Problems from category theory

Alexander Golys

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User guide

To make the problem-solving fun and productive, I recommend having in mind a few principles

- **Ignore applications in unfamiliar subjects**. I tried to mix into this collections pure abstract nonsense with various practical applications of category-theoretic concepts in other fields, including general and algebraic topology, ring and group theory, differential geometry or complex geometry. Naturally some of those examples will require some experience in these subjects.
- **Problems from the beginning should be easier and more relevant to the subject**. The order of problems in each sections is correlated with their difficulty, with a lot of exceptions coming from natural continuity of concepts introduced in previous exercises.
- **You don't need to solve every part**. A lot of problems are very long. If an example has 10 parts, I encourage you to try solving first 2 of 3 and continue to dive deeper if you had fun in the process.
- **Don't solve problems linearly**. Feel free to skip problems that looks tedious, boring or too hard.
- **nLab is your friend**. If you got stuck or don't understand the question, nLab of Google will most likely resolve your problems. Some exercises use results from previous ones and I tried to put them next to each other, so scrolling upwards and using ctrl+F is always a good idea.

Basics

1 Partial functions

Show that the category of pointed sets is equivalent to the category of sets and partial functions, but not to the category of sets.

2 Arrows in groupoids

Let G be a groupoid with n objects, each having m automorphisms. How many morphisms can G have in total?

3 Powers

Let $\mathcal{C}$ be a category of sets $\{1, \cdots, n\}$ and functions and X be any set.

a) Construct a functor $P_X : \mathcal{C}^{op} \to \mathbf{Set}$ such that $P_X(n) = X^n$

b) Does it extend to a functor $P : \mathbf{Set} \times \mathcal{C}^{\mathbf{op}} \to \mathbf{Set}$?

c) Can X be replaced by any object from category with finite products?

4 Functors of matrix groups

a) Show that GL_n forms a functor $\mathbf{Ring} \to \mathbf{Grp}$

b) Let E_n be a subgroup of GL_n generated by elementary matrices (with only one nonzero entry equal 1). Show that it is also a functor and that the inclusion $E_n(R) \hookrightarrow GL_n(R)$ is a natural transformation injective on objects.

c) A Steinberg group $St_n(R)$ is a group generated by r_{ij} indexed by $r \in R$ and pair of integers $i \neq j$ between 1 and n with relations

$$\begin{cases} [r_{ij}, s_{kl}] = e & i \neq l, j \neq k \\ [r_{ij}, s_{jk}] = (rs)_{ik} & i \neq k \\ [r_{ij}, s_{ki}] = (-sr)_{kj} & j \neq k \end{cases}$$

Show that St_n is a functor and construct natural transformation $St_n \Rightarrow E_n$ surjective on objects.

d) Show that taking the limit under canonical inclusion $E_n \hookrightarrow E_{n+1}$, $St_n \hookrightarrow St_{n+1}$ extends the surjection to transformation $\varinjlim St_n = St \Rightarrow E = \varinjlim E_n$.

e*) Show that the kernel $K_2(R) = \ker(St(R) \to E(R))$ is a covariant functor $\mathbf{Ring} \to \mathbf{Ab}$. Use the Euclidean algorithm to represent elements of $St(\mathbb{Z})$ and show that $K_2(\mathbb{Z}) \simeq \mathbb{Z}/2$.

5 Custom morphisms

Check if connected, compact smooth manifolds form a category with morphisms $X \to Y$:

a) having finite fibers

b) having finite fibers on a dense open subset

c) defined only on the complement of some finite subset (possibly empty)

d) with dense image

g) defined only on dense open subset and having dense image

h) identified with submanifolds of $X \times Y$ with surjective projections on both X and Y

i) of a form $X^n \to Y^n$, equivariant under S_n-action

j) having at least one finite fiber

k) having at least 2 fixed points

l) with image containing the Cantor set (in the subcategory of manifolds with positive dimension)

6 Center

a) Show that there is no functor $F : \mathbf{Grp} \to \mathbf{Ab}$ that $F(G) = Z(G)$.

b) Show that there is a functor $\mathbf{FinGrp} \to K\text{-}\mathbf{Alg} \to K\text{-}\mathbf{Mod}$ such that $F(G) = Z(KG)$

7 Natural transformations in groups

a) Show that there are exactly two natural transformations $1_{\mathrm{Grp}} \Rightarrow 1_{\mathrm{Grp}}$

b) Show that there are exactly three natural transformations $P \Rightarrow 1_{\mathrm{Grp}}$, where $P(G) = G \times G$.

c) Find four natural transformation $1_{\mathrm{Grp}} \Rightarrow P$

d) Does the answers change in the category of abelian groups?

8 Homogenisation

Let A be a finitely generated commutative algebra $A = R[x_0, \cdots, x_n]/I$. From any such ring we can canonically form a graded ring over $R[x_0, \cdots, x_n, t]$ by complementing polynomials to homogeneous ones, i.e. if $f \in I$ has order d, we multiply every monomial of f of order m by t^{d-m}. Show that such homogenisation does not extend to a functor $\mathbf{CAlg}_R^{fg} \to grCAlg_R^{fd}$.

9 Complex representations of $\mathbb{Z}$

Show that following categories are equivalent

1) Functors $[\mathcal{C}, {}^{fd}_{\mathbb{C}}]$ where $\mathcal{C}$ has one objects with arrows $\{f_n \mid n \in \mathbb{Z}\}$, $f_n \circ f_m = f_{m+n}, id = f_0$.

2) Finite dimensional complex representations of $\mathbb{Z}$ (equivalently: finite dimensional vector spaces with action of $\mathbb{Z}$ by linear automorphisms)

3) Finite dimensional torsion $\mathbb{C}[t, t^{-1}]$-modules

10 Equivalence relation

An equivalence relation in category $\mathcal{C}$ is a diagram $R \rightrightarrows U$ such that for any object $X \in \mathcal{C}$ the induced map of homsets is usual equivalence relation of sets.

$$\mathrm{Hom}(R, X) \to \mathrm{Hom}(U, X) \times \mathrm{Hom}(U, X)$$

a) Assuming $\mathcal{C}$ has finite limits and colimits, formulate the axioms of equivalence relation without invoking relations in sets.

b) A morphism $U \to X$ is a quotient if it is isomorphic to coequaliser of $R \rightrightarrows U$. The quotient is called effective if moreover it induces isomorphism $R \simeq U \times_X U$. Show that quotient is an epimorphism and is unique up to unique isomorphism.

c) Show that every quotient in **Set** is effective.

d) Find an equivalence relation having no quotient.

e) Show that if quotient of equivalence relation exists iff the diagram

$$\begin{tikzcd} R \arrow[r] \arrow[d] & U \arrow[d] \\ U \arrow[r] & X \end{tikzcd}$$

is a pushout and the quotient is effective iff it is also a pullback. Conversely, every pullback diagram is an equivalence relation with effective quotient.

f) In category $\mathbf{Alg}_{\Bbbk}{}^{\mathbf{op}}$ with $\mathrm{char}(\Bbbk) \neq 2$ find an equivalence relation

$$\Bbbk[t] \times \Bbbk[t, t^{-1}] \rightrightarrows \Bbbk[t]$$

that has quotient that is not effective.

g*) Show that the relation $R = \{a, b \in \mathbb{Z} \mid a - b \in n\mathbb{Z}\} \hookrightarrow \mathbb{Z} \times \mathbb{Z}$ is a relation in **Grp** with effective quotient $p : \mathbb{Z} \to \mathbb{Z}/n$, but $p^{\bullet}$ is no longer a quotient in $\widehat{\mathbf{Grp}}$.

11 Categorical center

A center of a category $\mathcal{C}$ is the group of automorphisms of identity functor.

a) Let $f, g : G \to H$ be a homomorphism of groups regarded as functors between one-element categories. Describe natural transformations $f \Rightarrow g$

b) Show that $Z(\mathbf{B}G) \simeq Z(G\text{-}\mathbf{Set}) \simeq \mathbf{Z}(\mathbf{G})$

c) Deduce that $\mathrm{Aut}(id)$ does not extend to a functor $\mathbf{Cat} \to \mathbf{Grp}$.

d) Show that $Z(\mathcal{C})$ is abelian.

12 Epi and monos

a) Show that $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in torsion-free abelian groups.

b) Show that $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a monomorphism in injective abelian groups.

c) Show that $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in **CRing**

d) Show that homomorphism in divisible abelian groups is mono iff has kernel with no non-trivial divisible subgroups.

13 Retracts

We say that Y is a retract of X if there are morphisms

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur] \arrow[rr, "id"'] & & X \end{tikzcd}$$

a) Show that in vector spaces $\Bbbk^n$ is a retract of $\Bbbk^m$ iff $n \leq m$

b) Show that functors preserve retracts

c*) Use the fact that for $n \geq 3$ the groups $PSL_n(\Bbbk)$ are simple to deduce that $V \mapsto GL(V)$ does not extends to a functor $\mathbf{Vect}_{\mathbb{R}} \to \mathbf{Grp}$ (hint: surjective homomorphism preserve centers and $SL_n(\mathbb{R})^{ab} = 0$).

14 Orthogonal complement

Let (V, Ω) be a finite dimensional $\mathbb{R}$-vector space with non-degenerate quadric form Ω. Let $\mathcal{G}(V)$ be a poset category of subspaces of V. If needed, we consider subspaces with restricted form $\Omega|W$ (not necessarily non-degenerate), however it does not provide new structure in $\mathcal{G}(V)$.

a) Show that the orthogonal complement $V^\Omega = \{v : \Omega(v, -) \cong 0\}$ defines a self-adjoint isomorphism of categories $I : \mathcal{G}(V) \to \mathcal{G}(V)^{op}$. Show that $\dim W + \dim W^\Omega$ is constant.

b) Show that if Ω is positive-definite, for any W the coproduct $I(W) \sqcup W$ is terminal and product $I(W) \times W$ is initial (i.e. elements W and $I(W)$ are always cofinal). Show that I has no fixed points.

c) If Ω is skew-symmetric, V is called symplectic. Subspace $W \subseteq V$ with $\Omega_{|W} = 0$ is called isotropic and when $2 \dim W = \dim V$ - Hamiltonian. Show that W is cofinal with $I(W)$ iff it is symplectic and $I(W) \geq W$ iff it's isotropic. Moreover, W is Hamiltonian iff W and W^Ω are isotropic iff W is a fixed point of I. For Hamiltonian subspace $W \subset V$, construct form Ω_0 on $W \oplus W^\vee$ making it isometric to (V, Ω) in a canonical way.

15 Covariant and contravariant

a) Realise powerset in two different ways: as covariant functor $\mathbf{Set} \to \mathbf{Set}$ and contravariant functor $\mathbf{Set}^{\mathbf{op}} \to \mathbf{Set}$.

b) Realise the poset of partitions $\mathbf{Set} \to \mathbf{Poset}$ in covariant and contravariant way.

16 Automorphisms

a) Show that the function $X \mapsto \mathrm{Aut}(X)$ does not form a functor $\mathcal{C} \to \mathbf{Grp}$

b) Show that $R \mapsto GL_n(R) = \mathrm{Aut}(R^n)$ form a functor $\mathbf{CRing} \to \mathbf{Grp}$. What is the difference between this and previous case?

c) Realise determinant as natural tranformation of functors. Can the same be done with the trace?

17 Groups

Let $\mathbf{B}G$ be a categorification of a group G.

a) show that categories $[\mathbf{B}G, \mathbf{Vect}_K]$, KG-$\mathbf{mod}$ and $\mathbf{Rep}_K(G)$ (K-linear representations of G) are isomorphic b*) show that G is finite and $\mathrm{char}(K)$ does not divide $|G|$ if and only if all KG-modules are projective (hint: consider the norm element to find a contradiction. In the other direction, consider the endomorphism $x \mapsto \frac{1}{|G|}\sum g^{-1}\pi(gx)$, where π is a projection to the complementary vector subspace)

18 Relations

Let **Rel** be a category of relations, where objects are sets and $\mathrm{Hom}(X, Y)$ are subsets of $X \times Y$.

a) give a reasonable formula for the composition of morphisms in **Rel**

b) show that **Set** is a subcategory of **Rel**

c) show that **Rel** is isomorphic to $\mathbf{Rel}^{op}$

19 Manifolds

a) Let **Eucl** be a category of spaces diffeomorphic to $\mathbb{R}^n$ with morphisms being smooth embeddings. Realise any smooth manifold as a presheaf on **Eucl**.

b) What happens when arrows are replaced by open immersions or arbitrary smooth maps?

c) Find a presheaf on **Eucl** that is not a manifold (preferably "finite dimensional one", determined by values on spaces of bounded dimension).

20 Symmetric powers

a) Find a natural partial order on the set of partitions of n. Denote category of this poset by $P(n)$.

b) A symmetric product of X is the quotient $X^{(n)} = X^x/S_n$ by natural action permuting coordinates. Prove that it is an endofunctor of **Top** (elements of $X^{(n)}$ can be identified with cycles of length n, a formal sums of $a_{\lambda_i}x_1 + \cdots + a_{\lambda_i}x_i$ for some partition $\lambda \in P(n)$)

c) Find a functor $S^n : \mathbf{Top} \to [\mathbf{P(n)}, \mathbf{Top}]$ such that $S^n(X)((n)) \simeq X$ and $S^n(X)((1, \cdots, 1)) \simeq X^{(n)}$ (stratifying symmetric power by the shape of cycles)

d) Characterise categories admitting symmetric products. Assuming all the quotients X^n/S_n exist, are they always a functorial?

