly different point, that taken as composition $M \times_{S^1} M \to M$ completes its Lie groupoid structure.

Note that the structure of M is not as symmetric as it may look. The action is almost transitive, with every point can be moved to another except the point x_0 from intersection of l and the line connecting S with T, which is always fixed, as cannot be reached by any line crossing only one distinguished point, thus M can be deconstructed into two orbits: $\mathbb{R}P^2 \setminus \{P, S, x_0\}$ and $\{x_0\}$.

Example 420 (Orbifolds)**.** *Orbifolds are another classical tools used to keep track of points in orbits of group actions. They are defined with direct analogy to manifolds - as topological spaces equipped with atlas containing charts $\varphi_i : U_i \to X$, but unlike in case of manifolds, ϕ_i are not homeomorphisms, but induce homeomorphisms of quotients under action of some finite groups G_i*

$$U_i/G_i \simeq \mathrm{im}(\varphi_i)$$

The idea behind these objects is to capture well some singular versions of manifolds, that are glued from orbit spaces of $\mathbb{R}^n$ acted on by finite groups. There are a lot of simple but satisfying examples, moreover often having funny names

- ***Mirror*** *is the half-space $\mathbb{R}^3/C_2$, reflecting the paths crossing a hyperplane. As topological space, its just $[0, \infty) \times \mathbb{R}^2$*
- ***Barber shop*** *is the strip $\mathbb{R}^3/(C_2 * C_2)$, reflecting rays with two parallel mirrors. It's homeomorphic to $I \times \mathbb{R}^2$*
- ***Billiard table*** *is $\mathbb{R}^2/(C_2 * C_2 \times C_2 * C_2)$. It reflects rays crossing a boundary of rectangle, and is indeed a rectangle as a manifold*
- ***Cone*** *is p times twisted plane - the quotient $\mathbb{R}^2/C_p$, as a topological space no different than $\mathbb{R}^2$*
- *the quotient $\mathbb{R}^2/C_2$ has underlying space not homeomorphic to any manifold - it is contractible, as its a cone over $\mathbb{R}P^2$, but after removing a point it is homotopy equivalent to $\mathbb{R}P^2$ - this is behaviour not possible for 3-manifolds, where removing a point cannot change a fundamental group!*

- ***(p, q)-teardrop*** *is an orbifolds not constructible as a quotient space whenever* $p \neq q$

The main feature of orbifolds is the canonical stratifications - decomposition of the space into standard and singular points of different kinds. This operation distinguish all type of singularities we can be interested in - for example the boundary in the mirror, called the mirror locus, is clearly of different nature than the corner of a cone, but corner of cone twisted 3 *times also looks different than when it's twisted* 4 *times. Orbifolds with properly defines smooth maps, preserving quotient structure by local lifts, form a category. We will skip the technical definition of these morphisms though, as we will construct this category starting from more categorical approach as a differentiable stack while generalising the notion of a sheaf. For now, we can take a look at a little spoiler of this construction and look at orbifolds as certain Lie groupoids.*

Example 421 (Orbifold groupoid)**.** *An orbifold can be realised also as a Lie groupoid. The idea is simple - orbifold is a manifold, but locally acted on by some finite group. Note that a group is finite if and only if discrete and compact. Groupoids capturing such structures are étale and proper - and indeed from any such groupoid* $\mathcal{G} \rightrightarrows M$ *we can make an orbifold by taking* $\pi_0(M)$*, where non-trivial stabilisers* $\mathcal{G}_x$ *act on some neighbourhood of* x*, forming together an orbifold atlas. Any such étale and proper Lie groupoid we'll call orbifold groupoid. Note that different, non-isomorphic orbifold groupoids may reduce to isomorphic orbifolds via* π_0*. The proper way of handling that problem is to consider only their Morita equivalence classes.*

Example 422 (Bisection)**.** *Bisection is a way of constructing global actions of a groupoid on the base space. It is such a section of the source map* $M \to \mathcal{G}$*, that composed with the target form a diffeomorphism of* M*. One can think about bisection as global continuous choice of arrows between points of* M*, combining together to its automorphism. Note that this definition has a disadvantage of breaking the symmetry between source and target maps. It can be fixed with equivalent definition of a bisection, as such submanifold of* $B \hookrightarrow \mathcal{G}$*, that both* s *and* t *restrict to diffeomorphism* $B \to M$*. Bisections can be composed in the obvious way, under which it form a group* $\Gamma(\mathcal{G})$*. Group of bisections comes with a natural homomorphism* $\Phi : \Gamma(\mathcal{G}) \to \mathrm{Diff}(M)$*, realising the bisection as diffeomorphism by composing restriction of* t *with inverse of restricted* s*, i.e.*

$$\Phi(S) = t_{|S} \circ s_{|S}^{-1}$$

Moreover, $\Gamma(\mathcal{G})$ *acts on* $\mathcal{G}$ *in three canonical ways - by left or right multiplication, as well as by the adjoint action, combining these two in a single action. The adjoint action is in fact just the extension of induced automorphism* Φ_S *to automorphism of entire* $\mathcal{G}$*, restricting to* Φ *on units. Let's look at a few examples of bisections*

- *bisection of Lie group is just multiplication by element of* G *and* $\Gamma(G) \simeq G$
- *Manifolds have only trivial bisection*
- *Bisections of the pair groupoid are graphs of automorphisms, with* $\Gamma(\mathrm{Pair}(M)) \simeq \mathrm{Diff}(M)$
- *Bisections of Atiyah groupoids correspond to principal bundle automorphisms*
- *Bisections of pair groupoids correspond to equivariant automorphisms*

Example 423 (Pseudogroups)**.** *Similarly as in case of any sections, local sections are as interesting objects as global ones. This distinction directly generalises to local bisections, defined only on the restriction to some open subset of* M *and appropriate pullback groupoid. The local bisections form a pseudogroup, an algebraic formalisation of operations on objects of local nature, generalises groups in similar way that sheaves generalise function spaces by including objects only locally defined. In fact most local sections or similar structures form a pseudogroup, for example*

- *With manifold we may identify pseudogroup of local diffeomorphisms*
- *Riemannian manifolds have pseudogroup of local isometries*
- *Foliated spaces have pseudogroup of local diffeomorphisms preserving leaves*

Note that it doesn't really make sense of formalising these as sheaves, even though their local nature can be expressed with sheaf-like axioms, put together in the definition of pseudogroup.

Example 424 (Affine group)**.** *Recall our toy example of a Lie groupoid constructed from projective plane without two points over a circle. To calculate it's group of bisections, it is more convenient to work over complex numbers, which does not change anything in the previous construction, as then we can help ourselves a bit using Bézout theorem, a classical and intuitive result from algebraic geometry, stating that every complex projective curve of degree* n *has exactly* n *intersections (with multiplicities).*

Note that a bisection is an embedding of the base space in the arrow manifold, in this case $\mathbb{C}P^1 \hookrightarrow \mathbb{C}P^2 \setminus \{S, T\}$. Since we work with complex manifolds, the possible embeddings of projective line into projective plane are quite limited, as these are classified as only lines and conics. Moreover, as a bisection, it must intersects fibers s^{-1}, t^{-1} exactly once, so the Bézout theorem excludes all the conics. It means that possible bisections are in bijection with lines in $\mathbb{C}P^2$ that do not cross points S, T. Recalling the coolest fact about projective geometry, the duality between lines and points, we may translate this to a set of points in projective plane not lying in two transversal curves, with common point x_0. This way we identify. A projective plane without a line is just an affine plane, which after removing the other line is diffeomorphic to $\mathbb{C} \times \mathbb{C}^\times$. We can identify this Lie group with the 2-dimensional subgroup of $\mathrm{PGL}_2(\mathbb{C})$ containing affine automorphisms of a form $az + b$, $a \neq 0$. This is result we should expect, as these are just automorphisms of projective plane fixing two lines - $x = 0$ and the line at infinity, corresponding to removed elements.

$$\mathrm{Bisec}(M) \simeq \mathrm{Aff}_2(\mathbb{C}) \hookrightarrow \mathrm{PGL}_2(\mathbb{C})$$

Note that such a result cannot be repeated in the real case, where the interplay between differential and algebraic geometry do not have a place and the embeddings of a circle in real projective plane form a surely infinitely-dimensional space. It may be pretty surprising considering the fact, that the constructions of both groupoids seemed completely indifferent.

Example 425 (C^∞ rings)**.** *The generalisation of differential geometry to more general spaces can be considered also in an opposite as in diffeological spaces, covariant direction, via approach generalising the notion of their function algebras instead of sheaves. There are a few ways of describing C^∞ rings. The simple, the formal and the elegant. Firstly, recall that any $\mathbb{R}$-algebra we have the product operation $A \times A \to A$ extending the usual product $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$. A C^∞ ring has an additional property, extending such lift such that any smooth function $\mathbb{R}^n \to \mathbb{R}^m$ lifts to n-ary operation $A^n \to A^m$. Note that this is just a direct analogy with the very definition of algebras, treating composition of functions equally to multiplication in a ring. The analogy is even cleaner in the formal definition, when we reformulate the definition of a ring in a slightly weird way*

ring	*set equipped with countably many n-ary operations $p_R : R^n \to R$ for any $n \in \mathbb{N}, p \in$*
C^∞-ring	*set equipped with infinite set of n-ary operations $f_A : A^n \to A$ for any $n \in \mathbb{N}, h \in$*

The operations f_A compose n elements of A with any real valued function on n-variables. Thus, similarly as we see rings as generalisation of numbers, these objects generalise functions.

Finally there is also a pure abstract nonsense definition - C^∞-rings are co-presheaves on Cartesian spaces preserving products.

C^∞-rings can be also used to generalise ringed spaces to C^∞-ringed spaces.

Example 426 (Differential spaces)**.** *C^∞ rings are often considered with some additional assumptions making them more traceable. The problem with them is similar to encountered with smooth spaces - we lack the notion of actual points, what makes them quite hard to work with. Pretty similar solution is contained in the notion of differential space - a topological space M together with family of real functions $\mathcal{F} \subset \mathrm{Hom}(M, \mathbb{R})$ satisfying moreover*

- *M has initial topology with respect to $\mathcal{F}$*
- *if locally $f \in \mathcal{F}$, then entire f is in $\mathcal{F}$ (equivalently: $\mathcal{F}$ coincides with the global sections of subsheaf of $C^0(M)$ generated by $\mathcal{F}$)*

Differential spaces form a wide family of spaces, including

- *smooth manifolds*
- *Tychonoff spaces with all real functions*
- $\mathrm{Hom}_{\mathbf{Diff}}(M, -)$
- *some singular spaces such as $\{xy = 0$, however not all in general*

Differential spaces form a subcategory of local C^∞-ringed spaces. Readers familiar with concepts of schemes probably see the potential of an abundant loan of ideas from algebraic geometry - indeed as you expected, this functor has a right adjoint C^∞-Spec, which is not however an equivalence of categories after restriction to the C^∞-schemes. It is not injective on objects, with many non-equivalent C^∞ admitting isomorphic spectrum.

Example 427 (Kähler differentials)**.** *Differential forms found a very fruitful and simple analogue in commutative algebra by the module of Kähler differentials. It satisfies the universal property, factorising all derivations of given A-algebra. Surprisingly, this construction turned out not to work in the classical setting on smooth manifolds, with the module of Kähler differentials of the ring of smooth functions having almost nothing in common with actual differential forms. The latter can be derived as Kähler differential of $C^\infty(M)$ regarded as C^∞-ring, where the construction of universal derivation directly generalises.*

The universal derivation extends also to a graded exterior algebra, extending the de Rham complex and the Cartan calculus.

8. Examples from algebra

Example 428 (Profinite group). *A topological product of infinitely many discrete finite spaces is far from discrete itself - its open sets are missing only finitely many points. Moreover, such a space is compact by Tichonov theorem. Similar case we observe in the case of direct limit of finite group - every such limit has a natural topology inherited from the inclusion* **Grp** $\hookrightarrow$ **TopGrp**. *Moreover, every such a group is compact and its open sets looks similar to the Zariski topology on the spectrum. We call them profinite group. The best known example of a profinite group are Galois group of the algebraic closure:*

$$\mathrm{Gal}(\bar{K}/K) \simeq \varprojlim_{L/K} \mathrm{Gal}(L/K)$$

The profinite topology on $\mathrm{Gal}(\bar{K}/K)$ *is called the Krull topology.*

Example 429 (Topology on p-adic numbers). *Similarly to groups, rings constructed as direct limit of finite rings also has naturally induced profinite topology. In particular, p-adic integers, as well as their fractions, p-adic numbers* $\mathbb{Q}_p$*, form a profinite ring and profinite field, due to the canonical constuction as*

$$\mathbb{Z}_p \simeq \varprojlim_{n} \mathbb{Z}/n\mathbb{Z}$$

Example 430 (Non-unital rings as slices). *A category of non-unital commutative rings* **CRng** *is equivalent to the slice category* **CRing**$/\mathbb{Z}$*, where the ring is identified with the kernel of the map to* $\mathbb{Z}$*, called the augmentation ideal. The inverse of this equivalence comes from a natural projection of a unitalisation of a ring (see Chapter 4)*

$$R \oplus \mathbb{Z} \to \mathbb{Z}$$

Example 431 (Localisation of a ring). *Recall that given any commutative ring* R *and multiplicatively closed subset, the localisation* $S^{-1}R$ *with explicit construction by fractions*

$$I_S = \{a \mid \exists_{s \in S} as = 0\} \lhd RS^{-1}R = \{\frac{a}{s} \mid a \in R/I_S, s \in S\}$$

or, more formally,

$$S^{-1}R = (RI_S \times S)/\sim$$

where the equivalence is the standard fraction construction

$$(a, s) \sim (a', s') \text{ iff } as' = a's$$

In most cases, the more convenient description of the localisation is the categorical one, specifying it via the universal property. Consider a category $R/\mathbf{CRing}_S$*, the category of arrows* $R \to X$*, such that the* *image of the set S is contained in units of X. Then the localisation together with the canonical map $R \to S^{-1}R$ is the initial object in $R/\mathbf{CRing}_S$. It means that given any map from such a category, it has an unique lift*

$$\begin{tikzcd} R \arrow[r] \arrow[d] & X \\ S^{-1}R \arrow[ur, dotted, "\exists!"] & \end{tikzcd}$$

Alternative description of $S^{-1}R$ can be found in the section about adjunctions.

Example 432 (Non injective monomorphism in divisible abelian groups)**.** *Let* **Div** *be a category of injective objects in* **Ab***, i.e. divisible groups. The quotient morphism*

$$\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$$

is a monomorphism in **Div***, even though is not an injection as sets. Indeed, consider a map $G \xrightarrow{f} \mathbb{Q} \xrightarrow{p} \mathbb{Q}/\mathbb{Z}$ such that $p \circ f = 0$. Since for each $g \in G$ $p(f(g)) = 0$, it must be that $\mathrm{im}(f) \subseteq \mathbb{Z}$. But if G is divisible, the only possible such a map is $f = 0$.*

Example 433 (Epimorphisms of rings)**.** *Epimorphisms in the category* **CRing** *have quite surprising characterisation - a map $A \to B$ is an epimorphism if and only if the induced map*

$$B \to B \otimes_A B$$

is surjective on underlying sets. This condition has also an equivalent statement on elements of the tensor product: f is an epimorphism if and only if , the equality

$$1 \otimes b = b \otimes 1$$

holds in $B \otimes_A B$ for every $b \in B$.

Example 434 (Valuation ring)**.** *All valuation rings can be characterised as colimits of some special types of diagrams. For any field K ans some other algebraically closed field $\Bbbk$, a valuation ring of K always is a colimit in* **CRing**

$$\begin{tikzcd} A_1 \arrow[r, hook] \arrow[dr] & A_2 \arrow[r, hook] \arrow[d] & \cdots \arrow[r, hook] & K \\ & \Bbbk & & \end{tikzcd}$$

Example 435 (Auslander algebra)**.** *Consider some finite dimensional K-algebra A and let $\{M_i\}$ be representants of isomorphic classes of its* *indecomposable modules. Then the Auslander algebra* $B = \mathrm{End}(\bigoplus M_n)$. *It has a following categorical property: the category* $\mathbf{AddFunct^{op}}(A)$ *of* K*-additive constravariant functors* $A - \mathbf{mod} \to K - \mathbf{mod}$ *is naturally equivalent to* $B - \mathbf{mod}$. *In other words, considering all categories from above as* $K - \mathbf{mod}$ *enriched, there is a bijection between enriched contravariant functors forgetting* A*-module structures and endomorphisms of the Auslander algebra of* A. *The isomorphism* $\mathbb{H}$ *takes an functor* $F : A - \mathbf{mod} \to K - \mathbf{mod}$ *to* B*-module* $\mathbb{H}(F)$ *- a vector space* $F(M)$ *with right multiplication*

$$v \cdot f = F(f)(v)$$

Example 436 (Specker group)**.** *There is a functor* $\mathbf{Top} \to \mathbf{Ab}$ *sending a space to its Specker group* $S(X)$ *of bounded not necessarily continuous functions* $X \to \mathbb{Z}$. *Note that* $S(X)$ *is always a free abelian group generated by points of* X. *Restricting ourselves to quasi-compact spaces, there is a natural transformation of functors* $\mathrm{Hom}(-, \mathbb{Z}) \Rightarrow S$, *realising* $\mathrm{Hom}(X, \mathbb{Z})$ *as a subgroup of the Specker group. It shows in particular that that* $\mathrm{Hom}(-, \mathbb{Z})$ *in that case is always free abelian.*

Example 437 (Burnside ring)**.** *Let* G *be a finite group. Isomorphism classes of finite* G*-sets form a commutative semiring* $M(G)$ *under disjoint union as* $+$ *and product as multiplication. Note that since every* G*-set is a disjoint union of orbits and every orbit is isomorphic to the coset* G/H *for some subgroup* $H \leq G$, *additively* $M(G)$ *is always a free abelian monoid* $\mathbb{N}^r$ *generated by cosets* G/H. *Moreover,* $G/H \simeq G/H'$ *as* G*-sets if and only if* H *and* H' *are conjugate, so* r *is a number of conjugacy classes of subgroups of* G. *The semi-ring completion of* $M(G)$ *is a commutative ring* $A(G)$ *called the Burnside ring of* G. *Moreover, the forgetful functor* G-$\mathbf{Fin} \to \mathbf{Fin}$ *induces a counting morphism* $M(G) \to \mathbb{N}$, *making* $A(G)$ *naturally augmented with* $\epsilon : A(G) \to \mathbb{Z}$.

There are two different functorial structures realising A. *Firstly, it can be constructed as a contravariant functor* $\mathbf{FinGrp}^{op} \to \mathbf{CRing}/\mathbb{Z}$ *by the* G*-structure on* $\varphi^*(X)$ *induced by pre-composition:* $g \cdot x := \varphi(g)x$. *Secondly, it can be constructed also in a covariant way, where a homomorphism* $\varphi : H \to G$ *induces takes a* H*-set* X *to* $\varphi_*(X) = G \times_H X = G \times X / \{(g, hx) \sim (g\varphi(h), x)\}$.

These two functors are connected by so called Frobenius reciprocity formula

$$\varphi_*(\varphi^*(X) \cdot Y) = X \cdot \varphi_*(Y)$$

Finally, let's calculate some simple example. Given a cyclic group of prime order $G = \mathbb{Z}/p\mathbb{Z}$, $A(G)$ *is generated by two elements* $[1], [G]$. *The multiplication has a form* $[G][G] = [G \times G] = p[G]$, *thus we end* *up with the isomorphism*

$$A(G) \simeq \frac{\mathbb{Z}[x]}{(x^2 - px)}$$

with the augmentation $\epsilon(x) = p$.

Example 438 (Representation ring)**.** *Let G be a finite group. Isomorphism classes of complex representations of G form an abelian semiring under direct sum and tensor product over $\mathbb{C}$. The theorem of Mashke implies that the group algebra $\mathbb{C}[G]$ is semisimple, thus the additive monoid of $\mathrm{Rep}_{\mathbb{C}}(G)$ is isomorphic to $\mathbb{N}^{cl(G)}$, where $cl(G)$ is the number of conjugacy classes of G. The semiring completion of $\mathrm{Rep}_{\mathbb{C}}(G)$ is called the representation ring $R(G)$. It's underlying abelian group is $\mathbb{Z}^{cl(G)}$.*

Note that this construction coincides with the ring of complex characters of G, thus can be realised as a subring of $\mathrm{Hom}(G, \mathbb{C})$.

Example 439 (K_0 of rings)**.** *Note that we can generalise the representation ring for more general class of algebras, not necessarily appearing as group rings. Since for finite groups and field of characteristic zero KG is a semisimple algebra, finite dimensional representations of G are finitely generated modules of KG, which are all projective. The right way of generalising this construction is to consider the completion of monoid $P(R)$ of classes of finitely generated projective modules of R. It is closed under binary direct sums and has identity 0. Moreover, if the ring is commutative, the tensor product of projective modules is still a projective module. This operation is distributive with respect to direct sums, thus we obtain a commutative ring structure on the completion, known as $K_0(R)$. Thus we obtain functors*

$$K_0 : \mathbf{Ring} \to \mathbf{Ab}$$

$$K_0 : \mathbf{CRing} \to \mathbf{CRing}$$

Example 440 (Partially ordered abelian group)**.** *A connected partially ordered abelian group is an abelian group A with submonoid P, such that*

- *P generates A*
- *$P \cap -P = \{0\}$*

Every such pair provides an partial order on A invariant under addition, derived as

$$a \geq b \text{ iff } a - b \in P$$

Conversely, every such A-invariant partial order gives raise to a submonoid $P = \{a \in A \mid a \geq 0\}$. However, in general (A, P) need not be *a connected partially ordered abelian group - it is true precisely when P generates A. Connected partially ordered abelian groups form naturally a category* **PoAb**.

The completion of abelian monoids $M^{-1}M$ often gives raise to a partially ordered abelian group $(M^{-1}M, M)$, however this construction fails in full generality. In particular, both Burnside rings and representation rings associated to a finite group always are partially ordered in this way, yielding three functors

$$\begin{aligned} A_{op} &: \mathbf{FinGrp}^{op} \to \mathbf{PoAb} \\ A &: \mathbf{FinGrp} \to \mathbf{PoAb} \\ R &: \mathbf{FinGrp} \to \mathbf{PoAb} \end{aligned}$$

Note that such a partial ordering indeed provide some significant extra structure on the completion. To see that in action, assuming $0 \in \mathbb{N}$ consider for example two abelian semirings $\mathbb{N}^2$ and $\{0,0\} \cup \mathbb{N}^2_+$. Both of them have the same completion $\mathbb{Z}^2$, however the induced partial ordered differs - in the first case we get $(0,1) \geq (0,0)$, which is not true in the second completion.

9. Examples from homological algebra

9.1. Abelian categories.

– TODO: exactness of colimits (Weibel 57) – – Mod op is not Mod –

Example 441 (Additive but not abelian category)**.** *The category* **AbF** *of filtered abelian groups is additive but not abelian. Its objects are chains of subgroups, and morphism is a morphism of groups, with compatible restrictions on the filtration.*

$$\begin{tikzcd} \cdots \arrow[r, hook] & A_2 \arrow[r, hook] \arrow[d, "\varphi_{|A_2}"] & A_1 \arrow[r, hook] \arrow[d, "\varphi_{|A_1}"] & A \arrow[d, "\varphi"] \\ \cdots \arrow[r, hook] & B_2 \arrow[r, hook] & B_1 \arrow[r, hook] & B \end{tikzcd}$$

The kernel and cokernels of morphisms in **AbF** *are some filtrations of kernel or cokernel of the "top map", thus the morphism identity morphism between on A with trivial and non-trivial filtration must have trivial kernel and cokernel, but is not an isomorphism.*

Example 442 (Topological abelian groups)**.** *The example of non-abelian but additive category are topological abelian groups. In a similar fashion to the previous case, the bad behaviour can be easily exposed by* *considering automorphisms of groups with different topology, for example identity on $\mathbb{R}$ with discrete and euclidean topology.*

Example 443 (Non-abelian category of modules)**.** *A category of finitely generated R-modules is not abelian if the ring R is not Noetherian. The problem lies in kernels: if R is not Noetherian, it must have some not finitely generated ideal $I \lhd R$. But both R and R/I are finitely generated abelian modules, thus the kernel of the quotient map $R \to R/I$ cannot be I. But since the inclusion of $\mathcal{C}$ into all $R-\mathbf{Mod}$ is fully faithful, it must be exact, what implies that the kernel of such a map does not exist.*

Example 444 (Injective sheaf)**.** *Consider the category $\mathbf{Shv}(X)$ of sheaves of abelian groups over X (or with values in any category with enough injectives). For any point $x \in X$ the sky-scrapper sheaf $\mathcal{F}_x$ given by*

$$\mathcal{F}_x(U) \begin{cases} \mathbb{Q}/\mathbb{Z} & x \in U \\ 0 & x \notin U \end{cases}$$

is injective. Moreover using similar argument as in the case of $\mathbb{Q}/\mathbb{Z}$ in $\mathbf{Ab}$, every sheaf has an injective map to the product of sky-scrapper sheaves, which shows that the category $\mathbf{Shv}(X)$ has enough injectives. It is a fundamental fact providing a construction of a sheaf cohomology as a derived functor to global sections.

Example 445 (Homology)**.** *For every abelian category $\mathcal{C}$, the category of chain complexes $\mathbf{Ch}(\mathcal{C})$ is also abelian. The nth homology of a complex, defined classically as $\ker\partial_n/\mathrm{im}\partial_{n+1}$, form an additive functor $\mathbf{Ch}(\mathcal{C}) \to \mathcal{C}$.*

Example 446 (Connecting homomorphism)**.** *The "short exact sequence of chain complexes induces long exact sequence of homology" principle can be formalised by considering the category of short exact sequences (as a full subcategory of $\mathbf{Ch}(\mathcal{C})$). Given a sequence of chain complexes*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

the connecting homomorphism from the induced sequence

$$\cdots \to H_{n+1}(B) \to H_{n+1}(C) \xrightarrow{\delta_n} H_n(A) \to H_n(B) \to \cdots$$

comes from the natural transformation of functors $H_{n+1}(C_\bullet) \Rightarrow H_n(A_\bullet)$

Example 447 (Semisimple categories)**.** *The idea of semisimple rings has a natural correspondence in abelian categories. Abelian category is called semisimple if every short exact sequence of its objects splits. An standard of an semisimple category are vector spaces over any field.*

On the other hand, the category of abelian groups is not semisimple, as well as most abelian categories often considered in practice. At the first glance one can hope to restate for example the Serre-Swan theorem in this language and say that the category of finite dimensional vector bundles over compact space is semisimple, as equivalent to the category of finitely generated projective modules over $C(X)$. Unfortunately, as discussed in earlier sections, such a category in general fails to have kernels, so is only additive (and as such is semisimple)

Example 448 (Maschke theorem)**.** *Using the notion of semisimple categories, one can reformulate categorically the Mascke theorem as stating that for any finite group G and a field of characteristic not dividing $|G|$, the category of finite dimensional linear* k*-representations of G is semisimple.*

9.2. Derived functors.

Example 449 (Tor)**.** *The tensor product $-\otimes_R M : R-\mathbf{Mod} \to \mathbf{Ab}$ is a right exact functor. Its left derived functor is $\mathrm{Tor}_*^R(-, M)$. It Tor_1 has a natural interpretation of measuring additional relations that are added to the module after changing its coefficients to from R to M, that do not follows from the relations that was already there. Take for example the group $\mathbb{Z}/4\mathbb{Z}$ as a $\mathbb{Z}$-module. Its relations, captured by its free resolution, are $\langle 4z = 0\rangle$. Now consider its tensor product $\mathbb{Z}/4\mathbb{Z} \otimes \mathbb{Z}/6\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}$. Now the relations extended to $< 2z = 0 >$. This additional restriction comes from relations of the new coefficients and is captured via their* Tor *group:* $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}/4\mathbb{Z}, \mathbb{Z}/6\mathbb{Z}) \simeq \mathbb{Z}/2\mathbb{Z}$*. On the other hand $\mathbb{Z}/4\mathbb{Z} \otimes \mathbb{Z}^2 \simeq \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$, where the new relations are $\langle 4z = 0, 4w = 0\rangle$ leads to trivial* Tor*, since even though there are some additional relations in the product, all of them are the consequence of previous relations appearing in $\mathbb{Z}/4\mathbb{Z}$, not caused by relations of new coefficients, coming from a free module, so not having any relations at all. In some cases relations of some non-free module does not lead to additional restrictions as new coefficients. Such modules are called flat, and are characterised by vanishing all the* Tor *groups. Example of such modules are localisations of R. The sense of this claim is not surprising from the perspective of suggested interpretation of* Tor *- changing coefficient of a module by inverting some of them clearly does not make any elements vanish, we would be rather willing to say that the new conditions are more relaxed.*

Example 450 (Ext)**.** *The right derived functor of the internal $Hom(A, -)$ or, equivalently, left derived functor of $Hom(-, A)$ (which is not an obvious fact!) is the functor $\mathrm{Ext^n_R(\text{-}, A)}$. Its first functor Ext^1 also comes* *with a natural interpretation, even more natural than* Tor_1*. Consider the following problem: given objects* A *and* C*, what are possible short exact sequences of a form (called extensions of* C *by* A*)?*

$$0 \to A \to ? \to C \to 0$$

Clearly always we have a trivial one, called a splitting

$$0 \to A \to A \oplus C \to C \to 0$$

Sometimes, however, there are more possibilities, such as the classical

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

while sometimes, for example for $A = C = \mathbb{Z}$ *(or in general whenever* C *is projective or* A *is injective), there are no other possibilities. The space of possible extensions of* C *by* A *is exactly captured in* $\mathrm{Ext}^1(C, A)$*. In fact this naturally provides us the abelian group structure on equivalence classes of extensions, however it can be also constructed more directly and even provide a way to slightly generalise the* Ext *itself, as it is shown in the next example.*

Example 451 (Yoneda Ext)**.** *Firstly let's take a closer look at the classes of extensions of* R*-module* C *by* A*, denoting these as* $\mathrm{ext}(C, A)$*. Two extensions are considered equivalent if there is a commutative diagram of a form*

$$\begin{tikzcd}
 & & B \arrow[dd] \arrow[dr] & & \\
0 \arrow[r] & A \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\
 & & B' \arrow[ur] & &
\end{tikzcd}$$

with both horizontal sequences exact. $\mathrm{ext(C,A)}$ *has a structure of abelian group with the addition given by the so called Baer sum, and with the split exact sequence serving as* 0*. The Baer sum is constructed by a pullback: given two extensions* E_1, E_2 *with middle space* B *and* B' *the extension* $E_1 + E_2$ *comes from*

$$\begin{tikzcd}
A \arrow[r] & A \oplus A \arrow[dr, dotted] \arrow[drr] \arrow[ddr] & & \\
 & & B'' \arrow[r, dotted] \arrow[d, dotted] & B' \arrow[d] \\
 & & B \arrow[r] & C
\end{tikzcd}$$

Where B'' is a pullback of the square and A is embedded in $A \oplus A$ via $x \mapsto (x, -x)$. This abelian group structure provide a way to talk about Ext_6 *functor even in some wild categories that do not have neither enough projectives nor injectives, thus the construction of derived functors is impossible in general. Due to the Freyd-Mitchell embedding theorem, providing an exact embedding of any abelian category in $R-\mathbf{Mod}$, the Ext^1 functor to abelian groups exists in every abelian category.*

Example 452 (Meaning of higher Ext functors)**.** *The analogy with extensions does generalize nicely also to all the higher opExts. Consider the group of equivalence classes of extensions (with analogous equivalence condition as in the first case) of a form*

$$0 \to A \to B_1 \to \cdots \to B_n \to C \to 0$$

The Baer sum of two extensions is represented by

$$0 \to A \to X \to B_2 \oplus B_2' \cdots \to B_{n-1} \oplus B_{n-1}' \to Y \to C \to 0$$

where X is a pushout of B_1 and B_2 under A quotiened out by the skew diagonal map $xmapsto(x, -x)$ and Y be the pullback of B_n and B_n' under C. Such a construction not only has a natural $1-1$ correspondence with the derived Ext *functors, but also provide the construction of Yoneda* Ext *functors for all n in every abelian category.*

9.3. **Simplicial methods.** Recall that the category of simplicial sets is a presheaf category $[\Delta^{op}, \mathbf{Set}]$, where Δ is just a poset of non-negative integers.

Example 453 (Singular set)**.** *A first example of a simplicial set is a singular set, which could be familiar if you've seen some algebraic topology, as it is used to define singular homology and cohomology. Let $|\Delta^n|$ denote the standard n-simplex with Euclidean topology, defined as a convex hull of a basis vectors of $\mathbb{R}^n$. Then the singular set is a presheaf*

$$SX = n \mapsto \mathrm{Hom}_{\mathbf{Top}}(|\Delta^n|, X)$$

You can think about a singular sets as grouping together all possible ways, in which some subspace of X can be presented as a simplex, so keeping track of all possible pairs of points, loops, and their higher-dimensional equivalents, that X can possibly have, all in a functorial way, meaning that all such "hyperloops" have naturally distinguished faces.

Example 454 (Presentation of Δ)**.** *In previous examples we've seen that every small category has a presentation, i.e. can be constructed as* *a coequaliser of free categories on some graphs. Such a presentation, as in the algebraic case, defines $\mathcal{C}$ with generators and relations. Generators of the category Δ comes from 2 distinguished families of maps, the cofaces $d^k : [n-1] \to [n]$ and codegeneracies $s^k : [n+1] \to [n]$. The relation satisfied by generators are called cosimplicial identities, and have a form*

$$\begin{aligned}
d^k d^m &= d^m d^{k-1} \quad \text{if } k < m \\
s^k d^m &= d^m s^{k-1} \quad \text{if } k < m \\
s^k d^k &= 1 \\
s^k d^{k+1} &= 1 \\
s^k d^m &= d^{m-1} s^k \quad \text{if } k > m+1 \\
s^k s^m &= s^m s^{k+1} \quad \text{if } k \leq m
\end{aligned}$$

The presentation of Δ has a form

$$F(6 \times \mathbb{N}) \underset{\mathbb{1}}{\overset{R}{\rightrightarrows}} F(2 \times \mathbb{N}) \longrightarrow \Delta$$

Example 455 (Simplicial abelian group)**.** *Given a simplicial set X_n, one can naturally construct its free simplicial abelian group $\mathbb{Z}X_n$, being an object of $\mathbf{sAb} = [\Delta^{op}, \mathbf{Ab}]$. To do so, it is enough to map each n to the free abelian group of X_n.*

Example 456 (Moore complex)**.** *For any simplicial abelian group $\mathbb{Z}X_n$ there is a natural construction of associated chain complex over $\mathbf{Ab}$, called the Moore complex. The construction has a form*

$$\cdots \longrightarrow \mathbb{Z}X_2 \xrightarrow{\partial} \mathbb{Z}X_1 \xrightarrow{\partial} \mathbb{Z}X_0$$

where the boundary map is the alternating sum of faces:

$$\partial(\sigma) = \sum_{i=0^n} (-1)^i d_i(\sigma)$$

Example 457 (Singular homology)**.** *The Moore complex provides a categorical construction of the singular homology of any topological space, which can be done by composing 3 previously considered functors*

$$\mathbf{sSet} \xrightarrow{\mathbb{Z}-} \mathbf{sAb} \xrightarrow{Moore} \mathbf{Ch_{Ab}} \xrightarrow{H_n} \mathbf{Ab}$$

Example 458 (Classifying space)**.** *Similarly as in case of topological spaces, also every small category has a standard construction as a simplicial set, called the classifying space $B\mathcal{C}$. The construction is also* *pretty similar to singular sets:*

$$B\mathcal{C}_n = \mathrm{Hom}_{\mathbf{Cat}}(\boldsymbol{n}, \mathcal{C})$$

where n *is the subcategory of* Δ *consisting of ordinals less or equal to* n*. The* n*-simplices of a classifying space can be described explicitly as chains of* n *compatible morphisms from* $\mathcal{C}$

$$a_1 \to a_2 \to \cdots \to a_n$$

Example 459 (Classifying spaces of groups)**.** *The classifying spaces of groups (considered as categories with single object)* BG *has a surprising property of being homotopy equivalent to Eilenberg-McLane spaces. Eilenberg-McLane space* $K(G, n)$ *is such a space (or rather a homotopy class of spaces, as these are mostly considered in the category* **Htop***) that its* n*-th homotopy group* $\mathrm{Hom}_{\mathbf{Htop}_*}(S^n, X)$ *is isomorphic to* G*, while all the other vanish. It can be shown, that every realisation of a classifying space* BG *is homotopic to the space* $K(G, 1)$*. Using this neat trick, one can reduce the homology of groups, considered in the category of its group algebras, to the singular homology of Eilenberg-McLane spaces, which is a really surprising connection between topology and group theory.*

Example 460 (Standard simplex)**.** *Finally, the simplices itself also have, unsurprisingly, a canonical representation as simplicial sets - these are just the representable presheaves* $\Delta^n := h_{[n]}$*.*

Example 461 (Simplex category and coYoneda lemma)**.** *Using standard simplices, the coYoneda lemma provides a nice construction of any simplicial set. The coYoneda lemma is a theorem in some sense dual to the Yoneda lemma, stating that every presheaf in* $[\mathcal{C}^{op}, \mathbf{Set}]$ *can be constructed as a colimit of representable functors. To apply this construction to simplicial sets, it is handy to firstly define the simplex category associated to* $X \in \mathbf{sSet}$*, which is just a comma category* $\Delta \downarrow X$*, having elements of the form*

$$\begin{tikzcd} \Delta^n \arrow[rr] \arrow[dr] & & \Delta^m \arrow[dl] \\ & X & \end{tikzcd}$$

The alternative, conceptually simpler description of simplex category is as the category of elements $\int X$*, which can be identified with a set of simplices with additionally specified face maps, which makes them not just a set of simplices freely floating in aether, but rather a set of simplices freely floating in aether, but with some construction plan. Any* *simplicial set X (as well as every presheaf in general) can be constructed as a colimit of natural projections of Δ^n's from $\Delta \downarrow X$ onto* **sSet**

$$X \simeq \varinjlim_{\Delta \downarrow X} \Delta^n$$

Example 462 (Boundary of a simplex)**.** *A boundary of a simplex $\partial\Delta^n$ can be canonically realised as a simplicial set*

$$\partial\Delta^n(k) = \{f \in \Delta^n \mid f \text{ is not surjective } \}$$

Example 463 (Horn)**.** *Another important subfunctor of Δ^n is the horn Λ_i^n corresponding to a simplex missing single face. Denoting the set corresponding to the i-th face as $[\hat{i}] = \{0, \cdots, i-1, i+1, \cdots, n\}$, the i-th horn can be constructed as*

$$\Lambda_i^n(k) = \{f \in \Delta^n \mid [\hat{i}] \notin \operatorname{im}(f)\}$$

Example 464 (Realisation)**.** *A slight modification of the construction considered previously provides a way of constructing a realisation, a functor $|\cdot| :$ **sSet** $\to$ **Top** left adjoint to the singular complex.*

$$|X| := \varinjlim_{\Delta \downarrow X} |\Delta^n|$$

where $|\Delta^n|$ is the standard topological simplex, considered already at the beginning of the section.

Example 465 (Kelley product)**.** *The realisation does not behave well with respect to some pathological topological spaces. The formula*

$$|X \times Y| \simeq |X| \times |Y|$$

making it commutative with finite products does not hold in general, but only in the category **CGH** *of compactly generated Hausdorff spaces. The categorical product in* **CGH** *differs from the product in* **Top***, and is called the Kelley product $X \times_{Ke} Y$. It forms a functor* **CGH** $\to$ **Top***, and the corrected formula*

$$|X \times Y| \simeq |X| \times_{Ke} |Y|$$

does indeed hold for all topological spaces, but since it is no longer the categorical product, it becomes pretty useless.

Example 466 (Simplicial complex)**.** *A simplicial complex is simpler, but confusingly similar object to a simplicial set. Consider some (gentle - face to face, vertex to vertex) construction made out of standard simplices in $\mathbb{R}^n$. It's easy to see that there is a lot of redundant information is such a construction - after labeling all the pieces and denoting which pairs of simplices where glued to each other by simple binary relation,* *we can delete all the topology and reconstruct homeomorphic space in some functorial way. Thus any such a space with some triangulation (built out from simplices) can be decoded in a purely combinatorial way as an element of a category which objects are abstract simplicial complexes: its object K consists of*

- *A set of vertices $V(K)$*
- *A set of simplices $S(K)$, where each simplex is determined by its vertices - some non-empty, finite subset of $V(K)$*

Moreover, it must satisfy two additional conditions

- *$S(K)$ must be closed under subsimplices - so if a simplex exists, so must all its faces*
- *Every vertex of K must be also a simplex.*

The main difference between simplicial sets and simplicial complexes is their directedness - while in the simplicial set there is a clear face map corresponding to inclusion, in case of simplicial complexes, only the fact that these two are connected is encoded. Such a subtle difference leads to surprisingly significantly larger capacity of storing structures of simplicial sets, which we will try to explore studying some more concrete examples.

Example 467 (Simplicial complex vs simplicial set)**.** *The first non-trivial example of structural difference between simplicial complexes and simplicial sets is the case of a single edge - the* 1*-simplex. The simplicial complex of this space has 2 vertices and one simplex connecting them, so can be visualised as a simple graph*

$$\begin{tikzcd} & \bullet \arrow[dl, dash] \arrow[dr, dash] & \\ \bullet & & \bullet \end{tikzcd}$$

On the other hand, the standard simplicial set Δ^1 is the representable functor $\mathrm{Hom}_\Delta(-,[1])$*. There are* 3 *possible maps to* $[1]$ *in* Δ*. We can represent them as a directed graph*

$$\begin{tikzcd} {[0]} \arrow[r, shift left] \arrow[r, shift right] & {[1]} \arrow[loop right] \end{tikzcd}$$

Example 468 (Ordering the complex)**.** *The undirectedness of simplicial complexes is in fact the only feature distinguishing them from simplicial sets. Every simplicial complex can be turned into simplicial set by choosing some total order on its vertices. Directing every arrow in a* *graph, makes automatically into a functor $\Delta^{op} \to \mathbf{Set}$, as it is equivalent to specify all the required face maps. However such an approach is very dirty, as there is no canonical or functorial way of picking such an order, which in categorical world is obviously absolutely unacceptable.*

Example 469 (Not ordering the complex). *There is another, also awful but at least functorial way of turning complex into a simplicial set. Instead of choosing some arbitrary ordering on simplices, we can just use all of them at once. To do so, to every n-simplex from a complex we associate $(n+1)!$ different simplices in our simplicial set, each with different possible configuration of face map directions. Obviously, computing such a monstrum in practice in almost always impossible, but due to naturality, but no one cares about practice in this field anyway.*

Example 470 (Complex from poset). *A flag in a poset P is just any finite chain of incidence relations*

$$x_0 < x_1 < \cdots < x_n$$

This notion provide a canonical way of turning any poset into a simplicial complex by taking the set itself as vertices, and as simplices - the set of all the flags existing in P. On the other hand, we have also the functor in the other direction, taking the simplices into a poset ordered by inclusion. Sadly, these two functors are not adjoint.

$$\mathbf{Poset} \underset{U}{\overset{\text{Flag}}{\rightleftarrows}} \mathbf{sComp}$$

Example 471 (Barycentric subdivision). *Barycentric subdivision is a way of constructing refinement of a complex, with preserving all its geometrical structure. The most common way is to do that via the barycentric subdivision. There are two, completely different constructions of such. First one is the brut-force solution, using the embedding of a simplex in euclidean space. The idea is to refine every Δ^n symmetrically into $n+1$ new simplices, cutting it through its center of mass, called barycenter. Barycenter of a simplex σ of vertices $(v_0, \cdots, v_n)$ is given by the formula*

$$g(\sigma) = \sum_{i=0}^{n} \frac{v_i}{n+1}$$

Collecting all the baricenters, we set them to be vertices of the subdivision $\mathrm{Sd}K$, while the new simplices are all non-empty collections of barycenters that can be assembled into a chain of face relations.

Happily, by exploiting further the last step we can construct a subdivision functor without touching any topology whatsoever. In fact, the steps we've done so far can be boiled down to following construction:

- $V(\mathrm{Sd}K) = S(K)$
- $S(\mathrm{Sd}K)$ *are all totally ordered chains of baricenters*

And now, recalling the functors considered in the previous example, the construction become almost trivial:

$$\mathrm{Sd}K = \mathrm{Flag} \circ U$$

Surprisingly, the construction of a subdivision of simplicial set is doable, but much harder. It provides a cool usage of Kan extensions, so you can find it in Chapter 10.2.

Example 472 (Nerve)**.** *Every small category can be realised as a simplicial set called its nerve. The nerve of $\mathcal{C}$ is defined as the functor*

$$N\mathcal{C}(n) = [\mathbf{n}, \mathcal{C}]$$

The elements of nerve has a pretty straightforward interpretation: $N\mathcal{C}_0$ corresponds to objects of $\mathcal{C}$, $N\mathcal{C}_1$ to arrows, $N\mathcal{C}_2$ to pairs of composable arrows and so on. The nerve functor is fully faithful, thus provides the embedding of full subcategory

$$\mathbf{Cat} \hookrightarrow \mathbf{sSet}$$

By the Yoneda lemma, the standard simplex Δ^n is isomorphic to $N\mathbf{n}$.