21 Fundamental groupoid

a) Show that the fundamental group $\pi_1 : \mathbf{Top}_{\bullet} \to \mathbf{Grp}$ extends to a basepoint-free fundamental groupoid

$$\Pi_1 : \mathbf{Top} \to \mathbf{Grpd}$$

having points of X as objects and homotopy classes of paths as morphisms.

b) Show that if X is path-connected, $\Pi_1(X) \simeq \mathbf{B}\pi_1(X, x_0)$

22 Semi-direct products

a) From any G-set X we may form an action groupoid $X//G$ with objects X and morphisms $x \to gx$. Extend this definition to action of groups on categories to construct the semi-direct products $G \rtimes \mathcal{C}$

b) Show that S_2 acts on **Cat** by taking opposite category and that $S_2 \rtimes$ **Cat** is the category of small categories and both covariant and contravariant functors.

c) Show that in case of groups, the $G \rtimes H$ coincides with usual semi-direct product.

23 Smooth rings

a) Show that the structure of commutatuve $\Bbbk$-algebra on A is equivalent to defining a lift of every polynomial $f : \Bbbk^n \to \Bbbk$ to $f_* : A^k \to A$ in a way preserving compositions.

b) A smooth ring (C^∞-ring) is an set such that the lift is defined for any smooth function $\mathbb{R}^k \to \mathbb{R}$ in a way preserving compositions and projections. Show that the category of C^∞-rings is equivalent to product-preserving functors **Eucl** $\to$ **Set** where **Eucl** is a full subcategory of smooth manifolds with objects $\mathbb{R}^n$.

c) Show that $a(x, y) = x + y, m(x, y) = xy$ and $\lambda(x) = \lambda x$ gives the structure of commutative $\mathbb{R}$-algebra to any smooth ring.

d) Show that $\mathbb{R}$ has a smooth ring structure of an initial object in C^∞**Ring**

e) Realise the category of smooth manifolds as subcategory of C^∞**Ring** by the ring of smooth functions $M \mapsto C^\infty(M)$

f) Let $\otimes_\infty$ be the coproduct in C^∞**Ring**. Show that $C^\infty(\mathbb{R}^n) \otimes_\infty C^\infty(\mathbb{R}^m) \simeq C^\infty(\mathbb{R}^{n+m})$ and that $\otimes_\infty$ differs from ordinary tensor product of $\mathbb{R}$-algebras.

g) Show that $C^\infty(\mathbb{R}^n)$ is a free smooth ring on n generators, is a sense of adjunction with forgetful functor C^∞**Ring** $\to \mathbb{R}$-**Alg**

$$\mathrm{Hom}_{C^\infty\mathbf{Ring}}(C^\infty(\mathbb{R}^n), A) \simeq \mathrm{Hom}_{\mathbb{R}-\mathbf{Alg}}(\mathbb{R}[x_1, \cdots, x_n], A)$$

h) Generalise modules to smooth rings. Show if $E \to M$ is a vector bundle, $C^\infty(E)$ is a smooth $C^\infty(M)$-module. Show that this construction is functorial under pullback:

$$(f^\bullet)_*(C^\infty(E)) \simeq C^\infty(E) \otimes_{C^\infty(Y)} C^\infty(X) \simeq C^\infty(f^\bullet E)$$

i) Generalise derivations to smooth modules. Show that exterior derivative yield a derivation $C^\infty(M) \to C^\infty(T^\vee M)$.

j*) Let M be a smooth manifold. Show that the module of smooth differential 1-forms $\Omega^1(M)$ over a ring $C^\infty(M)$ is not isomorphic to the module of Kähler differentials $\Omega^1(C^\infty(M))$ (universal derivations) as in the case of polynomial rings. Show that it is indeed true when $\Omega^1(M)$ is regarded as the C^∞-module.

k) Show that germs of smooth functions at a point form a local C^∞-ring.

24 Complex tori

a) Show that the category of compact connected complex Lie groups and continuous homomorphisms is equivalent to finite dimensional complex vector spaces together with lattice $L \subset V$ (a free abelian subgroup spanning V over the reals) and lattice-preserving linear maps. (Hint: consider universal covers).

b) Show that under this equivalence the functor $H^2(-;\mathbb{Z})$ is isomorphic to $\mathrm{Hom}^{(}\bigvee^2 L, \mathbb{Z})$.

c) Show that we have also the isomorphism between following three functors

$$\{\text{holomorphic line bundles on } X\}/\text{iso}$$

$$\{\omega : \bigvee^{2} V \to \mathbb{R} \mid \omega(L, L) = \omega(L, L) \subset \mathbb{Z}, \omega(v, w) = \omega(iv, iw)\}$$
$$\{f : X \to X^\vee \mid f = f^\vee\}$$

25 Arrow category

a) Let I be an interval category with 2 objects $0, 1$ and one single non-identity morphisms $0 \to 1$. Show that the functor category $[I, \mathcal{C}]$ is equivalent to the arrow category $\mathbf{Arr}(\mathcal{C})$ where objects are morphisms of $\mathcal{C}$ and arrows - commutative triangles.

b) Show that functors $\mathcal{C} \to \mathbf{Arr}(\mathcal{D})$ are in natural correspondence with natural transformations between functors $\mathcal{C} \to \mathcal{D}$.

26 Over categories

For $S \in \mathcal{C}$, the over category $\mathcal{C}/X$ is a category with objects being morphisms $X \to S$ and morphisms - commutative triangles.

a) Show that if 1 is terminal, $\mathcal{C}/1 \simeq \mathcal{C}$.

b) Show that a slice category of a poset P/x is the down-set $\downarrow(x)$.

c) Show that covering spaces of X are full subcategory of $\mathbf{Top/X}$ and vector bundles - subcategory that is not full.

d) Show that the slice category is functorial and that specifying the functor $\mathcal{C}/(-) : \mathcal{C} \to \mathbf{Cat}$ is determined by "codomain fibration" $\mathbf{Arr}(\mathcal{C}) \to \mathcal{C}$.

e) Identify $\mathbf{Ring}/\mathbb{Z}$ with augmented rings. Show that group rings are augmented.

27 Under categories

Dual notion to over categories are under categories $X \downarrow \mathcal{C}$ with objects $X \to Y$.

a) Realise pointed sets and pointed spaces as under categories. Generalise this construction to bi-pointed and n-pointed objects.

b) Show that $\mathbf{Alg}\text{-}R \simeq \mathbf{Ring/R}$.

c) Describe pointed abelian groups and show that pointed abelian additive group of a ring has multiplicative unit as distinguished element.

d) Show that categories of pointed abelian groups and abelian groups are not equivalent (what are initial and terminal objects?)

28 Comma categories

Given two functors $F : \mathcal{C} \to \mathcal{E}$ and $G : \mathcal{D} \to \mathcal{E}$ we can form the comma category (F/G) with objects consisting of triples $(c, d, F(c) \to G(d))$.

a) Come up with natural description of morphisms in (F/G)

b) Realise $\mathbf{Arr}(\mathcal{C})$ as comma category.

c) Realise $\mathcal{C}/X$ as comma category.

d) Realise $X \downarrow \mathcal{C}$ as comma category.

e) Realise natural transformation $F \Rightarrow G$ as functor $\mathcal{C} \to (F/G)$

f) Show that (F/G) has following universal property: functors $T : \mathcal{C} \to (F/G)$ that are sections of both projections functors $(F/G) \to \mathcal{C}$, $(F/G) \to \mathcal{D}$ correspond to unique natural transformation $F \Rightarrow G$.

29 Graphs

a) Define morphisms in the category of graphs

b) Realise **Graphs** as comma category of two endofunctors of **Set**

c) Deduce that **Graphs** is complete and cocomplete.

30 Twisted arrow category

The twisted arrow category $\mathbf{Tw}(\mathcal{C})$ is the comma category $(\bullet/\operatorname{Hom})$ of the hom functor $\mathcal{C} \times \mathcal{C}^{op} \to \mathbf{Set}$ and terminal functor.

a) Show that $\mathbf{Tw}(\mathcal{C})$ can be describe as category with arrows as objects and morphisms being commutative diagrams of a form

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \\
C \arrow[r] & D \arrow[u]
\end{tikzcd}$$

b) Show that if $\mathbb{R}$ is a poset category of real numbers, objects of $\mathbf{Tw}(\mathbb{R})$ can be identified with closed intervals.

c) Describe the projection $\mathbf{Tw}(\mathcal{C}) \to \mathcal{C} \times \mathcal{C}^{op}$

31 Category of elements

The categories of a form $(\bullet/F)$ for a copresheaf $F : \mathcal{C} \to \mathbf{Set}$ and terminal functor are called categories of elements and denoted as $\int F$. We've seen already a few special cases, for example the twisted arrow category.

a) Describe elements and morphisms of $\int F$

b) Show that the category of pairs of sets - with objects with distinct subset - is the category of elements of the contravariant powerset functor.

c) Show that the action groupoid $X//G$ (with objects X and arrows $x \to g \cdot x$) is the category of elements of functor $BG \to \mathbf{Set}$ corresponding to G-set X.

d) Show that if X is a simplicial set, $\int X$ is its category of simplices X and morphisms between them.

e) Consider the constant singleton functor $s : \mathcal{C}^{op} \times \mathcal{D} \to \mathbf{Set}$. Show that $\int s$ is the join category $\mathcal{C} \star \mathcal{D}$, having objects $ob\mathcal{C} \sqcup \mathrm{ob}\,\mathcal{D}$ and arrows

$$\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(x,y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(x,y) & x, y \in \mathcal{C} \\ \mathrm{Hom}_{\mathcal{D}}(x,y) & x, y \in \mathcal{D} \\ \{\bullet\} & x \in \mathcal{C}, y \in \mathcal{D} \\ \varnothing & x \in \mathcal{D}, y \in \mathcal{C} \end{cases}$$

f) Construct the category of colored graphs as category of elements.

g) Describe the category of pathed spaces - category of elements of the fundamental groupoid.

h*) Show that $\left(\int F\right)^{op} \simeq (y/i_F)$ where y is the Yoneda embedding and i_F is the functor $1 \to \widehat{\mathcal{C}}$ peaking F.

i) Show that $\widehat{\int F} \simeq \widehat{\mathcal{C}}/F$

32 Vector bundles

a) Show that vector bundles over topological space X form an additive category $\mathbf{Vect}/X$. Find spaces X, Y such that $\mathbf{Vect}/X$ is abelian, but $\mathbf{Vect}/X$ is not abelian.

b) Let (E, X) be a vector bundle over topological space X. The morphisms $(E, X) \to (F, Y)$ are continuous maps $X \to Y$ together with an isomorphism of vector bundles $E \to f^{\bullet}F$. Show that vector bundles over arbitrary base space defined this way do not form a category.

c*) Try to came up with a definition of a bicategory, having objects, morphisms and 2-morphisms between morphisms such that vector bundles defined as above can be constructed as a bicategory. Use the "model example" of a bicategory $\mathbf{Cat}_2$ of small categories (encoding categories as objects, functors as morphisms and natural transformation as 2-morphisms). Compare your construction with definition from nLab.

33 Dual rings

a) Show that assigning ring to its dual is not functorial.

b) Show that rings and antihomomorphism do not form a category, but rings with homomorphisms and antihomomorphism do.

c) Show that dual ring is functorial in the category of rings with homomorphisms and antihomomorphism.

d) A ring is self-dual iff there exist anti-automorphisms. Show that matrices over commutative rings and group rings are self-dual. Show that if the ring is not not self-dual, we still have $M_n(R^{op}) \simeq M_n(R)^{op}$.

e) Show that for self-dual rings cardinalities of automorphism and anti-automorphisms groups are equal.

d) Show that the ring is commutative iff is equal to its dual.

e) Find a not self-dual ring.

34 Ring categories

A rring (or ring category) over ring R is an additive category with tensors distributive over direct sums and homsets having a structure of R-modules with composition maps R-linear. A rring is Karoubian is every idempotent endomorphism has an image.

a) Show that finitely generated projective R-modules $P(R)$ form a Karoubian rring.

b) Let G be a group and $\mathcal{R}$ a Karoubian rring over R. Show that functor category $[\mathbf{B}G, \mathcal{R}]$ has a structure of Karoubian ring over $R[G]$ induced by conjugation.

c) Construct a natural external product bifunctor

$$\boxtimes : [\mathbb{B}G, P(R)] \times [\mathbb{B}G, \mathcal{R}] \to [\mathbb{B}G, \mathcal{R}]$$

d) Show that

$$R[G]^n \boxtimes X \simeq \bigoplus_G X^n$$

e) Suppose G is finite of order divisible in R. Show that idempotent $\frac{1}{|G|}\sum_G g \in R[G]$ induce an idempotent isomorphism of $\mathcal{R}$ and the a functor of G-invariants

$$(-)^G : [\mathbb{B}G, \mathcal{R}] \to \mathcal{R}$$

f) Show that given two Karoubian rings $\mathcal{R}, \mathcal{S}$ over R, the category $\mathcal{R} \otimes \mathcal{S}$ with objects being formal sums of tensors of objects, is also Karoubian rring.

g*) (Peter-Weyl theorem) Show that the following composition is an equivalence of categories

$$[G, P(R)] \otimes \mathcal{R} \hookrightarrow [G, P(R)] \otimes [G, \mathcal{R}] \xrightarrow{\boxtimes} [G, \mathcal{R}]$$

35 Plethysm*

The symmetric category $\mathbb{S}$ is a disjoint union of symmetric groups $\coprod \mathbf{B}S_n$. Suppose $\mathcal{R}$ is a Karoubian rring over a field $\Bbbk$ of characteristic 0.

a) Let H be a subgroup of finite group G. Show that the construction of induced representation extends to $\mathcal{R}$, i.e. the formula $Ind_H^G X = (\Bbbk[G] \boxtimes X)^H$ form a functor

$$Ind_H^G : [H, \mathcal{R}] \to [G, \mathcal{R}]$$

b) Identify the category $[\mathbb{S}, \mathcal{R}]$ with category $\bigoplus [S_n, \mathcal{R}]$ of finite sequences $(X(n))$ of objects of X acted on by S_n.