Example 473 (Čech nerve)**.** *A nerve is a way of turning a small category into simplicial set. A variation on this idea provides similar construction for topological spaces with some distinguished open cover $\mathcal{U}$. From such a cover we can form a simplicial set $\check{C}_\bullet(\mathcal{U})$, called the Čech nerve. Its points correspond to sets $U_i \in \mathcal{U}$. Every non-empty intersection $U_i \cap U_j$ establish an edge between vertices U_i, U_j, with face maps induced by inclusions. Similarly every n-fold intersection $U_{i_1} \cap \cdots \cap U_{i_n}$ form an n-simplex of $\check{C}_\bullet(\mathcal{U})$. The Čech nerve can be constructed also with an intermediate step - instead of working directly with simplices, we can use the covering $\mathcal{U}$ to form a small category from its poset of inclusions. Then the Čech nerve is nothing more than the nerve of such poset. Below we present a simple example of such diagram and* *its Hasse diagram*

$$\begin{tikzcd}
 & U_{123} \arrow[dl] \arrow[d] \arrow[dr] & \\
U_{12} \arrow[d] \arrow[dr] & U_{13} \arrow[dl] \arrow[dr] & U_{23} \arrow[dl] \arrow[d] \\
U_1 & U_2 & U_3
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \bullet \arrow[dl] \arrow[d] \arrow[dr] & \\
\bullet \arrow[d] \arrow[dr] & \bullet \arrow[dl] \arrow[dr] & \bullet \arrow[dl] \arrow[d] \\
\bullet & \bullet & \bullet
\end{tikzcd}$$

Example 474 (From space to nerve and back)**.** *The key theorem connecting algebraic topology with simplicial category theory is the so called nerve lemma, saying that each paracompact topological space can be constructed up to homotopy as realisation of a simplicial set via its Čech nerve. The construction heavily relies on the existence of a good cover, where all the sets U_i and their finite intersections are contractible.*

The key in the proof of nerve lemma lies in a cute observation that in case of any good cover, up to weak homotopy equivalence the sets $U_{i_1,\cdots,i_n}$ represents n-simplices of some simplicial complex, and the colimit of such diagram is exactly its geometric realisation. Moreover, the nerve of such incidence diagram of any simplicial complex is canonically isomorphic to its barycentric subdivision. Thus, up to weak equivalence, every paracompact space is a realisation of its nerve with respect to any good cover.

Note that this correspondence fails if the cover is not good - it fails in fact on the most trivial example, where X is not contractible and $\mathcal{U} = \{X\}$, as realisation of such Čech nerve is always just a single point.

9.4. **Cubical sets.**

Example 475 (Pre-cubical sets)**.** *With only a bit more effort, we can build sets of a cubical nature instead of triangular analogously to simplicial sets. A cubical set is a sequence of sets $K[n]$, this time with two face maps d_n^0 and d_n^1 satisfying similar relations as in the simplicial case. Those two face maps adds additional space for choosing one of two possible faces of a cube on given coordinate, instead of a single face as in the simplex. Pre-cubical sets form a category* $\square$**Set** *with functions commuting with face maps. Not every pre-cubical set in fact corresponds to a geometric construction we have in mind, since the faces could happen to be doubled. The most interesting for us are those with no such anomalies, called proper. A cubical set is proper if it is uniquely determined by extremal vertices of its geometric realisation.*

More intristicaly, it is equivalent to a condition that the map

$$\coprod_{n=0}^{\infty} K[n] \to 2^{K[0]}$$

mapping a cube to its vertices is injective, or more functorially, if every its face maps uniquely corresponds to choosing a subset of vertices of a cube together with their directions.

Example 476 (Non-looping length covering)**.** *Given a pre-cubical set, there is a functorial construction untying it a little bit, however paying the price of making it larger (in particular infinite!). It is an endofunctor* $\square\mathbf{Set} \to \square\mathbf{Set}$

$$\widetilde{K}[n] = K[n] \times \mathbb{Z}$$

where the face maps sends the "floor changing" arrows to the other layer of a complex:

$$d_k^{\epsilon}(x, m) = (d_k^{\epsilon}(c), m + \epsilon)$$

Example 477 (Cubical chains)**.** *Let* K *be a* $\square$*-set. A cube chain on* K *from* v *to* w *is a sequence of cubes* $(c_0, \cdots, c_k)$ *where* $c_k \in K[n_k]$ *such that:*

- $d^0(c_1) = v$
- $d^1(c_k) = w$
- $d^1(c_i) = d_0(c_{i+1})$

Moreover, with every such a chain we can associate:

- ***type****:* $(n_1, \cdots, n_k)$*; decoding its shape as a sequence of dimensions of all the cubes*
- ***dimension****:* $\sum(n_k - 1)$*; decoding the number of parameters needed for its description*
- ***length*** $\sum n_k$*; total number of edges needed to cross from* v *to* w

There is a neat way of categorifying the cube chains: if we contract any cube in a cubechain, we obtain a cubechain that is thinner, but longer, as the contracted cube will split into two degenerated ones. We can do that by choosing an index of this cube $i \in \{1, \cdots, k\}$ *and any subset* $A \subseteq \{1, \cdots, n_i\}$*. Denoting its complement as* A'*, the contraction can be constructed explicit as a cube chain*

$$(c_1, \cdots, c_{i-1}, d_A^0(c_i), d_{A'}^1(c_i), c_{i+1}, \cdots, c_k)$$

Such a contraction operation defines a partial order on a set $\mathrm{Ch}(K)$ *of cubical chains, which naturally form a category. The cool fact about cubical chains is that they decode nicely the homotopy type of directed* *path space of a complex, which we will see in a moment, after we discuss the construction of directed paths themselves.*

Example 478 (Geometric realisation of a pre-cubical set)**.** *In case of pre-cubical sets, we can consider eometric realisations on a similar fashion to simplicial sets, defining it as a colimit of canonical euclidean cubes I^n*

$$|K| = \left(\coprod_{n=0}^{\infty} K[n] \times I^n \right) / ((d_k^\epsilon(c), x) \sim (c, \delta_k^\epsilon(x)))$$

where $\delta_k^\epsilon(s_1, \cdots, s_{n-1}) = (s_1, \cdots, s_{i-1}, \epsilon, \cdots, s_{n-1})$

Example 479 (Geometric realisation of a pre-cubical set)**.** *In case of pre-cubical sets, we can consider eometric realisations on a similar fashion to simplicial sets, defining it as a colimit of canonical euclidean cubes I^n*

$$|K| = \left(\coprod_{n=0}^{\infty} K[n] \times I^n \right) / ((d_k^\epsilon(c), x) \sim (c, \delta_k^\epsilon(x)))$$

where $\delta_k^\epsilon(s_1, \cdots, s_{n-1}) = (s_1, \cdots, s_{i-1}, \epsilon, \cdots, s_{n-1})$

10. Examples from probability theory

Example 480 (Distribution monad)**.** *Another monad $\mathbb{D}$ in* **Set** *is a distribution monad, assigning a set to its distribution with finite support* $\mathbb{D}(X) = \{p : X \to I \mid \mathrm{supp}(p)$ *is finite and* $\sum_{x \in X} p(x) = 1\}$*, with unit given by indicator* $\eta_X(x)(y) = \mathbb{1}_x(y)$ *and multiplication evaluating the formal sum, multiplying the coefficients-distribution by a convolution (which should be pretty natural after one recalls that the sum of distribution corresponds to convolution on its densities).*

Example 481 (Giry monad)**.** *This example comes from the probability theory, providing a few elegant categorical constructions in this area, more of which can be found in the next subsection. Given any measure, the measurable spaces together with some σ-algebra form a category* **Meas** *of measurable spaces, with morphisms being measurable maps. The Giry monad G sends the space X to* Prob(X)*, space of all possible probability measures on X. From such a set of probability measures on X there is a natural evaluation map $\epsilon_U : G(X) \to I$ for any open subset $U \subseteq X$. We can use these family of maps to generate the σ-algebra on $G(X)$ in a functorial way (which means that we choose the smallest σ-algebra on which all these evaluation maps are measurable). The unit of a monad $\eta : X \to G(X)$ associate the point to its Dirac δ-measure. The cutest part is the product, merges a set of measures into a single* *measure, that can be thought as their mean. In the analythical language it can be formalised via integration:*

$$\mu_X(P)(U) = \int_{p\in G(X)} \mathrm{ev}_U(q)dP$$

—- TODO: Kantorovich monad, Radon monad ——

Example 482 (Extended probabilistic powerdomain)**.** *A valuation of a distributive lattice L with bottom element 0 is a function $\nu L \to [0,\infty]$ satisfying*

- *(monotonicity) $x > y$ implies $\nu(x) > \nu(y)$*
- *(unitality) $\nu(0) = 0$*
- *(modularity) $\nu(x) + \nu(y) = \nu(x \vee y) + \nu(x \wedge y)$*

Since every every topology is a distributive lattice of open subsets, we may consider valuations on topological spaces. Moreover, valuation is called ***continuous*** *if it commutes with supremum operation on bounded sets and* ***extendable*** *if it can be extended to a Borel measure. The set of extendable continuous valuations on a X, denoted as VX, has a natural topology generated by the sets*

$$B(U,\alpha) = \{\nu : \nu(U) > \alpha\}$$

It makes V to an endofunctor of **Top** *with induced maps being naturally defined pullback valuations. Moreover, the operation of integral over a valuation, similar as in Giry monad defines the multiplication $E : VVX \Rightarrow VX$ as*

$$E_\nu(\eta)(U) = \int_X \nu(U)d\eta(\nu)$$

Together with Dirac δ distribution, it makes V into a monad (V,δ,E) on **Top**.

11. Sheaves

11.1. **Sheaves and bundles.** We've previously seen some example coming from sheaf theory, but only as some illustration. In this section I tried to gather some examples illustrating some aspects of sheaf theory itself.

Example 483 (Stalk and germ)**.** *Recall that the germ of functions at x is classically as an equivalence class of functions that coincide at some neighbourhood of x. This term has more elegant definition via sheaves - given any sheaf $\mathcal{O}$, the stalk at p, denoted as $\mathcal{O}_p$ (or $\mathcal{O}_{X,p}$, where we want to specify the space) is defined as*

$$\mathcal{O}_p = \varinjlim \Gamma(U_p, \mathcal{O})$$

where the colimit varies over all the neighbourhoods of x and is taken in the category $\mathbf{Top}(X)$*. The germs are just elements of the stalk* $\mathcal{O}_p$*.*

Example 484 (Bundle)**.** *A bundles over a set X (not necessarily locally trivial) are just all the maps to X, so elements from* $\mathbf{Set}/X$*. For X with a discrete topology, these are equivalent to* $\mathbf{Shv}(X)$*, where a map $Y \to X$ is associated with its sheaf of sections.*

Example 485 (Sheaf of holomorphic functions)**.** *Given a Riemann surface X (complex curve, for example the complex plane $\mathbb{C}$ or the Riemann sphere $S^2 \simeq \mathbb{C}P^1 \simeq \mathcal{C} \cup \{\infty\}$), $\mathcal{O}_X$ is the sheaf of holomorphic functions on X. Let's take a look at the case of the Riemann sphere and denote $\mathcal{O} = \mathcal{O}_X$. As a projective space, it is a compact manifold, so by Louisville we know that the only holomorphic functions on X are constant*

$$\Gamma(X, \mathcal{O}) \simeq \mathbb{C}$$

On the other hand, for any point $x \in X$, we have

$$\Gamma(X \setminus \{x\}, \mathcal{O}) \simeq \Gamma(\mathbb{C}, \mathcal{O}_{\mathbb{C}}) = \left\{ \sum^{\infty} a_n x^n \mid a_n = o(r^n) \textit{ for all } r > 0 \right\}$$

by the Taylor series convergence theorem.

Example 486 (Constant functions)**.** *The presheaf of constant functions on a manifold is not a sheaf - for example for $M = \mathbb{R}$ the sections $1 \in \Gamma((-1, 0), \mathcal{O})$ and $2 \in \Gamma((0, 1), \mathcal{O})$ cannot be glued to any constant function on $(-1, 0) \cup (0, 1)$. The sheafification of this presheaf is a sheaf of locally constant functions, where such a problem is resolved by the definition.*

Example 487 (Integrable functions)**.** *Integrable continuous functions on $\mathbb{R}$ naturally form a presheaf $\mathcal{F}$, but not a sheaf, since integrability is a global condition, so the gluing axiom is not satisfied. To construct some explicit counter-example lets cover $\mathbb{R}$ with the intervals $\{(n, n+2)\}$ and consider the function constantly equal 1 on each of them. Each of them is a section from $\Gamma(U_i, \mathcal{F})$ and all of them are obviously equal on intersections, but there is no global section restricting to all of them, as $f = 1$ is not integrable. The sheafification of of $\mathcal{F}$ is just a sheaf of continuous functions on M, since every continuous function is locally integrable.*

Example 488 (L^1 functions)**.** *The case of measurable integrable functions is similar to the previous one in a sense that it does form a presheaf which is not a sheaf. However in this case the sheafification produce something new - the sheaf of locally L^1 functions, which is different from the sheaf of measurable functions, that we've started with.*

Example 489 (Holomorphic functions not only on manifolds)**.** *Let choose some subset of $\mathbb{C}^n$, for example $X = \{(x, y) \mid xy = 0\}$. It does not form a manifold, since it is not diffeomorphic to $\mathbb{C}^2$ in the neighbourhood of 0. However, we can still consider holomorphic functions at X by considering the sheaf $\mathcal{O}_X$, such that $\mathcal{O}_X(U)$ consist of restrictions of holomorphic functions from all the open subsets of $\mathbb{C}^2$ containing X.*

Example 490 (Solutions to linear differential equation)**.** *Let's take again at the sheaf of holomorphic functions, now on a set $X = \mathbb{C} \setminus \{0\}$. Given any linear differential equation $L(f) = 0$, it defines a map of sheaves, which kernel $\mathcal{S}$ is the sheaf of solutions to the equation*

$$0 \to \mathcal{S} \to \mathcal{O} \xrightarrow{L} \mathcal{O} \to 0$$

As an example let's take a look at the equation

$$L(f) = \frac{d^2 f}{dz^2} + \frac{df}{zdz}$$

The solutions to this equation are functions $c_1 \log(z) + c_2$, where any branch of the logarithm does the job. The logarithm appearing in the solution make it a very interesting sheaf, as at any convex open set we have

$$\Gamma(U, \mathcal{S}) \simeq \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{Z}$$

but if the set is not contractible, the only sections are constant functions.

Example 491 (Modern definition of a manifolds)**.** *Classically topological/smooth/analytic manifolds were defined using atlases. Sheaf theory offers a different approach, using models. Let's stick to the smooth case, as the other are analogous. The model of C^∞ is a family of sheaves of smooth functions on open subsets of $\mathbb{R}^n$. Given some Hausdorff, second-countable locally ringed space $(M, \mathcal{O}_M)$, we say that M is a smooth manifold if $(M, \mathcal{O}_M)$ is locally isomorphic (as a ringed space) to some sheaf from the model, so explicitly: every point $x \in M$ has some open neighbourhood U, such that the ringed space $(U, \mathcal{O}_M|U)$ is isomorphic to some $(V, \mathcal{O})$ from the model of C^∞.*

Example 492 (Category of manifolds)**.** *We've seen that manifolds can be constructed as locally ringed spaces, so the natural question: can we consider the category* **Diff** *as a full subcategory of* **LRS***, or do we have some weird additional maps? The answer is: there are, but only if morphisms are not morphisms of $\mathbb{R}$-algebras. To see that, we can use the following trick: if morphism θ from $(\varphi, \theta) : (M, \mathcal{O}_M) \to (N, \mathcal{O}_N)$ is a morphism of $\mathbb{R}$-algebras on all the open subsets, then it is identity on constant functions. Denoting $y = \varphi(x)$, given any $f : U \to \mathbb{R}$, if*

$f(y) = 0$, then we obviously have $\theta_y(f) = 0$. Thus $0 = \theta_y(f - f(y))$, so the value $\theta_y(f)$ is uniquely determined by f. However, if we do not require θ to be the identity on constant functions, we get some additional maps????

Example 493 (Tangent space without an atlas)**.** *Consider some smooth manifold M. The tangent space at a point $x \in M$ is a vector space spanned by tangent vectors to M at x, which correspond to all possible gradients of smooth functions at x. Classically to calculate such a thing we need to differentiate f in the local coordinates, but there is a neat way to get rid of the local coordinates - we can notice that the directional derivative of a function is exactly some linear operator δ satisfying the Leibniz rule*

$$\delta(fg) = \delta(f)g(x) + f(x)\delta(g)$$

Example 494 (Contangent space)**.** *We can take the previous abstraction much further. Consider some locally ringed space $(M, \mathcal{O})$, for a moment supposing it is a smooth manifold. When m is a maximal ideal of a stalk $\mathcal{O}_x$, the ring $\mathcal{O}_x/\mathrm{m}^2$ splits as $\mathbb{R} \oplus \mathrm{m}/\mathrm{m}^2$ via $f \mapsto (f(x), f - f(x) + \mathrm{m}^2)$ - it is just a Taylor expansion with linear term and the Peano reminder. From such an perspective you will probably believe that mapping f to the second term $f - f(x) + \mathrm{m}^2$ is a linear derivation, which constitute the isomorphism*

$$TM_x \simeq \mathrm{Hom}(\mathrm{m}/\mathrm{m}^2, \mathbb{R})$$

It allows us to construct canonically the cotangent space as $T^*M_x = \mathrm{m}/\mathrm{m}^2$ for any locally ringed space, and the tangent space as its dual.

Example 495 (Real functions)**.** *Consider some topological space X and its trivial $\mathbb{R}$-bundle $\mathbb{R} \times X \to X$. Then the real valued functions on X correspond exactly to the global sections $\Gamma(X, X \times \mathbb{R})$. In general for any topological ring R the global sections of the trivial bundle $\Gamma(X, R \times X)$ is the ring of R-valued functions. It works equally well for topological groups or just topological spaces, but the bundle must be considered as a sheaf in a suitable category of groups or sets.*

Example 496 (Orientation bundle)**.** *For every differentiable manifold M, we can construct a bundle of its local orientations (it can be done also for topological manifolds using some homological methods). Since locally every set has exactly two orientations, such a bundle is in fact always a double cover. The interesting feature of the orientation bundle is that it recovers the orientability of the manifold - the orientation bundle is trivial (homeomorphic to disjoint union) if and only if M is orientable. Such an interpretation of orientability is probably the closest* *possible to the intuitive picture that everyone has in mind while talking about it, that an orientable manifold has 2 sides, while non-orientable only one, as sides of a manifold are exactly the connected components of orientation bundle,*

Example 497 (Vector fields)**.** *Given a smooth manifold M, we can consider naturally its tangent bundle $TM \to M$, structured by the derivative of its local parametrization. The global sections $\Gamma(M, TM)$ is exactly the space of vector fields on M.*

Example 498 (Differential 1-forms)**.** *Dually to the construction of vector fields, one can construct the space of differential 1-forms by considering the global sections of the cotangent bundle $\Gamma(M, T^*M)$.*

Example 499 (Bundle of Riemann surfaces)**.** *Consider a sheaf of analytic functions on the Riemann sphere $\bar{\mathbb{C}}$. We can topologize such a bundle via its Etalé space, but also the explicit construction: $\mathcal{F}$ is a set containing points*

$$E = \{(z_0, f) \in \mathbb{C} \times \mathbb{C}[[z - z_0]] \mid f \text{ converges in some neighbourhood of } z_0\} \cup \{(\infty, f) \mid f \in \mathbb{C}[[z]], f \text{ converges for all } |z| > R \text{ for some } R > 0\}$$

The topology on E is given by a basis consisting all the germs of functions. Considering such a bundle we can in fact construct all the Riemann surfaces. Given any section $f \in \Gamma(U, E)$, we associate to f its Riemann surface as a maximal path connected component of E, containing $\mathrm{im}k$*. For example taking any complex logarithm on some convex open subset of $\mathbb{C}$, its associated Riemann surface is the logarithmic spiral.*

Example 500 (Cokernels differ in **Shv** and **PShv**)**.** *Let's again consider $\mathcal{O} = \mathcal{O}_{\mathbb{C}\setminus\{0\}}$. The differentiation of a function form a map of sheaves*

$$\mathcal{O}_X \xrightarrow{d/dz} \mathcal{O}_X$$

This map is an isomorphism on every simply connected region. Since every point has a convex neighbourhood, it is isomorphism on all stalks. However, on $U = \mathcal{C} \setminus \{0\}$ the function $f(z) = \frac{1}{z}$ is not in the image, so the cokernel taken in the category of presheaves is not a sheaf itself.

Example 501 (Borel's Theorem)**.** *In the course of real or complex analysis you could have seen the Borel's theorem, stating that for every sequence (a_n), there exists an analytic function on the neighbourhood of 0 having the Taylor series $\sum^\infty a_n x^n$. In the context of sheaves it means that every stalk of the sheaf of analytic functions on $\mathbb{R}$ is isomorphic to $\mathbb{R}[[x]]$ as a ring. This statement fails for the sheaf of smooth functions,* *as there exist some smooth (but not analytic) functions which are not equal to its Taylor series at any neighbourhood of* 0. *The most classical example of such a function is*

$$f(x)\begin{cases} 0 & x=0 \\ e^{-1/x^2} & x\neq 0 \end{cases}$$

Example 502 (Looking at stalks is not enough). *Consider a sphere* $X = S^2$ *and some point* $x_0 \in S^2$. *Let* $U = S^2 \setminus \{x_0\} \xrightarrow{i} X$. *By adding together the sheaf* $_!i\mathbb{Z}_X$, *which sections are constant functions vanishing at* x_0 *and the skyscrapper sheaf* $\mathrm{sk_{x_0}}\mathbb{Z}$, *we obtain a sheaf*

$$\mathcal{F} = {}_!i\mathbb{Z}_X \oplus \mathrm{sk_{x_0}}\mathbb{Z}$$

Even though every stalk of $\mathcal{F}$ *is isomorphic to* $\mathbb{Z}$, *the sheaf is far from constant - in fact there are is no even a single morphism from* $\mathcal{F}$ *to the constant sheaf. To see why, consider some two connected open sets* U, V *with* $x_0 \in U$, $x_0 \notin V$ *and* $U \cap V$ *connected and non-empty. Clearly we have* $\mathcal{F}(U) \simeq \mathcal{F}(V) \simeq \mathcal{F}(U\cap V) \simeq \mathbb{Z}$. *However, the section generating* $\mathcal{F}(U)$ *comes from the skyscrapper, while generator of* $\mathcal{F}(V)$ *lies in extension by zero. Suppose that there is a nonzero morphism of sheaves*

$$\mathcal{F} \to \mathbb{Z}_X$$

Then the diagram

$$\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "0"] \arrow[d] & \mathcal{F}(U\cap V) \arrow[d] & \mathcal{F}(V) \arrow[l, "1"'] \arrow[d] \\
\mathbb{Z}_X(U) \arrow[r, "1"] & \mathbb{Z}_X(U\cap V) & \mathbb{Z}_X(V) \arrow[l, "1"']
\end{tikzcd}$$

should commute, which is clearly not the case due to the observation made about generating sections.

Example 503 (Local system). *Consider connected and locally contractible pointed topological space* (X, x_0) *and its universal cover* $\tilde{X} \to X$. *Since the fundamental group* $\pi_1 := \pi_1(X, x_0)$ *acts freely on* $\tilde{X}$, *it can be used to locally twist a space, creating non-trivial bundle. To get also some interesting fiber, consider some group representation* $\rho : \pi_1 \to \mathrm{Aut}(M)$ *for some module* M. *As now we have two independent* π_1 *actions on the product* $\tilde{X} \times M$, *the most natural way of connecting them together to get a real twist is by simply taking the pullback of them* π_1*-spaces* $\tilde{X} \times_{\pi_1} M$, *which explicitly is expressed as quotient under the equivalence relation*

$$(x \cdot g, m) \simeq (x, \rho(g) \cdot m)$$

The resulting bundle $\mathcal{L} = \tilde{X} \times_{\pi_1} M \to X$ form a locally constant sheaf under sections. It's local triviality follows from local contractibility of X, as on each contractible open subset the bundle is trivial, as all the fiber bundles with contractible bases. In fact the other implication also does hold, as all the locally trivial bundles over connected spaces can be constructed this way.

Example 504 (Homology with local coefficients)**.** *Given a local system $\mathcal{L}$ on simplicial complex X (or just a triangulated topological space), the simplicial chain complex $C_n(X, \mathcal{L})$ can be constructed with coefficients with $\mathcal{L}$ instead of classically considered constant module. Elements of $C_n(X, \mathcal{L})$ are, as always, formal sums of a form*

$$\sum m_i \sigma_i$$

where σ_i are n-simplices of X and m_i is an element of some stalk $\mathcal{L}_{x_i}$ with $x_i \in \sigma_i$. The rest of the construction is exactly identical to the standard simplicial homology, however such a twisted version can have some additional nice features impossible to get with the classical homology, which we'll explore with more depth in a moment.

Example 505 (Simplicial sheaf)**.** *A quite easy to visualise while remaining rich structure example of a sheaf is the simplicial sheaf associated to some simplicial complex K. Given any n-simplex $\sigma \in K_n$, one can naturally construct its open star $\mathrm{St}(\sigma)$, playing a role of its interior (however, it's not an interior in the subspace topology on K - considering such one, all simplices of dimension less than K would have an empty star). The simplicial sheaf $\mathcal{S}$. Since open stars of all simplices of K form a basis closed under intersections, they can be used to define a sheaf on the entire space K by assigning to each star some abelian group associated to σ.*

$$\mathcal{S}(\mathrm{St}(\sigma) = S(\sigma)$$

The choice of abelian groups of course cannot be completely random, as to get finally an actual sheaf one must specify restrictions $S(\sigma) \to S(\tau)$ for any subsimplex τ of σ, which induced by boundary maps makes $\S$ to be a proper functor. Simplicial sheafs on a fixed simplicial complex form an abelian category.

Example 506 (Simplicial sheaf cohomology)**.** *The construction of the simplicial sheaf on finite simplicial complex naturally provides a tool to calculate its homology, however one annoying additional has to be done first, which is choosing the orientation of simplices (equivalently, choosing an ordering of vertices). As usual, there is no functorial way* *to do that. More about this problem and possible solutions you can find in the section about simplicial sets. After choosing the ordering, we may consider cochain complex associated to § freely generated by functions F, which associate simplices σ with some elements of $S(\sigma)$. The boundary maps are constructed as in the classical case.*

Example 507 (Flabby sheaf)**.** *The sheaf over X is called flabby (or flasque) if every local section extends to a global section, or that every restriction*

$$\Gamma(X, \mathcal{O}) \twoheadrightarrow \Gamma(U, \mathcal{O})$$

is a surjection for all open sets U. The standard example of a flabby sheaf is a sheaf of sections of bundle (or any surjective function), where we do not require the section to be continuous.

Example 508 (Not flabby sum of flabby sheaves)**.** *Consider a space $X = \{0, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\} \subset \mathbb{R}$ with a subspace topology and a sheaf $\mathcal{F}$ assigning to every U the set of its subspaces. Such a sheaf can be alternatively seen as the sheaf of functions (not necessarily continuous) $U \to \{0, 1\}$. Such a sheaf is obviously flabby. But let's consider its infinite direct sum $\mathcal{F}^{\oplus} = \bigoplus^{\infty} \mathcal{F}$ and the discrete open subset $U = \{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$. The family of sections $s_k = \{\{\frac{1}{k}\}\} \in \Gamma(U, \mathcal{F})$ assembles into the section $s = (s_k)_{k \in \mathbb{N}} \in \Gamma(U, \mathcal{F}^{\oplus})$, since each point $\frac{1}{n}$ has a neighbourhood, where the restriction of s has only one non-zero summand s_n. But such a section does not extend to a global section, since this property does not hold in the neighbourhood of* 0*.*

Example 509 (Soft sheaf)**.** *The sheaf over a paracompact space X is soft if every section over a closed set X extends to some global section. Such a property, similar to this from the definition of flabby sheaves, is in fact strictly weaker (every flabby sheaf is soft). The standard example of a soft sheaf is the sheaf of continuous or smooth functions on a manifold.*

Example 510 (Sheaf of smooth functions is soft)**.** *The sheaf of smooth functions on any manifold is soft, since the bounded function from any subset can be extended to some globally smooth function.*

Example 511 (Sheaf of holomorphic functions is not soft)**.** *Even though the sheaf of smooth functions is soft, this property fails for holomorphic or analytic functions. Since the analytic continuation of a holomorphic function is unique, we can produce a lot of examples, for example the Riemann ζ, the Γ function, logarithm of even simple $\frac{1}{z}$.*

Example 512 (Fine sheaf)**.** *The notion of a fine sheaf can be understood as the sheaf-theoretic construction of a partition of unity with* *respect to given sheaf. We say that the sheaf $\mathcal{F}$ over a paracompact space is fine, when for every two disjoint closed sets D_1, D_2 there is such an automorphism φ of $\mathcal{F}$, that is 0 on some neighbourhood of D_1 and identity on some neighbourhood of D_2. Every fine sheaf is soft, but the other implication does not hold.*

Example 513 (Soft sheaf that is not fine)**.** *Consider $\mathbb{C}$ with the Zariski topology, so where the closed sets are all finite ones. Then any non-trivial constant sheaf of abelian groups is soft but not flabby.*

Example 514 (Direct and inverse image)**.** *Every map $f : X \to Y$ naturally induces the direct image functor $f^\bullet : \mathbf{Shv}_Y \to \mathbf{Shv}_X$. The sheaf $f^\bullet(\mathcal{F}_Y)$ is given as by the formula*

$$f^\bullet(\mathcal{F}_Y)(U) = \mathcal{F}(f^{-1}(U))$$

The direct image functor has a left adjoint functor - the inverse image $f_\bullet$. To since the map f need not to be open, to define a functor in the covariant direction we need —– TODO ——

Example 515 (Global sections as direct image)**.** *The direct sections functor Γ can be constructed as a direct image associated to the map $X \to \bullet$, since $f^\bullet(\bullet) = \mathcal{F}(f^{-1}(\bullet)) = \mathcal{F}(X) = \Gamma(X, \mathcal{F})$.*

Example 516 (Sheafification as inverse image)**.** *The inverse image always has a structure of a sheaf, even evaluated at arbitrary presheaves. In particular, the formula defining the inverse image of an identity map $X \to X$ exactly coincides with the sheafification construction, as we get $\mathbb{1}^\bullet(U_x) = \varinjlim \mathcal{F}(U_x) = Shv(\mathcal{F})_x$*

Example 517 (Inverse image of a module)**.** *Given a map of ringed spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ and $\mathcal{O}_X$-Module $\mathcal{F}$, the inverse image functor $f^\bullet(\mathcal{F})$ fails to have a structure of a $\mathcal{O}_Y$-module, as even though have a natural map $f^\bullet\mathcal{O}_X \to \mathcal{O}_Y$, $f^\bullet$ is still carrying the structure of $f^\bullet\mathcal{O}_X$-Module. Given a previous map it's pretty straightforward how to fix such a problem, as we can just use to change the coefficients of the module. This defines the "enhanced inverse image" as*

$$f^*\mathcal{F} = \mathcal{O}_Y \otimes_{f^\bullet\mathcal{O}_X} f^\bullet\mathcal{F}$$

The only drawback is that such a functor loses the exactness property, and is only right exact, as opposed to $f^\bullet$ in the sheaf of abelian groups. This lose is obviously a direct consequence of composition with the tensor, which is right exact itself, so we can expect f^* to have a similar behaviour, in particular that its acyclic objects are flat ones.

Example 518 (Sheaf classifier)**.** *A category of sheaves of sets $\mathbf{Shv}(X)$ has subobject classifier Ω. For any open subset $U \subset X$, $\Omega(U)$ is just a* *set of all open subsets of U. Using such a sheaf we can construct every subsheaf via the pullback*

$$\begin{tikzcd} \mathcal{S} \arrow[r, dotted] \arrow[d, dotted] & 1 \arrow[d, hook] \\ \mathcal{F} \arrow[r, "\varphi"] & \Omega \end{tikzcd}$$

The map $\varphi_U(x)$ assign to x a maximal open subset, in which the restriction belongs to $\mathcal{S}(U)$. Obviously such a maximal element exists, as it is expressed by the union of open subsets. This way we can identify $\mathcal{S}(U)$ with a subset of $\mathcal{F}(U)$, consisting all the points x for which $\varphi_U(x) = U$ (so $\mathcal{S}(U) = \varphi_U^{-1}(U)$).

Example 519 (Sheaf classifier)**.** *A category of sheaves of sets $\mathbf{Shv}(X)$ has subobject classifier Ω. For any open subset $U \subset X$, $\Omega(U)$ is just a set of all open subsets of U. Using such a sheaf we can construct every subsheaf via the pullback*

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The map $\varphi_U(x)$ assign to x a maximal open subset, in which the restriction belongs to $\mathcal{S}(U)$. Obviously such a maximal element exists, as it is expressed by the union of open subsets. This way we can identify $\mathcal{S}(U)$ with a subset of $\mathcal{F}(U)$, consisting all the points x for which $\varphi_U(x) = U$ (so $\mathcal{S}(U) = \varphi_U^{-1}(U)$).

Example 520 (Etalé spaces)**.** *As already mentioned in examples of adjoints, the category $\mathbf{Shv}(X)$ is equivalent to category $\mathbf{Etal\acute{e}}(X)$, which is a full subcategory of $\mathbf{Top}/X$, consisting of all the local homeomorphisms. This correspondence produce in fact 3 pairs of adjoint functors, presented in the following diagram*

$$\begin{tikzcd} \mathbf{Etal\acute{e}}(X) \arrow[r, "\simeq"] \arrow[d, hook, bend left] \arrow[d, phantom, "\dashv"] & \mathbf{Shv}(X) \arrow[d, hook, bend left] \arrow[d, phantom, "\dashv"] \\ \mathbf{Top}/X \arrow[u, bend left] \arrow[r, bend left, "\Gamma"] \arrow[r, phantom, "\perp"] & \mathbf{Psh}(X) \arrow[u, bend left] \arrow[l, bend left, "\Lambda"] \end{tikzcd}$$

The functors between presheaves ans sheaves come from the inclusion-sheafification adjunction, functors connecting $\mathbf{Top}/X$ and $\mathbf{Psh}(X)$ correspond to the germs-sections adjunct pair, and the last pair, which *we've not seen before, is the inclusion of Etalé maps with all bundles together with it's adjoint, which cut the bundle into local homeomorphisms via sheafification of its sections. The resulting Etalé space was aptly compared by McLane to the shish kebab machine, but in case you're a vegetarian, the onion pictures it just as fine.*

Example 521 (Locally constant sheaf as a covering space)**.** *An covering spaces, considered as Etalè maps, correspond bijectively to loally constant sheaves. Indeed, a sheaf if locally constant iff every point x has a neighbourhood U_x, such that on all its open subsets the restrictions induced by a sheaf are bijective. Thus from the Etalè perspective, the inverse image of such a neighbourhood must be just a disjoint union of U_x, which is exactly the condition characterising covering spaces.*

Example 522 (Odd line)**.** *Sheaves provide a way of constructing weird objects, which algebraically encoded behaviour is often not possible to realise with ordinary topology or analysis. For instance the odd line $\mathbb{R}^{0|1}$ is an object encoding the idea of infinitesimal values. It is a ringed space on a point, which sections take values in $\mathbb{Z}/2$-graded algebra $\mathbb{Z} \oplus \theta\mathbb{Z}$. The grading is a way of encoding the idea similar to the little o notation, that the infinitesimal linear approximation is irrelevant after squaring, while not exactly equal to zero. The ideal of 0-graded element is often called irrelevant ideal by algebraic geometers, as we see not without a reason.*

11.2. **Sites.** To understand the motivation behind sieves, make sure you're comfortable with the examples of subobject classifiers.

Example 523 (Subfunctor)**.** *Consider some functor $F : \mathcal{C}^{op} \to \mathbf{Set}$. Its subfunctor is a functor G, such that for any object X $G(X)$ is a subset of $F(X)$, and the induced morphism $G(f)$ is a restriction of the morphism induced by F. Alternatively, a subfunctor is just a subobject in the category of presheaves $[\mathcal{C}^{op}, \mathbf{Set}]$.*

Example 524 (Sieve)**.** *Let $\mathcal{C}$ be a small category. There are several ways of describing sieves. Firstly, we have an axiomatic approach: sieve on X is a set of morphisms with codomain X closed under compositions. That's pretty meaningless, but let's look at some motivations. We may ask: does subobject classifier exist in the category of presheaves $\hat{\mathcal{C}} = [\mathcal{C}^{op}, \mathbf{Set}]$? Suppose it does and find some of its properties. By Yoneda lemma we can see that*

$$\mathrm{Sub}_{\hat{\mathcal{C}}}(h_X) = \mathrm{Hom}_{\hat{\mathcal{C}}}(h_X, \Omega) \simeq \Omega(X)$$

Thus $\Omega(X)$ must be a functor of all the subfunctors of the hom functors h_X. And such subfunctors are exactly the sieves on X.

Example 525 (Subobjects in presheaves). *Now using the notion of sieves we can construct the subobject classifier in $\hat{\mathcal{C}}$. To do that, we expect to obtain every sieve by some pullback. A pullback sieve has a very simple structure: given any map $X \to Y$ and a sieve $Q \hookrightarrow h_X$, the pullback sieve*

$$\begin{tikzcd} f \cdot Q \arrow[r, dotted] \arrow[d, dotted] & Q \arrow[d, hook] \\ h_Y \arrow[r, "\circ f"] & h_X \end{tikzcd}$$

has a form $f \cdot Q = \{g \mid f \circ g \in Q\}$. The maximal sieve of $t(X)$ on X is just a sieve consisting of all possible maps to X, so $t(X) = \mathrm{Hom}(-, X)$. Thus the morphism

$$1 \xrightarrow{t(X)} \Omega(X)$$

is indeed the subobject classifier in $\hat{\mathcal{C}}$. Consider some monomorphism of functors $Q \hookrightarrow P$ making Q a subfunctor of P. Then its value at any object X can be constructed as a pullback

$$\begin{tikzcd} Q(X) \cdot Q \arrow[r, dotted] \arrow[d, dotted] & 1 \arrow[d, hook] \\ P(X) \arrow[r, "\varphi_X"] & \Omega(X) \end{tikzcd}$$

where $\varphi_X : P(X) \to \Omega(X)$ is a natural transformation mapping $x \in P(X)$ to all arrows $Y \to X$, that takes x to $Q(Y)$.

Example 526 (Ideals as sieves). *Consider some monoid M. The sieves in the corresponding category BM with one object are exactly the right ideals of M. Since the morphisms in BM are the elements of M, a sieve I on $\bullet$ is a subset of M closed under composition, which translates into the condition that*

$$a \in I \Rightarrow am \in I$$

which is, by definition, an ideal of M.

Example 527 (Downwards closed sets as sieves). *Consider some partial order category. Then a sieve on x is a set of elements less or equal to x that is downwards closed - so whenever some element belongs to S, all the smaller elements also do. As an example consider $\mathbb{R}^2$ with the standard partial order. Then given any non-increasing function, the region under the curve is is a sieve. But not all sieves have such a form - for example the first quadrant is also a sieve.*

Example 528 (Principal sieve)**.** *In this example we will classify all sieves in the category* $\mathbf{Top}(X)$*. Since a sieve is just a subfunctor of* $\mathrm{Hom}(-, U)$ *for some open set* U*, every sieve must be some subset of open subsets of* U*. Moreover, the axioms of a sieve require it to be downward closed, so every open set belonging to* S *comes together with all its subsets. Since open sets are closed under unions, every sieve have some unique set of maximal open sets* $\{U_i\}$*, which spans* S*. A sieve with single generator* U_0 *is called a principal sieve, which is just a hom functor* $\mathrm{Hom}(-, U_0)$*. Principal sieves in fact classify all the subsheaves of* $\mathrm{Hom}(-, U)$*, as by the gluing condition every open subset of* $\bigcup U_i$ *must belong to* S*. This fact in fact shows us that homs can be seen as a family of "simple" sheaves, in a sense that it is minimal with respect to taking subsheaves. It is surprising is a sense that skyscrappers looks minimal at a first glance, but these two families, as we've just seen, are incomparable in this partial order.*

Example 529 (Finite sieves)**.** *In this example we will consider what happens when our presheaves are valued in some different category than sets, such as finite sets. As long as we consider only finite categories* $\mathcal{C}$*, the subobject classifier in* $[\mathcal{C}^{op}, \mathbf{Fin}]$ *is exactly the same as in ordinary presheaves* $\hat{\mathcal{C}}$*, since all the sieves must be finite anyway. Thus* $1 \xrightarrow{t(X)} \Omega(X)$ *is a subobject classifier in both* $[\mathcal{C}^{op}, \mathbf{Set}]$ *and* $[\mathcal{C}^{op}, \mathbf{Fin}]$*. However, in case of category with infinitely many objects, such a reasoning fails. Consider for example* $[\mathbb{N}, \mathbf{Fin}]$*. Now the set* $\Omega(X)$ *of sieves on* X *fails to be finite, so do not form a functor* $\mathbb{N} \to \mathbf{Fin}$*. A category* $[\mathbb{N}, \mathbf{Fin}]$ *is an example of a category that do not have a subobject classifier.*

Example 530 (No classifier of abelian groups)**.** *The other example of a category without subobject classifier is* $\mathbf{Ab}$*. Suppose such an object exists. Then every abelian subgroup must have been a pullback of the diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[d, hook] & 0 \arrow[d] \\ A \arrow[r, "\varphi"] & \Omega \end{tikzcd}$$

But it means exactly that we would have an exact sequence

$$0 \to S \to A \to \Omega \to \mathrm{coker}\varphi \to 0$$

Thus if such Ω *exists, it has a subspace isomorphic to* A/S *for any abelian group* A*, in particular in the case of* $S = 0$*, every abelian group is a subgroup of* Ω*, which is clearly an absurd.*

Definition 11.1 (Site)**.** *A site is a small category with a Grothendieck topology J, which associate to each object X a set of sieves $J(X)$ and satisfies following 3 axioms:*

- ***Maximal sieve***: *A maximal sieve $t(X)$ belongs to $J(X)$ (S_X covers all its elements)*
- ***Stability***: *$J(X)$ is closed under pullbacks (if S covers f, then also all its compositions)*
- ***Transitivity***: *If all the pullbacks of some sieve S on X belongs to $J(-)$, then the sieve itself belongs to $J(X)$ (if S covers f and R covers all $s \in S$, then R also covers f)*

As in case of topologies, it is usually more convenient to describe a site by a basis.

Definition 11.2 (Basis of a site)**.** *Given a small category with pullbacks, a basis of a Grothendieck topology J on $\mathcal{C}$ is a function $K : \mathrm{ob}(\mathcal{C}) \to \mathrm{mor}(\mathcal{C})$ satisfying following axioms:*

- ***Isomorphisms***: *A set of all isomorphisms $Y \to X$ belongs to $K(X)$*
- ***Stability***: *$K(X)$ is closed under pullbacks*
- ***Transitivity***: *Given $S = \{A_i \to X\} \in K(X)$ and any collection of $R_i \in K(A_i)$, the set of all compositions $\{f_i \circ g_i \mid f_i \in S, g_i \in R_i\}$ also belongs to $K(X)$.*

A basis generates a site via taking a upward closure, meaning that

$$S \in J(X) \text{ iff } S \supseteq R \in K(X)$$

Example 531 (Trivial site)**.** *A most trivial example of a topology on $\mathcal{C}$ is a trivial topology, where each set $J(X)$ consists only single element, which is the maximal sieve $t(X)$.*

Example 532 (A topology need not be its basis)**.** *As opposed to classical topologies, a basis of a Grothendieck topology in general need not to be its own basis. The only problem is the first axiom requiring the set of all isomorphisms to be in a basis, which need not to be satisfied by a site in general. For example a trivial site, where $J(X)$ consists only the maximal sieve, is not a basis unless $\mathcal{C}$ is a grupoid. It might seems at the first glance that it must imply that not every topology has a basis, but it is not true - the set of all the isomorphisms of X just do not form a sieve!*

Example 533 (Open cover site on fixed space)**.** *A standard site on* $\mathbf{Top}(X)$ *comes from the open cover topology. Recall that a sieves in this categories are downward closed families of open sets. Every such* *a family is generated by some sets $\{U_i\}$, meaning that it contains $\{U_i\}$ together with all its open subsets. The family $\{U_i\}$ can be associated with open covers of some subspaces of X, and following this trope the open cover site is defined - $J(U)$ is a family of all the sieves generated by open covers of U (in other words, $J(X)$ contains all the sieves, in which union it is contained).*

Example 534 (Open cover topology)**.** *An open cover topology on* $\mathbf{Top}(X)$ *can be generalised to all the small subcategories of* **Top** *closed under finite limits and open subspaces, for example separable Hausdorff spaces. This time the more convenient description is obtained by a basis - $K(X)$ can be just set to contain all the open coverings of X (considered as suitable inclusions).*

Example 535 (Open cover topology on smooth manifolds)**.** *In the category* **Diff** *of smooth manifolds we can also consider the open cover topology constructed above, even though it does not have all the pullbacks (for example the pullback of two projections $S^1 \to I$ is $S^1 \vee S^1$, which fails to be a manifold). Since the pullbacks are needed only to satisfy the stability axiom, in case of this particular topology we only need the pullbacks of a form*

$$\begin{tikzcd} f^{-1}(U) \arrow[r] \arrow[d] & U \arrow[d, hook] \\ Y \arrow[r, "f"] & X \end{tikzcd}$$

to exist. Since such a pullback is always an open subset, it is indeed a manifold.

Example 536 (Finer open coverings)**.** *Instead of open cover topology on a suitable category (in which such a site exists), one can consider finer topology, in which we drop the condition that a covering must come from inclusions, and only require the maps to be open. In this setting the coverings from a basis $K(X)$ are all the sets*

$$\{f_i : A_I \to X \mid f : \coprod A_i \to X \text{ is an open surjection}\}$$

Clearly in this case a topology $J(X)$ is strictly bigger - for example all the (reasonably small) covering spaces of X are also considered as its open covers. The condition of being open makes the sieves still having similar properties to open covers, while the stability axiom is satisfies, since being an open surjection is closed under pullbacks.