c) Show that $[\mathbb{S}, \mathcal{R}]$ is a rring with pointwise sums and products, find its unit

$$(X \otimes Y)(n) = \bigoplus_{i+j=n} {}^{S_n}_{S_i \times S_j}(X(i) \otimes Y(j))$$

d) If $\lambda = (\lambda, \cdots, \lambda)$ is a partition of n, we say that it has length $l(\gamma) = k$ and associated subgroup $S_\lambda = \prod S_{\lambda_i} \leq S_n$. Show that its normaliser $N(S_\lambda)$ has a natural action on

$$X(l(\lambda) \otimes Y(\lambda_1) \otimes \cdots \otimes Y(\lambda_{l(\lambda)})$$

e) This action provides different (non-symmetric) monoidal product on $[\mathbb{S}, \mathcal{R}]$ called plethysm

$$(X \circ Y)(n) = \bigoplus_{\lambda \in \mathrm{part}(n)} \bigoplus_{k=0}^{\infty} \mathrm{Ind}_{N(S_\lambda)}^{S_n} \left((X(l(\lambda) + k) \otimes \bigotimes_{i=1}^{m(\lambda)} Y(\lambda_i) \otimes Y(0)^k)^{S_k} \right)$$

Verify that this operation is indeed associative and find its unit.

f) Denote the trivial S_n module by 1_n. Show that

$$1_n \circ (X \oplus Y) \simeq \bigoplus_{p+q=n} (1_p \circ X) \otimes (1_q \circ Y)$$

g) Show that $\circ$ is distributive on the left with respect to both sums and products

$$(X \oplus Y) \circ Z = (X \circ Z) \oplus (Y \circ Z)$$

$$(X \otimes Y) \circ Z = (X \circ Z) \otimes (Y \circ Z)$$

36 Morita equivalence

We say that rings (not necesserily commutative) R, S are Morita equivalent if have equivalent categories of right modules. Let $T : R\text{-}\mathbf{Mod} \to S\text{-}\mathbf{Mod}$ be such additive equivalence with quasi-inverse U. Set $P = T(R)$ and $Q = U(S)$. Note that P and S are naturally $R - S$ and $S - R$ bimodules. Show that:

a) $P \otimes Q \simeq R, Q \otimes P \simeq S$

b) $T(M) \simeq M \otimes P, U(N) \simeq N \otimes Q$

c) P and Q are finitely generated and projective

d) $\mathrm{End}(P) \simeq \mathrm{End}(Q)^{op} \simeq R$

e) $\mathrm{Hom}(P, -)$ is faithful

f) Any ring R is Morita equivalent with $M_n(R)$. Find associated modules P, Q.

g) Show that it is no longer true in general for $\varinjlim M_n(R)$

37 Stratification

Let X be a topological space with decomposition into connected disjoint subspaces $X = \bigsqcup X_i$.

a) The exit path preorder $\mathrm{Exit}(X)$ on (X_i) is given by relation $X_i \leq X_j$ iff $\overline{X_i} \cap X_j \neq \emptyset$. Show that $\mathrm{Exit}(X)$ is indeed a preorder.

b) A stratified space is a decomposition $X = \bigsqcup X_i$ such that $\mathrm{Exit}(X)$ is a poset. Show that CW decomposition is a stratification. More generally, you can construct canonical stratification of any filtered space. Find a decomposition into connected subspaces that is not a stratification.

c) Construct a characteristic function $X \to \mathrm{Exit}(X)$. It can be used to construct a category of stratified spaces, with stratified maps being continuous functions $X \to Y$ factoring lifting through characteristic functions to morphisms of posets $\mathrm{Exit}(X) \to \mathrm{Exit}(Y)$. Show that $\mathrm{Exit} : \mathbf{Str} \to \mathbf{Poset}$ is a functor.

d) (Alexandroff topology) Every poset has canonical Alexandroff topology where upward-closed subset of a poset are open. Show that it defines a functor $\mathbf{Poset} \to \mathbf{Top}$.

e) Show that a characteristic function is not continuous unless stratification is locally finite but that its restriction to the preimage of any full finite subposet is continuous. Find stratified space with not continuous characteristic map.

f) (Poset stratifications) Show that a continuous map $f : X \to P$ where X is a topological space and P is a poset with Alexandroff topology yields a unique stratification $c(X)$ such that f factors uniquely through characteristic map $X \to \mathrm{Exit}(c(X)) \to P$.

g) Construct fully-faithful embeddings of categories

$$\mathbf{Poset} \hookrightarrow \mathbf{Top} \hookrightarrow \mathbf{Str} \hookrightarrow \mathbf{Arr}(\mathbf{Str})$$

h) Construct a canonical stratification of a cone over stratified space to embed the simplex category Δ into $\mathbf{Str}$.

i) Describe products, subobjects and isomorphisms in $\mathbf{Str}$. Find 3 different stratifications of a sphere, exactly 2 of them isomorphic.

j) Show that taking iterated singular locus of the zero set $X = V(f)$ of polynomial $f \in \mathbb{C}[x_1, \cdots, x_n]$ yields a stratification of X by complex manifolds.

k) Show that an orbit space of finite group acting on smooth manifold is stratified by smooth manifolds with strata being components of subspace with fixed stabiliser subgroup. Show that strata of codimension 1 are stabilised by subgroup of order 2. Find stratification of $\mathbb{R}^2/D_6$ and $\mathbb{R}^3/S_3$ under canonical linear actions. What happens if G is not finite (consider for instance $\mathbb{R}^2/\mathbb{R}_+$)?

d) (classifying stratification) Construct a functor $\mathbf{Poset} \to \mathbf{Str}$ that is a right-inverse of Exit.

38 Profunctors

Similar idea to spans can be also realised by profunctors, i.e. functors $\mathcal{C} \times \mathcal{C}^{op} \to \mathbf{Set}$.

a) Show that profunctors on discrete categories can be composed and form a category equivalent to **Rel**

b) Show that spans can be realised as profunctors.

c) Construct bi-modules as profunctors on abelian categories (with values in **Ab**)

d) Show that in bicomplete category double hom form a functor $\mathbf{Set} \mapsto [\mathcal{C} \times \mathcal{C}^{\mathrm{op}}, \mathbf{Set}]$ that has a right adjoint called an end and denoted by integral

$$\int_{x \in \mathcal{C}} F(x, x)$$

Use this construction to define composition of profunctors.

e) Show that the hom profunctor has also left adjoint, called coend $\int^{x \in \mathcal{C}} F(x, x)$. Use coend to construct a trace of a bimodule.

39 More on ends*

Ends and coends are very cool and powerful tools, but also are as abstract and nonsense as it can get, so feel free to skip them if it doesn't feel like fun to you.

a) Show that given two functors $F, G : \mathcal{C} \to \mathcal{D}$, composing them with double hom yields a profunctor and that

$$\int_{x \in \mathcal{C}} \mathrm{Hom}(F(x), G(x)) = \mathrm{Nat}(F, G)$$

b) Let $\mathcal{V}$ be a closed monoidal preorder $([0, \infty], 0, +)$. Show that a category with hom functor $d : X \times X^{op} \to \mathcal{V}$ valued in $\mathcal{V}$ is a non-symmetric metric space and that for a functors f, g between such metric spaces (what are such functors?)

$$\int_{x \in X} d_Y(f(x), g(x)) = \sup_{x \in X} d_Y(f(x), g(x))$$

$$\int^{x \in X} d_Y(f(x), g(x)) = \inf_{x \in X} d_Y(f(x), g(x))$$

40 Canonical form

a) Show that tangent bundle form an endofunctor on **Diff**

b) Show that 1-forms form a functor $\mathbf{Diff}^{op} \to \mathbf{Vect}_{\mathbb{R}}$

c) Construct a canonical (non-trivial) 1-form on cotangent bundle of a smooth manifold

d) Show that every cotangent bundle is a symplectic manifold, i.e. it admits a non-degenerate closed 2-form.

e) Show that cotangent bundle of n-dimensional compact smooth manifold is orientable and has non-trivial cohomology with real coefficients in all even degrees less than $2n$.

f) Show that not every symplectic manifold can be constructed this way.

41 Internal hom

An internal hom $\mathcal{H}om(-,-)$ is a functor $\mathcal{C} \times \mathcal{C}^{op} \to \mathcal{C}$ such that for all objects $\mathcal{H}om(\mathcal{X},-)$ is right adjoint to product $- \times X$ (or any monoidal product in monoidal category). Category with internal hom is called closed.

42 Arrows as modules

Let $\mathbf{ArrowFinVect}_K$ be the arrows category finite dimensional vector spaces over K (where objects are arrows and morphisms are commutative squares). Show that $\mathcal{C}$ is naturally equivalent to a category of modules over some 3-dimensional K-algebra

43 Segre's Γ

Segre's Γ is a category of finite sets, where morphism $X \to Y$ is a function $X \to \mathcal{P}(Y)$ such that images of all points are disjoint. Proof that Γ is naturally equivalent to the opposite category of finite pointed sets.

44 Group objects

A group object in a category $\mathcal{C}$ is an object G with multiplication morphism $G \times G \to G$, unit $1 \to G$ and inversion $G \to G$, satisfying usual group axioms.

a) Formulate associativity of multiplication, right and left neutrality of unit and inversion property of inverse by commutativity of appropriate diagrams (hint: you will need 3 of them)

b) Show that G is a group object iff the representable functor h_G lifts to the category of groups.

c) Show that group object in a category with finite products form a category

d) Find the category of group objects in **Set**, **Grp**, **Ab**, **Top**, **SmoothMan**, **CRing**

e) Find the category of group objects in $\mathbf{CRing}^{op}$, G-**Set**

f) Show that **Grp**(**Grp**(**Set**)) and **Grp**(**Grp**((**Fin**)) are abelian categories, but **Grp**(**Grp**(**Top**)) is additive but not abelian

45 Eckmann-Hilton argument

a) A cogroup object is a group object in $\mathcal{C}^{op}$. Show that $\mathbf{CoGrp}(\mathcal{C}^{op})^{op} \simeq \mathbf{Grp}(\mathcal{C})$

b) Show that cogroup in **Top** is only an empty space, cogroups in **Grp** are the free groups, while in **Ab** every object has unique group and cogroup structure.

c) Show that if endofunctor takes values in group objects, it's left adjoint takes values in cogroup objects.

d) Show that in **HTop**. any loop space is a group object.

e) Show that two binary operations $\bullet, \circ$ with neutral elements satisfying interchange law $(a \circ b) \bullet (c \circ d) = (a \bullet b) \circ (c \bullet d)$ are associative, commutative and equal to each other.

f) Deduce that for $n \geq 2$, the homotopy groups $\pi_n(X, x_0) = [(S^n, 1), (X, x_0)]$ are abelian.

46 Composing circle actions

Consider action of circle on itself given by left multiplication. With any group action we may associate action groupoid $s, t : G \times X \rightrightarrows X$, where $a \in G \times X$ is an isomorphism between source $s(a)$ and target $t(a)$. Composition of arrows in groupoid $\mathcal{G} \rightrightarrows X$ can be computed from pullback diagram (you may think of $\mathcal{G} \times_X \mathcal{G}$ as about the space parameterising pairs of composable arrows)

$$\begin{tikzcd}
\mathcal{G} \times_X \mathcal{G} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{G} \arrow[d] \\
\mathcal{G} \arrow[r] & X
\end{tikzcd}$$

Give explicit description of composition $(S^1 \times S^1) \times_{S^1} (S^1 \times S^1) \to S^1 \times S^1$

47 Gauge groupoid of orthogonal groups*

Recall that a principal G-bundle is a fiber bundle $P \to B$ with G acting continuously, freely and transitively on fibers.

a) Construct a Gauge groupoid $\frac{P \times P}{G} \rightrightarrows B$ associated to principal G-bundle $P \to B$.

b) Identify the group $SU(2)$ with unit quaternions to show that the map $p(x, y)(v) = xvy^{-1}$ gives a double cover

$$SU(2) \times SU(2) \to SO(4)$$

c) Use the previous covering to identify the Gauge groupoid of double cover of $SO(3)$

$$0 \to \mathbb{Z}/2 \to SU(2) \to SO(3) \to 1$$

with groupoid

$$SO(4) \rightrightarrows SO(3)$$

d**) Can you explain the multiplication in this groupoid in terms of linear algebra? (As far as I know this problem is mostly open)

48 Rigidity lemma

Let $\mathcal{C}$ be a category with finite products.

a) Formulate universal description (without invoking elements of the objects) of: point, constant morphism, constant morphism along the fiber, property of morphism $X \times Y \to Z$ being independent of X.

b) We say that $\mathcal{C}$ obeys the rigidity lemma if any morphism $X \times Y \to Z$ constant along some point y of Y is (globally) independent on X. Show that connected compact complex manifolds obey the rigidity lemma.

From now let $\mathcal{C}$ obey the rigidity lemma.

c) Show that every group in $\mathcal{C}$ is abelian.

d) Show that every morphism between groups preserving identity is a group homomorphism.

e) Show that the group structure on object of $\mathcal{C}$, if exists, is unique.

f) Show that the categories of **Monoids**, **Grp**, modules over commutative ring and compact complex manifolds obey the rigidity lemma

g) Show that every compact complex Lie group is isomorphic to $\mathbb{C}^n/\mathbb{Z}^{2n}$ for some n.

49 Lie groupoids

a) Show that any small groupoid can be identified with sets G_0, G_1 of objects and maps specifying source, target, identities, composition and inversions under coherence conditions on associativity and composition of identities and inverses. Write down their diagrams and show that this is data uniquely determines a small groupoid.

b) Using description from a), describe a groupoid object in a small category.

c) A Lie groupoid is a pair of smooth manifolds $G_1 \rightrightarrows G_0$ that is a groupoid with composition submersive. Show that it is not equivalent to being groupoid object in smooth manifolds, but the subcategory of Lie groupoid has the advantage of being closed under pullbacks, while pullbacks does not exist in general in all smooth groupoids.

d) For any smooth manifold construct the trivial groupoid $1 \rightrightarrows M$ and the pair groupoid $M \times M \rightrightarrows M$. Describe the composition map.

e) Construct a groupoid $G \times X \rightrightarrows X$ from any smooth action of a Lie group. Describe the maps on some example, for example a circle acting on a plane or itself.

f) Construct two different types of fibers over each point of G_0. Show that their intersection is a Lie group that can be understood as stabiliser. Describe the orbit space over a point O_x and show that it is a submanifold with projection from both type of fibers being principal O_x-bundles.

g) Show that every groupoid have canonical unit subgroupoid $U(X) = i(\mathcal{G}_1) \rightrightarrows G_0$ and the inertia subgroupoid $IG = \{f \in \mathcal{G} \ : \ s(f) = t(f)\}$ (usually not smooth). Find them on your favourite Lie groupoid.

h) Formalise the weaker notion of groupoid (Morita) equivalence mimicking the definition in discrete case. Show that groupoids associated to actions with equivalent quotient spaces (in particular, well defined) are Morita equivalent but not isomorphic as Lie groupoids. Note that this refined notion makes a groupoids a tool to handle quotients of group actions. Show that topological quotient and stabiliser subgroups (or more generally, fibers) are invariants under Morita equivalence. Deduce that groupoids associated to non-isomorphic Lie groups (acting on point) are all non-equivalent, and even such subtle differences as actions of $SO(n)$ and $O(n)$ on the vector space are detected (which is not the case for some alternative frameworks such as diffeological spaces).

i) Construct a Čech groupoid associated to an open cover. Show that it is Morita equivalent to a trivial groupoid of a manifold (with both arrows identities), but not isomorphic. This is once again a reason to prefer isomorphism in the weaker sense.

j) Show that sections of bundles generalise to bisections on groupoids and that they form a functor (**Lie**)**Grpd** → **Grp**

$$
\begin{tikzcd}
 & M \arrow[dl, "1"'] \arrow[d, "b"] \arrow[dr, "1"] & \\
M & \mathcal{G} \arrow[l, "s"] \arrow[r, "t"'] & M
\end{tikzcd}
$$

i) A fundamental group in a basepoint-free settings can be realised as Lie groupoid (you may take that as a known fact). Give an interpretation of constructions and maps described above on $\Pi_1(X)$ defined as

$$\Pi_1(X) = \left\{ \begin{matrix} (x_0, x_1, [\gamma]_{ho}) \mid x_0, x_1 \in X, \\ \gamma : (I, 0, 1) \to (X, x_0, x_1) \end{matrix} \right\} \rightrightarrows X$$

j*) (For those familiar with foliations) Describe monodromy and holonomy groupoid on a foliated manifold, generalising the construction of fundamental groupoid. Show that they may have non-Hausdorff arrow space (consider the case of $\mathbb{R}^3 \setminus \{\bullet\}$)

k) Construct a frame groupoid associated to any surjective submersion, where the arrows are all isomorphisms between fibers. Describe the frame groupoid of a Möbius strip double covering a cylinder.

l) Construct the associated groupoid from a principal bundle. Describe its structure on a finite cover of circle.

$$\frac{P \times P}{G} \rightrightarrows B$$

m**) Using associated groupoid of a double cover of $SO(3)$ construct a groupoid of a form. The meaning of its multiplication maps is still an open question.

$$SO(4) \rightrightarrows SO(3)$$

50 Monoid objects

a) Modify the definition of group object to define a monoid object

b) Find monoid objects in **Set**, **Monoid**, **Grp**, **Cat**

c) Define monoid object internal to monoidal category $\in$ **Mon**(**Cat**)

d) Find monoids in the monoidal category of vector spaces with tensor product

e*) Find monoids in End($\mathcal{C}$)

f) Define commutative monoid object in a symmetric monoidal category. Find commutative monoids in $(\mathbf{Set}, \times), (\mathbf{Ab}, \otimes), (\mathbf{Ch}(\mathbf{Ab}), \otimes), (\mathbf{CMonoid}, \times)$

51 Monoidal categories

A monoidal category is a monoid object in **Cat**. Equivalently, it is a category with unit object 1, functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ being unital and associative up to natural isomorphisms $(A \otimes B) \otimes C \Rightarrow A \otimes (B \otimes C)$, $1 \otimes A \Rightarrow A \to A \otimes 1$ called associators and unitors satisfying coherence conditions of monoid objects. A category is called strong monoidal if these natural isomorphisms are identities, and otherwise - weak monoidal. A symmetric monoidal category is equipped with additional isomorphisms $A \otimes B \Rightarrow B \otimes A$.

a) Show that product and coproduct if exist, yield a monoidal (called respectively cartesian and cocartesian) structure. Find weak and strong Cartesian category.

b*) Find 3 different monoidal structures on **Set** and **Top**..

c) Show that the join of categories $\mathcal{C} \star \mathcal{D}$ having as objects disjoint union of objects from $\mathcal{C}$ and $\mathcal{D}$ and homs

$$\mathrm{Hom}_{\mathcal{C}\star\mathcal{D}}(x,y) = \begin{cases} \mathrm{Hom}_{\mathcal{C}}(x,y) & x,y \in \mathcal{C} \\ \mathrm{Hom}_{\mathcal{D}}(x,y) & x,y \in \mathcal{D} \\ \{\bullet\} & x \in \mathcal{C}, y \in \mathcal{D} \\ \emptyset & x \in \mathcal{D}, y \in \mathcal{C} \end{cases}$$

yields a monoidal structure on **Cat**

d) Find two symmetric monoidal products on poset $0 \leq 1$.

e) Find symmetric monoidal products on powerset poset.

f) Find symmetric monoidal product on positive integers ordered by divisibility.

g) Find 3 different monoidal products on closed unit interval with standard order.

f) Find a symmetric monoidal structure on coproduct on groupoid form from disjoint union of symmetric groups and non-symmetric one on the union of braid groups.

52 Cohomology with support

Consider category 2**Man** of closed immersions $A \hookrightarrow M$ of smooth manifolds and pullback squares.

a) Show that 2**Man** is a category.

b) Show that cohomology with support $(M, A) \mapsto H^\bullet_A(M)$, constructed from chain complex of cycles supported in A, form a functor $2\mathbf{Man}^{op} \to \mathbf{grAb}$.

Yoneda

53 Posets

Use the Yoneda lemma to show that in any partially ordered set $x = y$ if and only if for any z $x \leq z \Leftrightarrow y \leq z$

54 Closure

Show that the Yoneda embedding in **Poset** is the downward closure.

55 Involutions

Let M be a monoid with 2 elements, both idempotent. Identify presheaves on $\mathbf{B}M$ with sets together with an involution. Explicitly describe the Yoneda embedding.

56 Coverings

a) Let $E \to X$ be covering map of degree 3 with no structure group. Let T be a space parameterising functions $\{a, b, c\} \to E$ that are bijective with some fiber of $E \to X$. Find canonical topology on T.

b) Show that $T \to X$ is a degree 6-cover and canonically a principal S_3-bundle.

c) Find a functor representable by T.

57 Open subfunctors

Let R be an integral domain and I and ideal. Show that the functor $F : \mathbf{CRing}^{\mathrm{op}} \to \mathbf{Set}$

$$F(S) = \{f : S \to R \mid f^{-1}(I)S = S\}$$

is representable. Describe the ring representing F.

58 Tangent space to a functor

Let F be a functor $\mathbf{Alg}_{\mathbb{k}} \to \mathbf{Set}$ preserving pullbacks.

59 Additive polynomials

Consider the functor $\mathbb{G}_a : \mathbf{Alg}_{\mathrm{R}} \to \mathbf{Ab}$ forgetting multiplicative structure.

a) Use Yoneda lemma to identify endomorphisms of $\mathbb{G}_a$ with additive polynomials, i.e. $f \in R[x]$ satisfying $f(0) = 0$ and $f(x + y) = f(x) + f(y)$.

b) Describe the ring $\mathrm{End}(\mathcal{G}_a)$ for torsion-free ring R.

c) Describe the ring $\mathrm{End}(\mathcal{G}_a)$ for $R \in \mathbf{Alg}_{\mathbb{F}_p}$

d) Use the previous results to construct a representable functor $\mathbf{Alg}_{\mathbb{F}_p} \to \mathbf{Ab}$ that cannot be lifted to $\mathbf{CRing} \to \mathbf{Ab}$.

60 Lifting homs

Let $\mathcal{C}$ be a small category with finite products and coproducts. Suppose $\mathcal{C}$ has isomorphic initial and terminal object and all the canonical maps $X \sqcup Y \to X \times Y$ are isomorphism. Show that then the hom functors are canonically commutative monoids and the Yoneda embedding lifts to

$$\mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \mathbf{CMon}]$$

61 Wreath product of groups

In this exercise let's denote the permutation group of set X as $\mathrm{Sym}(X)$. Fix groups $H \leq \mathrm{Sym}(X), G \leq \mathrm{Sym}(Y)$

a) Use the Yoneda lemma to realise group G as symmetry group of its underlying set, giving the embedding

$$G \hookrightarrow \mathrm{Sym}(|G|)$$

b) Construct a canonical action of G on H^Y permuting the components.

c) Construct a canonical action of H^Y on $X \times Y$ and a semidirect product $H \wr G := H^Y \rtimes G$.

d) Show that $\mathbb{Z}/2 \wr \mathbb{Z}/2 \simeq D_4$

e*) Show that given Sylow p-subgroup $P_k < S_k$, the Sylow p-subgroup of larger symmetric groups can be constructed recursively as $P_n \simeq \mathbb{Z}/p \wr P_k$ for $n = kp + r$, $0 \leq r < p$.

62 Open subsets

Check if following functors $F : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$ are representable if F assign to X the set of

a) Pairs of disjoint open subsets

b) Pairs of open subsets

c) Triples of open subsets with $U_1 \subseteq U_2 \cap U_3$

d) Ascending sequences of open subsets

e) Descending sequences of open subsets

f) Sequences of open subsets, almost all empty

63 Cone

Let $\mathcal{C}$ be a small category and $F : \mathcal{I} \to \mathcal{C}$ be a functor. A cone is a natural transformation $F \Rightarrow c$ where c is a constant diagram and $\mathrm{Cone}(F, c)$ is the set of cones. Use the Yoneda embedding to show that $\mathrm{Cone}(F, -)$ is a representable functor.

64 Epi and mono

Let $h : \mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$ be the Yoneda embedding.

a) Show that in $\widehat{\mathcal{C}}$ morphism $F \to G$ is monomorphism (epimorphism) iff $F(X) \to G(X)$ is injective (surjective) for all $X \in \mathcal{C}$

b) Show that f is a monomorphism iff $h(f)$ is.

c) Show that if $h(f)$ is epi, then f is epi

d) Show that $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in **Ring**, but not in $\widehat{\mathbf{Ring}}$

e) Show that every morphism in $\widehat{\mathcal{C}}$ factors as the composition of an epimorphism and a monomorphism

65 Tangent space

Show that the tangent space is a non-representable functor in pointed smooth manifolds.

b*) Show that there is a representable functor in pointed locally ringed spaces coinciding with tangent space on smooth manifolds.

66 Representability of symmetric powers

Fix some category **Top** of nice topological space, e.g. CW complexes.

a) Show that unordered n-tuples of maps to X is a functor representable by symmetric powers $S^n X := X^n / S_n$.

b) Construct a free commutative topological monoid and describe functor it represents.

c) The flag symmetric power, with subspace topology, is the space

$$S^(n, m)X = \{a \subset b \mid a \in S^n X, b \in S^m X\} \subset S^n \times S^m$$

Describe the functor representable by $S^(n, m)X$ (as a subfunctor of $h_{S^n} \times h_{S^m}$).

d) Describe universal families corresponding to identity maps in $h_X(X)$ for symmetric powers and flag symmetric powers.

67 Presheaves of slice

Show that $\widehat{\mathcal{C}/X} \simeq \widehat{\mathcal{C}}/y(X)$, where y is the Yoneda embedding $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}}$.

68 Representability criterion

Show that:

a) the covariant (contravariant) functor is representable if and only if its category of elements has initial (terminal) objects.

b) initial (terminal) objects of category of elements correspond to representants of a functor together with their universal elements.

69 Jet rings

In this examples we will work in the category $\mathcal{C}$ of commutative $\mathbb{C}$-algebras.

a) Show that the functors $R \mapsto R[t]/(t^m)$ ($R \mapsto R[[t]]$) have left adjoints $R_m(R_\infty)$ called m-order jet (arc) spaces.

b) Construct functorial truncation maps π_m^k for $\infty \geq k \geq m \geq 0$

$$\pi_m^k : R_m \to R_k$$

c) Find jets and arcs of $\mathbb{C}[x_1, \cdots, x_n]$ and $\mathbb{C}[x, y]/(xy)$.

d) Show that for $R = \frac{\mathbb{C}[x_1, \cdots, x_n]}{(f_1, \cdots, f_k)}$ there is a natural bijection between homomorphism $R_m \to \mathbb{C}$ and solutions

$$p_i = \sum_j = 0^m a_{ij} t^j$$

to the problem

$$\begin{cases} f_1(p_1, \cdots, p_n) \cong 0(\mathrm{mod} t^{m+1}) \\ \cdots \\ f_k(p_1, \cdots, p_n) \cong 0(\mathrm{mod} t^{m+1}) \end{cases}$$

e) Show that the m-jet (resp. arc) functors

$$J_{m,R}(S) = \mathrm{Hom}(R, \frac{S[t]}{(t^m)})$$

$$J_{\infty,R}(S) = \mathrm{Hom}(R, S[[t]])$$

are representable by R_m (resp. R_∞)

f) Show that $X_\infty = \varinjlim X_m$

g) Show that the algebra of first order jets is the symmetric algebra on the module of Kähler differential

$$R_1 \simeq \mathrm{Sym}\,\Omega^1_R$$

h) Show that if R is an algebra of Krull dimension n, R_m has dimension at least $n(m+1)$, with equality in case of regular rings. Show that this equality is false in general by considering the quotient of the ideal $(x^k + y^k + z^k)$ for $k > 3$ and dividing $m - 1$.