Example 537 (Ètale topology)**.** *Drastically different site on* **Top** *than the standard one comes with Ètale topology, where the covering sieves consist all jointly surjective families of local homeomorphisms.*

Example 538 (Sup topology)**.** *The real idea behind considering sites is to construct some analogue of topology on some abstract categories, providing a tremendous generalisation of sheaves. Finally we will look at such an example in the case of complete Heyting algebras. Recall that a complete Heyting algebra is a lattice (poset closed under infs and sups), which is distributive (so such that d'Morgan laws holds) and has an implication operator $a \Rightarrow b$, being an exponential object (adjoint to the product), thus satisfying*

$$x \leq (a \Rightarrow b) \text{ iff } x \wedge a \leq b$$

Implication $a \Rightarrow b$ is the biggest element, which intersected with a is less than b. A model example of an Heyting algebra is a poset of open sets in some space X. A sup topology on a Heyting algebra has a basis $K(x)$, made out of all the sets of supremum x. Since a sieve on a Heyting algebra is exactly a downwards closed set, the topology is easy to describe explicitly, as $J(X)$ is formed by all the downwards closed sets of supremum x. You may notice, that in a case of $\mathcal{C} = \mathbf{Top}(X)$, a sup topology is just an open cover topology.

Example 539 (Dense topology)**.** *A dense topology generalizes a sup topology, as can be constructed on any poset. A subset D of such a poset is called dense below x if $x \in D$ and for each element p smaller than x, there is some even smaller element $p \geq d \in D$. A dense topology can be explicitly constructed by taking $J(x)$ to consist all the dense sets below x, which elements are all less or equal to x.*

Example 540 (Atomic topology)**.** *In every small category, in which every pair of maps f, g can be completed to the commutative square (a property much weaker than the existence of pullbacks)*

$$\begin{tikzcd} A \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ Y \arrow[r, "g"] & Z \end{tikzcd}$$

we can consider the atomic topology - the maximal one, where $J(X)$ is a set of all non-empty sieves.

Example 541 (Minimal topology)**.** *Sites in some cases can be generated by only a single family of arrows. It is possible whenever such a family $\mathcal{A}$ is*

- *closed under composition*
- *interpolative, i.e. every arrow in $\mathcal{A}$ can be factor as two composed arrows*

In such a case, we can construct a site $J_{\mathcal{A}}(X)$ containing all the sieves on X containing all the arrows from $\mathcal{A}$ with codomain X (and possibly more).

For example, such a site can be always constructed from any full subcategory.

Grothendieck topologies of a form $J\mathcal{A}$ are always minimal in a sense, that they have the smallest covering sieves $J(X)$ on every object

Example 542 (Zariski site)**.** *Recall that for each commutative ring we have a natural Zariski topology on its spectrum, thus naturally also a site on* $\mathbf{Top}(\mathrm{Spec}R)$*, called a small Zariski site. A big Zariski site generalizes this topology to the category $C\mathbf{CRing}^{\mathbf{op}}_{\mathbf{FG}}$ opposite to finitely generated rings (only since we need $\mathcal{C}$ to be small) similarly to generalising open cover topology on fixed topological space to category of some bigger family of spaces, like separable Hausdorff. Its Grothendieck topology $J(X)$ contains all the sieves, that after identifying morphisms in $C\mathbf{CRing}^{\mathbf{op}}_{\mathbf{FG}}$ with maps induced on their spectra, contains canonical inclusions induced by localising elements $R \to R_x$*

$$S(R) = \{\mathrm{Spec}R_{a_i} \hookrightarrow \mathrm{Spec}R \mid a_i \textit{ generates } R\}$$

Example 543 (Sheafification reversed)**.** *Classically given a presheaf, we got a natural operation of making it into a sheaf. Considering sheaves on sites, we can also consider the operation of opposite nature. Given some presheaf F, there exist a largest Grothendieck topology on $\mathcal{C}$ in which F is a sheaf. Its construction just takes as covering sieves all making F satisfying the sheaf axiom. By intersecting such topologies, we may also construct a site generated by some family os presheaves.*

Example 544 (Canonical topology)**.** *Since, as previously considered, one can generate a Grothendieck topology on any small category with some family of presheaves and representable presheaves can be considered as somehow canonical, the topology generated by them got such a name. Any smaller topology, on which representable presheaves are sheaves are called subcanonical.*

11.3. **Sheaves on sites.**

Example 545 (Smooth manifolds as sheaves on Cartesian spaces)**.** *Consider a category* **Cart** *of Cartesian spaces - full subcategory of smooth manifolds with objects diffeomorphic on to $\mathbb{R}^n$. Considering* *the open cover topology on* **Cart***, we can identify smooth manifolds M with sheaves $\bar{M}$ on* **Cart** *of a form*

$$\bar{M}(U) = C^\infty(U, M)$$

Using such a trick we obtained an embedding

$$\mathbf{Diff} \hookrightarrow Sh(\mathbf{Cart})$$

It is very handy, as while **Man** *is an ugly category, rarely having limits and colimits, the category $Sh(\mathbf{Cart})$ is a topos - a nicer category is almost impossible to find.*

Example 546 (Differential forms from Yoneda)**.** *The presheaf of differential forms Ω^k obviously also form a sheaf on* **Cart***, as they do in* **Diff***. Using the Yoneda lemma we can construct from them differential forms on manifolds working only in the category* **Cart** *using the natural isomorphism*

$$\Omega^k(M) = \mathrm{Hom}(\bar{M}, \Omega^k)$$

Example 547 (Smooth sets)**.** *Among sheaves on Cartesian spaces we have a lot more objects than only manifolds called smooth sets. For example two lines crossing at a point, i.e. zero set of polynomial $xy \in \mathbb{R}[x, y]$, is a smooth set. It's sheaf X, that for $n > 1$ do not distinguish X from disjoint lines $X(\mathbb{R}^n) = C^\infty(\mathbb{R}, \mathbb{R}) \times C^\infty(\mathbb{R}, \mathbb{R})$, but it does distinguish the common point - $X(\bullet) = \mathbb{R} \vee \mathbb{R}$. Many other, much more wild spaces also appear in $\mathbf{Shv}(\mathbf{Cart})$ - for example, as any topos, it has internal hom functor. In case of smooth manifolds it is in fact given by the simple formula* $\mathrm{map}(M, N)(U) = C^\infty(U \times M, N)$.

Example 548 (Diffeological spaces)**.** *The problem with smooth sets is that it's quite hard to work with objects without any sensible notion of points. We can get such a notion by restricting ourselves to concrete sheaves - admiting the injective map*

$$X(U) \hookrightarrow \mathrm{Hom}(U, X(\{\bullet\}))$$

We want to think about $X(\{\bullet\})$ as an object parametrising points of X, and $X(U)$ - as embeddings of U in X. Thus if such a map is injective, we can say that each element of $X(U)$ is some actual inclusion of U into points of X at the level of underlying sets. For example all the smooth manifolds are concrete sheaves, as every smooth embedding is also embedding of undelrying sets

$$C^\infty(U, M) \hookrightarrow \mathrm{Hom}(U, M)$$

However, the sheaf of differential forms is not concrete, as $\Omega^n(\{\bullet\})$ is a trivial vector space, so we can't talk about points of such smooth set in a *reasonable way. On every concrete sheaf we can reconstruct the notion of a chart and an atlas and retrieve some generalised version of classical construction of manifolds. Such atlases are called diffeologies and their charts - parametrisations or plots. Concrete sheaves with diffeology are called diffeological spaces - we think that they model smooth topological spaces in a way mimicking classical manifolds. Diffeological spaces form a category* **Diff**, *with functions preserving precompositions with plots. We get the fully faithful inclusions*

$$\mathbf{Cart} \hookrightarrow \mathbf{Man} \hookrightarrow \mathbf{Diff} \hookrightarrow \mathbf{SmoothSet}$$

Moreover, the category **Diff** *has fantastic properties:*

- *it is complete and cocomplete*
- *products are computed pointwise*
- *subspace diffeomology is restriction of plots cintained in subspace*
- *quotient diffeology contains plots composed with quotient projection*
- *every function $X \to Y$ from diffeological space X induces pushforward diffeology on Y*
- *every function $X \to Y$ to diffeological space Y induces pullback diffeology on X*
- *there is an internal hom $C^\infty(X, Y)$ with so called functional diffeology*
- *Banach manifolds have diffeological structure*
- *Frèchet manifolds have diffeological structure*
- *complex and analytic manifolds have diffeology encoding their extra structure*
- *manifolds with boundary have diffeological structure*
- *orbifolds have diffeological structure*
- *sections of fiber bundles have diffeological structure*
- *has a homotopy theory*
- *has differential forms and deRham cohomology*

Example 549 (Spaghetti diffeology)**.** *Among such beautiful family of interesting spaces as mentioned above, we get also some weird ones. Consider for example the set $\mathbb{R}^2$. Besides the canonical manifold structure, it admits also different, not equivalent so called spaghetti diffeology. Its plots are all the maps $p : U \to \mathbb{R}^2$ that locally factors through the real line - thus for any $x \in U$ it has some neighbourhood V such* *that restriction of p factors through some plots f and q as*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[dr, "p|V"'] & \mathbb{R} \arrow[d, "q"] \\ & \mathbb{R}^2 \end{tikzcd}$$

Notice in particular, that the identity $\mathbb{R}^2 \to \mathbb{R}^2$ is not a plot with such diffeology.

Example 550 (Irrational torus)**.** *Consider the diffeological real line (as a smooth manifold). Topologically, the quotient $\mathbb{R}/(\mathbb{Z} + \pi\mathbb{Z})$ is the uncountable codiscrete space, however it is not the case for diffeological structure - the diffeology is trivial (codiscrete) if every map $U \to X$ is a plot, but not every such a map araises as the composition $U \xrightarrow{p} \mathbb{R} \to \mathbb{R}/(\mathbb{Z}+\pi\mathbb{Z})$. For example the map $\mathbb{R} \to \mathbb{R}$ which is the bijection between $\mathbb{R}$ and all the irrational numbers can't be constructed this way, as every smooth map $\mathbb{R} \to \mathbb{R}$ is either constant or contains some rational number in the image.*

Example 551 (Continuous diffeology and diffeological topology)**.** *There is an adjoint pair of functors*

$$\begin{tikzcd} \mathbf{Top} \arrow[r, bend right, "cdiff"'] \arrow[r, phantom, "\perp"] & \mathbf{Diff} \arrow[l, bend right, "dtop"'] \end{tikzcd}$$

The functor dtop generates a topology on a diffeological space, called the diffeological topology or simply D-topology. It is the final topology making all the plots continuous maps. Its adjoint right adjoints constructs the continuous diffeology from a topological space - its plots are just all the continuous functions from open subsets of $\mathbb{R}^n$. Similarly, we can form a diffeological realisation of simplicial set, as well as singular diffeological complex from diffeological space, which also provide an adjoint pair, just as in the topological case. Notice that none of these four functors are faithful.

Example 552 (Tangent spaces)**.** *The notion of tangent spaces of manifolds can be quite easily extended to diffeological spaces, however it demands a bit of caution. In differential geometry tangent spaces could be defined in several ways. The internal definition regards T_xM as germs of smooth curves at point x. It can be also defined externally - as derivations of germs of functions. Both of these two definitions makes perfect sense in* **Diff***, however are not equivalent in general. A weird counterexample is for example the set $\mathbb{R}^n/O(n)$ with all the plots locally factoring through the norm $\mathbb{R}^n \to \mathbb{R}^n/O(n)$. The diffeologic spaces of* *this form are pretty interesting on its own - even though for each n as topological spaces are homeomorphic to $[0, \infty)$, all of their diffeological structures differ. However, the spaces $\mathbb{R}^n/O(n)$ and $\mathbb{R}^n/SO(n)$ are no longer distinguished by quotient diffeologies, even though are distinguishable by action Lie groupoids.*

12. Examples from algebraic geometry

In this chapter, we assume all rings to be commutative.

Example 553 (Schemes)**.** *Among the sheaves of rings, called ringed spaces, particularly interesting are locally ringed spaces, having stalks with unique maximal ideal (called local rings). Locally ringed spaces form a not full subcategory of ringed spaces, as we consider additional restriction on their morphisms to preserve unique maximal ideals of all stalks. Two most important classes of locally ringed spaces are sheaves of real functions on topological spaces and schemes. In this chapter we will focus on the latter class.*

The spectrum of commutative ring form naturally a locally ringed space called an affine scheme. More generally, a scheme is a locally ringed space that is locally affine. The most important property of schemes is the adjunction between spectrum functor and global sections. Moreover, the spectrum is a fully faithful functor, making $\mathbf{CRing}^{op}$ *a reflective subcategory of schemes. Its essential image, i.e. schemes isomorphic to spectrum of some ring, is the category of affine schemes*

$$\begin{tikzcd}
CRing^{op} \arrow[rr, hook, bend left=10, "Spec"] \arrow[rr, phantom, "\perp"] \arrow[dr, "\simeq"'] & & Sch \arrow[ll, bend left=10, "\Gamma"] \\
& AffSch \arrow[ur, hook] &
\end{tikzcd}$$

Example 554 (Pullbacks)**.** *The reflective inclusion of affine schemes simplifies a lot of operations to purely algebraic operations on rings. For example the terminal object in* **Sch** *we can identify with initial object in* **CRing***, as reflective inclusion preserves colimits. Surprisingly, this terminal object* $\operatorname{Spec}\mathbb{Z}$ *is not by any means one of the two simplest one, usually corresponding to initial or terminal.*

Particularly interesting objects are pullbacks of schemes. These reflecting pushouts of rings, corresponding to the operation of gluing. Since schemes are defined to be locally affine, pullbacks can be reduced to gluing together pullbacks of affine schemes, which corresponds to tensor products of algebras.

Example 555 (Base scheme)**.** *In algebra its often to consider algebras over a field instead of ordinary rings, as they are often much better behaved. As $\mathbb{Z}$-algebras, rings are naturally equipped with canonical map $\mathbb{Z} \to R$, and A-algebras are under this perspective nothing more than rings together with factorisation of this map*

$$\mathbb{Z} \to A \to R$$

So the category of A algebras is just the slice category $A/\mathbf{CRing}$. Similar reasoning can be applied to schemes, but under contravariant correspondence - any scheme has a natural map $S \to \operatorname{Spec}\mathbb{Z}$, restricted on each affine subscheme to unique induced morphism on corresponding rings. If S is any scheme, a category of S-valued schemes S-$\mathbf{Sch}$ is the over category $\mathbf{Sch}/S$. When $S = \operatorname{Spec} R$ is an affine scheme, by abuse of notation we call Spec *R-valued scheme just R-scheme.*

Among most notable examples are K-valued schemes where K is a field, admitting best properties if K is algebraically closed.

13. Abstract nonsense

13.1. **Enrichment.** Enrichment of a category is a motive we've already seen a lot of times during inspecting the internal Hom and related concepts. For arbitrary category, by default every hom-set is, by definition, a set. However, in some cases it is possible to equip them with some additional structure, changing the codomain of Hom functors to some category $\mathcal{V}$ different than sets. In such a case we say that our category $\mathcal{C}$ is $\mathcal{V}$-enriched (or, to makes things shorter, we'll say that $\mathcal{C}$ is a $\mathcal{V}$-category). For example by definition pre-additive categories are $\mathbf{Ab}$-enriched. Each category with internal hom functor, for example Cartesian monoidal category, is self-enriched, as the internal hom functor has value in $\mathcal{C}$ itself. Sometimes categories have multiple enrichments - for example the category of sheaves of abelian groups $\mathbf{ShvAb}(X)$ is self-enriched via the hom-sheaf functor, having a role of the internal hom, as well as $\mathbf{Ab}$-enriched, since it is an abelian category. As we will see in this chapter, enrichments in different categories also can provide some interesting constructions. A rich family of such examples are categories enriched with some preorder. We'll start with a construction of a few such preorders with additional structure, and then use them to make some interesting enrichments.

Example 556 (Rings)**.** *Similarly as we considered monoids to be categories with single object, rings can be similarly constructed as* $\mathbf{Ab}$-*categories with single object.*

Example 557 (Locally finite categories)**.** *Similarly as all locally small categories can be seen as* **Set**-*categories, locally finite categories (with finite hom-sets) are* **FinSet**-*categories.*

Example 558 (Cost)**.** *The other natural example of a symmetric monoidal preorder is the standard order on* $[0, \infty]$ *(meaning obviously* $[0, \infty) \cup \{\infty\}$*), with the monoidal structure given by addition. The entire construction* $([0, \infty], \geq, 0, +)$ *is called* **Cost** *(from a reason that will be clear in a moment)*

Example 559 (Topological categories)**.** *Let* **Top** *be the category of compactly generated weakly Hausdorff spaces, which is Cartesian closed. A* **Top**-*enriched category has a topological hom-space* $\mathcal{C}(a, b) \in \mathbf{Top}$ *between each pair of objects.*

A basic example of a topological category is **Top** *itself, as since it is Cartesian closed,* $\mathbf{Top}(X, Y) = \operatorname{map}(X, Y)$ *with the compact-open topology provides a nice enriched structure.*

Example 560 (Algebroids)**.** *A category is called linear, or an algebroid, if is enriched over* $\mathbf{Vect_K}$*. Many commonly considered categories has this structure, for example*

- $\mathbf{Vect_K}$
- *modules over* k*-algebra*
- k*-linear representations of groups*
- k*-linear representations of quivers*
- *delooping of* k*-algebras*

Example 561 (Lawvere metric space)**.** *Consider some symmetric monoidal preorder* $\mathcal{V}$*. A category* $\mathcal{C}$ *is* $\mathcal{V}$*-enriched, when its hom-objects* $\mathcal{C}(x, y)$ *have values in* $\mathcal{V}$*. It provides some additional features on them coming from the symmetric structure on* $\mathcal{V}$*:*

- *for every* $x \in X$ $\mathcal{C}(x, x) \leq 1$
- *for every* $x, y, z \in X$ $\mathcal{C}(x, y) \otimes \mathcal{C}(y, z) \geq \mathcal{C}(x, z)$

These two axioms looks similar to the definition of a metric, mimicking the axioms

- *for every* $x \in X$ $d(x, x) \leq 0$
- *for every* $x, y, z \in X$ $d(x, y) + d(y, z) \geq d(x, z)$

In fact, the **Cost**-*enriched category is a clear generalisation of a metric space, as it satisfies the axioms*

- *for every* $x \in X$ $\mathcal{C}(x, x) = 0$
- *for every* $x, y, z \in X$ $\mathcal{C}(x, y) + \mathcal{C}(y, z) \geq \mathcal{C}(x, z)$

Note that not all the axioms of a metric space are satisfied there - we lack the condition $d(x, y) = 0 \Rightarrow x = y$ *and the symmetricity* $d(x, y) =$

$d(y, x)$. Moreover, the value $d(x, y) = \infty$ is allowed in a **Cost***-enriched category, as opposed to the classical metric space, where the metric is valued in $\mathbb{R}$.*

Example 562 (Counting metric)**.** *The lack of symmetry axiom in Lawvere metric spaces leads to many interesting examples that we can treat as metric spaces in this broader sense. For example the cardinality of subsets now form a metric space. One can consider any set X, but we'll restrict our attention to $X = \mathbb{N}$ already providing all the interesting properties, as only the finite subsets really plays some role here. The powerset $\mathcal{P}(X)$ form a Lawvere metric space with the metric*

$$d(A, B) = |B \setminus A|$$

identifying the distance from A to B as the number of points lying from B outside A.

Example 563 (Lebesgue measure)**.** *The previous example can be substantially generalised. Given any Boolean algebra M together with outside measure μ, it forms the Lawvere metric space given by analogous formula*

$$d(A, B) = \mu(B \setminus A)$$

In particular, taking μ as the outer Lebesgue measure and $X \subset \mathbb{R}^n$, we get the metric structure on X measuring the set-theoretic difference. We can also refine this construction to more civilised subsets, as every σ-algebra is Boolean, thus the space of measurable subsets also form the metric space, this time with the proper Lebesgue measure.

Example 564 (Floor is lava)**.** *The other interesting Lawvere metric can be constructed on convex spaces. To make the formula less messy, lets denote the generalised interval, i.e. the convex hull of two points, as $[x, y]$. Moreover, for any point a lying in such interval, it can be presented as $a = (1-t)x + ty$, $t \in [0, 1]$. Let's denote such coefficient as t_a. Then the metric is given by the mysteriously looking formula*

$$d(a, b) = \inf_{x \in K \ : a \in [x, b]} -\log t_a$$

The interpretation is following - the coefficient t_a minimizing by the point lying on the boundary of the closure of K, or the metric takes value 0 if there is no such point. If it does exist, the interval $[x, b]$ is the unique line beginning at the boundary, containing a in between and ending at b. The coefficient t_a is the ration between the distance between points and the length of entire line $[x, b]$. Thus imagine that you travel from the boundary to b and you're currently at the point a. Then the distance measured by considered metric between you and the *endpoint of your journey decreases by a constant value* $\log 2$ *each time you double the distance already traveled, and more generally, each time you increase the distance covered* n *times, you are* $\log n$ *closer to the end.*

Clearly this category does not comes from a metric space in the classical sense, as the distance between a *and* b *might be completely different than from* b *to* a *- it is calculate with respect to the wall from opposite direction, thus in general there are no relations between these two.*

Example 565 (Non-symmetric Hausdorff metric). *In general given a metric space* X *we can define the Hausdorff metric on its subsets, measured by infimum of distances between their points. In case of Lawvere metric spaces we can also consider the non-symmetric version of such measure - the distance from closed sets* A *to* B *(distance between subsets is the distance between closures anyway) is then the line from point* $a \in A$ *maximising its length to* b*, that minimise it*

$$d(A, B) = \sup_{a \in A} \inf_{b \in B} d(a, b)$$

Example 566 (Weighted Hasse diagram). *From each finite weighted graph* G *one can construct a Lawvere metric space, treating it as a weighted Hasse diagram (after adding all zero-weighted self-loops). Points of such an associated space has are vertices of the graph, and the distance* $d(x, y)$ *is the minimum over total weight of paths connecting* x *and* y*.*

The same result can be achieved with more computation-friendly way, by presenting the graph via the incidence matrix M_G*. Using the variation of matrix multiplication given by the formula*

$$(AB)(x, y) = \min_{y \in \mathcal{G}} (M(x, y) + M(y, z))$$

The n*-th power* M_G^n *encodes exactly the minimal weight of path from* x *to* y *with length at most* n*. Since* G *is finite, the matrix* $M_G^{|G|}$ *is then a valid incidence matrix of a Lawvere metric space.*

Example 567 (Preorders as an enriched categories). *Consider a category* $\mathcal{C}$ *which is* $\mathbb{2}$*-enriched (note that the notations can be slightly confusing here - a* $\mathbb{2}$*-enriched category can be also denoted as* $\mathbb{2}$*-category, but it has nothing to do with* 2*-categories considered in later chapters). The properties of* $\mathcal{C}(-, -)$*s provided by the monoidal structure of* $\mathbb{2}$ *are*

- *for every* $x \in X$ $\mathcal{C}(x, x) = true$
- *for every* $x, y, z \in X$ $\mathcal{C}(x, y) \wedge \mathcal{C}(y, z) \geq \mathcal{C}(x, z)$

Understanding the truth value of $\mathcal{C}(x, y)$ *as indicating the existence of arrow* $x \to y$*, these conditions corresponding precisely to the axioms of* *a preorder, thus preorder categories can be identified with* 2*-categories (or better, it allows to define a preorder as a* $\mathbb{2}$*-enriched category!)*

Example 568 (Paraconsistent logic)**.** *A special case of a powerset construction is a Belnap's lattice* A_4*, called also the "knowledge lattice" or "approximation lattice", constructed simply by taking* $S = \{0, 1\}$*. The idea behind such names is its interpretation as a model of paraconsistent logic, with four truth values*

$$\begin{tikzcd}
& \bot & \\
true \arrow[ur] & & false \arrow[ul] \\
& \mathbf{Top} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The symbol $\bot$ *can be given a meaning of neither true nor false, while* **Top** *denotes both true and false. The operation* $\otimes$ *is a meet operation of this lattice (the infimum). With such self-imposing description of elements, the operation* $\otimes$*, being a meet of a lattice (meet of a lattice denotes the infimum, while the join is the supremum) has an obvious interpretation as an equivalence of the AND gate in this weird looking logic. After a second thought, such a logic isn't weird at all - it just adds in the most natural way two additional answers of "yes/no" questions - the not enough information provided* $\bot$ *option and the provided information is inconsistent* **Top** *option. The logical system constructed above can be realised on concrete objects as* A_4*-enriched category.*

Example 569 (Powerset enrichment)**.** *Recall that a powerset of any set has a natural partial ordered induced by inclusion. Considered as a Cartesian monoidal category* $(P(X), \subseteq, X, \cap)$*, it is also strict and symmetric, so fits nicely as the base of enrichment. To understand how* $P(X)$*-categories look like, it's convenient to came up with some underlying story. Suppose for example some achievement tree* T *in a game and the set of gameplay modes* M*. Then* T *can be considered as* $P(M)$*-category, where* $T(x, y)$ *is a set of gameplay modes in which achievement* y *is available given* x *is already unlocked. The composition* $T(x, y) \times T(y, z) \to T(x, z)$ *is then a set of modes in which we can achieve* z *starting from* x *and completing* y *in the process.*

Example 570 (Bottlenecks)**.** *Another monoidal preorder describing nicely some features of graphs is* $(\mathbb{N} \cup \{\infty\}, \leq, \min)$*. Given some weighted* *directed graph, we can make out of it an associated* $\mathbb{N}$*-category where* $\mathcal{C}(x, y)$ *calculates the smallest possible bottleneck across all available paths from* x *to* y*, with no path understood as infinite bottleneck. It describes well for instance the situation in following story: consider a retired wealthy man travelling between cities presented as nodes of a graph. Between each city he needs to make a break in a hotel. As he has a lot of money and time, his biggest concern is to avoid hotels of low quality. Adding to the graph an edge for each hotel between each pair of cities weighted by its rating, it has associated* $\mathbb{N}$*-category* $\mathcal{C}$*, where* $\mathcal{C}(x, y)$ *is the rating of the worst hotel the man has to sleep in during the best possible travel from* x *to* y*.*

Example 571 (Symmetrisation)**.** *For any* $\mathcal{V}$ *category* $\mathcal{C}$ *there exist a canonical symmetrisation,* $\mathrm{Sym}(\mathcal{C})$*, with enrichment given by* $\mathrm{Sym}(\mathcal{C})(x, y) = \mathcal{C}(x, y) \otimes \mathcal{C}(y, x)$*. In particular, this way we can construct the classical real line (or rather half of it), symmetrising the category* $\mathbf{Cost}$*. Indeed since for each numbers* a, b *either* $d(a, b)$ *is zero of positive and equal to* $b - a$*, the metric induced on symmetrisation is exactly the standard euclidean* $d(a, b) = |a - b|$*. The main feature characterising symmetrised categories is that they are isomorphic to their duals.*

As usual when we consider some additional structure of an object, we'd like to consider it with structure-preserving morphisms. In this way we can (and should) consider enriched versions of almost all the previously considered operations - $\mathcal{V}$-enriched functors (analogue of additive functors considered with abelian categories) inducing the proper morphism on homs in $\mathcal{V}$, $\mathcal{V}$-enriched natural transformations, enriched functor categories and so on. All these concepts has boring and not very enlightening definitions that you can find on nLab.

Example 572 (Modules)**.** *Recall that using a presentation of a group as category with one object, we've been able to identify* G*-sets as functors* $[G, \mathbf{Set}]$ *and* K*-linear representations of* G *as* $[G, \mathbf{Vect}_K]$*. Similar constructions can be reconstructed in ring theory under the enrichment in* $\mathbf{Ab}$ *(we may consider algebras over any other ring as well by changing abelian groups to its modules). Considering a ring* R *as* $\mathbf{Ab}$*-category with one object, its representations (which are just modules) can be identified with analogous* $\mathbf{Ab}$*-functors*

$$[R, \mathbf{Vect}_K]_{\mathbf{Ab}}$$

Example 573 (Invariant subspace)**.** *Let* V *be a vector space and* φ *its endomorphism. Such a pair has a structure of a* $K[x]$*-module, as any polynomial can be evaluated on an endomorphism with multiplication* *defined by composition. Maximal φ-invariant subspace can be categorically constructed after identifying this module as an additive functor $\Phi : K[x] \to \mathbf{Vect}_K$ taking its (enriched) limit.*

$$V^{\varphi} = \{v \in V \mid \varphi(v) = v\} \simeq \lim_{\mathbf{Ab}} \Phi$$

Similarly, coinvariant quotient $V/\langle v - \varphi(v)\rangle$ can be constructed as the colimit.

Example 574 (Non-expanding functions)**.** *The enriched functors between Lawvere metric spaces preserve the preorder, thus satisfy*

$$d(x, y) \geq d(f(x), f(y))$$

It means that these are just 1-Lipschitz function, satisfying the Lipschitz condition with constant 1. This example it maybe becomes more clear why the category of metric spaces is considered only with short maps, satisfying the same relation.

Example 575 (Manhattan distance)**.** *The category $\mathcal{V}$-**Cat** has finite products. The hom-objects in the product category $\mathcal{C} \times \mathcal{D}$ have a form*

$$(\mathcal{C} \times \mathcal{D})((a, x), (b, y)) = \mathcal{C}(a, b) \otimes \mathcal{D}(x, y)$$

In case of Lawvere metric spaces, the product metric is an analogue of the Manhattan distance, as we have

$$d((x_1, x_2), (y_1, y_2)) = d(x_1, y_1) + d(x_2, y_2)$$

Example 576 (Closed functors)**.** *Similarly as in case of monoidal categories, the axioms of $\mathcal{V}$-functors can be weakened to induce only some morphism $F(A) \otimes F(B) \to F(A \otimes B)$, not necessarily isomorphism, with the property of respecting the internal homs still valid. Consider for example functors **Cost** $\to$ **Cost**. Since as an ordinary category it is a poset, ordinary functors are just non-decreasing positive functions. **Cost**-functors, as already mentioned, are additionally 1-Lipschitz. In the less restrictive version of closed functors, we obtain all non-decreasing subadditive functions, satisfying*

$$f(a) + f(b) \leq f(a + b)$$

Note that we do not need the category $\mathcal{C}$ to be enriched to consider closed functors $\mathcal{C} \to \mathcal{V}$ - only the closed monoidal structure suffice.

Example 577 (Measures)**.** *Consider some set X. The poset of its subsets form a monoidal category $P(X)$ with cocartesian structure, i.e. $A \otimes B = A \cup B$. Closed functors $P(X) \to$ **Cost** by definition satisfy axioms*

$$f(\emptyset) = 0$$

$$f(A \cup B) \leq f(A) + f(B)$$

So these functors are exactly all the outer measures on X.

Example 578 (Lipschitz functions)**.** *The interesting feature of closed functors $f : \mathcal{C} \to \mathcal{V}$ is that they always induce the structure of $\mathcal{V}$-category on $\mathcal{C}$, denoted as $f\mathcal{C}$. The internal hom in $f\mathcal{C}$ is obviously given by $\mathcal{C}(x, y) = \mathcal{V}(f(x), f(y))$. Note that it allows us to modify the internal hom in already enriched categories to obtain some new variants. Consider for example the endofunctor in* **Cost** *given by linear map $f(x) = kx$. It is not a* **Cost***-functor, but clearly is subadditive, thus defines a new metric space f***Cost** *with internal hom multiplied by the constant k. This construction provides a nice presentation of Lipschitz maps with arbitrary constant - these are, by the same argument as in non-modified case - all the* **Cost***-functors*

$$k\mathbf{Cost} \to \mathbf{Cost}$$

Example 579 (Enriched Yoneda lemma)**.** *The Yoneda lemma has a natural generalisation in the enriched case. For any $\mathcal{V}$-category $\mathcal{C}$ and $\mathcal{V}$-enriched functor $F : \mathcal{C}^{op} \to \mathcal{V}$, there is a $\mathcal{V}$-natural isomorphism*

$$\mathrm{Nat}_{\mathcal{V}}(h_X, F) \simeq F(X)$$

where $\mathrm{Nat}_{\mathcal{V}}(-, -)$ is the object (in $\mathcal{V}$!) of $\mathcal{V}$-natural transformations. The Yoneda embedding of a small category is also almost identical to the standard version. The proof and interpretation of the enriched Yoneda lemma is pretty much identical to the standard version, however it provides a few interesting new examples.

Example 580 (Yoneda embedding of posets)**.** *Since we've identified the preorder categories as the same as $\mathbb{2}$-enriched, we can apply it to interpret the Yoneda embedding of posets in the enriched, simpler version. Given a poset category P, its $\mathbb{2}$-presheaves are not hard to understand. A functor $F : P^{op} \Rightarrow \mathbb{2}$ is a $\mathbb{2}$-functor if it is a morphism of posets, so is monotone. It means that the $\mathbb{2}$-presheaves of P are just*

$$\mathbb{2} - \mathbf{Psh}(P) = \mathrm{Monotone}(P^{op}, \mathbb{2})$$

But $\mathrm{Monotone}(P^{op}, \mathbb{2})$ is just a poset category of downward closed subsets of P! This observation provides an explicit interpretation of the enriched Yoneda embedding, as identifying P with a full subcategory $\mathcal{D}(P)$ consisting principal downsets $\downarrow x$, since

$$\downarrow x = \{y \in P \mid y \leq x\} = \mathrm{Hom}(-, x)$$

Example 581 (Regular representation)**.** *Recall that a ring can be constructed as an additive category with one object. The Yoneda embedding*

$R \hookrightarrow [R^{op}, \mathbf{Ab}]$ map the single object of R to a functor $\rho : R^{op} \to \mathbf{Ab}$. Denoting $V = \rho(\bullet)$, on arrows it takes a form $R \to \operatorname{End}(V)$. Moreover the Yoneda lemma identifies this homomorphism with multiplication in R, thus this embedding is just regular representation of R.

Example 582 (Yoneda for metric spaces)**.** *The enriched Yoneda lemma applied to Lawvere metric spaces provides quite interesting piece of maths. Recall that* **Cost***-enriched functors can be identified as short maps. In particular the functor $A \to$* **Cost** *is a non-negative real* 1*-Lipschitz function on A, since it induces the arrow on internal homs in a poset, i.e. $d(x, y) \leq d(f(x), f(y))$ in* **Cost***. The representable functors have a form $h_x(-) = d(-, x)$. Note that the contravariancy of functors reverses the inequality, and the characterisation of presheaves as expanding maps can be directly deduced from enriched Yoneda lemma:*

$$d(h_x, h_y) = d(x, y)$$

And generally the Yoneda lemma is equivalent to the identities true for any expanding map $f : A^{op} \to \mathbf{Cost}$ and points $x, y \in A^{op}$

$$f(y) = d(h_y, f) = \sup_{a \in A^{op}} d(h_y(a), f(a)) \geq d(h_y(x), f(x)) = f(x) - d(x, y)$$

The other way of rephrasing the Yoneda embedding in this case is the existence of isometric embedding of each metric space in the function space $\mathbf{Cost}^{A^{op}}$ with the sup metric.

In some simple cases the presheaves and op-copresheaves can be identified with some quite simple metric spaces. Consider for example the metric space $X = N_{r,s}$ with two points $0, 1$ with distances r, s. Then explicit calculations shows that $\hat{X}$ and $\check{X}^{op}$ can be identified with subspaces of $\mathcal{V}^2$

$$\hat{X} = \{f : X \to \mathcal{V} \mid d_X(0, 1) \geq d_{\mathcal{V}}(f(1), f(0))\} \simeq$$
$$\{(a, b) \in \mathbb{R} \mid r \geq \max(0, a - b), s \geq \max(0, b - a)\} \simeq \{(a, b) \in \mathbb{R} \mid 0 \leq a \leq b \leq a + r \text{ or } 0 \leq b \leq$$

$$\check{X}^{op} = \{f : X \to \mathcal{V} \mid d_X(0, 1) \geq d_{\mathcal{V}}(f(0), f(1))\} \simeq$$
$$\{(a, b) \in \mathbb{R} \mid 0 \leq a \leq b \leq a + s \text{ or } 0 \leq b \leq a \leq b + r\}$$

Example 583 (Sequences of real numbers)**.** *Consider the discrete Lawvere metric on $\mathbb{N}$, i.e. with the distance given by*

$$d(n, m) \begin{cases} 0 & n = m \\ \infty & n \neq m \end{cases}$$

Then the presheaves $\mathbf{Cost}^{\mathbb{N}^{op}}$ we can identify with the space of real valued sequences with non-symmetric sup metric.

Example 584 (Zariski topology on metric spaces)**.** *In Lawvere metric spaces quite unexpectedly we can form topology of very similar nature to the Zariski topology on spectra of commutative algebras. Similarly to the latter, we start with construction of closed sets as zero sets. Instead of ideals, in this case we'll parametrise them with subsets of A. From each subset $F \hookrightarrow A$ we form the function*

$$F(x) = \inf_{a \in F} d(x, a)$$

The closed sets in A are closures of subsets, where closure of F we define as the zero-set of $V(F) = \{x \in A \mid F(x) = 0\} = \{x \in A \mid \inf_{a \in F} : d(x, a) = 0\}$. The family of associated functions $F(-)$ we'll denote as $\mathcal{F}(A)$. The magic begins when we realise that all these functions are expanding, thus by Yoneda lemma the presheaves, so that we may insert $\mathcal{F}(A)$ between the Yoneda embedding

$$A \hookrightarrow \mathcal{F}(A) \hookrightarrow \mathbf{Cost}^{A^{op}}$$

This realises $\mathcal{F}(A)$ as the subspace of $\mathbf{Cost}^{A^{op}}$, so it does inherit the supremum metric by restriction. This way we obtain the non-symmetric version of classical Hausdorff metric between subsets of A, however the metric on $\mathcal{F}(A)$.

Changing the base on enrichment to to **Bool** *constructs the poset of closed sets under inclusions. This is an analogue of the Nullstellensatz, as defines the Galois connection between poset of closed sets and functors the orders of $F(-)$, playing the role of functions in the same way as we think about subsets of the ring, with representable presheaves corresponding to its elements.*

$$F \subseteq G \textit{ iff } F \geq G \textit{ in } \mathbf{Cost}^{A^{op}}$$

Moreover, in the construction of spectrum we may as well consider only the ideals, not all possible subsets, as every subset generates the same zero set as the ideal it generates. In this case we can also replace the presheaves with such smaller class generating the zero sets equally well - these sort of refined closed sets, obtain from the left adjunct of the inclusion

$$\mathcal{F}(A) \underset{Z}{\hookleftarrow\!\!\longrightarrow} \mathbf{Cost}^{A^{op}}$$

Example 585 (Paths in metric spaces)**.** *Paths are very interesting objects of Lawvere metric spaces. Surprisingly, the categorical construction of a path requires some work, and involves a few interesting constructions itself. To problem with the obvious definition is to get an interval that is symmetric. Note that the* **Cost** *category looks as the obvious choice of a line, but its properties are far from intervals we* *know from topology. Firstly,* **Cost** *itself (which in this example we'll denote by* $\mathcal{V}$*, as it's more convenient with such a frequent usage) is by definition a preorder, with arrows going only from smaller to bigger numbers. Every preorder can be made into a Lawvere metric space by adding the missing arrows labeled as* ∞*, but such a structure is horribly non-symmetric, with such anomalies as* $d(1,2) = 1$ *and* $d(2,1) = \infty$*. To bring some symmetry we will restrict our line to a finite interval using the slice construction: the category* $\mathcal{V}/n$ *corresponds exactly to the points from the interval* $[0,d] \hookrightarrow \mathcal{V}$*. Applying to each element the internal hom with augmented element yields the dualising functor* $k \mapsto \mathcal{V}(k,n) = n-k$ *that reverses the ordering in* $\mathcal{V}/n$*, so takes value in the dual* $\mathcal{V}/n \to (\mathcal{V}/n)^{op}$*. Notice the similarity to the dual vector space - it is exactly the same construction, admitting very similar properties.*

After applying symmetrisation on $\mathcal{V}/d$ *we obtain the classical metric structure on interval, which we'll denote as* $\mathcal{V}(d)$*. We get in fact a functorial inclusion* $d \mapsto \mathcal{V}(d)$*, in other words - the functor of paths.*

$$\mathcal{V}^{op} \to \mathbf{Cat}(\mathcal{V})$$

This construction provides something even stronger than classical path space - each metric space can be analysed in terms of behaviour under all the paths of given length, and the family of d*-paths in* X*, constructed as functors*

$$\mathcal{V}(d) \to X$$

can be assembled into sort of a moduli space of paths, parametrising paths spaces of X *by their length*

$$P : \mathbf{Cat}(\mathcal{V}) \to \mathbf{Cat}(\mathcal{V})^{\mathcal{V}^{op}}$$

$$P(X)(d)$$

Example 586 (Self-enriched interval)**.** *Recall that the unit interval* $[0,1]$ *is a closed commutative monoidal preorder, having*

- *standard preorder* $\leq$
- *categorical product* $x \times y = \min(x,y)$
- *monoidal product* $x \otimes y = xy$
- *internal hom* $[x,y] = \min(\frac{y}{x}, 1)$*, right adjoint to* $\otimes$

Such a category can be naturally enriched over itself by replacing the binary hom-set from the preorder with the internal hom.

Example 587 (Fuzzy logic as enrichment over I)**.** *We know that we can represent a preorder as* **Bool***-enriched categories. In a similar spirit enriching over* $[0,1]$ *category can model fuzzy preorder, where the elements are connected by some probabilistic distribution. You can* *think about it as storing additional information in case of true sentences, measuring how far they are from being true. As an example take as objects some set of students and their exam results. A hom set* $\mathrm{Hom}(A, B)$ *models the answer to the question 'has student* B *better result than student* A*'. If* B *scored more or equal number of points on the test, it will be valued* 1*, but where the sentence is not true, it's value can be taken as the proportion of their points. Note that such a structure is in fact isomorphic to the enrichment over* $[0, \infty]$ *via exponential and logarithm mappings.*

Example 588 (Syntax category)**.** *This example shows a model of categorification of natural language. Consider a category* $\mathcal{L}$*, where objects are some expressions on alphabet* A*. Let's assume that it is a natural language, so that for any expression* X *there is some probability distribution of its possible extensions by longer expressions* Y*, understood as relative frequency of* Y *being continuation of the expression* X*. Let's denote as* $P(Y|X)$ *the probability of* Y *extending* X*, meaning that* $P(Y|X) = 0$ *if the concatenated* $X + Y$ *does not make sense, and that* $P(Y|X) = 1$ *if* X *is always followed by* Y*. Then* $\mathcal{L}$ *can be made into a* $[0, 1]$*-category by defining homs as*

$$\mathcal{L}(X, Y) = P(Y|X)$$

Example 589 (Distributional hypothesis in linguistics)**.** *In the previous example we've seen some simple model of syntax of a natural language, however much more interesting structure, capturing meaning of words, not only syntactic dependencies of the underlying language, can be achieved by looking at functors over the syntax category. By the enriched copresheaf we just mean an enriched covariant functor* $\mathcal{L} \to [0, 1]$*. Since* $[0, 1]$ *is closed, a condition for a copresheaf* F *being enriched can be expressed in a pleasant form using the internal hom* $[-, -]$

$$\mathcal{L}(X, Y) \leq [F(X), F(Y)]$$

A representable copresheaf h^X *is an object encoding the probability distribution of extensions of* X*, so* $h^X(-) = P(-|X)$*. Moreover, as we'll see in greater detail as an application of ends, every category of* $\mathcal{V}$*-functors has a natural induced enrichment, in particular our copresheaf category* $\hat{\mathcal{L}}$*. In this case, this enrichment has a form*

$$\hat{\mathcal{L}}(F, G) = \mathrm{Nat}_{\mathcal{V}}(F, G) = \inf_{X \in \mathcal{L}} [F(X), G(X)]$$

Note that by the enriched Yoneda lemma we can contravariantly identify expression X *with its distribution* h^X*. Representable copresheaves* h^X *has a natural interpretations as the meaning of* X*, as they capture* *how likely it is that X can appear in different contexts. Such a model in fact formalizes the distributional hypothesis from linguistics, stating that words in a language has similar meaning if their distributions of extensions are similar. Indeed, this kind of similarity is exactly measured by the enrichment in $\hat{\mathcal{L}}$. Finally, let's look at some concrete example, taking $\mathcal{L}$ to be a syntax category of English language. Considering expressions $X=$"black" and $Y=$"black car", a possible syntactic evaluation can be $\mathcal{L}(black,\ black\ car) = 0.03$, meaning that a word "black" is used to describe a car in 3% of all cases. The meaning evaluation provide us a reversed perspective*

$$\hat{\mathcal{L}}(h^{black\ car}, h^{black}) = 0.03$$

It tells us that a the word "black" has broader meaning than "black car", and that the details added to the meaning of "black" by specifying the context only to cars are correct with probability 0.03. The contravariantness of such an embedding reflects the duality between syntax and meaning, that adding additional details to the expression restricts the space of its possible interpretations in different contexts.