70 Ninja Yoneda lemma*

Let $\mathcal{C}$ be a small category and F be a functor $\mathcal{C} \to \mathbf{Set}$. Let $\Pi : (\mathcal{C} \downarrow F) \to \mathcal{C}$ be a projection $(X, F(X)) \mapsto X$ and $\Delta_X : (\mathcal{C} \downarrow F) \to \mathcal{C}$ be a constant functor with value X. Show that there is a natural bijection

$$\mathrm{Hom}_{[\mathcal{C},\mathbf{Set}]}(F, h^X) \simeq \mathrm{Hom}_{[(\mathcal{C}\downarrow F),\mathcal{C}]}(\Delta_X, \Pi)$$

71 Free cocompletion

Deduce from the previous exercise that:

a) every presheaf is a colimit of representable functors

b) Yoneda embedding of a small category is a free cocompletion (every functor $F : \mathcal{C} \to \mathcal{D}$ extends uniquely up to natural isomorphism through the Yoneda embedding to a cocontinuous functor $\bar{F} : \hat{\mathcal{C}} \to \mathcal{D}$ (cocontinuous = preserving small colimits)

72 Galois correspondence of ideals

Use Yoneda lemma to show that the Zariski topology on $\mathrm{Spec}(A)$ is closed under intersections (consider V and I as adjoint functors between poset categories).

73 Graph colorings

Show that the functor $\mathrm{Col_n} : \mathbf{Graph}^{op} \to \mathbf{Set}$ assigning to the graph sets of its n-colorings is representable. Find its representing object.

74 Projective representables

Let $\mathcal{C}$ be additive category.

a) Show that there is additive version of Yoneda embedding $\mathcal{C} \hookrightarrow [\mathcal{C},\mathbf{Ab}]_+$ in additive abelian presheaves.

b) Show that representable functors in $[\mathcal{C},\mathbf{Ab}]_+$ are projective.

75 Characteristic classes*

In this exercise there you can assume following 3 classical theorems from algebraic topology:

1) (Classifying bundle) There is a tautological vector bundle $EU(n)$ on the complex Grassmanian $BU(n) := Gr_n(\mathbb{C}^\infty)$ such that for every vector bundle over compact CW complex $V \to X$ there is unique up to homotopy map $X \to BU(n)$ such that $f^\bullet EU(n) \simeq V$

2) (Cohomology ring of Grassmannian) S_n naturally acts on $BU(n)$ by permuting coordinates and the "diagonal" map $BU(1)^n = (\mathbb{C}P^\infty)^n \to BU(n)$ is S_n-invariant. It follows that the cohomology ring $H^\bullet(BU(n))$ can be realised as is invariant subalgebra of $H^\bullet(BU(1)^n) \simeq \mathbb{Z}[t_1, \cdots, t_n]$, thus as symmetric polynomials in n-variables.

3) (Brown representability theorem) The singular cohomology functor $H^n(-, G) : HTop \to Ab$ is representable by Eilenberg-McLane space $K(G, n)$

The characteristic class of vector bundles is a natural transformation $\mathrm{Vect}_n(X) \Rightarrow H^\bullet(-, \mathbb{Z})$. Show that

a) Line bundles on compact CW complex X up to isomorphism are in natural correspondence (called the first Chern class) with elements of $H^2(X)$.

b) For each n there there are k-th Chern classes $c_i \in H^{2i}(BU(n))$ that can be identified with elementary symmetric functions on n-copies of fist Chern class of tautological line bundle on projective space.

c) All characteristic classes can be realised as polynomials in Chern classes generating $H^\bullet(BU(n)$ or symmetric polynomials in fist Chern classes generating $H^\bullet(BU(1)^n)$.

76 Principal bundles

Let G be a Lie group and X a smooth manifold. A principal G-bundle is a map $p : P \to X$ with action of G on E restricting to free and transitive action on fibers that is locally trivial, i.e. on some cover the restriction $P|U \to U$ is isomorphic to the trivial bundle $G \times U \to U$ with G acting on itself by right multiplication.

a) Show that category of principal G-bundles over X is a groupoid

b) Fix principal G-bundles P, Q over X. Show that the functor $\mathrm{Isom}_X(P, Q) : \mathrm{Diff}/X \to \mathbf{Set}$ assigning to $Y \to X$ the set of isomorphisms between $P \times_X Y$ and $Q \times_X Y$ is representable by G-bundle. In particular, for any G-bundle $P \to X$ there is a G-bundle $\mathrm{Aut}(P) \to X$.

c) Find the action of G on fibers of $\mathrm{Aut}(P) \to X$.

d) Show that the groupoid of vector bundles (subcategory consisting only isomorphisms) of rank n on X is isomorphic to category of principal GL_n-bundles.

e) A trivialisation of a vector bundle is an isomorphism with trivial bundle. Show that the frame functor $\mathrm{Fr(E)} : \mathbf{Diff}/X \to \mathbf{Set}$ sending $f : Y \to X$ to the set of trivialisations of $f^\bullet E$ is representable.

f) Show that the functor $\mathbb{P}\mathrm{Fr(E)} : \mathbf{Diff}/X \to \mathbf{Set}$

$$\mathbb{P}\mathrm{Fr(E)}(f : Y \to X) = \{\text{line bundle } \mathcal{L} \to Y + \text{trivialisation of } f^\bullet E \otimes \mathcal{L}\}/\sim$$

is representable by principal PGL_n-bundle

g) Show that $\mathrm{Hom}(E \times_X -,)$

77 Moduli spaces*

In this exercise we work in the category of smooth manifolds.

a) A family of labeled triangles up to similarity is a disc bundle with Riemannian metric such that on each fiber is isometric to a triangle with sides of length given by distances between points given by the sections. The isomorphism of families is isomorphism of bundles isometric on fibers. Show that up to isomorphism such families ale stable under pullback, thus their isomorphism classes form a presheaf on **Diff**. Show that it is representable, describe by explicit equation representing object $\mathcal{M}_l$ (called a fine moduli space) and the universal $\mathcal{F}_l$ family over it.

b) Find subobjects of $\mathcal{M}_l$ classifying equilateral and right triangles. Describe corresponding moduli functors as isomorphism classes of some families.

c) Find moduli functors classifying labeled triangles up to isometry and equality (as embedded subspace of the plane). Check if they are representable.

d) Find moduli functor of unlabeled triangles up to similarity. Show that it is not representable by constructing non-trivial family with isomorphic fibers, but there is a so called coarse moduli space $\mathcal{M}_{ul}$, satisfying the universal property on points.

e) Construct a non-trivial constant family over a circle for every object admitting non-trivial automorphism. Construct two non-isomorphic over an interval with the same moduli map (hint: fix two points on a circle and move the third one). Show that there are up to isomorphism at most 6 of such families over any map and base space.

f?) Show that M_{ul} has a representable subfunctor of scalene triangles (with all sides with different length). Show that it is dense and open in th sense that pullback of the inclusion with any representable functor is realised by an inclusion of dense open subset.

g) Show that the coarse moduli space $\mathcal{M}_{ul}$ is universal in a sense that it classifies unlabeled triangles regarded as families over a point and that every other natural transformation from a representable functor with that property factor through $h_{\mathcal{M}_{ul}} \to F_{ul}$.

h) Generalising the previous observation, explain how non-trivial automorphisms are an obstruction to representability of classifying functor. Relate that to the existence of non-trivial principal bundles.

i) Somewhat in between of labeled and unlabeled triangles we have also oriented triangles. Construct their families as 3-fold coverings with transitive $\mathbb{Z}/3$-action on fibers. Find its coarse moduli space and show that it is becomes fine after removing one point corresponding to equilateral triangle.

j) Show that every constant family of unlabeled triangles is trivial if the parameter space is simply connected

78 Grothendieck-Galois correspondence*

Let $\mathbb{k}$ be a field, $\bar{\mathbb{k}}$ its separable closure and $G_{\mathbb{k}} = \mathrm{Gal}(\mathbb{k}/\bar{\mathbb{k}}$ its absolute Galois group with profinite topology (constructed as topology of the limit over discrete subgroups of finite extensions with). Denote as $\mathcal{V}^0(\mathbb{k})$ the subcategory of finite $\mathbb{k}$-algebras and by $\mathcal{F}^0(G_{\mathbb{k}})$ - finite sets with continuous $G_{\mathbb{k}}$ action.

a) Show that $G_{\mathbb{k}}$ acts continuously on $\mathrm{Hom}(A, \bar{\mathbb{k}})$ for any $A \in \mathcal{V}^0(\mathbb{k})$

b) Show that functor representable by $\bar{\mathbb{k}}$ can be lifted to $\mathcal{F}^0(G_{\mathbb{k}})$

c) Reformulate the main theorem of Galois theory as equivalence of categories $\mathcal{V}^0(\mathbb{k})^{op} \simeq \mathcal{F}^0(G_{\mathbb{k}})$ representable by $\bar{\mathbb{k}}$ with quasi inverse sending X to the algebra of $G_{\mathbb{k}}$-invariant functions $X \to \bar{\mathbb{k}}$.

79 Representablility and algebraic groups

Show that following functors $\mathbf{Ring}^{\mathbf{op}} \to \mathbf{Grp}$ are representable and find rings representing them. Find the comultiplication homomorphisms $R \to R \otimes R$ lifting the functor h_X to the category of groups.

a) Group of units

b) Group of n-th roots of unity

c) Automorphisms of free module of rank n

A connected component of R is the component is a ring eR where e is idempotent and eR has no non-trivial idempotents. Every ring is a product of its connected components.

d) Free abelian group on connected components

e) Functions from connected components to some finite group G.

80 Familiar enrichments

Let $\mathcal{V}$ be a symmetric monoidal closed category (let's call them nice for clarity), i.e. having tensor product adjoint to internal hom.

a) Verify that **Set**, **Ab**, $\mathbf{Bool} := (0 < 1)$, $([0, \infty], \geq)$ are nice.

A $\mathcal{V}$-category (or enriched over $\mathcal{V}$, $\mathcal{V}$-enriched) is defined analogous to usual category, but with hom functors taking values in $\mathcal{V}$ with usual axioms (i.e. with composition maps $\mathcal{C}(a,b) \otimes \mathcal{C}(b,c) \to \mathcal{C}(a,c)$, unit map, associativity and unit axioms). Already familiar example are pre-additive categories, enriched over **Ab**.

b) Verify that ordinary category are the same as **Set**-categories and that pre-additive categories are pre-additive.

c) Show that **Bool**-categories are the same as posets. Identify internal hom in **Bool** with logical implication.

d) Formulate axioms of enriched functors and natural transformations. Show that category of $\mathcal{V}$-functors between $\mathcal{V}$-categories is also $\mathcal{V}$-enriched.

e) Show that Yoneda lemma holds for enriched categories. What does it say in case of **Bool**-categories?

f) Identify **Bool**-functors with order preserving functions.

g) Identify **Bool**-natural transformations $\mathrm{Nat}(f, g) \in \mathbf{Bool}$ as indicator if f dominates g.

h) Show that presheaves of **Bool**-categories are downward-closed subsets and copresheaves - upward closed.

i) Identify **Bool**-profunctors (**Bool**-functors $\mathcal{C} \times \mathcal{C}^{op} \to \mathbf{Bool}$) with relations

j) Define enriched adjunctions, classify **Bool**-adjoint pairs.

81 Lawvere metric spaces

a) Show that $\mathcal{V} = [0, \infty]$ with standard order and monoidal product $a \otimes b = a + b$. Find internal product in $\mathcal{V}$ to deduce that $\mathcal{V}$ is symmetric monoidal closed bicomplete poset.

b) Show that $\mathcal{V}$-category is generalisation of metric space, satisfying triangle inequality but with metric d possibly asymmetric or with infinite value (having in fact natural interpretation of living on different connected components).

c) Show that $\mathcal{V}$-functors are exactly 1-Lipschitz maps.

d) Identify $\mathcal{V}$-natural transformations of $\mathcal{V}$-functors with supremum metric on functions.

e) Write down inequality given by Yoneda lemma.

f) Show that $\mathcal{V}$-functors $f(x, y) = \min(x, y) : \mathcal{V}^2 \to \mathcal{V}$ and $\Delta(x) = (x, x) : \mathcal{V} \to \mathcal{V}^2$ are adjoint.

g) Pick two points x, y from a convex space K and denote their convex hull by $[x, y]$. By definition every point from this interval has a form $a = (1 - t_a)x + t_a y$. Define the distance function by

$$d(a, b) = \inf_{x \mid a \in [x, b]} -\log(t_x)$$

Show that d is a Lawvere metric on K, but not classical metric. Give an intuitive interpretation of d.

h) Show that the non-symmetric Hausdorff metric on subsets of metric space X

$$d(A, B) = \sup_{a \in A} \inf_{b \in B} d_X(a, b)$$

is a Lawvere metric space, but not classical one.

i) Find both adjoints of the discrete metric functor **Bool**-**Cat** $\to \mathcal{V}$-**Cat**.

j) Show that outer measure on boolean algebra or measure on σ-algebra has a non-symmetric Lawvere metric

$$d(A, B) = \mu(A \setminus B)$$

82 Enriched profunctors

Let $\mathcal{V} = [0,\infty]$. A $\mathcal{V}$-profunctor $A \nrightarrow B$ is an enriched functor $B^{op} \times A \to \mathcal{V}$. Define composition of profunctors $A \overset{g}{\nrightarrow} B \overset{f}{\nrightarrow} C$ by

$$(f \circ g)(c,a) = \inf_{b \in B}(f(c,b) + g(b,a))$$

a) Embed **Bool** in $\mathcal{V}$ and show that under restriction to **Bool**, composition of profunctors is just composition of relations. Write down the expression using quantifiers.

b) (co-design) Andrea Censi has proposed an elegant mathematical model for designing large production systems based on enriched profunctors. Show that function matching required and available resources can be encoded as **Bool**-profunctor $P^{op} \times R \to$ **Bool**. Show that every profunctor $\Psi : \mathcal{C} \nrightarrow \mathcal{D}$ induce canonical morphism

$$\mathcal{C}(b,a) \otimes \Psi(a,x) \otimes \mathcal{D}(x,y) \to \Psi(b,y)$$

and explain its role in the co-design.

c) Show that every $\mathcal{V}$-profunctor is generated by family of weighted arrows joining vertices of two full weighted graphs (in not unique way). Identify composition of profunctors with solution to path-finding problem minimising total weight.

d) Use the construction from above to identify profunctors with $\mathcal{V}$-valued incidence matrices. Show that under coproduct and tensor as addition and multiplication, composition of profunctors is equivalent to multiplication of matrices.