Example 590 (Underlying ordinary category)**.** *Every lax monoidal functor $F : \mathcal{V} \to \mathcal{W}$ induces the functor $(-)_F : \mathcal{V} - \mathbf{Cat} \to \mathcal{W} - \mathbf{Cat}$ changing the base of enrichment. The category $\mathcal{C}_F$ has the same objects as $\mathcal{C}$, with arrow functors composed with F, i.e. $\mathcal{C}_F(x, y) := F(\mathcal{C}(x, y))$.*

This way we can for instance remove the enriched structure from a category with the representable functor $\mathcal{V}(1, -) : \mathcal{V} \to \mathbf{Set}$, turning $\mathcal{C}$ to the ordinary category $\mathcal{C}_0$ with arrows $\mathcal{C}_0(x, y) = \mathcal{V}(1, \mathcal{C}(x, y))$

$$(-)_0 : \mathcal{V} - \mathbf{Cat} \to \mathbf{Cat}$$

Regarding a preorder as a **Bool***-category, its underlying category is just the same preorder considered as ordinary category. When $\mathcal{C}$ is a Lawvere metric space, $\mathcal{C}_0$ is a groupoid of an equivalence relation, indicating weather two objects has equal distance. In case of bicategories, $\mathcal{C}_0$ just forgets about its 2-arrows.*

Example 591 (Free enriched category)**.** *When the base of enrichment $\mathcal{V}$ is cocomplete, the functor $()_0$ has left adjoint free $\mathcal{V}$-category functor. This is an expression of a more general fact, that any monoidal adjunction induces also an adjoint change of bases. Since the functor $\mathcal{V}(1, -)$ in cocomplete monoidal categories are adjoint to the copower, taking a set X to $\coprod_X 1 \in \mathcal{V}$, the same thing happens on the level of enriched categories. For example each category assembles into a free pre-additive category, which homsets $F\mathcal{C}(x, y)$ are free abelian groups $\mathbb{Z}[\mathcal{C}(x, y)]$.*

Example 592 (Neighbourhood functor)**.** *The function* $min(\epsilon, x) : \mathbf{Cost} \to \mathbf{Bool}$ *induces the* ϵ*-neighbourhood functor* $\mathbf{Cost} - \mathbf{Cat} \to \mathbf{Pre}$*, measuring if two points of a Lawvere metric space are distant by less than* ϵ

Example 593 (Naive homotopy category)**.** *Consider the base change induced by* $\pi_0 : \mathbf{Top} \to \mathbf{Set}$ *on the self-enriched* $\mathbf{Top}$*. A path from* f *to* g *in the hom-space* $\mathrm{map}(X, Y)$ *is the function* $I \to \mathrm{map}(X, Y)$ *with values* f*,* g *on the boundary. Note that, since* $\mathrm{map}(I, \mathrm{map}(X, Y)) \simeq \mathrm{map}(X \times I, Y$*, such a path is just a homotopy between* f *and* g*. Thus the category induced by base change, having as morphisms* $\pi_0(\mathrm{map}(X, Y))$*, is just the naive homotopy category* $\mathbf{hTop}$*, which arrows are homotopy classes of functions.*

Example 594 (Poly-morphisms)**.** *The change of base functors produces interesting examples even in the world of ordinary categories. As a small category is the same as* $\mathbf{Set}$*-enriched one, any endofunctor of sets induces appropriate change of base* $\mathbf{Cat} \to \mathbf{Cat}$*. The categories constructed as change of base by the powerset functor has so called poly-morphisms. Denoting* $P_*\mathcal{C}$ *as* $\mathcal{C}_{poly}$*, the arrows in* $\mathcal{C}_{poly}$ *are collections of arrows from* $\mathcal{C}$ *with common domain and codomain. The name poly-morphism just reflects the fact that morphisms* $\mathcal{C}_{poly}(X, Y)$ *are any families of morphisms from* $\mathcal{C}(X, Y)$*.*

Example 595 (Linearisation)**.** *Consider some pre-additive category* $\mathcal{C}$ *and some field* k*. Then the category* $\mathcal{C}$ *can be canonically linearised by taking free vector space generated by homs and extend the composition bilinearly. Note that linearistion does not come from the hom tensor adjunction* $\mathbf{Ab} \to \mathbf{Vect_K}$*. For example in case of* $\mathbf{B}\mathbb{Z}/2\mathbb{Z}$*, the arrows of* $\mathbb{C}(\mathbf{B}\mathbb{Z}/n\mathbb{Z})$ *correspond to the group algebra* $\mathbb{C}[\mathbf{B}\mathbb{Z}/n\mathbb{Z}]$*, not the trivial tensor* $0 = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}/n\mathbb{Z}$*.*

Example 596 (Additive closure)**.** *Consider pre-additive category* $\mathcal{C}$*. From each such category we can canonically form an additive category by formally adding the zero object and formal biproducts* $A \oplus B$*, with morphisms from and into biproducts extended from biproducts of homs. Consider for example the category of irreducible representations of a finite group* G*. Its additive closure is then a category of all finite dimensional representations. Similarly, if* $\mathcal{C}$ *is the category of finite abelian cyclic groups, its additive closure is the category of all finite cyclic groups. Note that not always all the objects we might think are in fact generated in the additive closure - for example the additive closure of cyclic abelian groups is the category of finitely generated abelian* *groups, not the entire* **Ab** *- no infinitely generated group can be constructed as finite biproduct of cyclic groups.*

Example 597 (Connected components in Lawvere metric spaces)**.** *Consider the change of base in Lawvere metric spaces. The inclusion* **Bool** $\to$ **Cost** *preserving initial and terminal values has two adjoints* $p \vdash i \vdash \pi$. *Both* p *and* π *preserve extreme values mapping* 0 *to* 0 *and* ∞ *to* 1*, but differ on all the other points, where we have* $p(t) = 0$ *and* $\pi(t) = 1$. *Since both of them are closed functors, we get the classical adjoint triple, where the inclusion induces a functor of discrete space,* p *is the lazy functor - it admits that the two points can be connected if and only if the travel cost no effort. On the other hand* π_0 *constructs the poset of connected components, with arrows corresponding to distance being finite, so it is the hard-working functor, connecting points if only it is possible, no matter at what price.*

Example 598 (Isometries and translations)**.** *Since functors between Lawvere metric spaces are non-increasing distances between points, the isometries of such spaces are precisely its automorphisms in the enriched sense. In case of euclidean metrics, the group of isometries splits as the semi-direct product of linear automorphisms and affine translations. For example for* $\mathbb{R}^n$ *it is given by exact sequence*

$$0 \to T(n) \to E(n) \to O(n) \to 0$$

Since we've constructed the group $E(X)$ *for any symmetric Lawvere metric space without much effort, it begs the question if the entire sequence can somehow reconstructed as well. Classically there is an inartistic characterisation of translation as a transformation moving each points equally far away, so its an isometry satisfying* $d(x, f(x)) = const$. *Surprisingly, against our euclidean intuition, such family do not form a group in general - to get the group structure one needs to use some additional properties of underlying space, as it is done in case of* $\mathbb{R}^n$. *To fix that problem we can consider a different definition, demanding only that* f *satisfies*

$$\sup_{x \in X} d(f(x), x) < \infty$$

Now we can get maybe slightly different object than having in mind at the beginning, however admitting all the properties we want. Such defined translations $T(X)$ *form a subgroup of* $E(X)$ *due to the triangle inequality. Moreover for each isometry* f *and translation* t *we have*

$$d(f(t(f^{-1}(x))), x) = d(t(f^{-1}(x))), f^{-1}(x)) < \infty$$

so the subgroup is normal, providing the third fugitive - the rotation group $O(X) = E(X)/T(X)$. Note that

Example 599 (Lax functors preserve monoids)**.** *The base change of enrichment provides a nice proof that lax monoidal functors preserve internal monoids. Since monoids internal to $\mathcal{V}$ are in correspondence with $\mathcal{V}$-categories with single object, the base change induced by a lax monoidal functor $\mathcal{V} \to \mathcal{W}$ maps any monoid to monoid in $\mathcal{W}$, which in this language is a truly trivial statement.*

Example 600 (Co-design)**.** *Collaborative design (or co-design in short) is a theory of designing large production systems from subsystems independently producing and requiring resources. Andrea Censi has proposed an elegant mathematical model of co-design using enhanced profunctors. She used them to formalise so called feasibility relations, functions matching required resources with available. Amounts of produced and required materials, encoded in preorders P, R, assembles together to a functor*

$$\Phi : P^{op} \times R \to \mathbf{Bool}$$

indicating if given production of given resources is possible with supplies specified in second variable. Regarding preorders as **Bool***-categories, Φ is exactly a* **Bool***-profunctor. For a general closed symmetric monoidal category $\mathcal{V}$, a $\mathcal{V}$-profunctor is a $\mathcal{V}$-functor $\mathcal{C}^{op} \times \mathcal{C} \to \mathcal{V}$, denoted also as*

$$f : \mathcal{C} \nrightarrow \mathcal{C}$$

As enriched functor, each profunctor is equipped with an arrow

$$\mathcal{C}(b, a) \otimes \Phi(a, x) \otimes \mathcal{D}(x, y) \to \Phi(b, y)$$

Example 601 (Collage)**.** *Consider a* **Bool** *profunctor $f : A \overset{B}{\nrightarrow}$. The general arrow associated to a profunctor is just a relation*

$$b \leq a, x \leq y \Rightarrow \Phi(a, x) \leq \Phi(b, y)$$

Drawing Hasse diagrams of preorders A and B, we can present f as arrows connecting elements of A with elements of B that satisfy above relation — PIC — Note that any choice of such connections can be completed uniquely to satisfy the relations, thus one can draw only minimal set of connections.

By merging the entire picture to a single Hasse diagram, from any profunctor we obtain another preorder, called the collage of f.

Example 602 (Cost profunctors)**.** *The* **Cost***-profunctors admit similar presentation based on Hasse diagrams. Each Lawvere metric space can be presented as a full weighted graph, with each edge labeled by the* *directed distance from its source to the target. Any* **Cost***-profunctor* $f : X \nrightarrow Y$ *is induced from some set of weighted arrows connecting points of* X *with points of* Y*. Each such a family generates a profunctor* Φ*, where* $\Phi(x, y)$ *is the minimal total weight of path connecting* x *and* y *in the weighted collage diagram. Similarly, each profunctor is generated by such a graph containing a single edge between each pair of points weighted by the value of* Φ*. Note that this clearly is not unique - where* X *and* Y *are non-empty, each profunctor is generated by infinitely many different families of arrows.*

Example 603 (Profunctors as matrices)**.** *Recall that each each finite weighted graph with incidence matrix* M_G *represents Lawvere metric space with incidence matrix constructed from power* $M_X = M_G^{|G|-1}$ *under appropriate multiplication (or in general some smaller repeated power stabilising the sequence). In this language* **Cost***-profunctors also fit nicely, as writing down the incidence matrix* M_Φ *associated to its generating set of arrows (with size* $|X| \times |Y|$*), the action of* Φ *on* $X \times Y$ *can be reconstructed via matrix multiplication* $M_X \cdot M_\Phi \cdot M_Y$*. Moreover, similarly to the case of bilinear forms on vector spaces,* Φ *can be evaluated on specific points by multiplying both sides with the one-hot encoded vector.*

Example 604 (Size)**.** *Let* $\mathcal{V}$ *be monoidal with underlying monoid* $M\mathcal{V}$ *and* R *be a ring. Then the size of objects of* $\mathcal{V}$ *is given by monoid homomorphism* $|-| : M\mathcal{V} \to R_{mult}$*. Some examples of sizes include*

- *For* $\mathcal{V} = (\mathbf{Fin}, \{\bullet\}, \times)$ *and* $R = \mathbb{Z}$*,* $|X|$ *can be defined as cardinality of* X
- *For* $\mathcal{V} = (\mathbf{FinVect_K}K, K, \otimes)$ *and* $R = \mathbb{Z}$*,* $|X| = \dim V$
- *for* $\mathcal{V} = \mathbf{Cost}$ *and* $R = \mathbb{R}$*,* $|x| = e^{-x}$
- *For* $\mathcal{V} = \mathbf{FinCW}$ *and* $R = \mathbb{Z}$*,* $|X| = \chi(X)$

Example 605 (Magnitude)**.** *Let* $M \in M_{n,m}(R)$*,* u *be a column vector with constant values* 1 *and size* n*, and* u^* *a row vector filled with ones of size* m*. The* ***weighting*** *of* M *is a vector* w *such that* $Mw = u$*, while* ***coweighting*** *is a vector* v *with* $vM = u^*$*. The* ***magnitude*** $|M|$ *of* M*, if exists, is the sum of entries of any weighting or coweighting (which are always equal). The elements of (co)weightings are called* ***(co)weights*** *of* M*. If* M *is invertible, its weight can be calculated as sum of entries of its inverse. The notion of magnitude can be generalised to categories enriched over sized* $\mathcal{V}$ *with finitely many objects in a following way:*

- *the* ***similarity matrix*** $\zeta_\mathcal{C}$ *of* $\mathcal{C}$ *is defined as* $\zeta_\mathcal{C}(a, b) = |\mathcal{C}(a, b)|$
- ***(co)weighting*** *of* $\mathcal{C}$ *is a (co)weighting of* $\zeta_\mathcal{C}$

- ***(co)weights*** *of* $\mathcal{C}$ *are (co)weights of* $\zeta_{\mathcal{C}}$
- ***magnitude*** $|\mathcal{C}|$ *of* $\mathcal{C}$ *is* $|\zeta_{\mathcal{C}}|$
- $\mathcal{C}$ *has* ***Möbius inversion*** *if* $\zeta_{\mathcal{C}}$ *is invertible*
- ***Möbius matrix****, if exists, is the inverse of similarity matrix*

Note that these constructions directly generalise zeta and Möbius functions considered in chapter 1 *through incidence algebras. Finally, let's look at some concrete examples of magnitudes*

- *when* $\mathcal{V} = \mathbf{Fin}$ *and* $R = \mathbb{Q}$, $|\mathcal{C}|$ *is called the Euler characteristic of* $\mathcal{C}$ *and coincides with this notion considered in chapter* 1
- *Similar Euler characteristic can be considered over linear categories, enriched over* $\mathbf{FDVect}_K$.
- *When* $\mathcal{V} = \mathbf{Bool}$ *and* $R = \mathbb{Z}$, *the magnitude of a finite poset is sometimes also called its Euler characteristic*
- *For* $\mathcal{V} = \mathbf{FinCW}$, *magnitude directly extends the topological Euler characteristic to topological categories*

Example 606 (Dynamical systems on manifolds)**.** *The internal hom in the category of topological spaces makes it convenient for enrichment basis. Consider for example the dynamical systems on manifolds, defined as additive* $\mathbb{R}$*-actions on* M*, understood as the evolution in time or solutions to ordinary differential equations. Alternatively it can be seen as a one-parameter family of automorphisms constituting a smooth flow. Note that action of a group is as usual categorified as the functor* $\mathbf{B}\mathbb{R} \to \mathbf{Diff}$*. Moreover, its smoothness can be captured by considering simply the* $\mathbf{Diff}$*-enriched functors.*

Example 607 (Topological conjugation)**.** *Consider two dynamical systems* $\varphi : \mathbb{R} \to \mathrm{Aut}(M), \phi : \mathbb{R} \to \mathrm{Aut}(N)$*. The topological conjugation between* φ *and* ϕ *is a smooth function* $f : M \to N$ *forming a commutative square for all* $t \in \mathbb{R}$

$$\begin{tikzcd} M \arrow[r, "\varphi(t)"] \arrow[d, "f"] & M \arrow[d, "f"] \\ N \arrow[r, "\phi(t)"] & N \end{tikzcd}$$

This simple concept is extremely useful in the theory of dynamical systems, as it maps homeomorphically flows to flows, preserving a lot of their properties, reducing complicated maps to much simpler cases. In particularly simple but still interesting case where M *is a real line, systems are conjugate if and only if there is a bijection of their equilibria (sink/source/semi-stable), respecting their order. Conjugations in the enriched language are nothing more than* $\mathbf{Diff}$*-enriched natural transformations. It follows that dynamical systems and their conjugations* *can be described as just the* **Diff***-functor category*

$$\mathbf{Dynam} \simeq [\mathbb{R}, \mathbf{Diff}]_{\mathbf{Diff}}$$

Example 608 (Flow groupoid)**.** *The category* **Dynam** *of dynamical systems admits a construction similar to the fundamental groupoid, where paths between points are induced from flows. The functor*

$$\uparrow \Pi_1 : \mathbf{Dynam} \to \mathbf{Grpd}$$

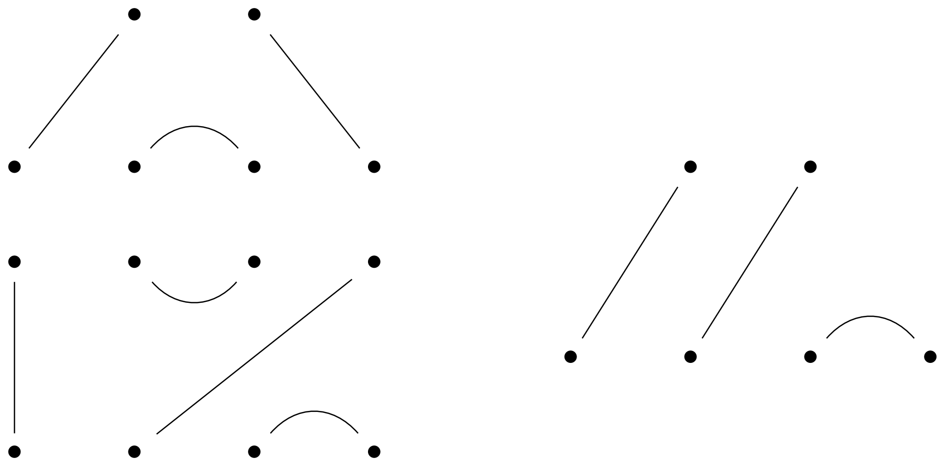
maps dynamical system $\phi : R \to \mathrm{Aut}(M)$ to its action groupoid, with objects M and arrows $x \to \phi(t)(x)$ for each $x \in M, t \in \mathbb{R}$. More interesting construction provides the category of elements $\int \uparrow \Pi_1$. Its objects are pointed dynamical systems, with arrows corresponding to topological conjugations $f : M \to N$ together with a flow in N connecting $f(x_0)$ with y_0. Such category has two natural interpretations: firstly it can be understood as the category of pointed dynamical systems, where its pointedness is defined up to a flow. Secondly, its objects represent dynamical systems with some initial conditions. Flows starting from this initial conditions are then solutions to associated differential equation, while morphisms are maps between manifolds, matching their distinguished flows after some time t.

Example 609 (Temperley–Lieb algebra)**.** *In chapter one we considered the category of string diagrams - its objects were natural numbers with arrows $\mathrm{Hom}(n, m)$ corresponding to classes of all the non-crossing planar paths from between rows of n and m dots. Now consider the linearisation of such a category in the field k and pick some element $\delta \in k$. Then the Temperley-Lieb category $\mathrm{TL}(\delta)$ is the linear category which homs are k-vector spaces generated by classes of tangles, but the chosen element δ modifies our addition - now instead of ignoring circles appearing in the concatenation, we replace them with multiplication of all elements by the element δ. The endomorphisms $\mathrm{Hom}(n, n)$ form so called Temperley-Lieb algebras $\mathrm{TL}_n(\delta)$, which dimensions are given by Catalan numbers. For some first few cases these algebras can be identified with*

- $\mathrm{TL}_2(\delta) \simeq k^2$ *for* $\delta \neq 0$
- $\mathrm{TL}_2(0) \simeq k[x]/(x^2)$

Moreover, $TL(\delta)$ forms also a strict monoidal category - the tensor product is given by addition, while on tangles it just stack them together side by side.

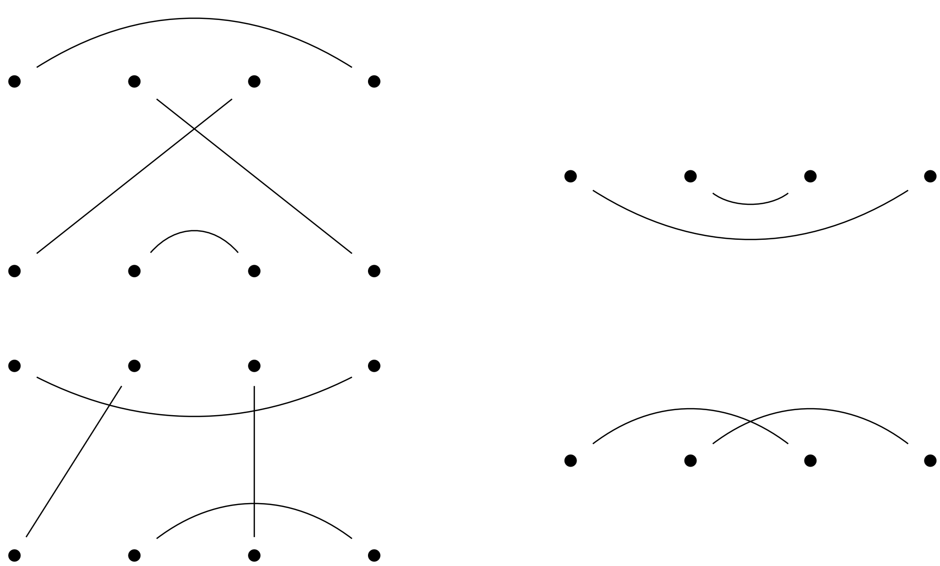
Here is an example of composition inducing one multiplication by δ



Example 610 (Brauer category)**.** *The construction of Temperley–Lieb category works equally well when we allow the strings to cross each other. Again choosing element of a field $\delta \in k$ and linearising the category replacing circles with multiplication, we end up with the Brauer category* $\mathrm{Br}(\delta)$*. Its endomorphisms form the family of Brauer algebras* $\mathrm{Br}_n(\delta)$ *with dimension*

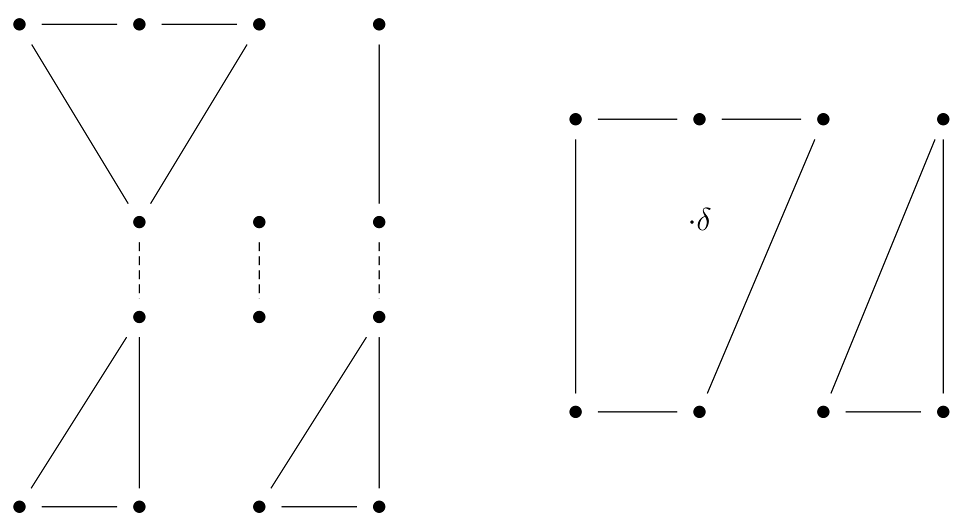
$$\dim \mathrm{Br}_n(\delta) = \frac{(2n)!}{2^n n!}$$

Example of composition in Brauer algebra $\mathrm{Br}_4(\delta)$ *- this time with no* δ *factor*



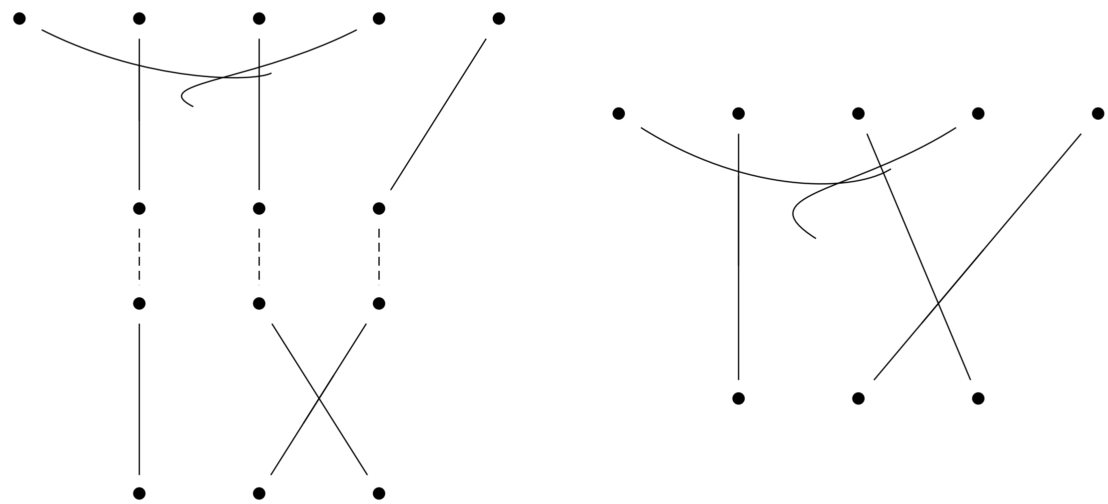
Example 611 (Partition category)**.** *The other variation of constructions presented above is the partition category. This time we consider the category* $\mathrm{Part}(\delta)$ *with objects* $\mathbb{N}$ *and arrows* $\mathrm{Hom}(n, m)$ *generated by all the partitions of set* $[n] \amalg [m]$*. The concatenation can be constructed similar as in previous cases - we concatenate the partitions containing common element of* $[m]$*, and delete all the partitions contained in* $[m]$*, for each of such multiplying the vectors by* δ*.*

Composition of partitions with one δ *factor:*



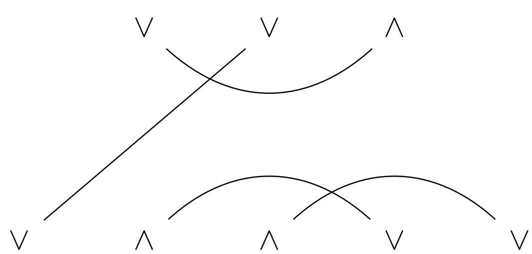
Example 612 (Tangles)**.** *The most complicated structure among categories of previously considered form has the tangle category* $\mathrm{Tang}(\delta)$*. Its morphisms are generated as in the Brauer algebra, but with distinction between the side of a crossing - its relation with Brauer category is analogous to the relations between symmetric and braid groups. The endomorphisms of tangles are always infinitely generated, and are called quantum groups.*

Composition of tangles:



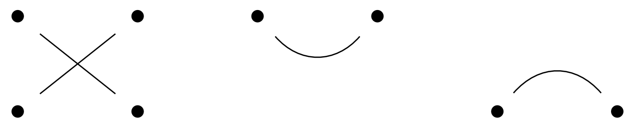
Example 613 (Oriented Brauer category). *We can consider also the oriented variant of Brauer category. Now each point in the sequence is given orientation up or down, so we get more objects - all the binary finite sequence. The morphisms this time are generated with diagrams satisfying additional restriction, that connected dots from the same row must have opposite orientation, while from opposite rows - consistent.*

Example of such oriented morphism:

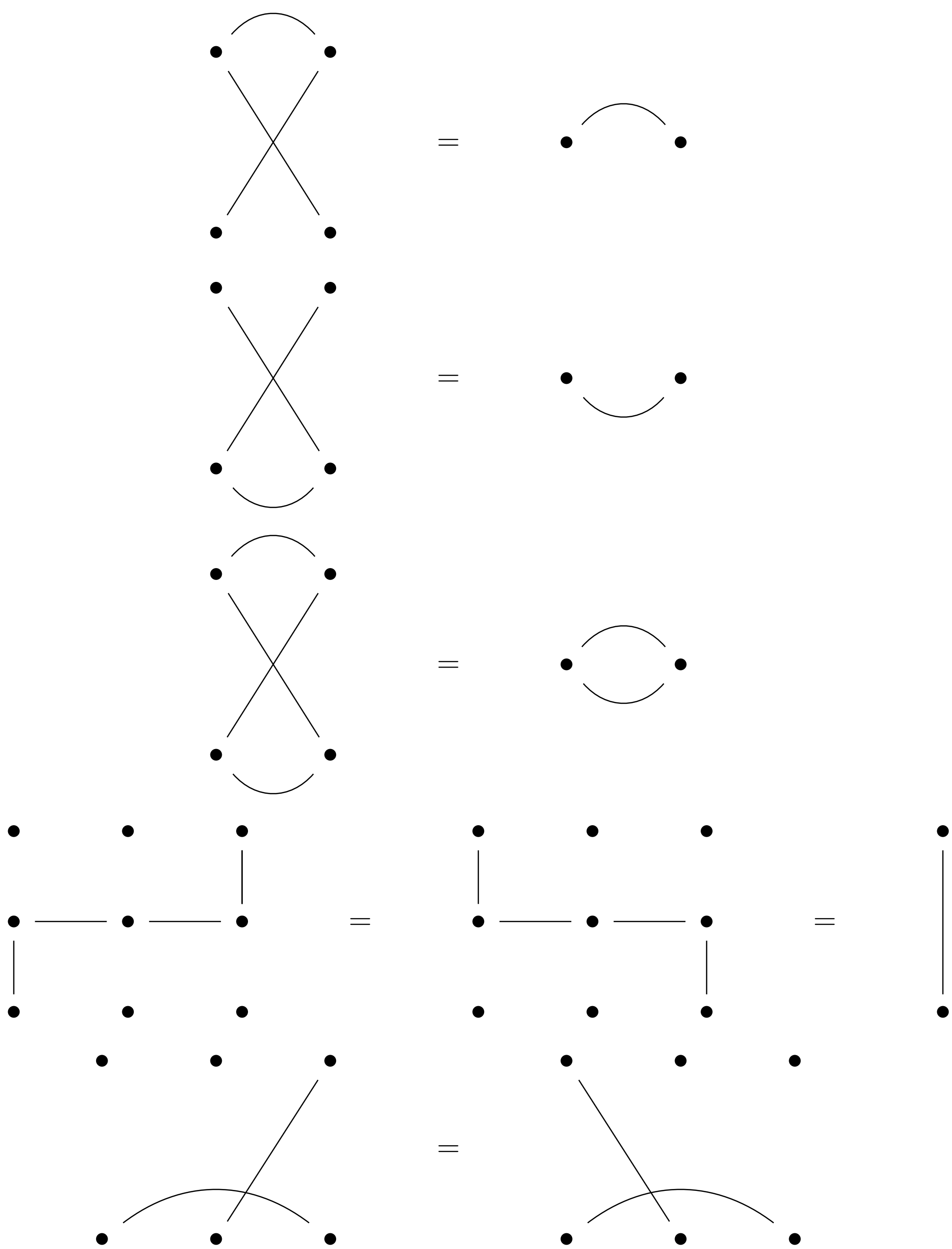


Example 614 (Tensor categories). *All the diagrammatic categories such as tangles, partitions, Brauer or Temperley–Lieb categories encodes some family of algebras as endomorphisms of objects. Such categorification of these algebras has a very handy property - similarly as algebras, it has a presentation with generators and relations but not only they generate all the algebras simultaneously, but often are significantly simpler than any single one of them. Such presentation can be done in any tensor category, i.e. symmetric monoidal and linear. Consider for example the Brauer category. Under monoidal product $\oplus$ and the linear structure on enriched hom, it is generated by the objects*

$0, 1$ *and three arrows* $2 \to 2, 0 \to 2, 2 \to 0$



subject to relations



The presentation of the Brauer algebras $\mathrm{Br}_n(\delta)$ *not only describe only a single object in this category, but are also significantly more complicated - each algebra* $\mathrm{Br}_{n+1}(\delta)$ *has minimally* $2n$ *generators and* $\frac{1}{2}(3n^2+25n-10)$ *relations!*

Example 615 (Adjoint metric spaces)**.** *Adjoint functors make sense in enriched case without any modifications. Let's understand for examp,e adjoints in Lawvere metric spaces. The hom formula identifies them as short maps satisfying*

$$d_Y(L(x), y) = d_X(x, R(Y))$$

The unit-counit formulas on the other hand shows that equivalently for all $x \in X$ *and* $y \in Y$ *holds*

$$d_X(x, RL(x)) = d_Y(LR(y), y) = 0$$

An example of such an adjunction is for example the subspace inclusion $i : [0, d] \hookrightarrow [0, \infty] = \mathcal{V}$ *with canonical, asymmetric metric. Its left adjoint is the truncation* $L(x) = \min(x, d)$*.*

Example 616 (Injective envelope)**.** *The injective envelope generalises the notion of convex hull of (symmetric) metric spaces, or more generally, of injective resolution. The construction starts from the space of extremal functions* $\mathrm{Aim}(X)$ *- real valued functions* $f : X \to \mathbb{R}$ *satisfying for every pair of points*

$$f(x) + f(y) \geq d(x, y)$$

the injective envelope $\mathcal{E}(X)$ *is the subset of minimal (in the poset* $f \leq g$ *iff on all points* $f(x) \leq g(x)$*) extremal functions with the supremum metric.*

The injective envelope can be also realised in different way, which happens to be isometric, called the tight span. It is the subspace of the presheaves $T(X) \subseteq \hat{X}$ *satisfying for all points*

$$f(x) = \sup d(f, h_x)$$

The presentation as tight spans is categorically a bit nicer, as has canonical inclusion in the presheaves - for example it's easy to see that the Yoneda embedding factors through $T(X)$

$$X \hookrightarrow T(X) \hookrightarrow \hat{X}$$

It means that we can think about $T(X)$ *as an extension of* X*. Now let's understand its points - suppose* $f \in T(X)$ *represents the point* p*, lying in the extension of* X*, which we'll informally treat as a virtual* *point and f - it's "virtual Yoneda embedding". Then the definition of $T(X)$ implies*

$$d(x,p) = \sup_{y \in X}(d(d(p,y), d(x,y))$$

in particular it means that for any $\epsilon > 0$ we can find such an $y \in Y$ that

$$d(x,p) + d(p,y) \leq d(x,y) + \epsilon$$

This property yields a reasonable interpretation of injective envelope - given such p, for any point $x \in X$ p lies arbitrarily close to some geodesic line connecting x with some other point, but not necesserily it belongs to such geodesic.

Let's analyse how does it look in the simplest case, where X has two points and $d(0,1) = d(1,0) = r$. We've seen that presheaves on X can be identified with a strip with corner. $T(X)$ then has a simpler form - it's just the convex hull of points, embedded via Yoneda in the plane

$$T(X) = \{(x,y) \in [0,\infty]^2 \mid x + y = r\} = \mathrm{conv}((0,r),(r,0))$$

Example 617.

13.2. **Kan extensions.**

Example 618 (Weak connected components and self-loops)**.** *Recall that the functor category $[\downarrow\downarrow, \mathbf{Set}]$ is isomorphic to the category of directed graphs. There is a natural functor sending a set X to the graph $\Delta(X)$ with vertices from X with single self-loops. Formally, it sends X to a functor $\Delta(X)$ by specifying values on both points and arrows from $\downarrow\downarrow$*

$$\Delta(X)(0) = \Delta(X)(1) = X \Delta(X)(V) = \Delta(X)(E) = \mathbb{1}_X$$

Such a functor has also a presentation as induced by precomposition with the final functor $p : \downarrow\downarrow \to 1$. Thus the adjoint functors of Δ can be identified as Kan extensions along p:

$$\begin{tikzcd}
\downarrow\downarrow \arrow[rr, "G"] \arrow[dr, "p"'] & & \mathbf{Set} \\
& 1 \arrow[ur, dotted] &
\end{tikzcd}$$

Simple calculation shows that the left Kan extension Σ maps a graph to its set of weak connected components (connected components of the underlying undirected graph), while right Kan extension Π maps a graph to set of all self-loops on vertices of G. It can be summarized as the adjoint triple

$$\Pi \vdash \Delta \vdash \Sigma$$

Example 619 (Homs). *Consider a small category $\mathcal{C}$ and its maximal discrete subcategory $D\mathcal{C} \hookrightarrow \mathcal{C}$. We get the functor $D\mathcal{C} \to \mathbf{Set}$ sending each object X to the singleton set $\{X\}$. Then the left Kan extension induced from the diagram*

$$\begin{tikzcd} D\mathcal{C} \arrow[rr, "sing"] \arrow[dr, hook] & & \mathbf{Set} \\ & \mathcal{C} \arrow[ur, dotted] & \end{tikzcd}$$

sends the object $X \in \mathcal{C}$ to the set of arrows with target X, i.e. $\mathrm{Hom}(-, X)$, *and arrows $X \to Y$ acts by composition inducing* $\mathrm{Hom}(-, X) \to \mathrm{Hom}(-, Y)$.

Example 620 (Products). *Let's work out one of the simplest examples, where we construct a product and coproduct in* $\mathbf{Set}$ *as right and left Kan extension. To do that, consider extending the functor from following diagram*

$$\begin{tikzcd} \{0,1\} \arrow[rr, "\Delta_F"] \arrow[dr, "p"] & & \mathbf{Set} \\ & \{0\} \arrow[ur, dotted] & \end{tikzcd}$$

Since we can explicitly identify the functor categories

$$[\{0\}, \mathbf{Set}] \simeq \mathbf{Set}$$

$$[\{0,1\}, \mathbf{Set}] \simeq \mathbf{Set} \times \mathbf{Set}$$

The left Kan extension satisfies the condition

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Lan}F_p(0), G(0)) \simeq \mathrm{Hom}_{\mathbf{Set}\times\mathbf{Set}}(F(0,1), (p\circ)G(0,1))$$

After fixing $X = F(0), Y = F(1), Z = G(0), p(0) = 0$ it just translates to

$$\mathrm{Hom}_{\mathbf{Set}}(\mathrm{Lan}F_p(0), Z) \simeq \mathrm{Hom}_{\mathbf{Set}\times\mathbf{Set}}(X \times Y, Z)$$

which is just a universal property of a product $X \times Y$. Similar situation we can encounter not only for coproducts, but for any limits and colimits, as we will see in the next example.

Example 621 (Limits and colimits)**.** *Consider a following setting for small category $\mathcal{C}$ and bicomplete category $\mathcal{D}$:*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "K"'] & & \mathcal{D} \\ & \mathbb{1} \arrow[ur, dotted] & \end{tikzcd}$$

Every functor $e : \mathbb{1} \to \mathcal{D}$ is constant, so the functor $e \circ K$ is also the constant functor Δ_e. The natural transformation $F \Rightarrow \Delta_e$ is some cone with summit e, thus the left Kan extension is a terminal object among these cones, which is the colimit of F. The same reasoning shows that a right Kan extension corresponds to its limit.

Note how this example generalizes all previously considered - the situation is the same as in the case of connected components, but we've replaced **Set** *with arbitrary category $\mathcal{D}$ and $\downarrow\downarrow$ with arbitrary small category $\mathcal{D}$ and got the classical adjunction*

$$\operatorname{colim} \vdash \Delta \vdash \lim$$

Example 622 (Adjoints)**.** *Consider a functor between small categories $F : \mathcal{C} \to \mathcal{D}$. The existence of its left/right adjoint functor can be checked via computing Kan extension, as well as the adjoint can be explicitly constructed. The left adjoint functor $G : \mathcal{D} \to \mathcal{C}$ exists and has a form $\mathrm{Lan}_F \mathbb{1}$ i and only the diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathbb{1}"] \arrow[dr, "F"'] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \\ & \mathcal{D} \arrow[u, dotted, "\mathrm{Lan}_F\mathbb{1}"] \arrow[ur, dotted, "\mathrm{Lan}_F F"'] & \end{tikzcd}$$

commutes up to isomorphism.

Example 623 (Skeleton and coskeleton)**.** *This example comes from the theory of simplicial sets. Let $\Delta_{\leq n}$ be a full subcategory of Δ with objects $\{0, \cdots, n\}$. Its inclusion induces the truncation functor via precomposition, acting forgetfully on simplicial set, removing its parts indexed by numbers bigger than n.*

$$\mathrm{tr}_n : \mathbf{sSet} \to \mathbf{sSet}_{\leq n}$$

Its left and right Kan extensions being the skeleton and coskeleton of X.

$$\begin{tikzcd}
\Delta^{op}_{\leq n} \arrow[rr, "X"] \arrow[dr, hook] & & \mathbf{Set} \\
& \Delta^{op} \arrow[ur, dotted, bend left, "\mathrm{sk}_n"] \arrow[ur, dotted, bend right, "\mathrm{cosk}_n"'] &
\end{tikzcd}$$

The resulting adjoint triple

$$\mathrm{sk}_n \dashv \mathrm{tr}_n \dashv \mathrm{cosk}_n$$

but by a slight abuse of notation we will denote $\mathrm{sk}_n = \mathrm{sk}_n \circ \mathrm{tr}_n : \mathbf{sSet} \to \mathbf{sSet}$ *and* $\mathrm{cosk}_n = \mathrm{tr}_n \circ \mathrm{cosk}_n : \mathbf{sSet} \to \mathbf{sSet}$*, which form an adjoint pair. Let's explore the idea behind these functors: the skeleton, as the left Kan extension, it factors every morphism between truncated complexes, meaning that*

$$\mathrm{Hom}(tr_n(X), tr_n(Y)) \simeq \mathrm{Hom}(sk_n(X), Y)$$

Intuitively, it means that the $n+1$ or more dimensional part must be maximally initial, without changing the n or less dimensional, thus the set $sk_n(X)$ will have no non-degenerate simplices of dimension bigger than n. The opposite story provide the coskeleton, which to satsify

$$\mathrm{Hom}(tr_n(X), tr_n(Y)) \simeq \mathrm{Hom}(X, \mathrm{cosk}_n(Y))$$

must approximate the terminal object on the new part. It means that every simplex of dimension less than n that can be placed in $\mathrm{cosk}_n(Y)$ *will be placed whenever its boundary is already there (the only necessary condition for it being placed), so taking the skeleton is something corresponding to "filling down all the holes of dimension $n+1$ or more". Using the more sheafy language, its construction can be also expressed as*

$$\mathrm{cosk}_n(X) = i^! i_* X$$
$$\mathrm{sk}_n(X) = i^* i_* X$$

Example 624 (Augmented simplicial sets)**.** *A category Δ can be seen as a full subcategory of a category Δ_+, which is the same as Δ with additional initial object -1. Presheaf category* $[\Delta_+^{op}, \mathbf{Set}]$ *is a category of augmented simplicial sets and the set $X[-1]$ is called the augmentation of X. Precomposing by the inclusion provides two Kan extensions* *augmenting the simplicial set.*

$$\begin{tikzcd} \Delta^{op} \arrow[rr, "X"] \arrow[dr, hook] & & \mathbf{Set} \\ & \Delta^{op}_{+} \arrow[ur, dotted, bend left, "\pi_0"] \arrow[ur, dotted, bend right, "triv"'] & \end{tikzcd}$$

The left Kan extension construct the augmentation out of all the path components of X, while the right Kan extension is just the trivial one, mapping $X[-1]$ to the singleton set.

Example 625 (Induced and coinduced representations)**.** *Let G be a group and H its subgroup. Recall that linear representations of G can be identified with functors $\mathbf{B}G \to \mathbf{Vect}_K$. Precomposition with the inclusion $H \hookrightarrow G$ has two natural adjoints, constructed as Kan extensions*

$$\begin{tikzcd} H \arrow[rr, "X"] \arrow[dr, hook] & & \mathbf{Vect}_K \\ & G \arrow[ur, dotted, bend left, "coind"] \arrow[ur, dotted, bend right, "ind"'] & \end{tikzcd}$$

These functors, called induced and coinduced representations, provide two ways of extending the representations of H to a representation of G. Whenever G is finite, or more generally compact, induced and coinduced representations coincide.