83 The zoo of enrichments

There is a lot more interesting enriched categories. Describe their morphisms, functors and natural transformation and Yoneda lemma. Try to find what can they model on some toy (or serious) example.

a) Topological categories are enriched over **Top** (a category of nice spaces, e.g. compactly generated weakly Hausdorff or locally compact).

b) Algebroids are categories enriched over vector spaces. Among examples are obviously modules of $\Bbbk$-algebras and group algebras, but there are a lot more interesting ones such as string diagrams, partitions or tangles.

c) Paraconsistent logic can be realised as enrichment over unique non-linear, complete lattice with 4 elements.

d) For set X, poset associated to powerset $\mathcal{P}(X)$ has closed symmetric monoidal structures, yielding interesting family of $\mathcal{P}(X)$-enriched categories.

e) Enrichment over $(\mathbb{N} \cup \{\infty\}, \leq, \min)$ can be used to describe bottlenecks of flow in a weighted graph. Fill the details of this construction.

f) Rings are **Ab**-categories with one object. How to construct the category of modules?

g) Free enriched categories can be constructed from left adjoint functor to forgetting the enrichment.

h) Show that every closed monoidal functor $\mathcal{V} \to \mathcal{W}$ induce the change of base functor $\mathcal{V}$-**Cat** $\to \mathcal{W}$-**Cat**. In particular, endofunctors of sets naturally induce endofunctors of **Cat**. Morphisms in categories constructed with base change by powerset $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ are called poly-morphisms.

84 Affine algebraic groups

a) Formulate axioms of group object in R-$\mathbf{CAlg}^{op}$ (it is called an affine algebraic group if is also finitely generated as algebra). Show that such ring has a structure of a bialgebra, with commutative multiplication (called Hopf algebra).

b) Show that a group ring is also Hopf algebra with cocommutative comultiplication. Show that affine algebraic group is abelian iff is cocommutative and a finite group is abelian iff its group algebra is commutative.

c) Show that $GL_n(R), SL_n(R), \mathbb{G}_a = R[x]$ and $\mathbb{G}_m = R[x, x^{-1}]$ are affine algebraic group. What group structures represent the last two? Write down explicit comultiplication.

d) Show that $SU(n), S^1$ are not complex algebraic groups.

e) (Artin-Schreier sequence) Let $R = \mathbb{F}_p$. Show that the map $f(x) = x^p - x$ is a group endomorphism of $\mathbb{G}_a$, calculate its kernel a_p.

f) (Kummer sequence) Show that the kernel μ_p of homomorphism $t^p : \mathbb{G}_m \to \mathbb{G}_m$ for $n > 1$ and is an algebraic group that is not reduced iff $char\mathbb{k} = p$.

g) An algebraic group is called finite if it is a finite-dimensional algebra over a field. Show that every finite group has a canonical structure of finite algebraic group.

h) (Cartier duality) Show that a Cartier dual of finite algebraic group $\hat{G} = [G, \mathbb{G}_m]$ is a finite algebraic group and that abelian finite algebraic group are self-dual. Calculate Cartier duals of μ_p, $\mathbb{Z}/p\mathbb{Z}$ and a_p over $\mathbb{F}_p$.

i) Show that linear action of $\mathcal{G}_m$ on vector space or algebraic action on R-algebra is equivalent with $\mathbb{Z}$-gradation.

Limits and colimits

85 Some explicit computations

In this exercise we work in **Ab**.

a) For prime p compute the pullback of canonical epimorphisms:

$$\begin{tikzcd}
 & \mathbb{Z}_{p^2} \arrow[d] \\
\mathbb{Z} \arrow[r] & \mathbb{Z}_p
\end{tikzcd}$$

b) For prime p compute the pushout of diagram

$$\begin{tikzcd} \mathbb{Z} \arrow[r, hook, "\cdot n"] \arrow[d, hook] & \mathbb{Z} \\ \mathbb{Q} & \end{tikzcd}$$

86 Some non-standard shape

Let $\mathcal{C}$ be a category with two objects $0, 1$ and arrows

$$\begin{cases} \mathrm{Hom}(0,0) = \{1, p\} & p \circ p = p \\ \mathrm{Hom}(1,1) = \{1, q\} & q \circ q = q \\ \mathrm{Hom}(0,1) = \{a\} & \\ \mathrm{Hom}(1,0) = \{b\} & a \circ b = q, b \circ a = p \end{cases}$$

a) Describe limits of functors $\mathcal{C} \to \mathbf{Top}$.

b) Describe limits and colimits of functors from subcategory with arrow a removed.

c) Describe limits and colimits of functors from subcategory with arrow a and p removed.

87 Limits under Yoneda

a) Show that the Yoneda embedding do not commute with colimits

b) Show that it does commute with limits

88 Topological constructions

Construct as limit or colimit in **Top**:

a) subspace

b) quotient space

c) union of subspaces

d) intersection of two subspaces

e) cylinder

f) mapping cylinder

g) cone

h) mapping cone

i) suspension

j) fiber

k) CW complex

l) geometric realisation of a simplicial set

89 Rings

a) Show that category of unital commutative rings have limits, but not coequalisers. On the other hand, show that coequalisers exist for non-unital rings.

b) Show that in unital commutative rings tensor products $\otimes$ are coproducts. Show that pushouts can be also computed as tensors, and coincide with coproducts in the category of appropriate R-algebras.

c) The category of "pointed algebras" is the category of $\Bbbk$-algebras with augmentation $R \to \Bbbk$. Let $\Bbbk[\varepsilon] := \Bbbk[x]/(x^2) \to \Bbbk[x]/(x)$ be such pointed ring. Show that

$$k[\varepsilon] \times k[\varepsilon] \simeq \frac{\Bbbk[x, y]}{(x, y)^2}$$

90 Free product

Let R be commutative unital ring and A, B - associative unital R-algebras.

a) Show that the tensor algebra

$$T(A, B). = \oplus(A \otimes A \oplus A \otimes B \oplus B \otimes A \oplus B \otimes B) \oplus \cdots$$

is a functor $R\text{-}\mathbf{Alg} \times R\text{-}\mathbf{Alg} \to R\text{-}\mathbf{Alg}$.

b) Let $I = (a_0 a_1 - a_0 \otimes a_1, b_0 b_1 - b_0 \otimes b_1, 1_A - 1_B)$ be ideal in $T(A, B)$. Show that the free product $A \cdot B = T(A, B)/I$ is a coproduct in R-**Alg**.

c) Show that for commutative algebras $A \cdot B \simeq A \otimes_R B$

d) Show that $T(A, A)$ is a bialgebra.

91 Semismall maps

Let $f : X \to Y$ be a proper surjection of smooth manifolds.

a) Show that $\dim X \times_Y X \geq \dim X$.

b) f is called semismall if $\dim X \times_Y X = \dim X$. Let $Y_n = \{y \mid \dim f^{-1}(y) = n\}$. Show that semismallness is equivalent to condition

$$\dim Y - \dim Y_n \geq n$$

for all n

c) Show that smooth manifolds with semismall maps do not form a category.

92 Push-pull property of image

a) Show that in the category of sets we can characterise the image of a function by the push-pull property: given a function $f : X \to Y$, its image is the unique monomorphism $f(X) \hookrightarrow Y$, with universal property of being the smallest subset with inverse image X, or more precisely such that for every monomorphism $f(X) \hookrightarrow Z \hookrightarrow Y$ the pullback $X \times_Y Z$ is an isomorphism.

b) Show that such universal image not always exists even if the category has all pullbacks and pushouts.

93 Regularity of groups

Show that the category of groups is regular and balanced, i.e.

a) That every monomorphism is an equaliser.

b) That every epimorphism is an coequaliser.

c) That morphisms that are epi and mono are isomorphisms.

94 Pushouts and epi/monos

Show that

a) Pushouts preserve epimorphisms.

b) In abelian categories pushouts preserve monomorphisms.

c) Pushouts need not preserve monomorphisms in general.

d) Show that regular monomorphisms (constructible as equalisers) are stable under pushouts. Deduce that monomorphisms are stable under pullbacks in the category of groups.

95 Combining pullbacks and pushouts

Let $\mathcal{C}$ be a category with pullbacks and pushouts.

a) Show that pushout of two projections from a pullback $X \times_S Y$ need not be isomorphic to S.

b) Show that if S is already a pushout of maps $f : A \to X, A \to Y$, then the pushout of projections from the pullback $X \times_S Y$ is isomorphic to S.

c) Show that taking in this manner the pushout of pullback of pushout is isomorphic to the first pushout.

96 Completed tensor product

In this exercise we will consider construction dual to free algebras over a field - the cofree coalgebra of some vector space A.

a) Show that the comultiplication $\Delta : T(A) \to T(A) \otimes T(A)$ in the completed tensor algebra $\hat{T}(A) = \prod_{k=0}^{\infty} A^{\otimes k}$ extends to linear map on the completed tensor product $\hat{\Delta} : \hat{T}(A) \to \hat{T}(A) \hat{\otimes} \hat{T}(A) := \prod T^i(A) \otimes T^j(A)$

b) Show that $\hat{T}(A) \otimes \hat{T}(A)$ is a proper subspace of $\hat{T}(A) \hat{\otimes} \hat{T}(A)$

c) Show that the cofree coalgebra (right adjoint functor to forgetful from algebras to vector spaces) is isomorphic to

$$C(A) = \{f \in \hat{T}(A) : \hat{\Delta}(f) \in \hat{T}(A) \otimes \hat{T}(A)\} \subset \hat{T}(A)$$

97 Pullbacks of manifolds

a) Find a map in the category of smooth manifold which pullback with itself does not exist.

b) Show that pullbacks always exist along submersions.

98 Short maps

Show that the category of metric spaces with 1-Lipschitz maps has no binary coproducts.

99 Relative separability

Show that in **Top** the criterion on the diagonal for being Hausdorff has following relative version: whenever the map ι from the left diagram is closed, equaliser of any right diagram is an inclusion of closed subspace.

$$
\begin{tikzcd}
Y \times_Y Y \arrow[dr, hook, "\iota"] \arrow[drr] \arrow[ddr] & & \\
& Y \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & Y \arrow[dl] \\
& S &
\end{tikzcd}
$$

100 Universally closed maps

Continuous function $f : X \to Y$ is called universally closed if for any space Z the map $f \times 1_Z : X \times Z \to Y \times Z$ is closed. Show that in the category of second countable locally compact Hausdorff spaces this is equivalent to being proper, i.e. preserving inclusions of compact subsets under pullback. Find a closed map that is not universally closed and a show that the this is no longer true for arbitrary topological spaces.

101 Hausdorff relations

a) Show that in **Top**, coequalisers $f, g : A \rightrightarrows X$ is the quotient space $X/R(f,g)$ by the equivalence relation $R(f,g) \subseteq X \times X$

b) Show that coequaliser of maps between Hausdorff spaces need not be Hausdorff.

c) Show that despite that, the category of Hausdorff spaces has all coequalisers, isomorphic to $X/\overline{R(f,g)}$.

d) Deduce that the inclusion $cnHaus \to$ **Top** has no left adjoint (it has right adjoint tho - the Cech-Stone compactification!)

102 Circle as cotrace

In the cocomplete category $\mathcal{C}$ consider cospans of a form

$$
\begin{tikzcd}
& X & \\
A \arrow[ur, "s"] & & A \arrow[ul, "t"']
\end{tikzcd}
$$

a) Note that such cospans can be regarded as endomorphisms in **Cospan** $:=$ **Cospan**$(\mathcal{C})$, the category with objects as in $\mathcal{C}$ and arrows cospans, composed via pullback.

b) Show that if 0 is initial in $\mathcal{C}$, composition of cospans yields a function **Cospan**$(0, X) \times$ **Cospan**$(X, 0) \to \mathcal{C}$

To any such cospan AA associate trivial and right dualising cospans

$$
\begin{tikzcd}
 & A & \\
0 \arrow[ur] & & A \sqcup A \arrow[ul, "\mathbb{1} \sqcup \mathbb{1}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
 & X & \\
A \sqcup A \arrow[ur, "s \sqcup t"] & & 0 \arrow[ul]
\end{tikzcd}
$$

the cotrace of an endomorphism $\mathrm{cotr}(f)$ is given by their composition.

c) Show that if I is the interval category, the cotrace of endpoints inclusion is $B\mathbb{N}$

$$
\begin{tikzcd}
 & I & \\
1 \arrow[ur, "0"] & & 1 \arrow[ul, "1"']
\end{tikzcd}
$$

d) Show that if we replace I with symmetric interval category, constructed by adjoining the inverse of non-identity arrow, the cotrace cotrace of endpoints inclusion is $B\mathbb{Z}$

e) Construct a topological circle as cotrace using only the unit interval.

103 Trace

a) Dualise the cotrace construction to traces of spans

b) Show that if we regard abelian group A as abelian category with one element, the trace can be coincide with usual trace of linear map.

c) Show under embedding $\mathbf{Top} \to \mathbf{Span}(\mathbf{Top})$ mapping function to its graph, trace of a map $X \to X$ is its fixed-point locus $X^f \subseteq X$.