Example 626 (Preorder classification)**.** *Consider the classical classification problem: given some already classified training set, we try to extrapolate the classification to a bigger set in order to make a prediction. It can be given following categorical model: all data are represented by some preorder D. Initial labeling of some portion of the training data from are given by some subset $T \subset D$ together with some arbitrary classifying functor $C : T \to \{0 < 1\}$ (where $\{0 < 1\}$ is preordered). Since the classification has no prior structure, we treat T as a discrete category, so that F is just a morphism of underlying sets. The classification problem is the extension of C to some monotonic function $D \to \{0 < 1\}$ (the preorder on D reflects some internal structure of the data, making the classification more elaborate than just guessing* *random labels). The training set can be identified as*

$$S = \{(x, C(x) \mid x \in T)\}$$

We can try to solve the problem by calculating Kan extensions of given two functors

$$\begin{tikzcd} D \\ T \arrow[u] \arrow[r, "C"] & \{0<1\} \arrow[from=1-1, dotted, "\mathrm{Lan}_i C"] \arrow[from=1-1, dotted, "\mathrm{Ran}_i C"'] \end{tikzcd}$$

Such a simple classifier can be computed on paper:

$$\mathrm{Lan}_i C(x) = \begin{cases} 1 & \exists t \in T : t \leq x, C(t) = 1 \\ 0 & otherwise \end{cases}$$

$$\mathrm{Ran}_i C(x) = \begin{cases} 0 & \exists t \in T : x \leq t, C(t) = 0 \\ 1 & otherwise \end{cases}$$

We can understand the left Kan extension as resolving all possible conflicts on behalf of 1 *(minimize false negatives), while the right extension favours* 0*, minimizing false positives. Points from* D *splits into* 3 *distinct parts: unambiguously classified as* 0*,* 1*, or ambiguous points, on which values of* $\mathrm{Lan}_i C$ *and* $\mathrm{Ran}_i C$ *differ.*

Example 627 (Clustering)**.** *The other example inspired by data science is solution to the problem of clustering. Consider a category of metric spaces* **Met** *with short maps and the category* **Part** *of sets with partitions, where objects are* (X, P_X)*, and map* $(X, P_X) \to (Y, P_Y)$ *are is a function* $f : X \to Y$ *that do not separate points (so that if two elements* $a, b \in X$ *lies in a common region* $S \in P_X$*, then so does their images). The problem of supervised clustering is extending some clustering functor defined on some subcategory* $T \subset$ **Met** *to a bigger subcategory* $T \subset D \subset$ **Met***. Similarly to classification, it can be solved using Kan extensions*

$$\begin{tikzcd} D \\ T \arrow[u] \arrow[r, "C"] & \mathbf{Part} \arrow[from=1-1, dotted, "\mathrm{Lan}_i C"] \arrow[from=1-1, dotted, "\mathrm{Ran}_i C"'] \end{tikzcd}$$

Note that this similarly looking example to the classification is in fact more elaborate - the category D *can have both more objects and more* *morphisms that T. This time the difference between left and right extension lies in the number of created clusters -* $\mathrm{Lan}_i C$ *essentially maximizes the number of clusters, while* $\mathrm{Ran}_i C$ *minimizes. The difference is similar to finding finest or coarsest topology on a set satisfying certain conditions.*

Example 628 (Geometric realisation)**.** *Given any functor $F : \Delta \to \mathcal{C}$, if $\mathcal{C}$ is cocomplete, its left Kan extension along the Yoneda embedding exists and naturally extends such a functor to $LF : \mathbf{sSet} \to \mathcal{C}$. Take for example the functor $G : \Delta \to \mathbf{Top}$ mapping $[n]$ to the standard n-simplex. The geometric realisation is just its extension LG. Note that we automatically deduce its adjunction to the total singular complex functor, arising from the precomposition.*

Example 629 (Nerve and homotopy category)**.** *The other functor that can be extended along Yoneda embedding is $D : \Delta \to \mathbf{Cat}$, assigning to $[n]$ the category $\mathbf{n}$ associated to the poset $0le \cdots len$. The functor constructed by the postcomposition is the nerve $N : \mathbf{Cat} \to \mathbf{sSet}$, computed as*

$$N\mathcal{C}_n = \mathrm{Nat}(\mathbf{n}, \mathcal{C})$$

Its left adjoint arising from the left Kan extension is interesting, as it turns out to correspond to the homotopy category $Ho : \mathbf{sSet} \to \mathbf{Cat}$. Its objects are the vertices of X and arrows are freely generated by classes of composable paths of edges, where composition of two edges are identified with the third one if together form a boundary of some 2*-simplex.*

Example 630 (Homotopy coherent nerve)**.** *The construction of a type nerve-realisation, involving the left Kan extension along Yoneda embedding and its left adjoint is quite universal, as it work for any functor with cocomplete domain. Now we'll consider more sophisticated version of the nerve construction, where the codomain will be the category of* **sSet***-enhanced categories, which we'll denote as $\mathbf{Cat}_\Delta$. The initial functor $\Delta \to \mathbf{Cat}_\Delta$ takes n to the category $\mathbf{n}_\Delta$ with objects $\{0, \cdots, n\}$. Since we want it to be* **sSet***-enhanced, morphisms $\mathbf{n}_\Delta(m, k)$ should form some simplicial set. The most convenient choice turned out to be the nerve of the poset category constructed from subsets of $\{m, \cdots, k\}$. After the weird start, the rest of the construction looks pretty much the same as in the case of classical nerve: left Kan extension S_Δ has a standard coend formula*

$$S_\Delta(X) = \int^n \mathbf{n}_\Delta \times X_n$$

and is right adjoint to the homotopy coherent nerve functor N_Δ*, again similarly constructed as the standard nerve, with the only difference that this time we consider only enhanced transformations.*

$$N_\Delta(\mathcal{C})_n = \mathrm{Nat}_{\mathbf{sSet}}(\mathbf{n}_\Delta, \mathcal{C})$$

Example 631 (Moerdijk generalised intervals)**.** *The other slight variation of the nerve construction is by changing the unit interval to any other ordered topological space* J *with minimum* 0 *and maximum* 1*. The geometric realisation with respect to* J *is identical to the standard one, induced by the functor*

$$|\Delta^n_J| = \{(x_0, \cdots, x_n) \mid x_0 \leq \cdots \leq x_n\} \subseteq J^{n+1}$$

Example 632 (Dold-Kan correspondence)**.** *The Dold-Kan correspondence is a central result in simplicial homotopy theory, stating the equivalence of categories of simplicial abelian groups (i.e. functors* $[\Delta^{op}, \mathbf{Ab}]$ *and chain complexes of abelian groups of non-negative indices*

$$\mathbf{sAb} \simeq \mathrm{Ch}^+(\mathbf{Ab})$$

The equivalence can be easily constructed using the nerve-realisation principle. Recall that given simplicial abelian group $A_\bullet$*, its Moore complex* $M(A_\bullet)$ *is a chain complex of abelian groups*

$$\cdots \to A_2 \to A_1 \to A_0 \to 0$$

with classical boundary map similar to the simplicial complex of a space

$$\partial_n = \sum_{k=0}^{n} (-1)^k d_k : A_n \to A_{n-1}$$

Using the Moore complex one easily construct a functor $d : \Delta \to \mathrm{Ch}^+(\mathbf{Ab})$ *by putting*

$$d(n) = M(\mathbb{Z}[\Delta^n])$$

Its left Yoneda extension $N : \mathbf{sAb} \to \mathrm{Ch}^+(\mathbf{Ab})$*, called the normalised chain complex, is given by the formula*

$$NA_n = \bigcap_{k=0}^{n-1} \mathrm{ker} d_k$$

with boundary maps $\partial_n = (-1)^n d_n$*. Its left adjoint have a simple form*

$$\Gamma(C_\bullet)_n = \bigoplus_{[n] \twoheadrightarrow [m]} C_m$$

and together with N *form an adjoint equivalence.*

Example 633 (Subdivision). *Kan extensions provide an elegant way of constructing in simplicial sets a barycentric subdivision, which turned out to be quite useful in case of simplicial complexes. It allows to skip construction for a general functor and work only with representable Δ^n instead, where the construction is trivial. This way without much work we get the subdivision endofunctor*

$$sd : \mathbf{sSet} \to \mathbf{sSet}$$

and its adjoint called extension $Ex = \mathrm{Lan}_d y$

Example 634 (Cocontinuous presheaves). *Let $\mathcal{C}$ be small and $\mathcal{D}$ cocomplete. The left Kan extension along the Yoneda embedding*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, hook, "y"'] & & \mathcal{D} \\ & \hat{\mathcal{C}} \arrow[ur, dotted] & \end{tikzcd}$$

has a property of classifying all the cocontinuous presheaves, which subcategory can be identified with $\mathrm{essim}(\mathrm{Lan_y}\text{-})$

Example 635 (Category of left adjoints). *Consider a special case of a Yoneda extension, where the codomain is also a category of presheaves on small category $\mathcal{D}$.*

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, hook, "y"'] & & \hat{\mathcal{D}} \\ & \hat{\mathcal{C}} \arrow[ur, dotted] & \end{tikzcd}$$

In this case not only does $\mathrm{essim}(\mathrm{Lan_y}\text{-})$ *classify all the preshaves, but all the left adjoint functors, thus can be identify with a full subcategory of $[\hat{\mathcal{C}}, \hat{\mathcal{D}}]$ consisting all the left adjoints.*

Example 636 (Smallness and completeness). *The notion of (co)completeness of categories measures the existence of (co)limits. One can restrict this notion also for some smaller class, for example consider small (co)completeness and small (co)limits. Completeness and smallness can be extended to much broader class of functors, where it forms a Galois connection. Consider some functor $F : \mathcal{C} \to \mathcal{D}$. The "(right/left) smallness" of F is measured by the class of categories $\mathcal{E}$, such that for any functor $\mathcal{C} \to \mathcal{E}$ the pointwise (right/left) Kan extension along F*

exists. Dually, the "(co)completeness" of category $\mathcal{E}$ is measured by the class of all such functors $F : \mathcal{C} \to \mathcal{D}$, that any Kan extension along F exists. These (large) preorders are then related via Galois connection. In case of $F : I \to 1$ we recover the existence of I-shaped (co)limits, thus a classical notion of completeness.

Example 637 (Orbit object)**.** *Consider an action of monoid M on small category $\mathcal{C}$. If $\mathcal{C}$ is complete with respect to functor $M \to 1$ (in a sense explained in the previous example), then we can construct the generalised orbit space as the image of its left Kan extension*

$$\begin{tikzcd} M \arrow[rr, "\mu"] \arrow[dr] & & \mathcal{C} \\ & 1 \arrow[ur, "orb(\mu)"'] & \end{tikzcd}$$

Example 638 (Partial periodicity)**.** *Recall that a continuous dynamical system can be modelled as a functor $\mathbf{B}\mathbb{R} \to \mathcal{C}$, where $\mathbb{R}$ is an additive monoid encoding the time translation (that can be chosen differently than real numbers, but we will stick to $\mathbb{R}$ to get more clear picture), acting on object $X \in \mathcal{C}$. We can identify N-periodic systems as admitting factorisation through the universal covering $\mathbb{R} \xrightarrow{T} S^1$*

$$\begin{tikzcd} \mathbf{B}\mathbb{R} \arrow[rr, "\mu"] \arrow[dr, "N"'] & & \mathcal{C} \\ & \mathbf{B}S^1 \arrow[ur] & \end{tikzcd}$$

Kan extensions along such covering

$$\begin{tikzcd} \mathbf{B}\mathbb{R} \arrow[rr, "\mu"] \arrow[dr, "N"'] & & \mathcal{C} \\ & \mathbf{B}S^1 \arrow[ur, dotted, "\Pi_N/\Sigma_N"'] & \end{tikzcd}$$

allows us to capture the partial periodicity in X. The right extension $\Pi_N X$, factorising each $\mathbb{R}$-equivariant morphism of a form

$$\begin{tikzcd} \mathbf{N}^*P \arrow[rr] \arrow[dr, "N"'] & & X \\ & \mathbf{N}^*\Pi_N X \arrow[ur, dotted, "N^*\varphi"] & \end{tikzcd}$$

where X is acted an by S^1 and in N^*X the action is given by $a \cdots [x] = (a + N\mathbb{Z}) \cdot x$. Thus $\Pi_N X$ can be identify with the biggest subspace of

X, where the action can be lifted through the covering map of S^1, i.e. its the largest subspace where the restriction of μ is N-periodic. Note that it can be seen as direct generalisation of the invariant G-module, obtain from induced representations. The left Kan extension on the other hand generalises the coinduction. Similar reasoning hints that it can be identified with the quotient space of X by all the periodic orbits. Unlike in the previous case, the form of such an object largely depends on the category $\mathcal{C}$, as the construction of quotient objects is far less universal than subobjects. Finally let's consider some example from topology. Let $X = \mathbb{C} \times \mathbb{R}$ be acted on by $\mathbb{R}$ as $\lambda(z, t) = (e^{2\pi i t\lambda}, t)$. Then the space $\Pi_1 X$ is a subspace $\mathbb{C} \times \mathbb{Z} \cup \{0\} \times \mathbb{R} \subset \mathbb{C} \times \mathbb{R}$, acted on by S^1. On the other hand the space $\Sigma_1 X$ is wild - each slice $\mathcal{C} \times \{t\}$ it is either fixed when t is an integer, finite branched covering if t is rational or uncountably many dense collections of lines glued together attached to the origin. It is obviously not Hausdorff.

Example 639 (Periodic spectrum)**.** *The partially periodic subobjects of continuous dynamical system $\mu : \mathbf{B}\mathbb{R} \to \mathcal{C}$ can be assembled together to a single functor encoding all the information about periodicity of μ. We say that a period K divides the period N whenever the factorisation of N can be further factored through K, as presented on the diagram below*

$$\begin{tikzcd}
\mathbf{B}\mathbb{R} \arrow[rr, "\mu"] \arrow[d] \arrow[drr, "N"] & & \mathcal{C} \\
\mathbf{B}S^1 \arrow[rr, "K"'] \arrow[urr, dotted, "\Pi_K"] & & \mathbf{B}S^1 \arrow[u, dotted, "\Pi_N"']
\end{tikzcd}$$

Note that periods with divisibility relation form a preorder, which associated category we'll denote as $\mathcal{Q}$. Then the partially periodic subobjects $N \mapsto \Pi_N X$ form a contravariant functor, called the periodic pre-spectrum

$$[\mathbf{B}\mathbb{R}, \mathcal{C}] \to [\mathcal{Q}^{op}, \mathcal{C}]$$

The periodic pre-spectrum is just a diagram of inclusions of periodic subspaces parameterised by their periods. The proper periodic spectrum however has a goal of encoding all the information about the periodicity of μ in a minimal fashion. The pre-spectrum in fact encodes a lot of additional data, that can be compressed to only the orbits of periodic states. A few examples ago we've constructed the orbit object as a functor $[\mathbf{B}\mathbb{R}, \mathcal{C}] \to \mathcal{C}$ for nice categories $\mathcal{C}$. Then parametrising the inputs and outputs of pre-spectrum by elements of $\mathcal{Q}$ we can construct *the spectrum as indicated in the diagram*

$$\begin{tikzcd}
 & {[\mathbf{B}\mathbb{R}, \mathcal{C}]} \arrow[d] \arrow[ddl] \arrow[ddr] & \\
 & {[\mathcal{Q}^{op}, [\mathbf{B}\mathbb{R}, \mathcal{C}]]} \arrow[dl, "\mathrm{preSpec}"'] \arrow[dr, "\mathrm{Spec}", "\mathrm{orb}"'] & \\
{[\mathcal{Q}^{op}, \mathcal{C}]} & & {[\mathcal{Q}^{op}, \mathcal{C}]}
\end{tikzcd}$$

Example 640 (Extending Lipschitz functions)**.** *Consider two Lawvere metric spaces $X \to Y$ and recall that every k-Lipschitz function we've realised as a **Cost**-functor $kX \to Y$, where kX is the metric on **Cost** induced by composition with $f(x) = nx$. Since the **Cost**-functor $i : X \to Y$ is fully faithful iff it is an isometric embedding, the nerve-realisation principle shows that every k-Lipschitz function $X \to \mathbb{R}$ can be extended along i with the same Lipschitz constant k.*

$$\begin{tikzcd}
kX \arrow[rr, "f"] \arrow[dr, hook] & & \mathbf{Cost} \\
 & kY \arrow[ur, dotted] &
\end{tikzcd}$$

Moreover, we can calculate these Kan extensions explicitly and obtain largest and smallest among all such extensions with respect to pointwise metric

$$\bar{f}(y) = \inf_{x \in X}(f(x) + k \cdot d(i(x), y))$$

$$\bar{f}(y) = \sup_{x \in X}(f(x) - k \cdot d(y, i(x)))$$

Example 641 (Riemann integral)**.** *The Riemann integration can be presented as similar extension problem. Consider $S(\mathbb{R}^n)$ as the space of non-negative step functions, embedded in $C(\mathbb{R}^n)$ of all non-negative functions with sup metric (non-symmetric sup metric - set $d(f, g) = 0$ iff $g \geq f$). The integration of step functions is trivial, forming a functor $S(\mathbb{R}^n) \to$ **Cost**. Then the right and left Kan extensions correspond to the lower and upper Riemann integrals*

$$\begin{tikzcd}
S(\mathbb{R}^n) \arrow[rr, "f"] \arrow[dr, hook] & & \mathbf{Cost} \\
 & C(\mathbb{R}^n) \arrow[ur, dotted, "\bar{\int}"] &
\end{tikzcd}$$

Moreover, a function $f \in C(\mathbb{R}^n)$ is Riemann integrable if and only if both extensions on f coincide.

Example 642 (Tangent spaces and differential forms)**.** *During the construction of diffeological spaces we've discussed that the sheaf of differential forms on a manifold have a structure of diffeological space. Let's go back to fill some details about this object. Differential forms, or rather the de Rham complex, putting them all together, form a sheaf on M, thus a functor $\Omega : \mathrm{Open}(M)^{op} \to \mathbf{Set}$. Its global sections form a space of differential forms on M. This structure can be easily made into diffeological space via*

$$\Omega(X) = \lim U \to X\Omega(U)$$

Similarly, the tangent space form a cosheaf on each smooth manifold M and also admits similar diffeology

$$T(X) = \operatorname{colim} U \to XT(U)$$

These constructions are functorial and can be extended to all diffeological spaces using left Kan extensions along the Yoneda inclusion of smooth manifolds in diffeological spaces

$$\begin{tikzcd} \mathbf{Man}^{op} \arrow[rr, "\Omega"] \arrow[dr, hook] & & \mathbf{Diff} \\ & \mathbf{Diff}^{op} \arrow[ur, dashed] & \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Man} \arrow[rr, "T"] \arrow[dr, hook] & & \mathbf{Diff} \\ & \mathbf{Diff} \arrow[ur, dashed] & \end{tikzcd}$$

Example 643 (Tangent space of groupoid)**.** *A natural generalisation of Lie groups in enriched category theory are diffeological groupoids - groupoids enriched in* **Diff**. *The Kan extension along Yoneda embedding provides a way of extending the most important features of Lie groups to Lie groupoids - the tangent map, connecting them with Lie algebras or - in this case - Lie algebroids. Unfortunately, the tangent extension is not particularly well-behaved functor, as it does not commute with limits, especially with pullbacks, which is a key feature of tangent maps. This problem can be fixed by restricting ourselves to a subcategory of* **DiffGrpd** *called elastic Lie groupoids* **Elst**, *behaving well under pullbacks of tangent functors. Among elastic groupoids we can find also*

- *diffeological groups*
- *manifolds*
- *manifolds with cusps*
- *pro-finite manifolds, such as infinite jet bundle*
- *sections of smooth fiber bundles (vector fields, differential forms, metrics, curvatures)*

- *function spaces of manifolds*
- *action groupoids*

In fact, such "nice behaviour" under pullbacks was defined using a pretty trick, once again involving Kan extension - this time to the pullback functors of tangent spaces $T_kX = TX \times_X TX \times_X \cdots TX$. After extending all these maps together with ordinary tangent functor, we may define elastic groupoids as spaces, where extended iterated pullback is naturally isomorphic with ordinary pullback of extended tangent space in the category **DiffGrpd**.

It turns out that elastic groupoids contains most examples considered in nature, in particular all the diffeological groups, providing a most important tool used in theory of Lie groups.

Example 644 (Algebroid of diffeomorphism groupoid)**.** *A group of diffeomorphisms of any smooth manifold is always an infinite-dimensional, thus do not form a Lie group. However, it can be given a smooth structure as diffeological group or groupoid. The algebroid, associated with* $\mathrm{Aut}(M)$ *by the tangent functor, turns out to be just a Lie algebra of vector fields, with usual Lie bracket (with opposite sign, hence the opposite category)*

$$\mathrm{Lie}(\mathrm{Aut}(M)) \simeq \mathfrak{X}(M)^{op}$$

13.3. **Ends and coends.**

Example 645 (Alternative constructions of ends)**.** *Ends and coends can be constructed in a number of ways using different tools. Firstly, end can be constructed* ***as limit****. Given a functor $F : \mathcal{C}^{op} \times \mathcal{C} {\to} \mathcal{D}$, if $\mathcal{C}$ is small and $\mathcal{D}$ complete, its end corresponds to the equaliser (similarly one constructs coends as the coequaliser)*

$$\int_{\mathcal{C}} F = \lim\left(\prod_{X\in\mathcal{C}} F(Y,X) \rightrightarrows \prod_{X\to Y} F(X,X)\right)$$

$$\int^{\mathcal{C}} F = \mathrm{colim}\left(\coprod_{X\to Y} F(X,X) \rightrightarrows \coprod_{X\in\mathcal{C}} F(Y,X)\right)$$

Alternatively, it also can be constructed ***as adjoint*** *to the hom functor*

$$\begin{tikzcd} {[\mathcal{C}^{op} \times \mathcal{C}, \mathcal{D}]} \arrow[r, bend left, "\text{end}"] \arrow[r, bend right, "\text{coend}"'] & \mathcal{D} \arrow[l, "\text{Hom}"'] \end{tikzcd}$$

Example 646 (Natural transformations)**.** *Consider two functors $F, G : \mathcal{C} \to \mathcal{D}$. We can form a bifunctor* $\mathrm{Hom}(F(-), G(-)) : \mathcal{C}^{op} \times \mathcal{C} \to \mathbf{Set}$.

The end of such a functor correspond exactly to natural transformations $F \Rightarrow G$:

$$\mathrm{Nat}(F, G) = \int_{X \in \mathcal{C}} \mathrm{Hom}(F(X), G(X))$$

Example 647 (Conjugacy classes)**.** *Let $\mathbf{B}G$ be a group delooping. The coend of the hom bifunctor $\mathbf{B}G \times \mathbf{B}G^{op} \to \mathbf{Set}$*

$$\int^{\bullet} \mathrm{Hom}(\bullet, \bullet) = \mathrm{coeq}\left(\coprod_G G \rightrightarrows G\right)$$

This coequaliser can be interpreted as the set of elements of G under relation $gh \simeq hg$, which is just the set of conjugacy classes of the group G.

Example 648 (Trace of a category)**.** *In a more general case when $\mathcal{C}$ is small, but not necessarily a delooping, the coend*

$$\int^{c \in \mathcal{C}} \mathrm{Hom}(c, c)$$

called the trace of $\mathcal{C}$ has more complicated structure. It can be seen as factorisation classes of endomorphisms - a disjoint union of endomorphisms of objects

$$\coprod_{c \in \mathcal{C}} \mathrm{Hom}(c, c)/\sim$$

under the relation gluing endomorphisms factorising as some two arrows $\varphi = f \circ g$ with the endomorphism $g \circ f$.

Example 649 (Trace of sets)**.** *Factorisations of endomorphisms in sets provide an analogue of the trace for ordinary functions. Assume that the category of sets is small (for instance fixing some Grothendieck universe). Let denote the factorisations as*

$$\mathrm{Tr}(\mathbf{Set}) := \int^{X \in \mathbf{Set}} \mathrm{Hom}(X, X)$$

It can be understood through a family of maps $\omega_X : \mathrm{Hom}(X, X) \to E$, corresponding to the quotient map from a coproduct with a property reflecting a key property of the trace:

$$\omega_X(fg) = \omega_Y(gf)$$

In particular, it means that the restriction of any endomorphism $f : X \to X$ to its image $R(f) = f_{|\mathrm{im}(f) \to \mathrm{im}(f)}$ we have $\omega_X(f) = \omega_{\mathrm{im}(f)}(R(f))$.

Moreover, the group $\mathrm{Aut}(X)$ acts on $\mathrm{End}(X)$ by conjugation $\varphi \cdot f = \varphi^{-1} \circ f \circ \varphi$. This operation is also ω-invariant, as $\omega_X(f) = \omega_X(f \circ \varphi \circ \varphi^{-1}) = \omega_X(\varphi^{-1} \circ f \circ \varphi)$

The trace $\mathrm{Tr}(\mathbf{Set})$ *can be fully described in terms of actions of restriction operator* R *and conjugations as their orbits in* $\coprod_{\mathbf{Set}} \mathrm{End}(X)$*, or in other words* $\mathrm{Tr}(\mathbf{Set})$ *is a maximal set of values admitted by universal traces, and the property defining traces is equivalent to conjugation and* R*-invariance.*

Example 650 (Young diagrams)**.** *The trace of the category of finite sets* **Fin** *has much simpler form and can be identified with set of multisets of positive integers. Recall that a trace is defined as*

$$\mathrm{Tr}(\mathbf{Fin}) = \int^{X \in \mathbf{Fin}} \mathrm{Hom}(X, X)$$

and it has been identified in the previous with a maximal labeling of endomorphisms which is invariant under conjugations and restrictions. If X *is finite, iterated restrictions of any endomorphism stabilises - for any* $f : X \to X$ *there is always such* $N \leq |X|$ *such that for any* $n > N$

$$\mathrm{core}(f) := R^N(f) = R^n(f) = \{x \mid \exists_k : f^k(x) = x\}$$

The value $\omega_X(f)$ *depends only on the bijection* $\mathrm{core}(f)$*. Moreover, any element* $x \in \mathrm{core}(f)$ *represents some* f*-cycle. Its cycle length,* $c_f(x)$*, is the smallest positive integer* k *such that* $f^k(x) = x$*. All the* f*-cycle lengths of elements of* X *are called its cycle type of* f*. Its easy to see that endomorphism with the same type are conjugate via an automorphism respecting the cycles, so* $\omega_X(f)$ *depends only on its cycle type. Moreover, conjugation fixes the cycle type, thus cores of endomorphisms* f, g *with different classes are not conjugate, so* $\omega_X(f) \neq \omega_X(g)$*. This establish a bijection between* $\mathrm{Tr}(\mathbf{Fin})$ *and the set of all possible cycle types of endomorphisms, which on the other hand are in bijection with multisets of positive integers or with classes of finite Young diagrams, corresponding to partition of* X *to stable subsets.*

Example 651 (Tensor product of modules)**.** *Consider (not necesserily commutative) ring* R *as* **Ab***-category with single object. The category of left/right* R*-modules has a natural construction as*

$$\mathbf{Mod}\text{-}R = \mathrm{Nat}_{\mathbf{Ab}}(R^{op}, \mathbf{Ab})$$
$$R\text{-}\mathbf{Mod} = \mathrm{Nat}_{\mathbf{Ab}}(R, \mathbf{Ab})$$

Using the tensor product of abelian groups, for any right module A *and left module* B *we get a functor (aka its value on single object)*

$$A \otimes_{\mathbb{Z}} B : R^{op} \times R \to \mathbf{Ab}$$

The tensor product of these modules over R can be constructed as the coend os this functor

$$A \otimes_R B = \int^R A \otimes_{\mathbb{Z}} B$$

By the colimit formula, one can easily check that this construction coincides with the canonical algebraic definition of $\otimes_R$.

Example 652 (Complete metric spaces)**.** *Every functor $X \to Y$ induces canonically two profunctors, $f_* : X \nrightarrow Y$ and $f^* : Y \nrightarrow X$, both obtain from composition with hom. Moreover, such induced profunctors are always adjoint in the bicategory of profunctors, i.e. there are transformations*

$$1_X \Rightarrow f^* \circ f_*$$

$$f_* \circ f^* \Rightarrow 1_Y$$

satisfying the usual triangle identities. The naturally araising question is if all the adjoint profunctors arise in this way, i.e. are induced by some functor $f : X \to Y$. In case of Lawvere metric spaces, such question has very surprising answer - all adjoint profunctors are induced if and only if Y is a complete metric space. Let's sketch the proof of this fact. Firstly, to show that each adjoint pair is induced from a map $f \to \mathrm{comp}(Y)$ it's enough to check this on all of the points separately, thus we may assume that $X = \{\bullet\}$. Then the adjointness conditions reduce to

$$\inf_Y (f_*(y) + f_*(y)) = 0$$

$$f_*(y_1) + f_*(y_2) \geq d(y_1, y_2)$$

Thus for any n, m we can find y_n, y_m satisfying

$$f_*(y_n) + f_*(y_m) \leq \frac{1}{n} + \frac{1}{m}$$

thus such pair of adjoint profunctors generates the Cauchy sequence in Y, and moreover each such sequence is equivalent, as we have

$$d(y_n, y_m) \leq \frac{1}{n} + \frac{1}{m}$$

Conversely also each Cauchy sequence (y_n) of complete space yields adjoint profunctors given by

$$f_*(y) = \lim_{n\to\infty} d(y, y_n)$$

$$f^*(y) = \lim_{n\to\infty} d(y_n, y)$$

Example 653 (Complete metric spaces)**.** *Every functor $X \to Y$ induces canonically two profunctors, $f_* : X \nrightarrow Y$ and $f^* : Y \nrightarrow X$, both obtain from composition with hom. Moreover, such induced profunctors are always adjoint in the bicategory of profunctors, i.e. there are transformations*

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$$f^*(y) = \lim_{n\to\infty} d(y_n, y)$$

Example 654 (Cauchy completeness of modules)**.** *finitely generated projective modules - TODO*

—- TODO —- : on sets

Example 655 (Center of a ring)**.** *Consider a ring R as* **Ab***-enriched category with one element. The multiplication $M : R \times R^{op} \to$* **Ab** *is* *then given by the enriched hom functor . The end of hom functor we've already identified as*

$$\int_R M \simeq \mathrm{Nat}_{\mathbf{Ab}}(\mathbf{1}_R, \mathbf{1}_R)$$

Every such natural transformation by definition satisfies $\phi(rs) = \phi(r)\phi(s)$, and since its additive, its a ring endomorphism, thus given by multiplication by $r = \phi(1)$. Moreover, the naturality of ϕ can be expressed by the condition $rs = \phi(1)s = s\phi(1) = sr$ holding for any arrow s, yielding the natural bijection

$$\int_R M \simeq Z(R)$$

Example 656 (Commutator of a ring)**.** *Consider again the multiplication functor $M : R \times R^{op} \to \mathbf{Ab}$ from the previous example. This time we are interested with its coend, which using the colimit formula we can identify with quotient S of R under relations $p(sr) = p(rs)$. This is just its commutator*

$$\int_R M \simeq \frac{R}{\langle rs - sr \rangle}$$

Example 657 (Optimal flow)**.** *Consider the enriched profunctors in Lawvere metric spaces. We've seen already their interpretation in finite case based on graph theory, but this time we won't use this construction and look at the story described by their compositions in the category $\mathbf{Prof}(\mathbf{Cost})$ computed as coends. As usual, the profunctor $f : B^{op} \times A \to \mathbf{Cost}$ we'll denote as $A \nrightarrow B$. Such profunctors generalise equivalence relations on $A \times B$ to admitting certain weights, or as suggests the base of enrichment, costs. Intuitively, given two spaces, profunctor f encodes the price list of moving from any point $a \in A$ to $b \in B$ as a value $f(a, b)$. Composition of profunctors $A \overset{f}{\nrightarrow} B \overset{g}{\nrightarrow} C$ is given by the coend integrating over the space in the middle*

$$(g \circ f)(a, c) = \int^{b \in B} f(a, b) \otimes g(b, c)$$

Since the monoidal product in $\mathbf{Cost}$ is addition and every colimit is just the infimum over indexing subset, it can be explicitly computed as

$$(g \circ f)(a, c) = \inf_{b \in B} (f(a, b) + g(b, c))$$

So composing profunctors chooses optimal detour in B during the travel from $a \in A$ to $c \in C$. As you can imagine, a lot of optimization tasks can be presented this way, for example finding optimal flow after connecting several circulation systems.

Example 658 (Tensor product of functors)**.** *Construction of the tensor product of modules via coend generalised to much larger class of functors. Given monoidal cocomplete category $\mathcal{V}$ and functors $F : \mathcal{C}^{op} \to \mathcal{V}$, $G : \mathcal{C} \to \mathcal{V}$, their tensor product can be defined by extending the special case of modules*

$$F \boxtimes_{\mathcal{C}} G := \int^{\mathcal{C}} F(c) \otimes G(c)$$

—— nie wiem czy to jest dobrze ———

Example 659 (Tensoring real functions)**.** *Consider the monoidal preorder $([1, \infty], \leq, 1, \cdot)$ and functors $g(x) = x + 1 : \mathcal{C} \to \mathcal{C}$, $f(x) = \frac{1}{x} : \mathcal{C}^{op} \to \mathcal{C}$. Its tensor product*

$$f \boxtimes g = \int^{x \in [1,\infty]} f(x)g(x)$$

by the coequaliser formula yields

$$f \boxtimes g = \sup_{x \in [1,\infty]} (f(x)g(x)) = 2$$

Example 660 (Kan extensions)**.** *Kan extensions also can be constructed as ends and coends*

$$\mathrm{Lan_K F}(Y) = \int^{X \in \mathcal{C}} \coprod_{F(X)} \mathrm{Hom}(K(X), Y)$$

$$\mathrm{Ran_K F}(Y) = \int_{X \in \mathcal{C}} \prod_{F(X)} \mathrm{Hom}(K(X), Y)$$

The notation can be simplified using the notion of a copower. Given a cocomplete category $\mathcal{C}$, a (co)power is a functor **Set** $\times \mathcal{C} \to \mathcal{C}$ *which is an iterated (co)product indexed by some set. Powers and copowers are denoted as*

$$X \odot I = \coprod_{i \in I} X_i$$

$$X^I = \prod_{i \in I} X_i$$

Using such a notation, the formula takes a form

$$\mathrm{Lan_K F}(Y) = \int^{c \in \mathcal{C}} F(c) \odot \mathrm{Hom}(K(c), X)$$

$$\mathrm{Ran_K F}(Y) = \int_{X \in \mathcal{C}} F(X)^{\mathrm{Hom}(K(X),Y)}$$

Example 661 (Yoneda lemma as Dirac delta)**.** *Using the identification of natural transformations with certain end, we can express the Yoneda lemma in the language of ends as*

$$F(X) = \int_{c\in\mathcal{C}} F(c)^{\mathrm{Hom}(X,c)}$$

Since the power operator can be identified with just a hom-set, it simplifies to well known formula

$$\int_{c\in\mathcal{C}} F(c)^{h_X(c)} = \int_{c\in\mathcal{C}} \mathrm{Hom}(h_X(c), F(c)) = \mathrm{Nat}(h_X, F)$$

Note that we can deduce this from the fact that F is isomorphic to its right Kan extension along identity functor, since the end expression can be computed as

$$\mathrm{Ran}_{\mathbb{1}}(F) = \int_{c\in\mathcal{C}} F(c)^{\mathrm{Hom}(X,c)}$$

Note that such a presentation of Yoneda lemma has an analogue in analysis, involving the dirac delta function. For compactly supported real function f we have the analytic Yoneda lemma

$$\int_{\mathbb{R}} f(x)\delta(x - x_0)dx = f(x_0)$$

Example 662 (coYoneda lemma)**.** *The coYoneda lemma, stating that every presheaf is a certain colimit of representable presheaves, can be deduced by using the observation dual to the previously mentioned, that a functor is isomorphic to its Kan extensions along identity. This time, applying this to the left extension, the result is dual to the Yoneda lemma, however due to the lack of the dual identification of coends with natural transformations, substantially less trivial.*

$$F(X) = \int_{c\in\mathcal{C}} h_c(X) \odot F(c)$$

The colimit formula for a presheaf F follows from the fact that the cotensor is symmetric in **Set***, so*

$$F(X) = \int_{c\in\mathcal{C}} h_X(c) \odot F(c) = \int_{c\in\mathcal{C}} F(c) \odot h_X(c) =$$
$$\mathrm{coeq}\left(\coprod_{a\to b}\coprod_{F(a)} h_X(b) \rightrightarrows \coprod_{c\in\mathcal{C}}\coprod_{F(c)} h_X(c)\right)$$

Example 663 (Decomposition of simplicial sets). *We can use the coY-oneda lemma to decompose any simplicial set $X : \Delta^{op} \to \mathbf{Set}$ into colimit of representable functors Δ^n. Notice that such a decomposition has a formula almost identical to the geometric realisation, only calculated in* **sSet** *instead of* **Top**

$$X = \int^{n\in\Delta} X_n \odot \Delta^n = \mathrm{coeq}\left(\coprod_{[n]\to[m] X_m \odot \Delta^n} \rightrightarrows \coprod_{[n]\in\Delta} X_n \odot \Delta^n\right)$$

Example 664 (Vector valued coends). *Similar to the case of analytic integrals, the coend of a functor*

$$F : \mathcal{C}^{op} \times \mathcal{C} \to \prod \mathcal{D}_i$$

can be calculated component-wise, treating it as a vector valued functor under the condition, that the coend of each composition with projection $\Pi_i \circ F$ exists and that every category $\mathcal{D}_i$ has initial and terminal objects. Then, considering $F_i = \Pi_i \circ F$ as components of F, we have

$$\int^{\mathcal{C}} F \simeq \prod \int^{\mathcal{C}} F_i$$

Example 665 (Pullbacks and pushouts). *Denoting the category with 2 objects on single arrow by $\downarrow$, the end of every functor $T :\downarrow^{op} \times \downarrow \to \mathcal{C}$ can be identified with the pullback*

$$\begin{tikzcd}
\int_{\mathcal{C}} T \arrow[r, dotted] \arrow[d, dotted] & T(0,0) \arrow[d] \\
T(1,1) \arrow[r] & T(0,1)
\end{tikzcd}$$

Similarly, the coend of T can be computed from similarly constructed pushout diagram.

Example 666 (Supremum metric). *The construction of natural transformations as ends generalizes to enriched categories as well. Given two $\mathcal{V}$-enriched functors, we can define $\mathcal{V}$-enriched natural transformations between them as the end*

$$\mathrm{Nat}_{\mathcal{V}}(F, G) = \int_{X\in\mathcal{C}} \mathrm{Hom}(F(X), G(X))$$

This way we can form a category of $\mathcal{V}$-arrows, which is also $\mathcal{V}$-enriched. A nice example illustrating this is the case of metric spaces. Considering metric spaces X, Y as $[0,\infty]$-categories, we get also a natural *induced metric structure on maps $X \to Y$, which coincides with the standard supremum norm:*

$$d(f,g) = \int_{x\in X} \mathrm{Hom}(f(x), g(x)) = \sup_{x\in X} d(f(x), g(x))$$

Example 667 (Stokes theorem)**.** *The interplay of coends and integration cannot be better expressed but with categorifying the Stokes theorem. Let M be n-dimensional compact smooth orientable manifold with boundary and N be any smooth manifold. Then we can form a functor $C_n : \mathbb{N}^{op} \to \mathbf{Vect}_{\mathbb{R}}$ by taking C_k to be a freely generated vector space by smooth maps $M \to N$, $C_{k+1} \to C_k$ be a restriction to the boundary of M. Moreover, we can form a covariant functor $\Omega : \mathbb{N} \to \mathbf{Vect}_{\mathbb{R}}$ from the de Rham complex of N i.e. taking Ω_n to be a space of differential n-forms on N with exterior derivative operation as the induced morphism $\Omega_n \to \Omega_{n+1}$. By tensoring these two functors, we obtain the functor*

$$C \otimes \Omega : \mathbb{N}^{op} \times \mathbb{N} \to \mathbf{Vect}_{\mathbb{R}}$$

The Stokes theorem asserts that the coend of this functor is the constant $\mathbb{R}$ functor. Since the integration along the pullback can be seen as natural transformation

$$\int : C \otimes \Omega \Rightarrow \mathbb{R}$$

The Stokes' formula

$$\int_{\partial M} f^*_{|\partial}\omega = \int_M f^* d\omega$$

is equivalent to the commutativity of the square

$$\begin{array}{ccc}
C_{n+1} \otimes \Omega_n & \xrightarrow{\partial \otimes 1} & C_n \otimes \Omega_n \\
\downarrow{\scriptstyle 1\otimes d} & & \downarrow{\scriptstyle \int_n} \\
C_{n+1} \otimes \Omega_{n+1} & \xrightarrow{\int_{n+1}} & \mathbb{R}
\end{array}$$

Example 668 (Tannaka duality)**.** *Even though there is no natural isomorphism between a finite dimensional K-vector space and its dual and only the evaluation gives the correspondence between V and its double dual V^{**}, using coends we can say something about the evaluation on V^* as well, since there is a natural isomorphism*

$$\int^V V^* \otimes V \simeq K$$

The easiest proof of this formula uses a Kan extension formula

$$\int^{V} \mathrm{Hom}(V, K) \otimes V \simeq \mathrm{Lan}_{\mathbb{1}}\mathbb{1}(V) = V$$

Example 669 (Trace)**.** *The coYoneda lemma in the category of finite dimensional K-vector spaces provides a categorical description of a trace. Using the isomorphism*

$$W \simeq \int^{V} \mathrm{Hom}(V, W) \otimes V$$

in the coend square with respect to V and K fills the diagram

$$\begin{tikzcd} \mathrm{Hom}(V,V) \otimes V \arrow[r] \arrow[d] & \mathrm{Hom}(K,V) \otimes V \arrow[d] \\ \mathrm{Hom}(V,V) \arrow[r, "\mathrm{tr}"] & K \end{tikzcd}$$

and writing down explicit formulas shows, that the lower horizontal arrow can be identified with a trace map.

Example 670 (Geometric realisation)**.** *Geometric realisation of a simplicial set has a nice coend formula*

$$|X| = \int^{n \in \Delta} X_n \times \Delta^n$$

Example 671 (Analytic functors)**.** *Consider a groupoid $\mathcal{B}$ constructed from the coproduct of all symmetric groups. A functor $F : \mathbf{Set} \to \mathbf{Set}$ is called analytic if it is isomorphic to the left Kan extension of some functor $f : \mathcal{B} \to \mathbf{Set}$ along the natural inclusion (i.e. forgetful functor on each component)*

$$\begin{tikzcd} \mathcal{B} \arrow[rr, "f"] \arrow[dr, dotted] & & \mathbf{Set} \\ & \mathbf{Set} \arrow[ur, "F"] & \end{tikzcd}$$

The reason behind the name, as well as all the buzz about these functors, comes from their coend formula having a form of a "Taylor expansion"

$$F(X) = \int^{n} X^n \times f(n)$$

Similarly as in the case of generating functions, analytic functors capture a lot of combinatorial properties.

Along most basic analytic functors we can find representable ones by considering a functor $f(n) = S_n$ for some fixed n and $f(k) = 1$ if $k \neq n$. A simple calculation involving coequaliser formula shows that

$$F(X) = \int^n X^n \times f(n) \simeq X^n$$

A free monoid functor is also analytic, coming from $f(n) = S_n$, this time for all n with induced inclusions and group multiplication. Its coend formula has a form

$$M(X) = \coprod^{\infty} X^n = \int^n X^n \times S_n$$

Example 672 (Symmetrized representables)**.** *A bit more sophisticated example of analytic functors involves a construction similar to representable functors, but restricted to some subgroup $H \leq S_n$. In this case the functor $f(n) = S_n/H$ (regarded as cosets, as H need not to be normal) and $f(k) = 1$ for $k \neq n$, the functor F has a form*

$$F(X) = X^n/G$$

Functors from such family are called symmetrized representables. Its elements can be understood as n-tuples of elements from X, but with some permutations of coordinates being identified under symmetries represented by action of a finite group.

Example 673 (Finite multisets)**.** *A special case of symmetrized representable is the case of $H = S_n$, when the functor f takes a trivial form $f(n) = 1$ for all n. The inconspicuously looking coend formula*

$$F(X) = \int^n X^n$$

in fact constructs quite an interesting functor, as $F(X)$ can be identified with all finite multisets with elements from X.

Example 674 (Directed trees)**.** *A more elaborate analytic functor comes from an observation, that S_n naturally acts on directed trees with n nodes. It provides a functorial way of constructing them from the symmetric groupoid, as there is a functor $f : \mathcal{B} \to \mathbf{Set}$ taking n to isomorphism classes of trees on $[n]$. The resulting Kan extension $F(X)$ extends f unsurprisingly mapping X to the set of classes of directed trees on X.*

Example 675 (Tensor algebras, symmetric and alternating powers)**.** *A notion of analytic functors can be considered also in different codomains than sets. We will take a closer look at K-linear analytic functors. This* *time such a functor does not have a Kan extension representation, but instead is defined by the coend*

$$F(X) = \int^n V^n \otimes f(n)$$

when again f is a functor $\mathcal{B} \to \mathbf{Vect}_K$. Note that functors coming from Kan extensions would have a form

$$G(X) = \int^n V^n \times f(n)$$

which has less interesting properties, since the natural monoidal product on module categories is the tensor.

Some functors previously constructed on sets has a natural equivalents in $\mathbf{Vect}_K$. The free monoid induced by $f(n) = S_n$, now as permutation representation, this time is equivalent to the tensor algebra functor, which shouldn't be surprising, as it's a free algebra over a monad coming from free-forgetful adjunction.

$$T(V) = \int^n V^{\otimes n} \otimes S_n = \oplus^\infty V^{\otimes n}$$

The functor $f(n) = 1$, which induced finite multisets, now produces the symmetric algebra. However, there are some functors on $\mathbf{Vect}_K$ which does not have equivalents in sets. For example consider a functor with values $f(2) = K^2$ and $f(k) = 0$ for $k \neq 2$ with σ generating S_2 acts on K^2 by

$$\sigma \cdots (x, y) = (x, -y)$$

Analytic functor induced by f is the alternating square

$$F(V) = \Lambda^2(V) = V \otimes V / \langle x \otimes x \rangle$$

Similarly one can construct the external algebra Λ^n using analogous functor with $f(n) = K^n$.

Example 676 (Pre-Lie algebras)**.** *Pre-Lie algebra is a vector space V with bilinear map $[-,-] : V \times V \to V$ satisfying relation*

$$[[x, y], z] - [x, [y, z]] = [[x, z], y] - [x, [z, y]]$$

Those familiar with Lie algebras can probably see the similarity to the Jacobi identity, which justifies the name (the other justification comes from the fact, that even though pre-Lie algebras are not associative, their commutators still do generate a Lie algebra). The free pre-Lie algebra on V can be functorially constructed with the K-linear analogue of directed trees analytic functor from sets. To see that, consider a vector space $\mathbb{T}$ spanned by directed rooted trees. Given a vertex v of tree T_1, let $T_1 \circ_v T_2$ be a disjoint union of two trees with an additional *edge from v to the root of T_2. By iterating such operation one obtains the pre-Lie bracket*

$$[T_1, T_2] = \sum_{v \in V(T_1)} T_1 \circ_v T_2$$

which is isomorphic to the free pre-Lie algebra on one generator. The Kan extension of f, extends this free construction to any generating vector space.

Example 677 (Ètale spaces via Kan extension)**.** *—–TODO : Loregian p.86 ——– The equivalence of sheaves on X and Ètale bundles, i.e. local homeomorphisms $E \to X$ has a nice perspective via the Kan extensions and the variation of nerve-realisation principle applied outside the world of simplicial sets. From the category* $\mathbf{Top}(X)$ *there is a natural map inclusion to the category of bundles* $\mathbf{Top}/X$*, mapping U to its inclusion $U \hookrightarrow X$.*

Example 678 (Extending augmentation)**.** *Consider the category enriched with $\mathcal{V}$, where $\mathcal{V}$ has the additional property that the monoidal unit is terminal. Most of basis we've extensively covered, such as* **Bool**, **Cost** *or* **Set** *satisfy this property, however it is not true in the categories of modules, as $R \neq 0$. We need this property to define the canonical augmentation $\mu : \mathcal{C}(x, y) \mapsto 1$. Now from any $\mathcal{V}$-functor $\mathcal{C} \to \mathcal{D}$ we obtain a family of profunctors $F(x, y)(-) = \mathcal{V}(x, y) \otimes \mathcal{D}(F(y), -)$. Its coend*

$$\varphi_F(d) = \int^{\mathcal{E}} F(e, e)(d)$$

is called an extension of f. To see more clearly what's going on, let's consider the case of Lawvere metric spaces. Given **Cost***-functor $f : X \to Y$, since the coequaliser in linear order is just infimum of the codomain, the extension of f takes a form*

$$\varphi_f(y) = \inf_{x \in X} d(f(x), y)$$

and is the distance between y and the image of f.

Example 679 (Comprehension scheme)**.** *The extension along augmentation considered previously defines a functor*

$$\varphi : \mathbf{Cat}(\mathcal{V})/\mathcal{D} \to [\mathcal{D}, \mathcal{V}]_{\mathcal{V}}$$

It always has a right adjoint called comprehension scheme $F \mapsto \mathcal{D}|F$. The category $\mathcal{D}|F$ has similar structure to the category of elements and coincides with it in case of $\mathcal{V} =$ **Set***. Its objects are arrows $x : 1 \to \phi(d)$*

and the internal hom $(\mathcal{D}|F)(1 \xrightarrow{x} \phi(d_1), 1 \xrightarrow{y} \phi(d_2))$ *is given by the following pullback in* $\mathcal{V}$*:*

$$\begin{tikzcd}
(\mathcal{D}|F)(x,y) \arrow[rr] \arrow[d] & & 1 \arrow[d, "y"] \\
1 \otimes \mathcal{D}(d_1,d_2) \arrow[r, "x\otimes 1"] & F(d_1)\otimes \mathcal{D}(d_1,d_2) \arrow[r] & F(d_2)
\end{tikzcd}$$

In case of $\mathcal{V} = \mathbf{Set}$ *the objects* $\{\bullet\} \to F(X)$ *we may identify with pairs* $(X, F(x_0))$*, and the pullback defines the set* $\mathrm{Hom}((X,F(x_0)),(Y,F(y_0))$ *as subset of* $\mathrm{Hom}(X,Y)$ *containing such maps* $f : X \to Y$*, that* $F(x_2) = F(f)(x_1)$*, thus* $\mathcal{C}|F$ *is exactly the definition of* $\int F$ *with the canonical projection* $\int F \to \mathcal{C}$*.*

The cases of **Bool** *and* **Cost** *are even simpler. Comprehension scheme in posets retrieves just all the true values in the image*

$$P|\varphi = \{p \in P \mid \varphi(p) = \mathit{true}\}$$

while for Lawvere metric spaces it is the zero-set

$$X|f = f^{-1}(0)$$

Example 680 (Isbell duality)**.** *A Fubini trick of interchanging double ends provides an elegant proof of even more elegant, however non-trivial theorem due to Isbell. For any small* $\mathcal{V}$*-enriched category* $\mathcal{C}$*, where* $\mathcal{V}$ *is a good enriching category (called cosmos), i.e. bicomplete closed and symmetric, Isbell conjugation is a pair of adjoint functors*

$$\begin{tikzcd}
{[\mathcal{C},\mathcal{V}]_{\mathcal{V}}}^{op} \arrow[r, bend left, "\mathrm{O}"] & {[\mathcal{C}^{op},\mathcal{V}]_{\mathcal{V}}} \arrow[l, bend left, "\mathrm{Spec}"]
\end{tikzcd}$$

given by

$$\mathrm{O}(F)(X) = \mathrm{Nat}_{\mathcal{V}}(F, h_X)$$

$$\mathrm{Spec}(G)(X) = \mathrm{Nat}_{\mathcal{V}}(h_X, G)$$

A proof is pretty straightforward and satisfying, so I encourage you to trying to write it down yourself.

Example 681 (Reflexive completion)**.** *Instead of Isbell completion, we can consider also smaller reflexive completion formed from the fixed points of Isbell adjunctions. The inclusions of different completions can* *be pictured in following diagram*

$$
\begin{tikzcd}
 & I(C) & \\
\hat{C} \arrow[ur, hook] & & \check{C} \arrow[ul, hook'] \\
 & R(C) \arrow[ul, hook'] \arrow[ur, hook] & \\
 & \bar{C} \arrow[u, hook] & \\
 & C \arrow[u, hook] &
\end{tikzcd}
$$

where $\hat{\mathcal{C}}, \check{\mathcal{C}}$ are the free cocompletion and completion (equivalent to presheaves and co-presheaves in case of small categories), $\bar{\mathcal{C}}$ the Cauchy completion and $\mathcal{I}(\mathcal{C})$ is the bicategorical pullback. The reflexive completion $\mathcal{R}(\mathcal{C})$ is in fact the intersection $\hat{\mathcal{C}} \cap \check{\mathcal{C}}$.

Since the reflection completion is the subcategory of presheaves isomorphic to their dual, the conjugation restricts to equivalence on $\mathbb{R}(\mathcal{C})$. It follows that its elements we can consider equivalently as presheaves on $\mathcal{C}$ and as co-presheaves on $\mathcal{C}^{op}$. From this follow a few basic properties of reflexive completion:

- *representable functors belong to $\mathbb{R}(\mathcal{C})$, so the Yoneda embedding restricts to $\mathcal{C} \hookrightarrow \mathbb{R}(\mathcal{C})$*
- $\mathbb{R}(\mathcal{C})^{op} \simeq \mathbb{R}(\mathcal{C}^{op})$
- *elements of $\mathbb{R}(\mathcal{C})$ are exactly reflexive presheaves*
-

Example 682 (The other side of group action)**.** *Recall that the presheaves $\mathbf{B}G^{op} \to \mathbf{Set}$ we can identify with right group actions, while copresheaves $\mathbf{B}G \to \mathbf{Set}$ with left actions. Thus the Isbell conjugation interchanges right and left G-sets. In order to see the general formula of $X^\vee$, at first let's analyse some special cases. If X is acted by G freely and transitively, it is isomorphic to G acted on itself by right multiplication - we'll denote that as $X \simeq G_r$. Then the Isbell dual*

$$X^\vee = \mathrm{Hom}_G(X, G_r) \simeq \mathrm{End}(G_r) \simeq G_l$$

is isomorphic to G with right action, as every equivariant function can be uniquely identified with the image of e. Now suppose the action is not free. Then there is such element $g \neq e$ and $x \in X$ that $x = xg$.

Any equivariant map $\varphi : X \to G$ satisfies

$$\varphi(x) = \varphi(gx) = \varphi(x)g$$

it yields to a contradiction $g = e$, so

$$X^\vee = \mathrm{Hom}_G(X, G_r) = \emptyset$$

Now suppose the action is free but not transitive. Then it can be expressed as a coproduct of free and transitive actions on all the orbits, thus we obtain a general formula

$$X^\vee = \begin{cases} \emptyset & \textit{action is not free} \\ \prod_{X/G} G_l & \textit{action is free} \end{cases}$$

Example 683 (Upper bounds)**.** *Let's analyse the Isbell duality in some enriched context. A* **Bool***-category is a partially ordered set, with presheaves $F : P^{op} \to$* **Bool** *corresponding to downward closed families $X_F = \{a \in A \mid F(a) = 1\}$. Its Isbell dual is given as*

$$F^\vee(a) = \mathrm{Hom}(F, \mathrm{Hom}(-, a)) = \begin{cases} 1 & \forall_{x \in F_X} x \leq a \\ 0 & \textit{otherwise} \end{cases}$$

So $F^\vee$ can be identified with indicator of upper bounds of F_X, which is an upper closed set - the copresheaf on P. Similarly, a dual presheaf to any copresheaf corresponds to the set of lower bounds.

Example 684 (Dedekind–MacNeille completion)**.** *We've seen that the Isbell duality on posets interchange the lower and uppersets. It leads to the adjunction*

$$\begin{tikzcd} \mathrm{DownSubsets}(P) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & \mathrm{UpSubsets}(P)^{op} \arrow[l, shift left] \end{tikzcd}$$

The Isbell completion of a poset P consists of downsets of P which elements can be expressed as lower bounds of upper bounds of P. This includes for instance the famous construction of extended real line using Dedekind cuts, which is just the Isbell completion of $\mathbb{Q}$ with its natural total order.

Example 685 (Isbell envelope of metric spaces)**.** *Consider the Isbell duality of Lawvere metric spaces. Since the adjoint functors commute with Yoneda and coYoneda embeddings, for each metric space we have a triangle*

$$\begin{tikzcd} & X \arrow[dl, hook] \arrow[dr, hook] & \\ \hat{X} \arrow[rr, bend left, "L"] \arrow[rr, phantom, "\perp"] & & \check{X}^{op} \arrow[ll, bend left, "R"] \end{tikzcd}$$

Functors R, L are given by formulas

$$L(f)(y) = d(f, h_y) = \sup_{a \in X} d(f(a), d(a, y))$$
$$R(g)(x) = d(g, h^x) = \sup_{b \in X} d(g(b), d(x, b))$$

As for any adjoint pair, we can construct its envelope, in this case called the Isbell envelope - it contains all the invariant object, i.e.

$$I(X) = \{(f, g) \in \hat{X} \times \check{X}^{op} \mid L(f) = g, R(g) = f\}$$

The metric $I(X)$ can be expressed with two formulas, equivalent due to invariance of its elements

$$d((f_1, g_1), (f_2, g_2)) = d_{\hat{X}}(f_1, f_2) = d_{\check{X}^{op}}(g_1, g_2)$$

Moreover, $I(X)$ can be also realised as fix point of adjunction, both in presheaves and op-copresheaves - $\mathrm{Fix}_{RL}(X) = \{f \in \hat{X} \mid RL(f) = f\}$

$$\mathrm{Fix}_{RL}(X) \simeq I(X) \simeq \mathrm{Fix}_{LR}(X)$$

The space X can be embedded in $I(X)$ with Yoneda and coYoneda used simultanously: mapping x to the pair (h_x, h^x).

Now let's explore what the Isbell completion $I(X)$ actually represents. It can be understood in two different ways - firstly, as the generalisation of injective envelope, thus a convex hull, to the case of non-symmetric metrics. Secondly, it corresponds to a minimal bicompletion of a space, adding all the small limits and colimits in the enriched version, which we'll explore more deeply in near future. Similarly as injective envelope, $I(X)$ can be thought of as the closure of geodesic lines in the Yoneda embedding, but this time a bit different. Firstly, let's look how the Yoneda lemma gives us a nice formula of distances between any point $p = (f, g) \in I(X)$ and embedded point from X by double Yoneda $y : X \hookrightarrow I(X)$ described above:

$$d(y(a), p) = f(a)$$

$$d(p, y(a)) = g(a)$$

Now note that it means that for any point $a \in X$ such that $d(p, y(a)) > 0$, it implies that p must lies near some geodesic line connecting a with some other point from X by the same argument as we've made in the injective envelope, but this time in the non-symmetric metric of $\hat{X}$, resulting in a very different space than $\mathcal{E}(X)$ in general.

Consider for instance the space with two points $N_{a,b}$. The easiest way to construct $I(X)$ is done by the fixed points, as it realises it as subspace *of presheaves, realised as a strip of a plane. Calculating explicitly R and L is not hard for only two points, taking a simple form*

$$\begin{aligned} L(f)(0) &= \max_{\{0,1\}} d(f(-), d(-,0)) = \max(0, b - f(1)) \\ L(f)(1) &= \max_{\{0,1\}} d(f(-), d(-,1)) = \max(0, a - f(0)) \\ R(f)(0) &= \max_{\{0,1\}} d(f(-), d(0,-)) = \max(0, a - f(1)) \\ R(f)(1) &= \max_{\{0,1\}} d(f(-), d(1,-)) = \max(0, b - f(0)) \end{aligned}$$

It follows then that

$$\mathrm{Fix}_{LR} = \{(x,y) \in \mathcal{V}^2 \mid 0 \leq x \leq a, 0 \leq y \leq b\} \simeq [0,a] \times [0,b] \subset \mathcal{V}^2$$

Example 686 (Everything goes wrong for $p = 2$ even for groups)**.** *In this example we'll consider the Isbell completion of a finite group, which leads to a surprising exception in case of $p = 2$, not as usual caused by horrible identity $a + b = a - b$ true in characteristic 2, but this time resulting from $2^2 = 2 \cdot 2$. We've seen previously that for any G-set X we have*

$$X^\vee = \begin{cases} \emptyset & \textit{action is not free} \\ \prod_{X/G} G_l & \textit{action is free} \end{cases}$$

To identify all G-sets satisfying $X = X^{\vee\vee}$, note that if the action is not free, $X^{\vee\vee} = \emptyset$, and thus $\emptyset$ is the only invariant presheaf not having a form $X = \coprod_I G_l$. In the latter case, X is determined only by the cardinality of I, as by assumption G has finite order $|G| = n$ and thus $X \simeq Y$ iff $|X| = |Y| = n^k$ for some cardinality k - we'll see in a moment that it must be finite. Since cardinality is an invariant of isomorphic G-sets, we must have

$$n^k = |X| = |X^{\vee\vee}| = |\prod_{X^\vee/G} G_l| = n^{|X^\vee|-1}$$

Since $|X^\vee| = |\prod_{X/G} G_l| = n^{|X/G|}$, it means that $|X^\vee/G| = n^{k-1}$ and thus

$$n^k = n^{n^{k-1}}$$

It now easy to see that obviously the regular representation G_r satisfies this equation for all groups with $k = 1$. Moreover, the equation

$$nk = n^k$$

Is also satisfied for $n = k = 2$ and no other natural numbers. It means that for any finite group $G \not\simeq \mathbb{Z}/2$ the Isbell envelope $I(\mathbf{B}G)$ contains *three presheaves - as in case of sets, initial, terminal and representable functors. But* $I(\mathbf{B}\mathbb{Z}/2)$ *has* 4 *elements*

$$\mathrm{ob}\, I(\mathbf{B}\mathbb{Z}/2) = \{\emptyset, 1, \mathbb{Z}/2, \mathbb{Z}/2 \oplus \mathbb{Z}/2\}$$

The different argument can be made that makes this phenomenon even clearer - one can note that for each group G *we have an isomorphisms*

$$(G \sqcup G)^\vee \simeq \mathrm{Hom}(G \sqcup G, G) \simeq G \times G$$

Since the number 2 *has the special property of* $2 + 2 = 2 \cdot 2$*, we have* $G \times G \simeq G \sqcup G$ *and thus the unique possibility of*

$$(G \sqcup G)^{\vee\vee} \simeq (G \times G)^\vee \simeq (G \sqcup G)^\vee \simeq G \times G \simeq G \sqcup G$$

Example 687 (Large vs small sets)**.** *In this example we'll consider the Isbell envelope of discrete categories. Surprisingly, the answer is pretty straightforward, depending only on the size of* $\mathcal{C}$*. The presheaves and copresheaves on* $\mathcal{C}$ *clearly coincides. Pick* $a \in \mathcal{C}$ *and suppose for some* $b \neq a$ $F(b) \neq \emptyset$*. Then*

$$F^\vee(a) = \mathrm{Nat}(F, h_a)\{\eta_x : F(x) \to \mathrm{Hom}(x, a)\} \ni F(b) \to \emptyset$$

so $F(a) = \emptyset$*. In case only the set* $F(a)$ *is possibly not empty,* $F(a)$ *contains just an unique arrow* $F(a) \to \{\bullet\}$*. It means that when* $\mathcal{C}$ *is small, the exact formula of* $F^\vee$ *depends on three different cases*

$$F^\vee \simeq \begin{cases} 1 & F(-) = \emptyset \\ \mathrm{Hom}(-, a) & \mathrm{supp} F = \{a\} \\ 0 & \textit{otherwise} \end{cases}$$

Now let's think when we have $F \simeq F^{\vee\vee}$*? Clearly applying this operation twice once again leads to similar formula*

$$F^{\vee\vee} \simeq \begin{cases} 1 & F^\vee(-) = \emptyset \\ h_a & \mathrm{supp} F^\vee = \{a\} \\ 0 & F^\vee = 1 \end{cases} \simeq \begin{cases} 1 & F^\vee = 1 \\ h_a & F^\vee = h_a \\ 0 & F^\vee = 0 \end{cases}$$

Thus presheaves invariant under Isbell conjugation are the representable ones, together with initial and terminal functors, so the Isbell completion takes a form

$$I(\mathcal{C}) = \mathcal{C} \sqcup \{0, 1\}$$

- the discrete category $\mathcal{C}$ *with freely added initial and terminal objects* $0, 1$*. In case of large categories the situation differs slightly however, as we restrict our attention only to small functors. This time the terminal functor has non-empty image on the entire class of objects* $\mathcal{C}$*, thus do not form a small functor* $\mathcal{C} \to \mathbf{Set}$*. It means that the initial presheaf* $F = 0$ *is not invariant anymore, as it's Isbell dual is not small.*

Moreover, the terminal functor, as its large, can not be even taken into consideration as invariant. It means that now only representable functors remain as valid invariants, so we've left with the smaller, trivial completion

$$I(\mathcal{C}) = \mathcal{C}$$

Example 688 (Reflexive modules)**.** *Let R be a ring regarded as 1-object additive category. We can regard the presheaves $[R^{op}, \mathbf{Ab}]$ as right R-modules, and the Isbell conjugation maps each such M to the dual left module* $\mathrm{Hom}(M, R)$*. It follows that the reflexive completion of R is the category of reflexive modules. For example in case of $R = k$, the completion $\mathcal{R}(k)$ is the category of finite dimensional vector spaces.*

13.4. **Weighted limits.**

Example 689 (Weighted colimits of metric spaces)**.** *Weighted colimits of Lawvere metric spaces have quite simple explicit form. A diagram in this case is a diagram in this case is just a short map $w : D \to X$, and the weighting is some presheaf $J : D^{op} \to \mathcal{V}$. It's weighted colimit can be identified with an object $c \in X$, unique if X is skeletal, satisfying for all $x \in X$ the equation*

$$d(c, x) = \sup_{a \in D} \{d(J(a), x) - w(a)\}$$

Informally, it can be identified with a weighted sum

$$c = \sum_{a \in D} w(a) J(a)$$

With this construction we can for instance generalise some constructions from orders to non-symmetric metric spaces. For example the infimum of X can be realised as the initial object, corresponding to the unique colimit of empty diagram. It is, as expected, a point $x_0 \in X$ satisfying

$$d(-x) = 0$$

Example 690 (Fat point)**.** *Weighted colimits of even very simple diagrams can correspond to pretty interesting constructions in generalised metric spaces. Consider for example a diagram $W = \{\bullet\}$ with weighting $w(\bullet) = R$. A functor $W \to X$ is just some point $x_0 \in X$, and its colimit has pretty interesting property of blowing up x_0 to a "fat point" $B(x_0, R)$, having property of a ball with center x_0 and radius R, but still being just a single point. It satisfies the universal property, expressed for all points y by the equation*

$$d(B(x_0, R), y) = \max(0, d(x_0, y) - R)$$

Note that except trivial cases classical metric spaces never has such points. However, all fat points can be found in the asymmetric line $\mathcal{V}$, where $B(x_0, R) = x_0 + R$.

Example 691 (Forward and backward convergence)**.** *Consider some generalised metric space X. A sequence in X is a diagram $f : \mathbb{N} \to X$, with discrete metric on $\mathbb{N}$. f is said to be forward-convergent with witness $g : \mathbb{N} \to \mathbb{R}$ if g is a decreasing sequence convergent to 0 and for all $n, k \in \mathbb{N}$*

$$d(f(n+k), f(n)) \leq g(n)$$

similarly we define forward convergent sequence, satisfying dual inequalities

$$d(f(n), f(n+k)) \leq g(n)$$

These two types of convergence can be naturally modelled as colimits and limits weighted by g, and in case of classical metric spaces they naturally correspond to the classical limit points of Cauchy sequences.

In case of X being the asymmetric line $\mathcal{V}$, these constructions can be identified with constructions know from analysis - for any g satisfying the conditions above, we have

$$\operatorname{colim}_g f = \limsup f(n)$$
$$\lim_g f = \liminf f(n)$$

13.5. **Groupoidification.**

Example 692 (Homotopy cardinality)**.** *The notion of cardinality of a groupoids has slightly different than the cardinality of sets. Groupoid cardinality, known also as homotopy cardinality, is a tool of counting objects only up to equivalence. Given groupoid, we define its cardinality as (possibly infinite) series*

$$|\mathcal{G}| := \sum_{[x] \in \pi_0(\mathcal{G})} \frac{1}{|\operatorname{Aut}(x)|}$$

For some known examples of groupoids, the cardinality can be understood as

- *cardinality of set, when $\mathcal{G}$ is discrete*
- $\frac{1}{|G|}$*, when G is a finite group*
- $|X//G| = \frac{|X|}{|G|}$ *- the class formula of group action*

We can also look at homotopy cardinality from another perspective - as an operations naturally associated with division. Consider for example that disjoint unions works well with addition, as we have

$$|A| + |B| = |A \sqcup B|$$

The Cartesian product on the other hand works well with multiplication

$$|A||B| = |A \times B|$$

But the operation of division is somehow flawed - while for free actions the formula

$$|X/G| = |X|/|G|$$

works fine, it causes problems where the action has fixed points. For example if G acts trivially on X, it leads to a complete absurd implying that all groups have order 1:

$$|X| = |X/G| = |X|/|G|$$

This problem can be fixed in two steps - firstly, we need to remember the information about fixed points, changing the quotient with the action groupoid. Now the groupoid cardinality do not cause troubles anymore, as we always have

$$|X//G| = |X|/|G|$$

independently of the action of G.

Example 693 (Categorical construction of e)**.** *A particularly interesting application of homotopy cardinality constructs the Euler number e as the cardinality of core groupoid of the category of finite sets and bijections* **FinIso***. Indeed directly from the definition we get*

$$|\mathrm{Core}(\mathbf{FinIso})| = \sum^{\infty} \frac{1}{|S_n|} = \sum^{\infty} \frac{1}{n!} = e$$

Example 694 (Stuff type)**.** *Groupoids constitute an important tool of categorification, as they are used to equip finite sets with extra stuff encoding their additional features. More precisely, this process (called groupoidification) often can be brought down to constructing so called stuff type - groupoid over the groupoid of finite sets, i.e. groupoid Ψ equipped with a forgetful functor $v : \Psi \to E$ to a groupoid E, where the extra structure is lost. Together with stuff type comes the generating function - the formal power series*

$$\bar{\Psi} = \sum^{\infty} |v^{-1}(n)|t^n$$

where we identify n with a finite set from E with n elements (as in E, with extra stuff forgotten, these are indistinguishable). The generating function is then a way of measuring the groupoid $v^{-1}(n)$, characterising the extra stuff admitted by sets of order n by its cardinality.

Example 695 (2-colored sets)**.** *Consider the groupoid of* 2*-colored finite sets. A* 2*-colored set is a set with coloring functions* $X \to \{\bullet, \circ\}$*. The morphisms between colored sets preserve colorings, which means exactly that it is so called slice category* $\mathbf{FinIso}/\{\bullet, \circ\}$ *or, in other words, its morphisms are commutative triangles - maps preserving coloring*

$$\begin{tikzcd} X \arrow[rr] \arrow[dr, "c_X"] & & Y \arrow[dl, "c_Y"] \\ & \{\bullet, \circ\} & \end{tikzcd}$$

Now consider the forgetful functor $v : \mathbf{Col}(2) \to \mathbb{N}$*. Note that there are* 2^n *ways to color every set of size* n*, so the groupoid* $v^{-1}(n)$ *has* 2^n *elements. Moreover, automorphism of each coloured set is given by action of* S_n *permuting the labels of elements in the set. Thus the viber is in fact the action groupoid* $[2^n]//S_n$*. Its generating function we can thus identify with*

$$\bar{\Psi} = \sum^{\infty} \frac{(2t)^n}{n!} = e^{2t}$$

Example 696 (Matrix multiplication)**.** *Let* $\mathbf{m}, \mathbf{n}$ *be any finite categories with respectively* m *and* n *elements. Consider some profunctor* $F : \mathbf{m}^{op} \times \mathbf{n} \to \mathbf{Set}$*. After choosing some total orders on elements of* $\mathbf{m}, \mathbf{n}$*, we can present its values with an* $m \times n$ *matrix* $M(i,j) \in \mathbf{Set}$*. Since each isomorphism class of a set is determined by its cardinality,* M *is up to isomorphism represented by a matrix with values in cardinal numbers. Using the standard cardinal arithmetic, where multiplication is given by the cardinality of a product, we can multiply such matrices. This operation corresponds to composition of profunctors understood as categorical bimodules. Such analogy correctly hints that it can be computed as the coend*

$$(MN)(i,j) = \int^{c \in \mathbf{m}} M(i,c) \times N(c,j)$$

Before considering more complex case, let's contemplate about the matrix nature of spans or profunctors for just a second. Let's think about the span

$$\begin{tikzcd} & M \arrow[dl, "t"'] \arrow[dr, "s"] & \\ S_0 & & S_1 \end{tikzcd}$$

as encoding some family of initial states by elements of S_0 *and terminal states by* S_1. *These two maps establish a partitions of* M *into subsets*

$$M_i^j = s^{-1}(i) \cap t^{-1}(j)$$

Thinking about M *as encoding all the ways of changing the state,* M_i^j *are all the possibilities of changing initial state* i *to terminal* j. *Note that such objects acting by changing the states are naturally described by matrices, in this case with natural coefficients.*

Example 697 (Matrix of groupoids)**.** *Now we'll generalise the linear maps constructed from groupoids over* $\mathbb{N}$ *to some arbitrary base, where the theory transposes to a familiar linear algebra of vector spaces. Consider enriching the system described above by adding some extra stuff to the elements of the sets. By principle of groupoidification, we will do it by replacing finite sets by groupoids, encoding additional features of objects by different classes. Every such a groupoid* $\mathcal{G}$ *can be equipped with a vector space of real valued functions of classes* $\mathbb{R}_{\mathcal{G}}$, *and when it is sufficiently nice (tame), the matrix* M *is in fact a linear operator* $\mathbb{R}_X \to \mathbb{R}_Y$ *- thus groupoidification resulted in the structure of real valued matrices. So in this case* M *still can be identified with a matrix, but now encoding more complicated relations between extra stuff of* X *and* Y, *at the same time gaining more convenient structure of linear operator, allowing us to use the tools of linear algebra failing for matrices with only natural coefficients. Now* M *does not only count the number of ways of changing the state, but also takes into account how it incidentally mixes up the extra stuff.*

In the general case of finite groupoids, the matrix coefficients are now calculated with the correction for groupoid cardinalities of fibers:

$$M_{ij} = \sum_{s^{-1}(i)\cap t^{-1}(j)} \frac{|\operatorname{Aut}(j)|}{|\operatorname{Aut}(s)|}$$

Example 698 (Two faces of partitions)**.** *Consider the groupoid* $\operatorname{Part}(6)$, *whose objects are all the partitions of sets with at most* 6 *elements and bijections preserving partitions. We have two different maps* $p, u : \operatorname{Part}(6) \to \mathbb{N}$ *- the map* p *is a map counting partitions, while* u *is forgetting the partitions and returns cardinality of a set. These two maps establish two different objects in* $\mathbf{Grpd}/\mathbb{N}$, *which we'll denote as* $\mathcal{G}_u$ *and* $\mathcal{G}_p$. *We've restricted the cardinalities of sets so that all the calculations can be made explicitly without much suffering, as the author does not* *know any nice formulas describing general case. The generating functions of both groupoids have forms*

$$\bar{\mathcal{G}_u} = \frac{1}{120}\left(120 + 120t + 180t^2 + 200t^3 + 235t^4 + 246t^5 + 267t^6\right)$$

$$\bar{\mathcal{G}_p} = \frac{1}{720}\left(720 + 1237t + 1511t^2 + 1560t^3 + 1380t^4 + 1080t^5 + 720t^6\right)$$

We can see these two different forgetful maps also as a span.

$$\begin{tikzcd} & \mathrm{Part}(6) \arrow[dl, "u"'] \arrow[dr, "p"] & \\ \mathbb{N} & & \mathbb{N} \end{tikzcd}$$

Assuming the only partition on the empty set has 0 *parts, consider a matrix induced by such a span. Note that for each i there is exactly one partition on the set of i elements with i distinct classes - all the classes must be singletons. It means that the diagonal of M contains only* 1*'s. Secondly, M is lower-triangular, as M_{ij} is a sum indexed by all partitions of $\{1, \cdots, i\}$ into j partitions, which can be done only for $j \leq i$. It's not hard to calculate all the other coefficients, as the number of automorphisms is just the inverse of product over factorials of partitions sizes.*

$$M = \frac{1}{720}\begin{pmatrix} 720 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 720 & 0 & 0 & 0 & 0 & 0 \\ 0 & 360 & 720 & 0 & 0 & 0 & 0 \\ 0 & 120 & 360 & 720 & 0 & 0 & 0 \\ 0 & 30 & 300 & 360 & 720 & 0 & 0 \\ 0 & 6 & 90 & 300 & 360 & 720 & 0 \\ 0 & 1 & 41 & 180 & 300 & 360 & 720 \end{pmatrix}$$

Note that since all points of $\mathbb{N}$ have trivial automorphisms, so the expression is symmetric. But consider the span

$$\begin{tikzcd} & \mathrm{Part}(6) \arrow[dl, "u"'] \arrow[dr, "p"] & \\ \mathrm{Core}(\mathbf{Fin}) & & \mathrm{Core}(\mathbf{Fin}) \end{tikzcd}$$

Example 699 (Eigenvalues)**.** *We can take a step further and categorify more linear algebra this way. In particular as special cases we obtain*

- *evaluation of M at vector v, as a vector we can identify with $n \times 1$ matrix, thus a profunctor $v : n \times 1^{op} \to \mathbf{Set}$. By the* *matrix multiplication formula it is then given by*

$$Mv(i) = \int^{c\in \mathbf{n}} M(i,n) \times v(n,\bullet)$$

- *similarly identifying scalars with* 1×1 *matrices, the scalar multiplication also has sense and is given by*

$$\lambda M(i,j) = \int^{\bullet\in 1} M(i,j) \times \lambda(\bullet,\bullet)$$

The equality of matrices after categorification correspond to representing isomorphic profunctors. Since its matrix is unique up to isomorphism of sets, it is just the equivalence class in the category **Prof** *of profunctors.*

With this tool we may consider such structures as eigenvectors and eigenvalues in such purely categorical context, as its definitions translates directly to such cardinals λ *(understood as trivial profunctors* $1 \times 1^{op} \to \mathbf{Set}$*) and profunctors* $\mathbf{n} \times 1 \to \mathbf{Set}$*, that there is an isomorphism in* **Prof** *of*

$$Mv = \lambda v$$

Thus in the coend language

$$\int^{c\in \mathbf{n}} M(i,n) \times v(n,\bullet) = \int^{\bullet\in 1} v(i,\bullet) \times \lambda(\bullet,\bullet)$$

Example 700 (Inner product)**.** *Considering groupoids over* X *as generalised vector spaces, the weak pullback provides a construction of their inner product*

$$\begin{tikzcd} & \langle\Phi,\Psi\rangle \arrow[dl] \arrow[dr] & \\ \Phi \arrow[dr] & & \Psi \arrow[dl] \\ & X & \end{tikzcd}$$

Note that not all such weak pullbacks are tame, but if it is the case, such an operation provides the inner product structure on the space of groupoids over X*. Groupoids admitting such a structure, where the inner product* $\langle\Phi,\Phi\rangle$ *are called square-integrable, and their space we'll denote as* $L^2(X)$*. It is the subspace of the entire space* $\mathbb{R}^X$ *of all groupoids over* X *with a structure of Hilbert space with inner product defined as above. Note that similarly as in ordinary linear algebra, in case of finite* X *all groupoids are square-inegrable and* $L^2(X) = \mathbb{R}^X$*, but it is no true in general, similarly as not all the infinitely-dimensional vector spaces* *are Hilbert spaces with inner product. The analogy with inner products of vector space is obvious from the basic properties of this construction:*

- $\langle \bar{\Phi}, \bar{\Psi} \rangle = |\langle \Phi, \Psi \rangle|$
- $\langle \Phi, \Psi \rangle = \langle \Psi, \Phi \rangle$
- $\langle \Phi + \Omega, \Psi \rangle = \langle \Psi, \Phi \rangle + \langle \Omega, \Phi \rangle$
- $\langle \lambda \times \Phi, \Psi \rangle = \lambda \times \langle \Psi, \Phi \rangle$

It follows that the degroupoidification can be seen as a functor forgetting the bicategory structure on spans:

$$\mathbf{Span}(X) \to \mathbf{Hilb}$$

The category of spans form a weak symmetric monoidal category with cartesian structure. With this setting, the degroupoidification has an additional structure of lax monoidal functor, as there is a natural map

$$\mathbb{R}^X \otimes \mathbb{R}^Y \to \mathbb{R}^{X \times Y}$$

which is not an isomorphism in the infinite case.

Example 701 (Classical dot product)**.** *In case of groupoids over discrete set $\mathbb{N}$, the induced inner product is just the classical dot product after degroupoidification, as elements mapping to n in $\langle \Phi, \Psi \rangle$ have a form (x, y), where $x \in \Phi$, $y \in \Psi$ and both maps to n. Since automorphisms of such element are all exactly products of automorphisms of x and y, on the generating functions we have*

$$\bar{\Psi}(t) = \sum^{\infty} a_n t^n$$

$$\bar{\Phi}(t) = \sum^{\infty} b_n t^n$$

$$\langle \Phi, \bar{}\Psi \rangle (t) = \sum^{\infty} a_n b_n t^n$$

So the Hilbert space associated to groupoid $\mathbb{N}$ are square integrable sequences,

$$L^2(\mathbb{N}) \simeq \ell^2$$

Note that in particular when Ψ is finite, coefficients of $\langle \Psi, \Psi \rangle$ are fractions of squares of integers.

Example 702 (Adjoint operator)**.** *Note that the spans, even though looks completely symmetric, do not play a symmetric role as linear operators. Swapping two arrows in a span S results in an adjoint operator $S^\dagger$, acting as usual adjoint linear function with respect to inner product, i.e.*

$$\langle S\Phi, \Psi \rangle = \langle \Phi, S^\dagger \Psi \rangle$$

Note also that the tame relation is also not symmetric, so the adjoint of tame groupoid need not be tame. As usually, it directly corresponds to the theory of Hilbert spaces - in particular, it is insignificant in finite case, where every span is tame.

Example 703 (Degroupoidification as (co)homology)**.** *Note that we have also an alternative way of degroupoidification, where instead of all real valued functions from isomorphism classes of objects $\mathbb{R}^X$ we consider just a vector space they generate, $\mathbb{R}[X]$. The duality between these approaches reflects the duality between 0-th cohomology and homology groups of topological spaces, where $H^0(X;\mathbb{R})$ is the vector space $\mathbb{R}^{\pi_0(X)}$ - constant real functions on connected components, and $H_0(X,\mathbb{R}) = \mathbb{R}[\pi_0(X)]$. Thus we can identify these two functors as zeroth (co)homologies of groupoids*

$$H^0(G;\mathbb{R}) := \mathbb{R}^G$$

$$H_0(G;\mathbb{R}) := \mathbb{R}[G]$$

Let's compare the result we've obtained for the groupoid of finite sets and bijections $\mathrm{Core}(\mathbf{Fin})$*. We've already established its cohomology as*

$$H^0(\mathrm{Core}(\mathbf{Fin});\mathbb{R}) = \mathrm{Hom}(\mathbb{N},\mathbb{R}) \simeq \mathbb{R}[[t]]$$

Its homology on the other we can identify with only such functions $\mathbb{N} \to \mathbb{R}$ with finite support, thus

$$H_0(\mathrm{Core}(\mathbf{Fin});\mathbb{R}) \simeq \mathbb{R}[t]$$

The biggest advantage of homological approach is the monoidal structure - this time degroupoidification form a weak monoidal functor, as we always have an isomorphism

$$\mathbb{R}[X] \otimes \mathbb{R}[Y] \simeq \mathbb{R}[X \times Y]$$

Example 704 (Degroupoidification with parameter)**.** *Note that the formula defining linear operators in degroupoidification is weirdly asymmetric. We can fix that problem in multiple ways, in fact there is a family of solutions parameterised by $\alpha \in [0,1]$. For each such a number we can define the α-degroupoidification by adding the weight to the equation, defining the multiplication as*

$$M_{ij} = \sum_{s^{-1}(i)\cap t^{-1}(j)} \frac{|\operatorname{Aut}(j)|^{\alpha} |\operatorname{Aut}(i)|^{1-\alpha}}{|\operatorname{Aut}(s)|}$$

Particularly interesting are the cases where $\alpha = 0, 1$ or $\frac{1}{2}$. The latter represents the fully symmetric version, resulting in some additional useful properties - for example the matrix of adjoint operator $S^\dagger$ is the transposition of matrix associated with S.

Example 705 (Asymmetric operators)**.** *Consider once again the example of the groupoid* Part(6)*, but this time with the same pairs of maps p, u not to $\mathbb{N}$, but to* **IsoFin** *- u is the forgetful functor, while p is the quotient map of a relation induced by partitions.*

$$\begin{tikzcd} & \mathrm{Part}(6) \arrow[dl, "u"'] \arrow[dr, "p"] & \\ \mathbf{IsoFin} & & \mathbf{IsoFin} \end{tikzcd}$$

In this case, contrary to the case over $\mathbb{N}$, the choice of α makes a difference in resulting matrices. Consider matrices M_α and their adjoints $M_\alpha^\dagger$

$$M_0 = (M_1^\dagger)^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{6} & 1 & 6 & 0 & 0 & 0 \\ 0 & \frac{1}{24} & \frac{5}{6} & 3 & 24 & 0 & 0 \\ 0 & \frac{1}{120} & \frac{1}{4} & \frac{5}{2} & 12 & 120 & 0 \\ 0 & \frac{1}{720} & \frac{41}{260} & \frac{3}{2} & 10 & 60 & 720 \end{pmatrix}$$

$$M_0^\dagger = M_1^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 3 & 10 & 15 & 41 \\ 0 & 0 & 0 & 6 & 12 & 50 & 180 \\ 0 & 0 & 0 & 0 & 24 & 60 & 300 \\ 0 & 0 & 0 & 0 & 0 & 120 & 360 \\ 0 & 0 & 0 & 0 & 0 & 0 & 720 \end{pmatrix}$$

$$M_{1/2} = (M_{1/2}^\dagger)^T = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{6}}{6} & \sqrt{3} & 6 & 0 & 0 & 0 \\ 0 & \frac{\sqrt{6}}{12} & \frac{5\sqrt{3}}{3} & 6 & 24 & 0 & 0 \\ 0 & \frac{\sqrt{30}}{60} & \frac{\sqrt{15}}{2} & 5\sqrt{5} & 12\sqrt{5} & 120 & 0 \\ 0 & \frac{\sqrt{5}}{60} & \frac{41\sqrt{10}}{60} & 3\sqrt{30} & 10\sqrt{30} & 60\sqrt{6} & 720 \end{pmatrix}$$

Example 706 (Quantum harmonic oscillator)**.** *In classical theory of electromagnetism the radiation in some region can be described as the* *mean over infinitesimal oscillations generated by each point from the box. The picture became to differ along with development of quantum physics. Since classical physics applies only to sufficiently large collections of particles, Planck introduced the quantisation of this theory by assuming that energy of elementary oscillators takes only discrete values, multiplicities of* $\hbar\omega$*. Einstein gave an physical interpretations of this approach, identifying the oscillators with energy* $n\hbar\omega$ *as coming from interferation of* n *elementary 'quanta'. Restricting ourselves to dimension* 1 *for the sake of convenience, the state of quantum harmonic oscillators is a resultant of states brought by oscillators admitting* n *quanta, thus can be expressed as some power series*

$$\sum^{\infty} a_n t^n$$

The physical model of interferation between such systems is modelled in the Fock space - the space of convergent power series with inner product defined as

$$\left\langle \sum^{\infty} a_n t^n, \sum^{\infty} b_n t^n \right\rangle = \sum^{\infty} n! a_n b_n t^n$$

The reasoning is following: only oscillators with the same number of quanta can affect each other, each state of n *quanta can be identified with each other in* $n!$ *different ways and each such identification contributes identically to the interferation of entire system.*

This model has a beautiful categorification - Fock space can be modelled as a Hilbert space associated to the groupoid of finite sets **IsoFin***. It's inner product can be calculated on basis* Ψ_n *- groupoid of* n *element sets and bijections, or equivalently the action groupoid* $S_n//S_n$*. The weak pullback*

$$\begin{tikzcd} & \langle \Psi_n, \Psi_m \rangle \arrow[dl] \arrow[dr] & \\ \Psi_n \arrow[dr] & & \Psi_m \arrow[dl] \\ & \mathbf{IsoFin} & \end{tikzcd}$$

is non-zero if and only if $n = m$*, and in this case contains bijections between* n*-element sets. Since each such pair has* $n!$ *automorphisms, we get*

$$\langle t^n, t^m \rangle = \left\langle \bar{\Psi_n}, \bar{\Psi_m} \right\rangle = |\left\langle \Psi_n, \Psi_m \right\rangle| = \begin{cases} 0 & n \neq m \\ \frac{1}{n!} & n = m \end{cases}$$

To connect the dots between this construction and physical theory, observe that the basis vector

$$\bar{\Psi}_n = \frac{t^n}{n!}$$

represents the state of single conglomerate of n quanta, and the overall effect such conglomerate have on the system is t^n, as each state is realised by each of its possible $n!$ distinct configurations.

Example 707 (Annihilation and creation)**.** *The categorification of quantum theory and Fock spaces can be taken much deeper by considering some operations that can take place there. Consider for instance the span*

$$\begin{tikzcd} & \mathbf{IsoFin} \arrow[dl, "X"'] \arrow[dr, "X\cup\{\bullet\}"] & \\ \mathbf{IsoFin} & & \mathbf{IsoFin} \end{tikzcd}$$

If a set represents the quanta in a system, the operator corresponding to such span describes the change caused by transition $X \cup \{\bullet\} \to X$, so by removing one particle from the system. Denoting this operator as $A : L^2(\mathbf{IsoFin}) \to L^2(\mathbf{IsoFin})$ (defined only on dense subset). On each n-element set $[n]$

$$A(t^n) = \sum_{m:|[m]|+1=|[n]|} \frac{|\operatorname{Aut}([n])|}{|\operatorname{Aut}([m])|} t^m = nt^{n-1}$$

Thus this annihilation operator is given by differentiation of formal power series

$$(A\phi)(t) = \frac{d}{dt}\phi(t)$$

Similarly, its adjoint, corresponding to creating new particle from the air and to the transposed span, can be similarly computed as

$$(A^\dagger\phi)(t) = t\phi(t)$$

These operators coincide exactly with annihilation and creation considered in physics, and moreover

- *explains why $\frac{d}{dt}$ and $tf(t)$ correspond to annihilation and creation*
- *explains why these operators are non-commutative with relation*

  $$AA^\dagger = A^\dagger A + 1$$

  The reason in terms of groupoids is obvious - given a set with n elements, there is exactly one more way to add new element and then take some out ($n+1$ ways) than to take out some element and then add new one (n ways).

14. Higher category theory

14.1. n**-categories.** Recall that all the small categories form a category **Cat** itself, where morphisms are functors. However, the structure of a category is not rich enough to capture all the dependencies between them, as such a construction is not capable of keeping the objects and morphisms. Enriching categories with these additional features leads to the notion of a bicategory.