104 Pointed spaces

a) Compute products and coproducts in $\mathbf{Top}_\bullet$.

b) The smash product is the quotient $X \bigwedge Y := (X \times Y)/(X \sqcup Y)$. Show that smash product is associative up to isomorphism for nice spaces (take for example CW-complexes; this can be weakened to the compactly generated weakly Hausdorff spaces), but not in general.

c) Show that in nice pointed spaces $X \wedge S^1$ is adjoint to the loops space $\Omega(X)$.

d) Show that $S^n \wedge S^m \simeq S^{n+m}$.

e) Compute left adjoint to forgetful functor $\mathbf{Top}_\bullet \to \mathbf{Top}$.

105 Short maps

Show that the category of metric spaces with short maps (functions not increasing distances between points, 1-Lipschitz) has no coproducts

106 Group action

Let G be a topological group acting on topological space X

a) Write down universal property of the orbit space X/G and express it as a colimit

b) Find non-transitive G-space X such that X/G is just a point.

c) Express orbit and stabiliser using limits and colimits

107 Homology

Show that singular homology commutes with coproducts, but does not commute with products and pushouts.

108 Algebraic closure of finite field

Let $\mathbb{F}_n$ be a finite field with n elements. Show that in the category of fields

$$\overline{\mathbb{F}}_p = \varinjlim\left(\mathbb{F}_p \hookrightarrow \mathbb{F}_{p^2} \hookrightarrow \mathbb{F}_{p^6} \hookrightarrow \mathbb{F}_{p^{24}} \hookrightarrow \cdots\right)$$

109 Localisation

Let M be a module and its M_x localisation at x. Show that

$$M_x = \varinjlim\left(M \xrightarrow{x} xM \xrightarrow{x} x^2M \xrightarrow{x} x^3M \hookrightarrow \cdots\right)$$

110 Suprema and infima

A lattice is a poset category with all binary products and coproduct. Show that, if exists

a) $\prod_{i\in I} x_i \simeq \inf_I\{x_i\}$

b) $\coprod_{i\in I} x_i \simeq \sup_I\{x_i\}$

111 (Co)kernel pairs

The kernel pair of a morphism is a pullback

$$\begin{tikzcd} K(f) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

while the cokernel pair is a pushout

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"] & Y \arrow[d] \\ Y \arrow[r] & C(f) \end{tikzcd}$$

Let define image of f as the equaliser of the cokernel pair and coimage as coequaliser of the kernel pair. Show that

a) image and coimage are isomorphic in **Ab**

b) image and coimage are in general not isomorphic in **Top**

112 Directed graphs

Show that the category of directed graphs is isomorphic to a category of presheaves over some small category $\mathcal{C}$. Show that the functor $\pi_0 : \mathbf{DirGraph} \to \mathbf{Set}$ is naturally isomorphic to the colimit of these presheaves.

113 Classifying object

Let $\mathcal{C}$ be a category with pullbacks. Subobjects of X are all monomorphisms $A \hookrightarrow X$. A subobject classifier is an arrow object $\bullet \to \Omega$ (Ω is called the classifying object) that there is a bijection between $\mathrm{Hom}(X, \Omega)$ and subobjects of X induced by the pullback.

a) show that classifying object, if exists, represents the functor $\mathrm{Sub} : \mathcal{C} \to \mathbf{Set}$ mapping X to the set of its subobjects

b) show that $\{0\} \hookrightarrow \{0, 1\}$ is the subobject classifier in **Set**

c) find the subobject classifier in **Top**

d) find the subobject classifier in subcategory of **Top** where morphisms are open maps

e) show that subobject classifier in **Ab** does not exist

114 Groupoids

Groupoid is a category with all morphisms being isomorphisms. Any set we identify with discrete groupoid, and any group with a groupoid with one object. Every construction below is done in the category of groupoids (e.g. group homomorphism = functor between groupoids with one objects)

a) give explicit description of pullbacks in the category of groupoids

b) Find an example of maps between groupoids with finitely many objects, which pullback groupoid has infinitely many objects

c) construct the embedding of category of G-sets in groupoids (Hint: the groupoid realising G-set X is called the quotient groupoid $[X/G]$)

d) realise the orbit of group action (as a set) via pullback of groupoids

e) Show that for any G-set X there is a natural projection functor $p : X \to [X/G]$. Find the pullback $X \times_{[X/G]} X$, describe its projections onto X (Hint: under the most natural description, these two are quite different)

f) find the pullback of two subgroup inclusions

g) realise any normal subgroup as a pullback of two group homomorphisms.

h) Given group epimorphism $G \to H$, realise the underlying set of H as a pullback of group homomorphism

i) Give explicit condition to a quotient groupoid to be naturally equivalent to a set or a group.

j) Show that in any groupoid $\mathrm{Hom}(x, y)$ is either empty or isomorphic to $\mathrm{Aut}(x)$. Use this to give a proof of the orbit-stabiliser theorem.

k*) Groups realised as one-element groupoids are called classifying spaces $\mathbf{B}G$ of principal bundles (in the topological sense, by that we mean their geometric realisations, but here we stick to just sets, i.e. principal bundle is a free and transitive action on fibers of some function), in a sense that every principal G-bundle can be realised as a pullback of $\bullet \to \mathbf{B}G$.

115 Under categories as pullbacks

Show that under category $X \downarrow \mathcal{C}$ can be realised as pullback in **Cat** of the diagram

$$\begin{tikzcd}
X \downarrow \mathcal{C} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "X"] \\
\mathbf{Arr}(\mathcal{C}) \arrow[r] & \mathcal{C}
\end{tikzcd}$$

116 Topological groups

a) Show that the category of topological groups is NOT a pullback of forgetful functors

$$\begin{tikzcd}
 & \mathbf{Grp} \\
\mathbf{Top} & \mathbf{Set}
\end{tikzcd}$$

b) Describe objects of this pullback $\mathscr{P}$ and construct embedding $\mathbf{TopGrp} \to \mathscr{P}$

c) Find an object of $\mathscr{P}$ that is not a topological group.

117 Profinite topology

Groups that are constructed as an inverse limit admit canonical structure of a topological group with profinite topology, obtained as the limit of discrete groups. If all the groups in the system are finite, such limit $A = \varinjlim A_i$ is called a profinite group.

a) Show that homomorphisms of profinite groups can differ from group homomorphisms considered in **Grp**

b) Show that every profinite group is compact, Hausdorff and totally disconnected

c) Show that if all the maps in the system are injective, profinite topology is unique topology on A such that A_i form a basis for open neighbourhoods of identity element and that addition is continuous.

d) Show that if A_0 is a ring and A_i are ideals, then A is a topological ring (i.e. multiplication is continuous)

e) Show that if all the maps are injective, every subgroup containing some A_i is open.

f) Describe the profinite topology on p-adic integers and formal power series over finite field associated to ring completion.

g) A profinite completion of a group is a profinite group associated to injective limit over quotients of normal subgroups of finite index. Describe the profinite completion of integers.

h) Show that absolute Galois group of number field is profinite.

i) Show that profinite group is isomorphic to its profinite completion iff every subgroup of finite index is open (hint: show at first that any profinite group is an inverse limit of quotients of its open normal subgroups of finite index). Groups of such property are called strongly complete.

118 Free coproduct completion

A free coproduct completion of a small category $\mathcal{C}$ is a category constructed by adjoining freely coproducts of elements from $\mathcal{C}$, i.e. a fully faithful inclusion $\mathcal{C} \to \mathcal{C}^c$ such that objects of $\mathcal{C}^c$ are disjoint union of objects $\mathcal{C}$ and coproducts of objects from $\mathcal{C}$ in $\mathcal{C}^c$.

a) Realise $\mathcal{C}^c$ as a category with objects $(I, (X_i)_{i\in I})$ where $I \in \mathbf{Set}$ and $X_i \in \mathcal{C}$

b) Realise $\mathcal{C}^c$ as a full subcategory of presheaves on $\mathcal{C}$

c) Show that Yoneda embedding factors as $\mathcal{C} \hookrightarrow \mathcal{C}^c \hookrightarrow \overline{\mathcal{C}}$. Show that both inclusions preserve limits, while only the second one preserve colimits. Show that Yoneda embedding is a free completion with respect to all colimits.

d) Find free coproduct completion of terminal category, **Grp** and the category $\mathrm{O}rb(G)$ of cosets of group G with equivariant functions.

119 Additive group functor

A group functor is a presheaf factoring through forgetful functor from the category of groups, (or simply just a functor $\mathcal{C}^{op} \to \mathbf{Grp}$). In this exercise we will consider the case of $\mathcal{C} = \mathbf{Alg_R}$.

a) Show that representable functor is a group functor iff it is represented by a group object.

b) Show that $R[x]$ represents a group functor $\mathbb{G}_a$, find its group structure.

c) Show that for any free module V of rank n the functor $V(S) = S \otimes_{\Bbbk} V \;:\; \mathbf{Alg_R}^{\mathbf{op}} \to \mathbf{Ab}$ is representable by $\mathbb{G}_a^n$.

d) Show that for every module of infinite rank, $V(S)$ is a not representable group functor.

e) Show that $\mathrm{Aut}(V)(-) \;:\; S \mapsto \mathrm{Aut}_S(V(S))$ is a group functor representable iff V has finite rank. Find rings representing $\mathrm{Aut}(\Bbbk^n)$.

f) Identify elements of $\mathrm{Aut}(V)(\Bbbk[x])$ with linear maps $V \to \mathrm{Sym}\, V$ and describe maps $V \to \mathrm{Sym}\, V$ corresponding to natural transformations of group valued functors $\mathcal{G}_a \Rightarrow \mathrm{Aut}(V)$ called representations of $\mathcal{G}_a$

g) Show representations of $\mathcal{G}_a$ are in natural bijections with modules over divided power algebra that are locally finite, i.e. every element is anihilated by all but finitely many x_i's.

$$\Lambda_R = \frac{R[x_1, x_2, x_3, \cdots]}{(\binom{i+j}{i} x_{i+j} - x_i x_j)}$$

h) Show that over $\mathcal{C}$, the matrix exponential $\exp(M \cdot t) : V \to V \otimes \mathcal{C}[t]$ gives a bijection between representations of $\mathcal{G}_a$ and nilpotent endomorphisms of V admitting a Jordan form (note that if V has infinite dimension, not all nilpotent endomorphisms are of this type; find a counter-example)

i*) Use the map $exp(x_1 t + x_p t^p + \cdots)$ to classify representations of $\mathcal{G}_a$ over $\mathbb{F}_p$.

120 Division map

Let G be a Lie group and X a smooth manifold. A principal G-bundle is a map $p : P \to X$ with action of G on E restricting to free and transitive action on fibers that is locally trivial, i.e. on some cover the restriction $P|U \to U$ is isomorphic to the trivial bundle $G \times U \to U$ with G acting on itself by right multiplication.

a) Show that if G is compact, P is principal iff p is a projection to an orbit space $P \to P/G$. Show that the compactness condition is essential.

b) Show that coequaliser of projection and action maps $P \times G \rightrightarrows P$ is isomorphic to X

c) Show that pullback of p with itself is isomorphic to $G \times X$, find induced map

d) Construct a division function $P \times_X P \to G$

e) Show that P is principal iff the shear map $P \times G \to P \times P$ is an isomorphic and the division function is continuous.

Adjoints

121 Galois connections

A Galois connection is a pair of adjoint (contravariant unless stated otherwise) functors between posets P, Q.

a) Show that f, g form a Galois connection iff they satisfy

- $f \circ g \circ f = f$
- $g \circ f \circ g = g$

b) (Galois correspondence) A Galois correspondence is a Galois connection when f and g are equivalences. Show that restriction of Galois connection to the images $f(P)$ and $g(Q)$ is a Galois correspondence.

c) Describe the unit and counit of Galois connections.

d) Recall that partitions can be constructed as a poset in both covariant and contravariant way. Show that induced functions $f^\bullet : \mathrm{Part}(Y) \to \mathrm{Part}(X)$ and $f_\bullet : \mathrm{Part}(X) \to \mathrm{Part}(Y)$ form a covariant Galois connection.

e) Show that the main theorem of Galois theory states that there is a Galois connection between intermediate Galois field extensions and subgroups of the Galois group.

f) Show that in a commutative ring Galois connection between poset of subspaces of spectrum and inclusion of ideals. What are the adjoint functors and the Galois correspondence of this connection?

g) Show that for nice connected topological space there is a Galois correspondence between subgroups of

122 Distributivity

A category is called (co)Cartesian if it has all (co)products and Cartesian closed if the product has a right adjoint.

a) Show that in a Cartesian closed category with coproducts binary products are distributive with binary coproducts

b) Show that categories **Top** and $\mathbf{Set}^{\mathrm{op}}$ are not distributive in this sense.

c) Find a category which is distributive but not Cartesian closed

123 Reflections

A subcategory is reflective if the inclusion has left adjoint, called reflector. Find reflectors of

a) $\mathbf{Ab} \hookrightarrow \mathbf{Grp}$

b) Fields in integral domains and injective homomorphisms

c) R/I-modules in R-modules, R commutative

d) $R[S^{-1}]$-modules in R-modules

e) **Top** in vector bundles

f) $\mathbf{CRing} \hookrightarrow \mathbf{Ring}$

g) Anti-commutative rings in **Ring**

124 Coreflections

A subcategory is coreflective if the inclusion has right adjoint, called coreflector. Find coreflectors of

a) Modules annihilated by come element of multiplicative subset S of commutative ring R.

b) $\mathbf{Grp} \hookrightarrow \mathbf{Mon}$

c) Connected topological groups in **TopGrp**

d) Locally connected spaces in **Top**

e) Locally path-connected spaces in **Top**

f) **Top** in vector bundles

125 Concrete examples

Find left and right adjoints or show that they don't exist for:

a) Abelianisation

b) Forgetful functor $R - \mathbf{Alg} \to R - \mathbf{Mod}$

c) Forgetting the unit of associative ring

d) Restriction of group action to a subgroup

e) $\mathbf{Grpd} \hookrightarrow \mathbf{Cat}$

f) Mapping groupoid to the underlying set of objects

g) Realising a set as discrete groupoid

h) Forgetful functors from $\mathbf{Grp}$ to $\mathbf{Monoid}/\mathbf{SemiGrp}/\mathbf{Set}$

i) Embedding $\mathbf{Set} \hookrightarrow \mathbf{Rel}$

126 Fundamental group of wedge sum

Use the fact that left adjoint functors preserve colimits to compute fundamental group of a wedge sum $X \vee Y$ of two pointed CW-complexes.