Definition 14.1. *A bicategory consists of*

- *(0-cells) objects*
- *(1-cells) for each two objects* X, Y *the category of morphisms between them* $\mathrm{Hom}(X, Y)$
- *(identity morphism) a distinguished identity morphism* $\mathbb{1}_X$ *in every* $\mathrm{Hom}(X, X)$
- *(2-cells) morphisms between morphisms - morphisms in a category* $\mathrm{Hom}(X, Y)$
- *(horizontal composition) For every triple of objects* X, Y, X *the horizontal composition functor* $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$
- *(unitor) Natural transformations* $(- \circ \mathbb{1}, - \circ \mathbb{1}) \simeq (-, -) \simeq (\mathbb{1} \circ -, \mathbb{1} \circ -)$ *between horizontal compositions making the identity morphism behaving as identity*
- *(associator) For every quadruple of objects* X, Y, Z, W *the natural isomorphism of composed horizontal compositions* $\mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \times \mathrm{Hom}(Z, W) \to \mathrm{Hom}(X, W)$ *making them behaving as compositions*

With additional requirements of unitors satisfying triangle identity and associators the pentagon identity, stated in the same way as in the construction of monoids, together with some 29 *more conditions, that you can look up on nLab.*

Example 708 (Strict 2-category)**.** *The most boring example of a bicategory is just* Cat*. In general the category where all the unitors and associators are identities are called strict and correspond to subcategories of* Cat*.*

Example 709 (Bicategory of modules)**.** *Modules over rings fits gently into a bicategory, where* 0*-cells are rings,* 1*-cells* $R \to S$ *are bimodules* ${}_RM_S$ *(which are just* $\mathbf{Ab}$*-enriched functors* $R \times S^{op} \to \mathbf{Ab}$*) and* 2*-cells are homomorphisms of modules.*

Example 710 (Bicategory of relations)**.** *Recall that for any object* $X \in \mathcal{C}$ *the subobjects of* X *form a poset, so also canonically a category. Moreover, for two fixed sets* A, B*, the set of relations between* A *and* B

coincides with the powerset $\mathcal{P}(A \times B)$, which makes the category **Rel** *into a bicategory, which* 0*-cells are sets,* 1*-cells* $\mathbf{Rel}(A, B)$ *are relations between them, and their composition*

$$\mathbf{Rel}(A, B) \times \mathbf{Rel}(B, C) \to \mathbf{Rel}(A, C)$$

serving as 2*-cells.*

Example 711 (Bicategory of ordered sets)**.** *Let* **Ord** *be a category of ordered sets with order-preserving maps. Since every set of order preserving maps also naturally ordered by $f \leq g$ iff $\forall_x f(x) \leq g(x)$,* **Ord** *is self-enriched. Identifying order with their canonical categories shows that indeed it can be considered as* **Cat***-enriched, with* 2*-cells constructed as ordered maps.*

Example 712 (Category of monoidal categories)**.** *Monoidal categories form a strict 2-category* **CatMon***. The 1-morphisms are all the strong monoidal functors - functors preserving the monoidal structure. 2-morphisms are monoidal natural transformations which, again, preserve monoidal structure.*

Example 713 (Lax monoidal functors)**.** *There is alternative way of constructing the category of monoidal categories, allowing more functors to exist. In the category* $\mathbf{CatMon}_{\updownarrow}$ *we require the functors only to have a natural transformation $\mu_{X,Y} : F(X) \otimes_{\mathcal{D}} F(Y) \Rightarrow F(X \otimes_{\mathcal{C}} Y)$ relating the monoidal structures of the image rather than strictly preserve it.*

Example 714 (Monoidal category)**.** *Just like we previously constructed any monoid M as a category BM with on element, the monoidal category $\mathcal{M}$ can be embedded in the same fashion in a bicategory $B\mathcal{M}$ with single object. The 1-morphisms represent the objects of $\mathcal{M}$ and 2-morphisms are morphisms from $\mathcal{M}$. The monoidal structure is captured by unitor and associator (so we can see that similar requirements that they satisfy, the triangle and pentagon identities, are not a coincidence)*

Example 715 (Adjunctions in relations are graphs of functions)**.** *The idea of relations can be generalised to bicategories different than sets. A very gentle example introduces this idea in the bicategory of relations. Recall that two functors, 1-cells in* **Cat***, were adjoint if there were unit and counit*

$$\mu : 1_{\mathcal{C}} \Rightarrow RL$$
$$\epsilon : LR \Rightarrow 1_{\mathcal{D}}$$

satisfying certain properties of interchange whiskerings. Similarly and even easier, in **Rel** *two relations* L, R *(*1*-cells) are considered adjoint whenever their compositions form an unit and counit, thus satisfy*

$$1_A \subseteq RL$$
$$LR \subseteq 1_B$$

To classify adjunctions in **Rel***, notice that to every function of sets* $f : A \to B$ *one can functorially associated its graph* $f^* = \{(a, f(a))\} \subseteq A \times B$*, which is a relation from* A *to* B*. Moreover, since* $A \times B \simeq B \times A$*, dually one can form a relation* f_* *from* B *to* A*. It can be easily checked that a composition of these relations form an adjoint pair*

$$f_* \dashv f^*$$

Moreover, it can be shown that every adjunction in **Rel** *has such a form.*

Example 716 (Adjunctions in ordered sets)**.** *Similarly as in relations, we may consider adjunctions in the bicategory* **Ord***, which coincide with Galois connections, extensively explained in chapter 4.*

Example 717 (Monad as bicategory)**.** *Every monad can be realised as* 2*-category in similar spirit as every group can be realised as* 1*-category. If* T *is a monad in some category* $\mathcal{C}$*, its associated category has a single object* $\mathcal{C}$*, single* 1*-cell* $T : \mathcal{C} \to \mathcal{C}$ *and two* 2*-cells - unit and evaluation of a monad,* $1 \Rightarrow T$ *and* $T^2 \Rightarrow T$*.*

Example 718 (Adjoint pair as bicategory)**.** *Similarly to monads, bicategories captures nicely adjunctions as well. Given adjoint pair* F, G *between categories* $\mathcal{C}, \mathcal{D}$*, it can be encoded in a* 2*-category with two objects* $\mathcal{C}, \mathcal{D}$*, two* 1*-cells* F, G *and two* 2*-cells - the unit and counit of adjunctions.*

Example 719 (Spans)**.** *Consider some category* $\mathcal{C}$ *with all pullbacks. The diagrams of a form, called spans (or roofs)*

$$\begin{tikzcd} & X \arrow[dl] \arrow[dr] & \\ A & & B \end{tikzcd}$$

add a new layer of morphisms to $\mathcal{C}$ *making it a double category* $\mathrm{Span}\mathcal{C}$*, where objects and vertical morphisms are the same as in* $\mathcal{C}$ *and additionally the horizontal morphisms are spoons with composition defined* *via pullbacks, while 2-cells correspond to maps between tops of the roofs $M \to N$.*

$$
\begin{tikzcd}
 & & X \times_B Y \arrow[dl, dotted] \arrow[dr, dotted] & & \\
 & X \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\
A & & B & & C
\end{tikzcd}
$$

Example 720 (Gluing subspaces)**.** *One may also consider the category of cospans on $\mathcal{C}$, defined dually to spans by using pushouts. Particularly pleasing special case is the category* cospan(**ETop**) *of a category of topological spaces with embeddings. In such a case a cospan $A \to B$ is a topological space containing both A and B as subspaces, and the composition of cospans correspond to gluing together these total spaces along subspaces pointed out by arrows.*

Example 721 (Cobordism)**.** *Cobordism theory can be gently encoded in a double category $\mathbf{Man}_n$. Its objects are n-dimensional manifolds without boundary, and their morphisms establish vertical cells. A horizontal cell $A \to B$ is an $(n{+}1)$-dimensional manifold M with boundary, together with homeomorphism $\partial M \to A \sqcup B$. A 2-cell*

$$
\begin{tikzcd}
A \arrow[r, "M"] \arrow[d] & B \arrow[d] \\
C \arrow[r, "N"] & D
\end{tikzcd}
$$

is a morphism of $(n+1)$-manifolds $\varphi : M \to N$, fitting into a commutative diagram

$$
\begin{tikzcd}
\partial M \arrow[r] \arrow[d, "\varphi"] & A \sqcup B \arrow[d] \\
\partial N \arrow[r] & C \sqcup D
\end{tikzcd}
$$

Horizontal composition on the other hand is done by gluing manifolds along boundaries induced from canonical pushouts, constructed dually to usually considered in this context pullbacks.

Example 722 (Matrices as spans)**.** *The bicategory* span(**FinSet**) *can be regarded as categorification of* $\mathbb{N}$*-valued matrices. To every span*

$$\begin{tikzcd} & M \arrow[dl, "g"'] \arrow[dr, "f"] & \\ {[n]} & & {[m]} \end{tikzcd}$$

we can associate $n \times m$ *matrix* $\hat{M}$*, which values* $\hat{M}_{i,j}$ *are cardinalities of the joint fiber under* $(i,j) \in [n] \times [m]$*. It is indeed a nice model, since composition of spans correspond exactly to multiplication of associated matrices.*

Example 723 (2-groupoid)**.** *2-groupoid are straightforward generalisations of ordinary groupoids, defined as bicategories with all (both 1- and 2-) arrows invertible.*

Example 724 (Fundamental 2-groupoid)**.** *Considering the 2-categorical version of the fundamental groupoid makes possible to get rid of relying only on homotopy classes, considered to fix the lack of associativity. In the fundamental* 2*-groupoid we obtain simpler structure with objects corresponding to points,* 1*-arrows to paths and* 2*-arrows to path homotopies. The associativity of composition is no longer needed, as while working with bicategories we only need it to be associative up to canonical isomorphism, obtained from reparametrisation.*

Example 725 (A lot of duals)**.** *Unlike in the case of classical categories, opposites of* n*-categories can be construction in* 2^n *different ways, as we can choose to inverse or not each arrows between* n*-cells independently. The convention resolving partially this problem is to denote as* $\mathcal{C}^{op}$ *the category where only the* 1*-cells has been inverted, while* $\mathcal{C}^{co}$ *denotes inverting all of the* 2*-cells.*

A common motive in higher category theory is the existence of two variants of structures satisfying some commutativity axioms, the strict case, often much less common, where the axioms require the stronger relation of equality, and the weak (or standard) one, where everything is considered only up to natural isomorphism. In this subsection we will take a closer look to the strict version of higher categories.

Example 726 (Enhanced construction)**.** *A strict* n*-categories can be constructed in a handy way by repeated enhancements. Starting from defining strict* 0*-categories to be sets, the category* $(n+1)$-**sCat** *of strict* $(n+1)$*-categories can be inductively constructed as the category* *of n-$\mathbf{sCat}$-categories (which, after denoting the category of $\mathcal{V}$-enriched categories as $\mathcal{V}\mathbf{Cat}$, can be expressed as $(n\text{-}\mathbf{sCat})\mathbf{Cat}$). First two cases are already familiar, and the third one, which happens to be also the most important, by the reason explained in the end of this section, has a reasonable characterisation as well*

$$0\text{-}\mathbf{sCat} = \mathbf{Set}$$
$$1\text{-}\mathbf{sCat} = \mathbf{Cat}$$
$$2\text{-}\mathbf{sCat} = \textit{subcategories of } \mathbf{Cat}$$

Example 727 (**Top** as 2-category)**.** **Top** *can be made into a strict* 2*-category after adding additional* 2*-morphisms between maps to be their classes of homotopies. More precisely, every hom-set* $\mathbf{Top}(X, Y)$ *can be also considered as category itself, with arrows between maps, i.e. elements of the hom-set* $\mathbf{Top}(X, Y)(f, g)$ *being isomorphism classes of homotopies between* f *and* g*.*

Example 728 (Chain complexes as 2-category)**.** *In a similar spirit the category of chain complexes* $\mathbf{Ch}(\mathcal{A})$ *in some abelian category* $\mathcal{A}$ *has a structure of strict* 2*-category, since as in* **Top** *we can use the notion of a chain homotopy to define arrows in* $\mathbf{Ch}(\mathcal{A})(C_\bullet, D_\bullet)$ *as isomorphism classes of chain homotopies.*

Example 729 (n-categories as presheaves)**.** *Yet another construction of strict higher categories uses the notion of globular sets. An* n*-globular set* X *is a diagram of sets of a form*

$$X(n) \underset{s}{\overset{t}{\rightrightarrows}} X(n-1) \underset{s}{\overset{t}{\rightrightarrows}} \cdots \underset{s}{\overset{t}{\rightrightarrows}} X(0)$$

with arrows s *and* t *(corresponding to the choice of source and target of an arrow) satisfying relations expected from actual domain and codomains:* $s \circ s = s \circ t$ *and* $t \circ s = t \circ t$*.*

A strict n*-category is equivalent to some* n*-globular set* X *equipped with additional* n*-dimensional composition map* $c_n : X(m) \times_{X(n)} X(m) \to X(m)$ *and identity* $1_n : X(n) \to X(n+1)$ *(note that in case of* 2*-categories* c_0 *is the composition along objects* $X(0)$*, so the horizontal one, while* c_1 *is the vertical composition along arrows* $X(1)$*) and satisfying additional* 6 *axioms encoding proper composition of source and targets, proper composing identities, associativity, identities, binary interchange and nullary interchange of identities.*

Moreover, since the category of globular can be defined as a presheaf category associated to the category $\mathbb{G}_n$ *corresponding to the diagram of opposite shape to previously considered, every strict* n*-category can be seen as a special type of presheaf over* $\mathbb{G}_n$*.*

Example 730 (Coglobular space)**.** *Characterising strict n-categories as presheaves over $\mathbb{G}_n$ provides some interesting examples, arising more naturally from this perspective. Let's start with the construction of the covariant functor $\mathbb{G}_n \to$ **Top**, called the n-coglobular space due to it's reversed direction (with respect to the standard contravariant one, of course). It can be assembled from the sequence of disks*

$$D^0 \underset{d}{\overset{u}{\rightrightarrows}} D^1 \underset{d}{\overset{u}{\rightrightarrows}} \cdots \underset{s}{\overset{t}{\rightrightarrows}} D^n$$

relying on the natural decomposition of disks into lower and upper half, with maps u and d denoting their inclusions. Applying the contravarint hom functor makes this space into a presheaf

$$\mathrm{map}(D^n, X) \underset{d}{\overset{u}{\rightrightarrows}} \mathrm{map}(D^{n-1}, X) \underset{d}{\overset{u}{\rightrightarrows}} \cdots \underset{s}{\overset{t}{\rightrightarrows}} D^0$$

naturally satisfying additional axioms making its to the strict n-category, sometimes called the singular globular set, and after taking the direct limit, strict ∞-category.

Example 731 (Globular complex)**.** *With a little more effort similar construction can be applied to non-negatively graded chain complexes* $\mathbf{Ch}^+(\mathbf{Ab})$. *This time the most natural choice of m-cells as C_m unfortunately fails apart from the first $X(0) = C_0$. To construct $X(1)$ we need to consider elements $\sigma \in C_1$ together with two zero cells σ_0^+ and σ_0^-, such that $d\sigma = \sigma_0^+ - \sigma_0^-$. Similarly, $X(m)$ will have a form of a $(2m+1)$-tuple*

$$\sigma = (\sigma, \sigma_{m-1}^+, \sigma_{m-1}^-, \cdots, \sigma_0^+, \sigma_0^-)$$

such that

$$d\sigma = \sigma_{m-1}^+ - \sigma_{m-1}^-$$
$$d\sigma_k^+ = d\sigma_k^- = \sigma_{k-1}^+ - \sigma_{k-1}^-$$

This artifically looking setting allows to define properly source and targets maps using the additional terms as

$$s()\sigma = (\sigma_{m-1}^-, \sigma_{m-2}^+, \sigma_{m-2}^-, \cdots, \sigma_0^+, \sigma_0^-)$$
$$t()\sigma = (\sigma_{m-1}^+, \sigma_{m-2}^+, \sigma_{m-2}^-, \cdots, \sigma_0^+, \sigma_0^-)$$

Again for every n such a presheaf has a natural structure of a strict n-category, as well as strict ∞-category after passing to the colimit.

Example 732 (Chain complexes as abelian groups in ∞-**sCat**)**.** *The construction of an strict ∞-category out of a chain complex, even though painful, turns out to be unexpectedly fruitful. The pointwise operations on abelian groups C_n makes its globular complex an abelian group internal to ∞-**sCat**. Moreover, all abelian groups turns out to* *have such a form, which provides a higher-categorical analogue of the Dold-Kan correspondence, i.e. the equivalence*

$$\mathbf{Ab}(\infty(-)\mathbf{sCat}) \simeq \mathbf{Ch}^+(\mathbf{Ab})$$

Example 733 (∞-groupoid cardinality as Euler characteristic)**.** *Recall that in first chapter we've constructed the homotopy cardinality of groupoids as the possible divergent series*

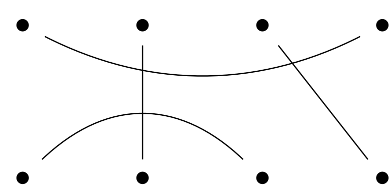
$$|\mathcal{G}| = \sum_{[x]\in\pi_0(\mathcal{G})} \frac{1}{|\operatorname{Aut}(x)|}$$

Such construction can be generalised to ∞-groupoids, where admits much more interesting properties. In this case we do not restrict ourselves only to counting automorphisms (1-arrows), but include in the formula all of them in the alternating sum, counted using the simplicial homotopy groups:

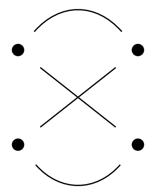
$$|\mathcal{G}| := \sum_{[x]\in\pi_0(\mathcal{G})} \prod \infty_{n=1} |\pi_n(\mathcal{G}, x)|^{(-1)^n} = \sum_{[x]\in\pi_0(\mathcal{G})} \frac{|\pi_2(\mathcal{G}, x)|}{|\pi_1(\mathcal{G}, x)|} \cdot \frac{|\pi_4(\mathcal{G}, x)|}{|\pi_3(\mathcal{G}, x)|} \cdot \ldots$$

By the correspondence between ∞-groupoids and homotopy types of spaces, this number corresponds to the Euler characteristic of associated topological space.

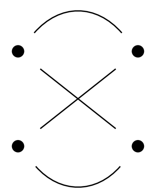
Example 734 (Presentation of algebroids)**.** *In the chapter about enriched categories we've seen a few examples of algebroids - linear categories generalising k-algebras. We've seen for example the Brauer category, with objects $\mathbb{N}$, homomorphisms - vector spaces generated by flat tangle diagrams such as*



We said that this category has a presentation - it is generated by 3 links subject to some relations. This presentation as generator and relations of any algebroid can be encoded in a 2-category. Its objects are the objects of algebroid. 1-arrows are generated with no additional relations - these form 2-arrows between proper combinations of generators specified in the relation. For example in case of Brauer category here is an example of 1-arrow, constructed as the composition of two arcs and a *cross*



The relations of Brauer categories provides the 2*-arrow identifying this tangle with a simpler one*



Example 735 (Higher monoidal categories)**.** *The notion of monoidal structures can also be considered with higher structure. The main motivation is following: a set is not monoidal at all, thus a* 0*-monoid. Monoid is a category with one object, with arrows given by a set (*0*-monoid). Let's call it* 1*-monoid. Now consider* 2*-category with one object and one morphism. By Eckelman-Hilton argument, its* 2*-arrows must form a commutative monoid -* 2*-monoid. Here the sequence stabilises, as in case of any n-category with one object, one* 1*-arrow, $\cdots$, one $(n-1)$-arrow are still just commutative monoids, thus n-monoid* = 2*-monoid* = *commutative monoid.*

Now consider a category. It is not monoidal, thus 0*-monoidal. A bicategory with one object is a monoidal category - we say:* 1*-monoidal. Now iterating this construction, in case of* 3*-category with one object and one* 1*-arrow we obtain braided monoidal category and a* 4*-category with one object,* 1*-arrow and* 2*-arrow is finally symmetric monoidal and the sequence stabilises. These constructions can be summarised in the following table*

	0-*category*	1-*category*	2-*category*
0-*monoidal*	*set*	*category*	*bicategory*
1-*monoidal*	*monoid*	*monoidal category*	*monoidal bicategory*
2-*monoidal*	*commutative monoid*	*braided monoidal category*	*braided monoidal bicatego*
3-*monoidal*	*commutative monoid*	*symmetric monoidal category*	*sylleptic monoidal bicateg*
4-*monoidal*	*commutative monoid*	*symmetric monoidal category*	*symmetric monoidal bicate*
5-*monoidal*	*commutative monoid*	*symmetric monoidal category*	*symmetric monoidal bicate*

Example 736 (Drinfeld center)**.** *Note that from every n-category $\mathcal{C}$ we can form an $(n+1)$-category $\mathcal{Z}(\mathcal{C})$ as follows*

- *$\mathcal{Z}(\mathcal{C})$ has one object*
- *1-arrows = objects of $\mathcal{C}$*
- *2-arrows = 1-arrows in $\mathcal{C}$*
- $\cdots$
- *$(n+1)$-arrows = n-arrows in $\mathcal{C}$*
- *composition = transformation $1_{\mathcal{C}} \to 1_{\mathcal{C}}$*

Note that the category $\mathcal{C}$ has additional level of single arrows, so it takes any k-monoidal n-category to a $(k+1)$-monoidal n-category. In particular

- *a center of a monoid $\boldsymbol{M}$ is has 2-arrows M with composition given by commutative squares*

$$\begin{tikzcd} M \arrow[r, "x"] \arrow[d, "y"] & M \arrow[d, "y"] \\ M \arrow[r, "x"] & M \end{tikzcd}$$

  so forming a commutative monoid $\{x \in M \mid \forall_{y\in M} xy = yx\} = Z(M)$
- *an (enriched) center of a category RR**Mod** is the ring $Z(R)$.*

Example 737 (Automorphism 2-group)**.** *A strict 2-group can be constructed in several equivalent ways:*

- *strict monoidal category with objects and arrows invertible*
- *strict monoidal bicategory with 1-arrows and 2-arrows invertible*
- *group in* **Cat**
- *internal category in* **Grp**
- *crossed module*

A family of examples of 2-groups can be found in any strict bicategory. From every its object X we can form the strict automorphism 2-group $\mathrm{Aut}_s(X)$.

- *objects of $\mathrm{Aut}_s(X)$ are automorphisms $X \to X$*
- *1-arrows are 2-isomorphisms between automorphisms $X \to X$*
- *monoidal product is done by horizontal composition*
- *monoidal 1 is the identity*

Consider the case of a monoid $\mathbf{B}M \in \mathbf{Cat}$. Objects of $\mathrm{Aut}_s(\mathbf{B}M)$ are elements of $\mathrm{Aut}(M)$. Natural transformation between such endofunctors of $\mathbf{B}M$ all comes from action of M on itself by right multiplication, thus we can identify them with some elements of $M^\times$. More precisely, *arrows between automorphisms $f \to g$ must fit into a diagram*

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "m"] & M \arrow[d, "m"] \\ M \arrow[r, "g"] & M \end{tikzcd}$$

so they satisfy the equation

$$g(x) = m^{-1} f(x) m$$

It means that the category $\mathrm{Aut}_s(\bullet_{\mathbf{B}M})$ encodes automorphisms of M and conjugations between them - we can think about them as symmetries of symmetries, if M represents just symmetries. Consider for instance the additive monoid of real matrices $M_n(\mathbb{R})$. Objects of its strict automorphism 2-group. Consider a special case of a group $M = G$. There is an exact sequence

$$1 \to Z(G) \to G \to \mathrm{Aut}(G) \to \mathrm{Out}(G) \to 1$$

The objects of $\mathrm{Aut}_s(\bullet_{\mathbf{B}M})$ are elements of $\mathrm{Aut}(G)$. Isomorphism classes of objects are in bijection with $\mathrm{Out}(G)$, while arrows between isomorphic objects with $G/Z(G) = \mathrm{Inn}(G)$. For example $\mathrm{Aut}_s(\bullet_{\mathbf{B}S_5})$ has two classes of objects - inner automorphisms $x \mapsto g^{-1}xg$ and outer automorphisms $x \mapsto (12)g^{-1}xg(12)$. Automorphisms of every object form a group isomorphic to A_5. Not that it is the case for every non-abelian symmetric group except S_6, where there are 4 components, as outer automorphisms of S_6 are isomorphic to $C_2 \times C_2$, not C_2 as in case of other symmetric groups. Objects of $\mathrm{Aut}_s(\mathbf{B}G)$ are also known as G-bitorsors.

Consider also some example when M is not a group. Automorphisms of monoids can be pretty surprising. Let $M = (\mathbb{Z} \setminus \{0\}, \times, 1)$. Note that, since every non-zero integer factors uniquely into primes + sign, M is isomorphic as monoid to $\mathbb{Z}/2 \oplus \bigoplus_{pprime} p\mathbb{N}$. It means that such an innocent monoid has in fact uncountably many automorphisms - each permutation of primes and choosing a sign establish such. Thus $\mathrm{Aut}_s(\bullet_{\mathbf{B}M})$ is a discrete category with uncountably many objects.

Example 738 (Tangent 2-group)**.** *2-groups can be considered also in different bicategories than* **Cat**. *Particularly interesting are Lie 2-groups - these can be describe either as internal categories in* **LieGrp** *or, more generally, as group object in* **DiffCat** *- category of smooth categories, constructed for instance as enriched over diffeological spaces (as we cannot enrich over* **Diff**, *which is not closed, so we use its convenient generalisation discussed more deeply in the chapter about sheaves). Recall also that for every smooth manifold (or diffeological* *space more generally) M we can construct the tangent groupoid from its tangent bundle*

$$TM \rightrightarrows M$$

with both s, t defined as the same projection and unit inclusion $M \hookrightarrow TM$ as the zero section. It defines a functor

$$T : \mathbf{Diff} \to \mathbf{DiffCat}$$

Restricting ourselves to Lie groups, the functor T sends such a group object in **Diff** *to group object in* **DiffCat** *- a Lie 2-group, called the tangent 2-group $\mathcal{T}G$. The object of $\mathcal{T}G$ correspond to the group G, and arrows have a form*

$$\mathcal{T}M_1 = \mathfrak{g} \rtimes G \simeq TG$$

where the Lie algebra $\mathfrak{g}$ acts on elements of G (objects) by adjoint representation. Similarly we can form a cotangent 2-group $\mathcal{T}^*G$ with arrows

$$\mathcal{T}M_1^* = \mathfrak{g}^* \rtimes G \simeq TG$$

acting by coadjoint representation.

Example 739 (Fundamental 2-group)**.** *In case of weak bicategories we can consider weak 2-groups. The automorphisms of an object in weak category establish an automorphism 2-group in two slightly different ways:*

- *weak automorphism 2-group* $\mathrm{Aut}_w(X)$ *- where 2-arrows have some quasi-inverse*
- *autoequivalence 2-group* $\mathrm{Aut}_{eq}(X)$*- where 2-arrows have specified quasi-inverse*

In particular this way we may categorise the second homotopy group in a similar way as we obtain the fundamental group from fundamental groupoid - the fundamental 2-group $\Pi_2(X, x_0)$ is the autoequivalence 2-group of a point x_0 in the fundamental 2-groupoid $\Pi_2(X)$.

14.2. **Cubical n-categories.** The notion of cubical of n-fold strict categories is an instance of the parallel study of internalisation instead of enrichment. The difference can be seen better than on the example of abelian groups. Categories enriched in abelian groups are the preadditive categories, where the abelian group structure is given to arrows. A different flavour, however similar philosophy, has the study of abelian group objects in categories. These are for example abelian Lie groups, such as $\mathbb{R}^n$. Similar things can be done with variety of objects - one may consider internal groups, monoids, fields, rings, intervals, path and loop spaces, and most importantly to us - categories.

Example 740 (Internal category)**.** *Given a category $\mathcal{C}$ with pullbacks, a category internal to $\mathcal{C}$ consists of:*

- *object of objects $\mathcal{C}_0 \in \mathcal{C}$*
- *object of arrows $\mathcal{C}_1 \in \mathcal{C}$*
- *source and target arrows $s, t : \mathcal{C}_1 \to \mathcal{C}_0$*
- *identity arrow $1 : \mathcal{C}_0 \to \mathcal{C}_1$*
- *composition $\mathcal{C}_1 \times_{\mathcal{C}_0} \mathcal{C}_1 \to \mathcal{C}_1$*

Together with a long list of axioms making it really behaving like an actual category. A categories internal to $\mathcal{C}$ form a category itself denoted as $\mathbf{Cat}(\mathcal{C})$. Some examples of internal categories include

- *small categories are precisely internal in sets: $\mathbf{Cat}(\mathbf{Set}) \simeq \mathbf{Cat}$*
- *crossed modules are categories internal in groups*
- *the arrow category $\to$ is internal in $\mathbf{Cat}^{op}$*

Example 741 (Strict cubical n-category)**.** *A strict cubical higher categories are constructed in the same way as classical strict categories, but iteratively considering internal categories instead of enriched ones. Again starting from $0\text{-}\Box\mathbf{Cat} = \mathbf{Set}$, inductively we may define all the rest as*

$$(n+1)\text{-}\Box\mathbf{Cat} = \mathbf{Cat}(n\text{-}\Box\mathbf{Cat})$$

The first few examples have their own specific names: $1\text{-}\Box\mathbf{Cat}$ are just small categories, $2\text{-}\Box\mathbf{Cat}$ are called double categories and respectively $3\text{-}\Box\mathbf{Cat}$ - triple categories.

Any cubical category can be represented with specifying appropriate classes of cells, with extra structure directing the arrows - for example a double category can be decomposed to

- *0-cells (objects) := objects of $\mathcal{C}_0$*
- *vertical 1-cells := arrows of $\mathcal{C}_0$*
- *horizontal 1-cells := objects of $\mathcal{C}_1$*
- *2-cells (squares) := arrows of $\mathcal{C}_1$*

and the decomposition of triple of higher categories also have a similar form.

Example 742 (Commutative squares)**.** *A natural example of a double category are commutative squares $\mathbf{Sq}(\mathcal{C})$ in some small category $\mathcal{C}$, constructed obviously from $\mathcal{C}_0 = \mathcal{C}$ and $\mathcal{C}_1 = \mathrm{ar}\mathcal{C}$, which is clear, as this way objects are objects, horizontal and vertical arrows are arrows and squares are squares.*

Example 743 (Bicategories)**.** *The fact that double categories have richer structure than bicategories become obvious after identification the latter as a special case of a double category, in which all the vertical arrows are identities.*

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d] & A \arrow[d] \\ B \arrow[r, equal] & B \end{tikzcd}$$

Example 744 (Strict 2-categories)**.** *Surprising break of symmetry between vertical and horizontal arrows provide the dual operations to changing all horizontal arrow to identity. This time, instead of bicategory, we obtain a strict* 2*-category* $\begin{tikzcd} A \arrow[r] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r] & B \end{tikzcd}$

Example 745 (Quintets)**.** *Apart from trivialising vertical arrows, bicategories can be doubled in two more ways. In both cases, instead of not using vertical arrows, one can made them just not distinguishable from horizontal by setting them both equal to* 1*-cells. Two choices appear in the last step, as reversing the arrows of* $\mathcal{C}_1$*, normally swapping vertical and horizontal compositions, now make no change at all. Such a realisation of a bicategory is called a quintet* $\mathbb{Q}\mathcal{C}$*.*

Example 746 (Relations)**.** *The bicategory of relations in a regular category* $\mathbf{Rel}(\mathcal{C})$ *can be non-trivially extend to a double category. Its objects and horizontal arrows coincide with* $\mathcal{C}$*, while vertical arrows* $A \to B$ *are relations - subobjects of the product* $A \times B$*. The square (on the left) in* $\mathbf{Rel}(\mathcal{C})$ *exists (and is unique) iff we have a commutative square in* **Cat** *(on the right)*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "R"] \arrow[dr, phantom, "\Downarrow\eta"] & C \arrow[d, "S"] \\ B \arrow[r, "g"] & D \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\eta"] \arrow[d, hook] & S \arrow[d, hook] \\ A \times B \arrow[r, "f\times g"] & C \times D \end{tikzcd}$$

Note that we have an interesting interplay between horizontal and vertical arrows - every horizontal arrow $f : A \to B$ *yields a relation corresponding to its graph* $f_* = \langle f, \mathbf{1}\rangle \subset A \times B$*, called its companion. Moreover, by symmetry its yields yet another relation - reverse of a graph, called conjoint* $f^* = \langle \mathbf{1}, f\rangle \subset B \times A$*. These two relations factorise each pair of horizontal arrow into two pairs, each one containing identity, and a vertical one in two complementary ways*

$$\begin{tikzcd} A \arrow[r, equal] \arrow[d, equal] & A \arrow[r, "f"] \arrow[d, "f_*"] & B \arrow[d, equal] \\ A \arrow[r, "f"] & B \arrow[r, equal] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[d, equal] \\ A \arrow[r, "f"] & B \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, equal] & B \arrow[r, equal] \arrow[d, "f^*"] & B \arrow[d, equal] \\ A \arrow[r, equal] & A \arrow[r, "f"] & B \end{tikzcd}$$

Moroever, similar behaviour takes place also in case of vertical composition

Example 747 (Conjoint and companion)**.** *The relation between vertical and horizontal arrows noticed in previous example generalises to all double categories, as the notion of conjoint and companion defined via factorisation condition is purely categorical (however, in some cases conjoints and companions may not exist).*

Example 748 (Companion span)**.** *In double category of spans* span($\mathcal{C}$) *every horizontal arrow, i.e. morphism $A \to B$ in $\mathcal{C}$, has both companion and conjoint span:*

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "1"] & A \arrow[dl, "1"'] \arrow[dr, "f"] & \\ A & B & B & A \end{tikzcd}$$

Example 749 (Companion profunctor)**.** *In double category of categories, functors and profunctors, each functor $F : \mathcal{C} \to \mathcal{D}$ has both conjoint and companion profunctors $F^* : \mathcal{D}^{op} \times \mathcal{C} \to \mathbf{Set}, F_* : \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$.*

$$F_* = \mathrm{Hom}(-, F(-))$$

$$F^* = \mathrm{Hom}(F(-), -)$$

Example 750 (Companions of metric spaces)**.** *Recall that on enriched categories we may consider enriched profunctors. For example in case of Lawvere metric spaces, the* **Cost***-profunctor $X^{op} \times Y \to$* **Cost** *encodes additional metric, measuring distances between points from X to points from Y. In the double category $\mathbf{Prof}_{\mathbf{Cost}}$, the companion Φ_F of $F : X \to Y$ measures the distance between the x and y by firstly applying the transformation F:*

$$\Phi_F(x, y) = d_Y(F(x), y)$$

Similarly, the conjoint Ψ_F does the same measurement in the other direction, specifying the distances from y to x by

$$\Psi_F(y, x) = d_Y(y, F(x))$$

Example 751 (Adjoints)**.** *A slightly more complicated factorisation condition makes possibly also a definition of adjoints in double categories, this type being arrows of the same type. We say that w is left adjoint to v if we have factorisation of a trivial diagrams of a form,* *with 2-arrows between non-square rectangles corresponding to unit and counit of the adjunction.*

$$
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "v"] & A \arrow[r, equal] \arrow[d, "v"] & A \arrow[dd, equal] \\
B \arrow[r, equal] \arrow[dd, equal] & B \arrow[d, "w"] & \\
 & A \arrow[r, equal] \arrow[d, "v"] & A \arrow[d, "v"] \\
B \arrow[r, equal] & B \arrow[r, equal] & B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, equal] \arrow[d, "v"] & A \arrow[d, "v"] \\
B \arrow[r, equal] & B
\end{tikzcd}
$$

$$
\begin{tikzcd}
B \arrow[r, equal] \arrow[d, "w"] & B \arrow[d, "w"] \\
A \arrow[r, equal] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
B \arrow[r, equal] \arrow[dd, equal] & B \arrow[r, equal] \arrow[d, "w"] & B \arrow[d, "w"] \\
 & A \arrow[r, equal] \arrow[d, "v"] & A \arrow[dd, equal] \\
B \arrow[r, equal] \arrow[d, "w"] & B \arrow[d, "w"] & \\
A \arrow[r, equal] & A \arrow[r, equal] & A
\end{tikzcd}
$$

The interplay between conjoints, companions and adjoints can be summarised in a theorem, stating that every two of the following conditions imply the third one:

- $w = f_*$
- $v = f^*$
- $w \dashv v$

Example 752 (Biadjunction of realisations)**.** *In case of a quintet double category $\mathbb{Q}\mathcal{C}$, all companions of horizontal arrows exist. Such a class of double categories form a sub-2-category of all double categories* $\mathbf{DoubleCat}_* \subset \mathbf{DoubleCat}$*. Under such a restriction, quintets, horizontal and vertical realisations of a strict bicategories fit form two biadjunctions (which are adjunctions between bifunctors, defined up to equivalence rather than isomorphism)*

$$\mathbb{Q} \dashv \mathrm{Hor} : \mathbf{DoubleCat}_* \to 2 - \mathbf{Cat}_{strict}$$

$$\mathrm{Ver}^{co} \dashv \mathbb{Q} : 2 - \mathbf{Cat}_{strict} \to \mathbf{strDoubleCat}_*$$

Example 753 (Homotopy double groupoid)**.** *Double categories allows to fully encode paths in topological space. The homotopy double groupoid of X is a double category with objects being points of X, both* *vertical and horizontal arrows paths between them, while squares corresponding to path homotopies (in this case not only their classes!).*

Example 754 (Double categories as presheaves)**.** *Similarly to strict n-categories, strict n-fold categories also can be constructed as presheaves with extra structure, now with the base category being even simpler, as to construct n-fold strict category it's enough to consider presheaves from the category $(\downarrow\downarrow)^n$ together with appropriate composition and identity arrows.*

Example 755 (Monoidal categories)**.** *Recall that a monoidal functor is lax if is equipped with natural transformation (satisfying coherence conditions)*

$$\eta : F(X) \otimes F(Y) \to F(X \otimes Y)$$

and colax if the transformation goes in the opposite direction

$$\eta' : F(X \otimes Y) \to F(X) \otimes F(Y)$$

Both these classes assembles together into a bicategory of monoidal categories, with lax horizontal arrows, colax vertical arrows and lax monoidal transformation

14.3. **Multicategories and operads.**

Example 756 (Monoidal multicategories)**.** *Every monoidal category produces canonically a multicategory by identifying n-arrows with arrows from monoidal product of n-elements. The best intuition provides of course the category of vector spaces or modules, where the same thing can be expressed with more elementary language - n-arrow, corresponding to n-ary linear map, is exactly a linear map (thus morphism in a category) from the tensor product of n objects.*

$$M\mathcal{C}(a_1, \cdots, a_n; b) := \mathcal{C}(a_1 \otimes \cdots, \otimes a_n, b)$$

Example 757 (Fake coproduct)**.** *Obviously more interesting examples of multicategories do not correspond to monoidal categories. Suppose $\mathcal{C}$ is a cocomplete category. Then we can make a multicategory from $\mathcal{C}$ using its cocartesian monoidal structure and get the n-arrows expressed with a nice form*

$$M\mathcal{C}(a_1, \cdots, a_n; b) = \mathcal{C}(a_1 \sqcup \cdots, \sqcup a_n, b) = \prod \mathcal{C}(a_i, b)$$

Now notice, that the final formula does not in fact use any colimits in $\mathcal{C}$, and makes sense in any category, even without coproducts. In such a case the n-arrows in multicategory $M\mathcal{C}$ captures the universal arrows that supposed to be produced by coproducts, even though these coproducts need not to exist in $\mathcal{C}$ itself.

Example 758 (Fake monoidal structure)**.** *A similar strategy of capturing the non-existing universal arrows can be used in case of monoidal products in a subcategory. Recall that a subcategory of a monoidal category need not to be monoidal itself - a counterexamples are for instance all subcategories not containing the unit object. In the classical category theory, all the information carrying by monoidal product is lost in non-monoidal subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$, but it can be captured by a multicategory, as the n-arrows can still be computed in $\mathcal{C}$:*

$$M\mathcal{D}(a_1, \cdots, a_n; b) := \mathcal{C}(a_1 \otimes \cdots, \otimes a_n, b)$$

Example 759 (Fake tensor product)**.** *The third instance of the "fake structure" captured by a multicategory cheats the tensor product in case of abelian groups. The classical abelian groups obviously have well-defined tensor product, encoding the n-linear maps. And given abelian groups with extra structure, i.e. internal abelian groups in some category $\mathcal{C}$ (forming a category $\mathbf{Ab}(\mathcal{C})$), the notion of n-linear maps still make perfect sense, but the tensor product need not exist - for example in the category of topological abelian groups there is no reasonable way of constructing tensor, however the idea of n-linear maps can still be captured by a multicategory in essentially the same definition as in the discrete case.*

Example 760 (Multivariable functions)**.** *Going back from pathological examples, let's consider some most important special cases of mostly monoidal multicategories. The Cartesian monoidal structure on sets produces the category of multivariable functions $M\mathbf{Set}$, where n-arrows are just functions of a form $X_1 \times \cdot X_n \to Y$. We will feel the vibe of moving from single- to multivariable calculus often in this chapter.*

Example 761 (Vector-valued posets)**.** *Restricting ourself to partially ordered sets we obtain the multicategory of posets with multivariable functions component-wise monotone, denoted as* **Prost**.

Example 762 (Grammar)**.** *Multicategories provides a nice structure for formal language models. Given some grammar of types (or just systems of numbers with specified arithmetic) and grammar of typed expressions with variables and substitutions (which are just rules of generating and evaluating sentences - for example relating addition with the expressions $\bullet + \bullet$ in the implementation of a calculator), they form a multicategory, which objects are types and n-arrows $(t_1, \cdots, t_n) \to s$ are expressions of type s with variables $t_1, \cdots, t_n$. Composition of arrows are just nested expressions, and evaluations of variables correspond to their identity arrows.*

Example 763 (Convex sets)**.** *The concept of a poset as a category with at most one arrow between each pair of objects egeneralises to multi-posets in the case of multicategories. Multi-posets can be described similar to posets with relations of generalised reflexivity and transitivity. The underlying idea is best captured by convex sets. Their relation can be realised as a multi-poset by taking as objects points from a suitable space, such as* $\mathbb{R}^m$*, and the* n*-variable relations indicating the convex hull of* n *points*

$$(x_1, \cdots x_n) \leq_n x \textit{ iff } x \in \operatorname{conv}(x_1, \cdots x_n)$$

Example 764 (Swiss cheese)**.** *Probably the coolest example of a multicategory is the Swiss cheese construction. Even though it's a bit technical as the beginning, the result is really satisfying. The construction starts with specifying the objects of* $\mathcal{C}$ *as* $\mathbb{N}$*, but our true objects of interest are the euclidean balls, or more precisely their slices. To avoid the upper bound on their dimension, we'll embed them in the countable dimensional space. Don't worry if you have a hard time visualising this step, as everything will reduce to the finite dimension in the end. We attempt to generalise the intuitive concept of slicing the* 3*-ball* $B^3 = \{(x, y, z) \mid x^2+y^2+z^2 \leq 1\}$*. Restriction to points with coordinate* $x \leq 0$ *cuts the ball in half, and similarly we can obtain the quarter of a ball by further restriction to* $x \leq 0, y \leq 0$ *or* $1/8$ *of a ball in case of adding the condition on* z*. While adding more conditions is not possible in dimension three, the infinite dimensional ball can be sliced freely. The* n*th slice of* B^∞ *we'll denote as*

$$B^{[n]} = \{(x_i) \mid \sum_i = 1^\infty x_i \leq 1, x_1, \cdots, x_n \leq 0\}$$

Each sliced ball can be divided into round part and flat faces by taking intersections with axes. Flat k*-faces of* $B^{[n]}$ *we'll denote as* F_k^n*. For* $k > n$ *the face has a form*

$$F_k^n = \{(x_i) \in B^{[n]} \mid x_k = 0\}$$

while when $k \leq n$ *this part of a ball is round, so we consider the face not to exist:* $F_k^n = \emptyset$*. To construct the cheese we need also a tool to move the slices away from each other to form holes in a cheese. Too allow the holes to have different sizes as well, the most handy will be the group of affine translations with scalar scaling*

$$G = \{\mathbf{x} \mapsto a\mathbf{x} + \mathbf{b} \mid a \in \mathbb{R} \setminus \{0\}, \mathbf{b} \in \mathbb{R}^\infty\} \leq \operatorname{Aut}(\mathbb{R}^\infty)$$

Now let's place the holes into the cheese. Given a cheese given as n*-dimensional slice, the* k*-dimensional hole can be properly placed in the* *cheese if the inclusion of flat faces holds as well, and all these inclusions form a set*

$$C(k,n) = \{\varphi \in G \mid \varphi(B^{[k]}) \subseteq B^{[n]}, \varphi(F_i^{[k]}) \subseteq F_i^{[n]}\}$$

The operations of placing more holes fit as multimorphisms of $\mathcal{C}$ by fake coproduct construction

$$\mathcal{C}(k_1, \cdots, k_m; n) = C(k_1, n) \times \cdots \times C(k_m, n)$$

To get a nice looking cheese we want the images of holes to be distinct, which can be conveniently achieved simply by considering such submulticategory $S\mathcal{C}$ of $\mathcal{C}$ where the condition is satisfied. It's easy to see that such class of multiarrows is closed under identities and compositions, establishing a perfectly fine multicategory.

Finally, since $S\mathcal{C}((k_i); n) = \emptyset$ if and only if some $k_i < n$, the infinite dimensional space can be replaced by $\mathbb{R}^{\max k_i}$, making the picture pretty, at least in dimension 3.

Example 765 (Monoids as multifunctors)**.** *In the obvious way we may consider multifunctors between multicategories, assembling into a category* **MultiCat***. The final object in* **MultiCat** *is the terminal multicategory 1 with single object and arrows $(\bullet, \cdots, \bullet)_n \to \bullet$. Since a monoid M in some monoidal category $\mathcal{C}$ can be expressed as a multicategory with object M and two arrows*

$$(M, M) \to_2 M : M \otimes M \to M$$

$$\to_0 M : 1 \to M$$

every monoid can be expressed uniquely as a multifuntor $F : 1 \to M\mathcal{C}$. However, obviously not every such multifunctor correspond to a monoid, as the images of arrows of dimension different than $0, 2$ need not to be empty. In particular, the identity $1 \to 1$ do not represent a monoid.

Example 766 (Algebras of multicategories)**.** *An algebra of a multicategory $\mathcal{C}$ ($\mathcal{C}$-algebra) is just a multicategorical version of copresheaves, defined as a multifunctor $\mathcal{C} \to M$***Set***. The name algebra reflects its more explicit description as a family as sets $X(c)$ acted on by the multicategory $\mathcal{C}$, i.e. every multiarrow $(a_1, \cdots, a_n) \mapsto a$ induces a function $\prod X(a_i) \to X(a)$. In the simplest cases $\mathcal{C}$-algebras are equivalent to already known objects:*

- *if $\mathcal{C}$ is unary, its algebras are just copresheaves, as multifunctors and ordinary functors of corresponding ordinary category coincide*

- *If $\mathcal{C}$ comes from a strict monoidal category, $\mathcal{C}$ algebras are lax monoidal copresheaves*

Example 767 (Endomorphism multicategory). *The idea of $\mathcal{C}$-algebra can be sort of inverted. Given any family of sets $\{X(i)\}_{i\in I}$, we can form a multicategory out of the indexing set I with multiarrows*

$$(i_n) \to i : \prod X(i_n) \to X(i)$$

This way any $\mathcal{C}$-algebra $X = \{X(c)\}_{c\in\mathcal{C}_0}$ can be represented as a multifunctor $\mathcal{C} \to \mathbf{End}(X)$ fixing the objects.

Example 768 (Forgetful algebra). *For any multicategory there exist a trivial $\mathcal{C}$-algebra, corresponding to the trivial $\mathcal{C}$ action. In this case its values are just $X(a) = \mathcal{C}(\emptyset, a)$. It generalises and sometime coincides with the forgetful functor, for example in case of multicategory of abelian groups or any R-modules.*

Definition 14.2 (Operads). *An operad is a multicategory with one object. Alternatively, it can be describe as a sequence of n-ary operations $P(n)$ together with composition $\prod^k P(n_i) \to P(\sum^k n_i)$ and with identity arrow $1 \in P(1)$. A $\mathcal{V}$-operad is an operad enriched over $\mathcal{V}$. A symmetric operad is an operad, such that each set $P(n)$ is acted on by S_n (in a compatible way with respect to all operations), which allows interchanging the variables in a canonical way.*

Example 769 (Monoids as operad algebras). *The most trivial example of an operad is the terminal one, 1, with exactly one operation in every dimension. Its algebras can be identified with the set $M = F(\bullet)$ together with action of 1 inducing functions*

$$p_n : M^n \to M$$

Denoting the value $p_n(x_1, \cdots, x_n)$ as $x_1 \cdot \cdots \cdot x_n$, it's easy to see that axioms of multifuntoriality translate directly to the axioms of a monoid, leading to the equivalence of categories

$$\mathrm{Alg}(1) \simeq \mathbf{Mon}$$

Similarly, we may consider the terminal symmetric operad with additional free and faithful action permuting the factors, and the same reasoning shows the equivalence of its symmetric algebras with commutative monoids

$$\mathrm{SAlg}(1) \simeq \mathbf{CMon}$$

Example 770 (Sets as operad algebras). *A few suboperads of 1 are interesting as well. The most trivial one, with only a single arrow, the identity in dimension 1, leads to most trivial algebras, which are* *just sets. This time the only morphism induced by the action is the morphism induces by identity arrow, really adding anything.*

Example 771 (Semigroups)**.** *Semigroups also can be identified with algebras over some suboperad of* 1*. Since a semigroup is a monoid without the identity element, which is determined with an action of element from* $P(0)$*, considering the operad with a single operation in positive dimensions and* $P(0) = \emptyset$ *leads to equivalence*

$$\mathrm{Alg}(P) \simeq \mathbf{SemiGrp}$$

Example 772 (Pointed sets)**.** *Yet another suboperad of* 1*, this one with only single* 0 *and* 1 *operations, has algebras equivalent to pointed sets, where the basepoint is identified with the action of* 0*-arrow, inducing the function*

$$x_0 : \{\bullet\} \to X$$

Example 773 (G-sets)**.** *Recall that an algebra of a unary multicategory is just a functor* $\mathcal{C} \to \mathbf{Set}$*. This way we may identify the category of* G*-sets with algebras over an operad* P*, where* P *is a multicategory associated to the ordinary delooping* $\mathbf{B}G$*.*

Example 774 (Lie operad)**.** *Algebras of operads capture even non-associative products, such as brackets in Lie algebras. To capture their underlying linear structure, we will work with enrichment over* $\mathbf{Vect}_K$*. The operation associated to Lie algebras is bilinear, so comes naturally from a single generator* b *of* $P(2)$*. In general,* $P(n)$ *are freely generated by compositions of identity arrow* 1 *and* 2*-arrow* b*. Moreover, to capture the Jacobi relations, we must add the symmetric structure on* P*, with* S_n *acting nontrivially in dimension* 2 *and* 3*. If* $\tau \in S_2$ *is a* 2*-cycle and* $\sigma \in S_3$ *a* 3*-cycle, the appropriate actions have a form*

$$b \cdot \tau + b = 0$$

$$b \circ (1, b) + b \circ (1, b)\sigma + b \circ (1, b)\sigma^2 = 0$$

Example 775 (Endomorphism operad)**.** *The construction of endomorphism multicategory can be generalised to any object from monoidal category by its endomorphism operad. Naturally, its* n*-ary operations* $\mathbf{End}(X)(n)$ *encodes just all* n*-ary endomorphisms*

$$\mathbf{End}(X)(n) = \mathcal{C}(X^{\otimes n}, X)$$

As in the case of set valued endomorphisms, each algebra of an operad P *correspond to some set* X *together with multifunctor* $P \to \mathbf{End}(X)$

Example 776 (Operad of curves). *A smooth curve is a smooth function of manifolds $\mathbb{R} \to M$. The curves of n-dimensional euclidean spaces form an operad of curves*

$$P(n) = C^\infty(\mathbb{R}, \mathbb{R}^n)$$

with composition constructed by substitutions. The operad P can be also identified with endomorphism operad in a weird way - it is isomorphic to $\mathbf{End}(\mathbb{R})$, however the real line $\mathbb{R}$ is taken as the element of the category $\mathbf{Diff}^{op}$, moreover with cocartesian structure.

Example 777 (Affine schemes). *Substitution leads to operad structure in several different settings, for example in polynomials $R[x_1, \cdots, x_n]$ over some commutative ring R. The operad $P(n) = R[x_1, \cdots, x_n]$ can be again identified with endomorphism of $R[x]$ in the opposite category of commutative rings, but this time this category is not wild at all, as the elementary tools of algebraic topology identify it with a category of affine schemes via the spectrum and global sections.*

Example 778 (Algebras and modules). *The idea of substitution is in fact well defined in case of any monad, providing a huge family of similar examples. Since the monoidal product in $\mathcal{V}$ coincides with the coproduct of $\mathcal{V}$-algebras, the Cartesian structure on opposite category $\mathcal{V}$-$\mathbf{Alg}^{op}$ translates exactly to evaluations of tensor products. A lot of algebraic theories fits into this pattern, with associated operad corresponding to endomorphism of free object on 1 generator. Among examples are:*

- *sequences of integers $\mathbb{N}^\mathbb{N}$ as endomorphism of $\mathbb{N}$*
- *polynomials in several variables as endomorphisms of $R[x]$*
- *free R-modules as endomorphisms of R*

Example 779 (Modular operad). *An interesting operad associated to complex cobordism theory can be found in topological quantum field theory. Its operations $P(n)$ are isomorphic classes of Riemann surfaces with boundaries homeomorphic to disjoint union of $(n+1)$ circles. Identifying first n circles with input and the last one as output, the morphism $(S_1, \cdots, S_n) \to S$ glue the outputs of surfaces S_i to appropriate inputs of S. Since automorphisms of components induce automorphisms of glued surface, it forms a well defined modular operad*

Example 780 (Fulton-MacPherson operad). *Fulton-MacPherson operad, known also as operad of planar linear sequences, forms yet another very geometric example. The construction starts with a group of affine linear automorphisms of $\mathbb{C}$, known also as the "ax+b" group.*

$$G = \{az + b \mid a \in \mathbb{C}^\times, b \in \mathbb{C}\}$$

Every group has associated free operad $P(n) = G^n$ with multiplication of all the elements together as n-arrows. The operad we're interested in is the suboperad of P encoding sequences of points from the plane by consecutive translations between them starting with some fixed points, let's say 0 and 1.

$$Q(n) = \{(\gamma_i) \in G^n \mid 0 = \gamma_0(0), \gamma_0(1) = \gamma_1(0), \cdots, \gamma_n(1) = 1\}$$

It is clear from construction that the set $Q(n)$ can be also identified with set of ordered $n+1$ points from the plane with distinct neighbours, i.e.

$$Q(n) \simeq \{(z) \in \mathbb{C}^{n+1} \mid 0 = z_0 \neq z_1 \neq \cdots \neq z_n = 1\}$$

The most interesting aspect of Q are its algebras. Consider for example the set $\mathcal{K}$ of compact subsets of $\mathbb{C}$. It can be given a Q-algebra structure, with Q acting on set of subsets by taking union of translation

$$(\gamma_1, \cdots, \gamma_n)^*(S_1, \cdots, S_n) = \gamma_1(S_1) \cup \cdots \cup \gamma_n(S_n)$$

The interesting aspect of $\mathcal{K}$ was found by Hutchinson - if all the transformations γ_i are contractions (or, in the language of points, the distance between consecutive points is less than 1), then their induced action $\mathcal{K}^n \to \mathcal{K}$ has unique fixed point among open subsets of $\mathbb{C}$. A lot of interesting fractals can be characterised this way, such as Sierpiński gasket, Koch snowflake or Péano space-filling curve, moreover with very simple configurations, using only 5 to 7 points.

Example 781.

Example 782 (Little disk operad)**.** *The idea behind Fulton-MacPherson operad has another, similarly geometrically pleasing realisation as the little disk operad. This time we start with the n-dimensional version of $ax + b$ group for fixed dimension n with associated free operad $P(n) = G^n$.*

$$G = \{\lambda \mathbf{x} + \mathbf{b} \mid \lambda < 0, \mathbf{b} \in \mathbb{R}^n\}$$

The suboperad of P of our interest are transformations moving the unit disk B^n to disjoint subsets, moreover all contained in the unit disk itself.

$$\mathbf{D}_n(k) = \{(\varphi_i) \in G^n \mid \varphi_i(B^n) \subseteq B^n, \varphi_i(B^n) \cap \varphi_j(B^n) = \emptyset\}$$

Compositions of arrows

$$\mathbf{D}_n(k) \times \prod^k \mathbf{D}_n(m_i) \to \mathbf{D}_n(\sum^k m_i)$$

Is the embedding of configurations in the disks, which are again itself embedded in the final disk. Interestingly, the loopspaces Ω^n are also $\mathbf{D}_n$ algebras - given n based maps $S^n \to X$, each configuration of disjoint disks inside B^n. Collapsing all the boundaries of disks into a single *point and choosing it as a basepoint determined the pointed map $S^n \to (S^n)^{\vee k}$. As in case of universal loop operads, composition with such a map leads to $\mathbf{D}_n$ action on the loopspace. In fact the universality of loopspace operad manifests in this example, as the construction of the map $S^n \to (S^n)^{\vee k}$, used to define the action on Ω^n, induces the natural embedding*

$$\mathbf{D}_n) \hookrightarrow \mathbf{U}_n$$

Example 783 (Associahedra)**.** *Recall that often associativity conditions on categorical structures are presented as commutativity of diagrams of some specific shape, depending only on numbers of components involved, for example a trivial diagram with no arrows in case of 2 components, always associative, single arrow $(X \times Y) \times Z \to X \times (Y \times Z)$ and pentagon relations in case of 4 objects. The geometry behind these relations can be expressed very literally as $(n-2)$-dimensional polytope, called associahedron. Its k-dimensional cells correspond to all meaningful inclusions of $n-k-2$ pairs of brackets into sequence of $n-2$ objects. A brilliant embedding of associahedra A_n into $\mathbb{R}^n$ was proposed by Loday. It relies on the combinatorial trick, ordering vertices of rooted finite binary tree as closest common ancestors of consecutive leaves, ordered from left to right, i.e. using the bijective correspondence $v_i = \mathrm{ClosesCommonAncestor}(l_i, l_{i+1})$. At next, to any tree we associate a single vector $m(T) \in \mathbb{R}^n$, which value at i-th coordinate is the product of left and right numbers of leafs lying under v_i. Then the associahedron A_n is realised by a polytope being a convex hull of vectors $m(T)$ associated to all classes of trees with $n+1$ leaves.*

Example 784 (Loop operad)**.** *Consider the cocartesian category of nice (i.e. Cartesian closed, for example CGWH) pointed spaces $(\mathbf{Top}_*, \vee, 1)$. The endomorphism operad of a sphere (classically in opposite Cartesian category), i.e.*

$$\mathbf{U}_k(n) = \mathrm{map}_*(S^k, \bigvee^n S^k)$$

is called the universal operad for k-fold loopspaces, since each loopspace $\Omega^k = \mathrm{map}_*(S^k, -)$ has a natural structure of $\mathbf{U}_k$-algebra with action coming from compositions

$$\mathbf{U}_k(n) \times \Omega^k(X)^n = \mathrm{map}_*(S^k, \bigvee^n S^k) \times \mathrm{map}_*(\bigvee^n S^k, X) \to \mathrm{map}_*(S^k, X) = \Omega^k(X)$$

Example 785 (Baratt-Eccles operad)**.** *The Baratt-Eccles operad assembles together all symmetric groups into an operad, and this way the free multiplication of elements $G^n \to G$ not used already in defining* *operations, provides the extra feature of simplicial enrichment. Recall that the action groupoid associated to a groupaction $X//G$ is the groupoid with objects $x \in X$ and arrows sending x to gx. We'll use this construction to define our* **sSet***-operad as nerves of groupoid of free and transitive action of S_n*

$$\mathcal{E}(n) = \mathcal{N}(S_n//S_n)$$

The category $S_n//S_n$ is just a codiscrete groupoid with $n!$ points, but presenting it as the action groupoid comes with additional action of S_n permuting its objects. The cells of the nerve $\mathcal{N}(S_n//S_n)$ have a simple form

$$N(S_n//S_n)_k = \prod S_n^{k+1}$$

with face d_i maps corresponding exactly to multiplication of consecutive elements σ_i and σ_{i+1}. The composition of operations

$$\mathcal{E}(k) \times \prod^k \mathcal{E}(n_i) \to \mathcal{E}(\sum^k n_i)$$

is just the composition of given k permutations with order specified by the first argument $\sigma \in \mathcal{E}(k)$. Moreover, action of S_n on $S_n//S_n$ gives $\mathcal{E}$ the symmetric structure.

Example 786 (Total orders operad)**.** *The classical* **Set***-operad lying under Baratt-Eccles operad has equivalent set-theoretic construction. Its operations $P(n)$ are total orders on n elements, which are composed under the lexicographic combinations.*

Example 787 (Operad of trees)**.** *The forgetful functor from multicategories to multigraphs (where edges are allowed to have multiple sources), leaving only the incidence multigraph of n-arrows compositions, in similar spirit to ordinary categories and ordinary graphs, has left adjoint free multicategory functor. The simplest free category generated by terminal multigraph is the operad of trees. Its n-operations* $\mathbf{tr}(n)$ *are finite rooted planar (with ordered leaves, or with specified embedding in $\mathbb{R}^2$) trees with n leaves. The compositions of threes just glue together the roots of list from the tuple to the leaves of the first tree in order consistent with labels of leaves. The freeness of* $\mathbf{tr}$ *is depicted by the recursive construction of* $\mathbf{tr}(n)$ *by freely adjoining tuples of smaller operations:*

- $\mathbf{tr}(1)$ *has a single element*
- *for any elements $t_1 \in \mathbf{tr}(k_1), \cdots, t_n \in \mathbf{tr}(k_n))$, there is an element $(t_1, \cdots, t_n) \in K(\sum^n k_i)$*

Example 788 (Classical trees)**.** *The operad of* $\mathbf{ctr}$ *of classical (binary) trees appears as suboperad of* $\mathbf{tr}$*, with* $\mathbf{ctr}(n)$ *containing only trees with exactly* 0 *or* 2 *vertices attached to each vertex. This operad also has a recursive presentation, which this time shows that it's not free, as it involves additional constrains on generating tuples:*

- $\mathbf{tr}(0)$ *has a single element - a single vertex*
- $\mathbf{tr}(1)$ *has a single element - a vertex with a leaf*
- *for any elements* $t_1 \in \mathbf{tr}(k_1), t_2 \in \mathbf{tr}(k_2)$*, there is an element* $(t_1, t_2) \in K(k_1 + k_2)$

Note the similarity to the linear functors, where similar construction were done.

Example 789 (Cartesian monad)**.** *A particular class of monad, called cartesian, are especially interesting from the multicategorical perspective. A category is called cartesian if it has pullback, a functor is cartesian if it preserves pullbacks and a natural transformation - if every induced square is itself a pullback. A monad* T *on* $\mathcal{C}$ *is cartesian if it satisfies all cartesian condition possible - i.e. if* $\mathcal{C}$ *is cartesian,* T *is cartesian as functor and natural transformations* $1 \Rightarrow T$*,* $T^2 \Rightarrow T$ *are cartesian. In general all monads induced from strongly regular algebraic theories are cartesian, but not all are of this form. Here are some already familiar examples of monads which are cartesian:*

- *identity monad*
- *free monoid monad (in any monoidal cartesian category - except classical monoids it includes also such monads as free topological monoids, free strict monoidal categories, free strict symmetric monoidal categories)*
- *induced from strongly regular theory (free semigroup, free monoid with involution)*

Among notable counterexamples one can find

- *free commutative monoid monad*
- *free symmetric monoidal category monad*

Example 790 (Free algebraic theory)**.** *An important example of cartesian monad produces free finitary algebraic theories - having a single* n*-ary operation for each* $n \in \mathbb{N}$ *with no additional relations. Its monad* $T : \mathbf{Set} \to \mathbf{Set}$ *can be described as a functor mapping* X *to all rooted trees with leaves labeled by* X*.*

Example 791 (Free category monad)**.** *Another cartesian monad that will be interesting for us is associated to free categories. It is induced from adjunction of free-forgetful pair from the category of directed* *graphs. Since directed graphs are just presheaves on* $\downdownarrows$, $FC : \widehat{\downdownarrows} \to \widehat{\downdownarrows}$ *takes a graphs and adds to it all the formal concatenations of edges (including self-loops, considered as concatenations of no edges). Similarly one can consider a free strict* n*-tuple category monad as a monad* $nFC : \widehat{\downdownarrows}^n \to \widehat{\downdownarrows}^n$

Example 792 (Free operad)**.** *Recall that any operad can be presented as a sequence of sets* $P(n)$ *with extra structure. The forgetful functor*

$$\mathbf{Operad} \to \mathbf{Set}^{\mathbb{N}}$$

has left adjoint, together assembling the free operad monad $FO : \mathbf{Set}^{\mathbb{N}} \to \mathbf{Set}^{\mathbb{N}}$*. This monad has a dual description to the case of free categories, as the sets* $FO(X)_n$ *can be identified again with trees labeled by* X_n*, but this time the labels are associated to vertices, not leaves.*

Example 793 (Bicategory of a monad)**.** *With any cartesian monad* $T : \mathcal{C} \to \mathcal{C}$ *one can associate a bicategory* $\mathcal{C}_{(T)}$*. Its objects coincides with objects of* $\mathcal{C}$*.* 1*-arrows* $f : X \to Y$ *are spans of a form*

$$\begin{tikzcd}
 & A \arrow[dl] \arrow[dr] & \\
T(X) & & Y
\end{tikzcd}$$

Composition of arrows uses the fact that T *is cartesian - composition of arrows* $X \xrightarrow{f} Y \xrightarrow{g} Y$ *is constructed by taking the pullback of the diagram*

$$\begin{tikzcd}
 & & & TA \times_{TY} Y \arrow[dl] \arrow[dr] & & \\
 & & TA \arrow[dl] \arrow[dr] & & Y \arrow[dl] \arrow[dr] & \\
 & T^2X \arrow[dl] & & TY & & Z \\
TX & & & & &
\end{tikzcd}$$

2-cells $f \Rightarrow g$ are functions $\eta : A \to B$, completing the commutative square

$$\begin{tikzcd}
& A \arrow[dl] \arrow[dr] \arrow[dd, dotted, "\eta"] & \\
TX & & Y \\
& B \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

Example 794 (*T*-multicategories)**.** *The category $\mathcal{C}_{(T)}$ provides a generalisation of a multicategory by specifying input shape of morphisms with a cartesian monad T. Formally, using the bicategory $\mathcal{C}_{(T)}$ one can define T-multicategories with a simple trick of identifying them with monads in $\mathcal{C}_{(T)}$. More concretely, a T-multicategory consists of an object C_0 of objects and C_1 of arrows. The domain and codomain of arrows are encoded in the so called T-graph. While classically codomains are just single objects from $\mathcal{C}$, domains of arrows are objects assembled together with the monad T*

$$\begin{tikzcd}
& C_1 \arrow[dl, "cod"'] \arrow[dr, "dom"] & \\
TC_0 & & C_0
\end{tikzcd}$$

Moreover, we have a composition of arrows, given with the functor

$$C_1 \times_{TC_0} TC_1 \to C_1$$

and the identity arrow $C_0 \to C_1$. Similarly, whenever $\mathcal{C}$ has terminal objects, one can think about T-operads as special cases of this construction, where $C_0 = 1$ is a terminal object. In degenerate special cases, we can identify a few T-categories with already known objects:

- *identity monad $T = \mathbb{1}_{\mathbf{Set}}$ yields T-***Multicat** $\simeq$ **Cat**
- *in case of operads, $T = \mathbb{1}_{\mathbf{Set}}$ yields T-***Operad** $\simeq$ **Monoid**
- *free monoid monad $FM(X) = \coprod^{\infty} X^n$ yields FM-***Multicat** $\simeq$ **Multicat**
- *FM-***Operad** $\simeq$ **Operad**

Example 795 (Contravariance as involution)**.** *A simple but non-trivial examples of T-multicategory can be found in case of a set-valued monad F^*M of a free monoid with involution. F^*M-multicategories have arrows similar to the case of free monoid, but now every element of*

$F^*M(C_0)$ is a tuple of objects, some of them possibly involuted, what can be identified with adding them additional sign σ

$$(D_1^{\sigma_1}, \cdots, D_n^{\sigma_n}) \in E^*M(C_0)$$

Similarly as in case of FM, arrows $((X_1^{\sigma_1}, \cdots, X_n^{\sigma_n}) \to Y)$ can be associated with functors $D_1^{\sigma_1} \times \cdots \times D_n^{\sigma_n} \to E$, but the invlution acts on categories by inverting the arrows:

$$D^\sigma = \begin{cases} D & \sigma = 1 \\ D^{op} & \sigma = -1 \end{cases}$$

Example 796 (Loops with going back)**.** *Similarly to invloutions, we can consider monoids with anti-involutions, which are operations reversing the order of objects:*

$$X^{\circ\circ} \simeq X, (X \otimes Y)^\circ \simeq Y^\circ \otimes X^\circ, 1^\circ \simeq 1$$

The monad associated with free monoid with anti-involution $F^\circ M$, even though they do not form a strongly regular theory, turns out to be cartesian. The previous example of categories with (possibly contravariant) functors is a $F^\circ M$-multicategory as well, since we have the natural isomorphism $\mathcal{C} \times \mathcal{D} \simeq \mathcal{D} \times \mathcal{C}$, making the conditions identical. The example of a $F^\circ M$-multicategory which is not a F^*M-multicategory is the loopspace. Given a pointed topological space, its loops naturally form a monoidal category $\Omega(X, x_0)$ having pointed loops as objects, classes of homotopies between them as arrows and a concatenation $\gamma * \eta$ as the monoidal product $\gamma \otimes \eta$. Since every loop can be considered with time inversed, we get the additional anti-involution given by $\gamma^\circ(t) = \gamma(1-t)$, which makes $\Omega(X, x_0)$ into a $F^\circ M$-multicategory. Clearly, unlike in the previous example, operation $^\circ$ is not an involution, since

$$(\gamma \otimes \eta)^\circ = \eta^\circ \otimes \gamma^\circ \neq \gamma^\circ \otimes \eta^\circ$$

Example 797 (Maybe multicategories, maybe single)**.** *Consider a cartesian maybe monad $T(X) = X \sqcup 1$. A similar motive seen in the isomorphism between pointed sets and sets with partially defined functions appears in the case of T-multicategories. Its arrows are shaped by T-graphs of a form*

$$\begin{tikzcd} & C_1 \arrow[dl, "cod"'] \arrow[dr, "dom"] & \\ C_0 \sqcup \{\bullet\} & & C_0 \end{tikzcd}$$

In particular, for any object Y, arrows $\mathcal{C}(-, Y)$ are all the hom-sets $\mathcal{C}(X, Y)$ with additional arrows going from the free point $\mathcal{C}(\bullet, Y)$. The function $Y \mapsto \mathcal{C}(\bullet, Y)$ is moreover functorial, thus any T-multicategory is just a small category together with some functor $\mathcal{C}(\bullet, -) : \mathcal{C} \to \mathbf{Set}$. It means, that we've established the equivalence of categories

$$T - \mathbf{Multicat} \simeq \mathbf{Cat}/\mathbf{Set}$$

Similarly, restricting ourselves to T-operads, we can associate them with monoids with set-valued functor, which are just sets acted by monoids.

Example 798 (Fields)**.** *The T-multicategories finally makes possible to establish some extra structure on the category of fields. However, to do that we need a little cheat in the form of considering some large categories, which hom-sets sometimes form a proper class. Allowing existence of large categories makes possible the existence of large monoids, for which the correspondence with algebraic monoids breaks - a large monoid is a large category with single object, without any algebraic counter-part. The hom-class $M(\bullet, \bullet)$ of our monoid of interest is a class of all ordinals, with composition defined by multiplication. This way we obtain a functor*

$$\mathbf{Field} \to M$$

which to every field inclusion $K \hookrightarrow L$ assigns its degree of extension. Finally, given any monoid M, the monad $T_M(X) = X \times M$ is cartesian, and its multicategories can be identified as

$$T_M - \mathbf{Multicat} \simeq \mathbf{Cat}/M$$

so the degree of extension provides the structure of T_M-multicategory on the infamous category of fields.

Example 799 (Opetopes)**.** *An interesting sequence of n-fold free objects can be assembled inductively starting from the set valued identity monad $T_0 = \mathbf{1}_{\mathbf{Set}}$. The T_0-operads we've already identified with monoids, also admitting a free monad $T_1 = FM$. FM-operads has been identified as well as just standard operads. Again, operads also admit free monad $T_2 = FO : \mathbf{Set}^{\mathbb{N}} \to \mathbf{Set}^{\mathbb{N}}$. The natural next step is obviously to take a look at T_2-operads. Recall that we've previously considered the structure of $FO(X)_n$, which are classes of trees with n leaves and vertices labeled by X_n. Denoting the classes of unlabeled* *trees with n leaves as $\mathbf{Tr}(n)$, the T_2-graph has a shape*

$$\begin{tikzcd} & P(n) \arrow[dl, "cod"'] \arrow[dr, "dom"] & \\ \mathbf{Tr}(n) \sqcup \{\bullet\} & & (1) \end{tikzcd}$$

So after identifying the elements of singletons from a terminal sequence $(\{\bullet\})_{\mathbb{N}}$ with n-leafed corollas, n-ary arrows in a T_2-operad P are morphisms of n-leafed trees to their corollas. A more tricky part is to describe the composition of arrows. It is given by a functor $P \times_{\mathbf{Tr}} T_2(P) \to P$. An arrow from a tree t with n vertices can be composed with n trees (t_i) labeling vertices of the tree t, as an element of $T_2(P)$, where the vertex labeled with n-leafed tree has degree n, looking locally as n-leafed corolla, making the substitution possible.

Example 800 (Tree-graded arrows)**.** *A very different structure provides the dual monad of free algebraic theories, which values $T(X)$ are all classes of trees, this time X labeling leaves. Domains of arrows in a T-multicategory $\mathcal{C}$ take shapes of all possible trees with leaves labeled by objects of $\mathcal{C}$, while a codomain is a simple object. The nice picture emerges when we consider the object from the codomain as labeling the root of a tree from the domain. This way the composition of arrows is just gluing roots and leaves with common labels together, leading to the same picture of compositions as in case of multicategories, only the shape of input is graded by trees instead of just corollas.*

Example 801 (Internal vs. enriched multicategories)**.** *Recall that if FM is a free monoid monad, FM-operads are just operads and FM-multicategories are just multicategories. Does the pattern continue in enriched versions? Let FT be a free topological monoid monad* $\mathbf{Top} \to \mathbf{Top}$. *It is indeed true that a FT-operad is the same as topologically enriched* **Top***-operads - both are just n-ary operations, acting on topological spaces by continuous functions. However, the correspondence breaks in case of multicategories. A FT-multicategory is equipped with a space C_0 of objects, together with spaces of n-operations between them. However, objects of* **Top***-multicategories form merely a set - only the arrows has a topological structure. Thus* **Top***-multicategory is just a special case of FT-multicategory, having discrete topology on its space C_0 of objects.*

Example 802 (Category of trees)**.** *Recall that since a morphisms of graphs are well-defined, we may consider a set of n-leafed trees not* *only as set, but as a category* $\mathbf{TR}(n)$. *This way the operad of trees* **tr** *can be turned into a* **Cat**-*operad* **TR**. *Similarly as in previous case, on the level of operads the internalisation and enrichment coincide -* **Cat**-*operads are exactly the same as* FC-*operads, where* FC *is the free strict monoidal category monad.*

14.4. **fc-multicategories.**

Example 803 (**fc**-multicategories)**.** *A special case of a* T-*category providing a lot of interesting structure is an* **fc**-*multicategory, where* **fc** *is the free category monad on directed graphs. The* T-*graph of* **fc**-*multicategory can be depicted by a mysterious diagram*

$$\begin{tikzcd} & A \rightrightarrows V \arrow[dl, "cod"'] \arrow[dr, "dom"] & \\ \bar{H} \rightrightarrows \mathcal{C}_0 & & H \rightrightarrows \mathcal{C}_0 \end{tikzcd}$$

It's convenient to depict A *as* 2-*arrows,* V *as vertical arrows,* H *as horizontal arrows,* $\mathcal{C}_0$ *as objects, while* $\bar{H}$ *as a set of chain of horizontal arrows, produced by the free category monad, adjoining all formal chains of composable arrows. This way a diagrams of arrows in* $\mathcal{C}$ *can be depicted as*

$$\begin{tikzcd} X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] & \cdots \arrow[r] \arrow[d, phantom, "\Downarrow"] & X_{n-1} \arrow[r] & X_n \arrow[d] \\ Y \arrow[rrrr] & & & & Z \end{tikzcd}$$

Behind each type of higher categories lies some common theme of objects it describes well. Similarly as n-*categories describe deeper levels of arrows between arrows, cubical* n-*categories has more parallel layers of possible compositions, or multicategories allow more sophisticated output, the theme behind* **fc**-*categories is the description of modules, generalised from rings to a variety of objects. Thus the fundamental example of an* **fc**-*multicategory is a multicategory of rings.*

Example 804 (Rings and modules)**.** *The entire idea behind* **fc**-*multicategories lies is copying constructions made on the category of rings and modules, thus we'll consider it as a first example.* $\mathbf{Ring}_2$ *(with index included to distinguish it from a classical category of rings) is a* **fc**-*multicategory. It's objects and vertical arrows coincide with the category* **Ring**, *while horizontal arrows* $R \to S$ *are* (S, R)-*bimodules. The general diagram* *in* $\mathbf{Ring}_2$ *has a form*

$$\begin{tikzcd} R_0 \arrow[r, "M_1"] \arrow[d, "f"'] & R_1 \arrow[r, "M_2"] & \cdots \arrow[r, "M_{n-1}"] \arrow[d, phantom, "\Downarrow\eta"] & R_{n-1} \arrow[r, "M_n"] & R_n \arrow[d] \\ Y \arrow[rrrr, "M"'] & & & & Z \end{tikzcd}$$

the 2*-arrow* η *is a homomorphism*

$$\eta : M_n \otimes_{R_{n-1}} M_{n-1} \otimes R_{n-2} \cdots \otimes_{R_1} M_1 \to M$$

which is coherent with vertical arrows, satisfying

$$\eta(a_n \cdot m_n \otimes \cdots \otimes m_1 \cdot a_0) = g(a_n) \cdot \eta(m_n \otimes \cdots \otimes m_1) \cdot f(a_0)$$

Note that a (S, R)*-bimodule is just a right* $S^{op} \times R$*-modules.*

Example 805 (Spans)**.** *From every bicategory one can canonically construct a* **fc***-multicategory, so it is not surprising that we can once again get a category of spans. However, the idea behind spans really nicely reflects the spirit of modules, thus we'll take a look at it once again. We consider the* **fc***-multicategory* $\mathbf{Set}_2$*, which objects and vertical arrows coincide with* **Set***, while horizontal arrows* $X \to Y$ *are spans*

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] & \\ X & & Y \end{tikzcd}$$

2*-arrows between spans are maps* $A \to B$*, fitting into a commutative diagram*

$$\begin{tikzcd} & A \arrow[dl] \arrow[dr] \arrow[d, dotted] & \\ X \arrow[d] & B \arrow[dl] \arrow[dr] & Y \arrow[d] \\ Z & & W \end{tikzcd}$$

and horizontal composition of spans is done by a canonical pullback.

Example 806 (Profunctors as bimodules)**.** *A probably most important generalisation of bimodules are profunctors, which are bimodules of categories. Similarly as a* (R, S)*-bimodule is an abelian group acted on by* $R^{op} \times S$*. Treating functors as action of categories, direct generalisation of this concept for (ordinary)* **Set***-categories hints the definition of a* $(\mathcal{C}, \mathcal{D})$*-bimodule category as a functor* $\mathcal{C}^{op} \times \mathcal{D} \to$ **Set***, called a profunctor. Similarly to previous examples, there is an* **fc***-multicategory of* *small categories, functors and profunctors. The last piece we need is to define the tensor product of modules to get its horizontal composition. Recall that that's something already considered during the study of coends - the composition $M_n \circ \cdots \circ M_1$ can be compressed to a single profunctor via*

$$(M_n \circ \cdots \circ M_1)(a_0, a_n) = \int^{c_1, \cdots, c_{n-1}} M_1(a_0, c_1) \times \cdots \times M_n(c_{n-1}, a_n)$$

Example 807 (Suspension)**.** *Every multicategory $\mathcal{M}$ is the same as an* **fc***-multicategory with one object and one vertical cell, called its suspension $\Sigma\mathcal{M}$. Objects of $\mathcal{M}$ are in correspondence with horizontal arrows of $\Sigma\mathcal{M}$, while n-arrows are captured by 2-arrows going from chains of n vertical arrows to the codomain.*

Example 808 (**fc**-operads)**.** *Dual to suspensions are* **fc***-operads, which are* **fc***-multicategory with single object and horizontal arrow.*

Example 809 (Double suspension)**.** *Note that a similar suspension construction is used to represent operads as multicategories with single object. Thus suspensions of operads can be seen as a double suspension, as an image of composition*

$$\mathbf{Operad} \xrightarrow{\Sigma} \mathbf{Multicat} \xrightarrow{\Sigma} \mathbf{fc}\text{-}\mathbf{Multicat}$$

Example 810 (Suspending monoidal category)**.** *A suspension of a monoidal category $\Sigma\mathcal{C}$ has a particularly simple form. It has, as always, single object and vertical arrow. It's horizontal arrows are exactly objects of $\mathcal{C}$, and 2-cells are arrows induced from monoidal product*

$$\eta : X_1 \otimes \cdots \otimes X_n \to Y$$

Example 811 (**fc**-algebras)**.** *A natural final step in exploring new type of higher categories is taking a look at algebras. A C-algebra is a morphism $\mathrm{Hom}_{\mathbf{fc}}(C, \mathbf{Set}_2)$. Particularly interesting are algebras of suspensions*

14.5. **Stacks.**

Example 812 (Bundles)**.** *A TODO*

15. Categorical homotopy theory

Definition 15.1 (Homotopy categories)**.** *A **homotopical category** is a category with a notion of weak equivalences, which are arrows in some wide subcategory $\mathcal{W}$, satisfying the 2-out-of-6 property. It can be* *easily remember through a diagram (where unlabeled arrows represent compositions)*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] \arrow[drr] & B \arrow[d, "g"] \arrow[dr] & \\ & C \arrow[r, "h"] & D \end{tikzcd}$$

The rule says that whenever two of the arrows from the diagram are morphisms in $\mathcal{W}$*, so must all the rest.*

Every homotopical category can be localised at $\mathcal{W}$ *by the standard trick of taking objects unchanged and morphisms constructed as classes of zigzags of arrows, where zigzags constructing from composing arrows, canceling weak equivalences pointing in opposite directions and removing identities are considered equivalent. Such a localisation is called the homotopy category* $\mathrm{Ho}\mathcal{C}$

Example 813 (Minimal homotopical category)**.** *Every category* $\mathcal{C}$ *can be considered homotopical by taking* $\mathcal{W}$ *to be a wide subcategory consisting only of isomorphisms. Clearly a homotopy category of such minimal structure is isomorphic to* $\mathcal{C}$*.*

Example 814 (Homotopy equivalences)**.** *The maps between topological spaces that are homotopy equvalences can be taken as weak equivalences in* **Top***, as they clearly satisfy the 2-ou-of-6 property. The resulting homotopy category* Ho**Top** *is unsurprisingly equivalent to the standard* **HTop***, where arrows are classes of homotopic maps.*

Example 815 (Weak homotopy equivalences)**.** *A model example of a homotopical category is* **Top** *with a subcategory* **WHTop** *of weak equivalences - maps inducing isomorphisms on all the homotopy groups. It follows from Whitehead's theorem that its homotopy category is equivalent to the category of CW complexes and homotopy classes of maps. Confusingly, both homotopy categories associated to homotopy equivalence and weak homotopy equivalence are called the homotopy category of spaces* Ho**Top***, even though are not equivalent.*

Example 816 (Homotopy category of chain complexes)**.** *In any abelian category* $\mathcal{A}$ *let* $\mathrm{Ch}(\mathcal{A})$ *be its category of chain complexes. Then a class of chain homotopy equivalences satisfy the 2-out-of-6 property, so can be considered as weak equivalences. The homotopy category of this pair* $K(\mathcal{A})$ *is the homotopy category of chain complexes.*

Example 817 (Derived category)**.** *Besides chain homotopy equivalences, there is a larger class of weak equivalences that can be considered on* $\mathrm{Ch}(\mathcal{A})$*, similar to the case of topological spaces. A chain map* *is called a quasi-isomorphism if it induces isomorphism on cohomology. Quasi-isomorphism is a weaker property than chain homotopy equivalence, as we will see in a concrete example in a moment. Its homotopy category $D(\mathcal{A})$ is called the derived category of $\mathcal{A}$.*

Example 818 (Classical homotopy invariants)**.** *A functor between homotopical categories is called homotopical if it preserves weak equivalences. Equivalently, this property can be expressed by the existence of factorisation*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d] & \mathcal{D} \arrow[d] \\ \mathrm{Ho}\mathcal{C} \arrow[r, dotted] & \mathrm{Ho}\mathcal{D} \end{tikzcd}$$

All the classical homotopy invariants from algebraic topology, such as homotopy groups, homologies and cohomologies are homotopical functors with respect to (any) homotopy category of spaces and a minimal homotopical structure on the category of groups or rings.

Example 819 (Cylinder)**.** *If a functor $F : \mathbf{Top} \to \mathbf{Top}$ has a natural transformation $F \Rightarrow \mathbb{1}$ being is weak equivalence on all the objects, it is homotopical. A standard example of such a functor is a cylinder*

$$M(X) = X \times I$$

Since the natural projection $X \times I \to X \times \{0\}$ is trivially a homotopy equivalence.

Example 820 (Path space)**.** *A functor $P(X) = \mathrm{map}(I, X)$ also is a homotopical functor in $\mathbf{Top}$, as the inclusion of constant paths $X \to P(X)$ is an equivalence. Note that this situation changes drastically in $\mathbf{Top}_*$ - here the path space is always contractible, as all the paths can be shrunken to the starting basepoint!*

Example 821 (Exact functors)**.** *Every additive exact functor between abelian categories is homotopical in the derived category, as well as in the homotopy category of chain complexes.*

Example 822 (Additive functors)**.** *Now consider some additive functor between abelian categories which is not exact. In this case, F is homotopical in $K(\mathcal{A})$, since map induced on the chain homotopy still remains chain homotopy due to additivity of F. However, F is never homotopical in the derived category. This fact is a key observation in the modern construction of classical derived functors such as* Tor *or* Ext*, as we will see in a moment. Finally, we can illustrate this with a* *concrete example. Consider a functor* $F = \mathrm{Hom}(\mathbb{Z}/2, -) : \mathbf{Ab} \to \mathbf{Ab}$ *and weakly equivalent chain complexes*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d] & \mathbb{Z}/4 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d] & \mathbb{Z}/4 \arrow[d] \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2
\end{tikzcd}$$

Applying the functor F *yields a chain map, which is clearly not a weak equivalence*

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}/2 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[r] \arrow[d, "0"] & \mathbb{Z}/2 \arrow[r] \arrow[d] & \mathbb{Z}/2 \arrow[d, "0"] \\
\cdots \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2 \arrow[r] & 0 \arrow[r] & \mathbb{Z}/2
\end{tikzcd}$$

Example 823 (Pullbacks are not homotopical)**.** *Pullbacks, considered as functors* $[\mathcal{I}, \mathbf{Top}] \to \mathbf{Top}$ *are not homotopical in* $\mathrm{Ho}\mathbf{Top}$. *An easy counterexample comes from the standard construction of a sphere by gluing two disks along boundary. Collapsing both disks into points yields a transformation*

$$\begin{tikzcd}
 & S^1 \arrow[dl, hook'] \arrow[d, "\mathbb{1}"] \arrow[dr, hook] & \\
D^2 \arrow[d, "*"] & S^1 \arrow[dl] \arrow[dr] & D^2 \arrow[d, "*"] \\
\bullet & & \bullet
\end{tikzcd}$$

Even though each vertical arrow is a homotopy equivalence, the upper pullback is S^2, *which is not homotopy equivalent to the lower one, which is just a single point.*

Example 824 (Nerve of a monoid)**.** *Since the nerve functor is fully faithful, the axioms of a monoid can be expressed with describing the properties of simplicial sets isomorphic to* $\mathcal{N}M$. *Fortunately, these conditions turns out to be reasonably simple. It decodes the identification of a tuple of elements by conditions depending on the face maps induced by* $\eta_1[n] \to [n+m], \eta_2 : [m] \to [n+m]$ *given by* $\eta_1(i) = i$ *and* $\eta_2(i) = n + i$. *The equivalent conditions*

- $X \simeq \mathcal{N}M$
- $(\eta_n \times \eta_m)^* : X[m+n] \to X[n] \times X[m]$ *is an isomorphism*
- $X[n] \to X[1]^n$ *is an isomorphism*

Example 825 (Simplicial knight in shining armor)**.** *Simplicial and homotopical methods comes to the rescue in the horrible mess of coherence conditions needed in the zoo of strict, weak or lax monoidal categories with strict, weak or lax monoidal functors, each variant additionally splitting into biased and unbiased versions. The idea relies on relations defined above for a monoid, which turns out to hold also after replacing the category of sets to any other category. In particular, replacing* **Set** *with* **Cat***, we obtain an easy description of monoids in* **Cat***, which are just stristly monoidal. The same thing can be generalised further, by replacing the isomorphism conditions to only equivalences. This way the equivalent conditions for a functor* $\Delta^{op} \to \mathbf{Cat}$

- $(\eta_n \times \eta_m)^* : X[m+n] \to X[n] \times X[m]$ *is an equivalence*
- $X[n] \to X[1]^n$ *is an equivalence*

Is a description of weak monoidal categories! Note that the category of such spaces is just the homotopy category, called homotopy monoidal category **HMonCat***.*

Example 826 (H-spaces)**.** *Pasting yet another category* **Top** *in place of sets yields a description of topological manifolds. The most interesting case is when again we relax the class of isomorphisms to homotopy equivalences. Simplicial spaces of this form are monoids up to homotopy, called H-spaces. A main example of an H-space is loopspace* Ω^n*.*

Example 827 (Augmented simplex category)**.** *More cool tricks describing monoidal zoo with simplicial methods provide the category* Δ^+*, constructed from* Δ *by adjoining the empty sequence.* Δ^+ *plays a role of an ultimate monoid, with unique free monoid classifying monoids in the same way as a singleton classify points from a set. Every weak monoidal functor* $\Delta^+ \to \mathcal{V}$ *extract some unique monoid in* $\mathcal{V}$ *as the image of* $\mathbf{1}$*. This way we get the equivalence*

$$[\Delta^+, \mathcal{V}]_{wk} \simeq \mathbf{Mon}(\mathcal{V})$$

In particular taking $\mathcal{V} = \mathbf{Cat}$ *it presents any strict monoidal category as a weak monoidal functor*

$$[\Delta^+, \mathbf{Cat}]_{wk} \simeq \mathbf{StrMonCat}$$

In the similar manner weak monoidal categories can be constructed from functors satisfying the weaker conditions. These functors are obviously not always weak, as weak functors correspond only to strict isomorphisms. However, choosing a suitable subcategory of colax functors satisfying coherence does the job, providing another construction of **HMonCat***.*

Example 828 (Morita equivalence of groupoids)**.** *Recall that in the category of groupoids isomorphisms correspond to equivalences of categories, but this correspondence breaks for Lie groupoids. In this case we can introduce the notion of Morita equivalence, conditions mimicking equivalence of categories but with additional assumptions taking into account their smooth structure - tldr functors essentially surjective are must now be also submersions, and pullback condition corresponding to fully faithfullness must be a good pullback, what happens if the diagram of tangent spaces and differentials is also a pullback square. Since the class of Morita equivalences form has decent properties of factorisation systems, we can localise the category of Lie groupoids and get its homotopy category* **HoLieGrpd***, obtained by the calculus of fractions. General morphisms* $X \to Y$ *in such category correspond to a spans of a form*

$$X \xleftarrow[\simeq_M]{} Z \longrightarrow Y$$

The category **HoLieGrpd** *has better properties than* **LieGrpd***, where isomorphisms conditions are too strong - for example the manifold is not isomorphic to its Čech groupoid, what is conceptually hurtful.*

16. Additional features

Example 829 (Conservative functors)**.** *The similarity of constructing topological structures and group structures on a set can be slightly misleading, as such categories in fact significantly differ in nature. The difference lies in the political views of underlying forgetful functors. We call a functor conservative if it preserve isomorphisms, meaning that morphisms* $F(f)$ *induced by* F *are isomorphisms if and only if* f *is an isomorphisms itself (in general case only one implication holds, as all the functor preserve isomorphisms by assumption). For example all reasonable forgetful functors from algebraic categories, such as groups, rings, monoids or fields are conservative, as generally if a function is a homomorphism and a bijection on underlying sets, it is also an isomorphism. In such cases its fibers have always a structure of a groupoid. Particularly, in case of groups, every two group structures on a set are either equivalent or incomparable. This is not the case, however, in topological spaces due to the existence of continuous bijections. This time a fiber has a weaker property of being only a poset. Intuitively, we can think about continuous bijections without an inverse as establishing an ordering on underlying topologies, since always in case of such functors the topology of a domain must be strictly finer that codomain, with trivial extreme cases of a function from anti-discrete to discrete category, corresponding to initial and terminal objects of the fibers.*