127 Funny abelian groups

For following subcategories of $\mathbf{Ab}$ show that:

a) Torsion-free abelian groups are reflective

b) Torsion-free injective abelian groups are reflective

c) Torsion abelian groups are coreflective

d) Divisible abelian groups are coreflective

e) p-groups are reflective

f) elementary groups (with non-identity elements of equal order) are reflective

128 Adjoints of monoids

Let M, N be monoids. Show that functor $\mathbf{B}M \to \mathbf{B}N$ has adjoint if and only is it is an isomorphism.

129 Galois correspondence

Describe explicit conditions of functors between posets being adjoint. Interpret the Strong Nullstellensatz as adjunctions between $\mathbf{Top}(SpecA)^{op}$ and poset of ideals of A.

130 Finite posets

Show that Hom($[n], [m]$) in finite posets is in bijective correspondence with functors $\mathbf{n} \to \mathbf{m}$. Show that such a functor has left adjoint if and only if $f(n) = m$. Give an explicit formula for its adjoint functor. Find similar condition for existence and formula for the right adjoint.

131 Limits and colimits

Show that limits and colimits are functors adjoint to the constant diagram functor

$$\mathcal{I} \hookrightarrow [\mathcal{C}^{op}, \mathcal{I}]$$

132 Self-adjoint

A self-adjoint endofunctor F is a functor adjoint to itself.

a) Show that taking the opposite category form an adjoint endofunctor in **Cat**

b) Show that if $\mathcal{C}$ is additive, composition of diagonal functor with biproduct is self-adjoint.

133 Group actions

Let $\mathcal{C}$ be a category of nice spaces and $\mathcal{C}/G$ - objects with of finite group G. Assume quotient spaces X/G exist (e.g. for CW complexes).

a) Find a left adjoint to functor $\mathcal{C} \to \mathcal{C}/G$ equipping objects with trivial action.

b) Find right adjoint to the functor $L : \mathbf{Psh}(\mathcal{C}/\mathbf{G}) \to \mathbf{Psh}(\mathcal{C})$ given by

$$L(F)(X) = \mathrm{colim}_{X \to Y/G} F(Y)$$

c) Let G be a subgroup of S_n. Show that $P : \mathbf{Psh}(\mathcal{C}) \to \mathbf{Psh}(\mathcal{C}/\mathbf{G})$

$$P(F)(X) = F((\coprod_n X)/G)$$

is a functor preserving colimits.

d) Show that P is representable.

134 Idempotent semirings and quantale

A commutative quantale is a bicomplete poset with symmetric monoidal product $\otimes$ (i.e. associative, unital, commutative binary operation) distributive over colimits.

a) Unpack the definition in set-theoretic language

b) Find a structure of commutative quantale on the unit interval.

c) Show that $- \otimes a$ has left adjoint $[a, -]$

d) Consider a commutative idempotent semiring, i.e. set with two structures of commutative monoids $(X, +), (X, \times)$, $+$ being idempotent ($a + a = a$) and distributive over $\times$. Show that the preorder P_X on X given by $a \leq b$ iff $a + b = b$ is a poset. Identify colimits and limits in P_X and show that this construction is functorial.

e) We say that X is complete if P_X is. Construct limits in complete P_X in terms of colimits. Show that P_X is commutative quantale, find tensor and internal hom.

f) Show that categories of commutative quantales and complete commutative idempotent semirings are equivalent.

135 Ambidextrous adjunction

An ambidextrous adjunction is functor with isomorphic left and right adjoints.

a) Find an inclusion with ambidextrous adjoint of finite discrete category n in $[\mathcal{C}^{op}, n]$ for any additive category $\mathcal{C}$.

b) Show that inclusion of **Top** in retractive spaces - spaces together with a retract - is an ambidextrous (bi)reflection.

136 Heyting algebras

A Heyting algebra is a lattice where the binary product has right adjoint functor $\Rightarrow$.

a) interpret $\Rightarrow$ as logical implication

b) show that poset of open subsets of a topological space is a Heyting algebra

137 Chain of adjoints

Find a functor that has two non-isomorphic adjoints, both also with that property.

138 Adjunctions of groupoids

a) Show that the functor forgetting arrows **Grpd** $\to$ **Set** have both adjoints - discrete and codiscrete groupoids.

b) Find two adjoints of discrete groupoid functor **Set** $\hookrightarrow$.

c) Show that there is a free groupoid functor associated to graph, left adjoint to forgetful **Grpd** $\to$ **Graph**.

d) Show that the category of small groupoids is cartesian closed, i.e. there is an endofunctor $\mathrm{Grpd}(G, -)$ right adjoint to $- \times G$

139 Idempotent adjunction

A subcategory $\mathcal{C} \to \mathcal{D}$ is called reflective if inclusion has a left adjoint and coreflective if it has right adjoint. For adjoint pair of functors L, R denote as ν, ε its unit and counit. An adjunction is idempotent if restricts to equivalence on essential images.

a) Show that idempotent adjunction factors through the subcategory FP of fixed points

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "L"] & FP \arrow[l, hook, shift left] \arrow[r, hook, shift left] & \mathcal{D} \arrow[l, shift left, "R"] \end{tikzcd}$$

b) Show that FP is reflective in $\mathcal{C}$ and coreflective in $\mathcal{D}$

c) Show that inclusion of (co)reflective subcategory is an idempotent adjunction

d) Show that adjunction is idempotent iff $L\nu, \varepsilon L, \nu R$ or (iff and) $R\varepsilon$ are isomorphisms.

e) Show that any adjunction between posets is idempotent.

f) Find two examples of not idempotent, reflective, coreflective, and both reflective and coreflective adjunctions.

g) Find idempotent adjunction neither reflective not coreflective.

140 Burnside ring

a) Construct left adjoint K of the forgetful functors from abelian groups to commutative monoids. Show that the natural map $M \to K(M)$ is injective iff M is a cancellation monoid (i.e. $a + b = c + b$ implies $a = c$). Compute K of $\mathbb{N}, \mathbb{Q}^{\times}$.

b) Show that K is a functor from symmetric monoidal categories to abelian groups. Calculate K_0 of finite sets with coproduct, product, and of non-empty finite sets with product.

c)When G is a finite group, $A(G) = K(G\text{-}\mathbf{Set})$ is called a Burnside ring of G. Show that Burnside ring of symmetric group (as a group) is free abelian of order $p(n)$ (number of partitions of n)

d) Show that for Cartesian closed categories with coproducts $K(\mathcal{C})$ is a commutative ring. Compute the Burnside ring of a finite cyclic group.

e) Show that $A(G) \otimes \mathbb{Q} \simeq \mathbb{Q}^k$ where k divides $|G|$

f) Show that the functor restricting G-set to a subgroup H has left adjoint called induced representation. Show that restriction and induction induce ring homomorphisms $i^\bullet : A(H) \to A(G)$ and $i_\bullet : A(G) \to A(H)$ that are related by the Frobenius reciprocity formula

$$i_\bullet(i^\bullet x \cdot y) = x \cdot i_\bullet(y)$$

141 K_0 or a ring

a) Show that for abelian categories the Grothendieck group $K_0(\mathcal{C})$ generated by objects of $\mathcal{C}$ with relations $[A] + [C] = [B]$ for SES $0 \to A \to B \to C \to 0$ form a functor $\mathbf{AbCat} \to \mathbf{Ab}$ under additive functors. Show that $K_0(\mathcal{C})$ is a quotient of $K(\mathcal{C})$

b) The group $K_0(R)$ we define as $K_0(P(R))$ where $P(R)$ is the category of finitely generated projective modules. Show that for commutative rings, $K_0(R)$ contains $\mathbb{Z}$ as a subgroup generated by free modules and that the inclusion splits.

c) Show that for $R = M_n(\Bbbk)$ we have $K_0(R) \simeq \mathbb{Z}$, but the inclusion of free modules is not split.

d) Find a ring with vanishing K_0

e) Show that K_0 commutes with finite limits and filtered colimits, but not with uncountable products.

f) Let R be the colimit of embeddings $M_{n!}(\Bbbk) \hookrightarrow M_{(n+1)!}(\Bbbk) \simeq M_{n!}(\Bbbk) \otimes M_{n+1}(\Bbbk)$ as $M_{n!}(\Bbbk) \otimes I$. Show that $K_0(R) \simeq \mathbb{Q}$. Construct a surjective analogue of the rank function with values in $\mathbb{Q} \cap [0, 1]$.

g) Show that K_0 of a category of countably generated projective modules is trivial for any ring.

h) Consider the category of faithfully projective modules $FP(R)$ - i.e. finitely generated, projective and non-zero under localisation in any maximal ideal. Calculate $K_0(FP(\Bbbk))$.

i*) Let $Az(\Bbbk)$ be a category of simple $\Bbbk$-algebras. Show that there is a monoidal functor $FP(\Bbbk) \to Az(\Bbbk)$ sending module P to its ring of endomorphisms. Show that the induced map on K_0 is injective. Its cokernel $Br(\Bbbk)$ is called the Brawer group $Br(\Bbbk)$. Compute $Br(\mathbb{R})$ (hint: by Wadderburn-Artin theorem simple k-algebras are precisely the matrix rings of division algebras).

142 λ-rings*

A λ-rings is a commutative ring together with functions $\lambda^i : R \to R$ for each $i \geq 0$ satisfying

- $\lambda^0(x) = 1$
- $\lambda^1(x) = x$
- $\lambda^k(x+y) = \sum_{i+j=k} \lambda^i(x)\lambda^j(y)$

a) Find λ-ring structure on $\mathbb{Z}, \mathbb{Q}[x]$ and the ring of symmetric functions

b) Find left and right adjoints to forgetful functor λ-**CRing** $\to$ **CRing**

c) Construct a canonical λ-ring of a small symmetric monoidal abelian category $(A, \oplus, \otimes, 0, 1)$ (if that's unfamiliar to you, check the definition on nlab or assume that you work with the category of finitely generated projective modules over a commutative ring; if you get stuck, check out https://ncatlab.org/nlab/show/Lambda-ring)

143 Topological abelian groups

In this exercise you will analyse the category of topological abelian groups **AbTop**

a) Show that the forgetful functor **Top** $\to$ **AbTop** has a left adjoint $\mathrm{Ab}(X)$.

b) Show that reduced abelianisation $\widehat{\mathrm{Ab}(X)} = \mathrm{Ab}(X)/\langle x_0 \rangle$ is a functor $\mathbf{Top}_{\bullet} \to$ **AbTop**

c) Show that topological abelian group that is a finite CW complex is a compact manifold

d) Show that $\widehat{\mathrm{Ab}(S^1)} \simeq S^1$, $\widehat{\mathrm{Ab}(S^2)} \simeq \mathbb{C}P^\infty$, $\widehat{\mathrm{Ab}(\mathbb{R}P^2)} \simeq \mathbb{R}P^\infty$

144 Isbell duality

a) Show that

$$\mathcal{O}(F)(X) = \mathrm{Nat}(F, h_X)$$

$$\mathrm{Spec}(G)(X) = \mathrm{Nat}(h^X, G)$$

Form adjoint functors between $[\mathcal{C}, \mathbf{Set}]^{\mathbf{op}}$ and $[\mathcal{C}^{op}, \mathbf{Set}]$

b) Using the fact that $\widehat{\mathcal{C}}$ is a free cocompletion, show that a functor $F : \mathcal{C} \to \mathcal{D}$ to cocomplete category $\mathcal{D}$ uniquely extends along Yoneda embedding to colimit preserving functor $\hat{F} : \widehat{\mathcal{C}} \to \mathcal{D}$

c) An functor is called Isbell self-dual if $\operatorname{Spec}\mathcal{O}(F) \simeq F$ or $\mathcal{O}\operatorname{Spec}(G) \simeq G$. Show that representable functors are Isbell self-dual.

d) Show that Isbell duality works also for additive categories, i.e. we have an adjunction $[\mathcal{C}, \mathbf{Ab}]_+{}^{\mathbf{op}}$ and $[\mathcal{C}^{op}, \mathbf{Set}]_+$ between categories of additive functors.

e) Fix a class of limits $I \to \mathcal{C}$ and denote by $[\mathcal{C}, \mathbf{Set}]_\times$ the category of functors preserving these limits. Show that Isbell duality descends to this inclusion, i.e. we have an adjunction

$$[\mathcal{C}, \mathbf{Set}]_\times{}^{\mathbf{op}} \underset{\operatorname{Spec}}{\overset{\mathcal{O}}{\longleftrightarrow}} [\mathcal{C}^{op}, \mathbf{Set}]_\times$$

145 Presheaf algebras

Let T be an additive category with all objects generated from products of $1 \in T$. The product-preserving functors $[T, \mathbf{Ab}]_\times$ are called T-algebras. We have a canonical forgetful functor $U_T : \mathbf{Alg_T} \to \mathbf{Ab}$ sending A to $A(1)$.

a) Show that Yoneda embedding yields the inclusion $F_T : T^{op} \hookrightarrow \mathbf{Alg_T}$.

b) Show that for any $X \in [T^{op}, \mathbf{Ab}]$ the Isbell functor $\mathcal{O}$ can be expressed with formula

$$\mathcal{O}(X)(t) = \operatorname{Nat}(X, \operatorname{Spec}(F_T(t)))$$

c) The presheaf $\mathbb{A}^1_T = \operatorname{Spec}(F_T(1))$ is called the T-line object. Show that $U_T \circ \mathcal{O}$ is representable by $\mathbb{A}^1_T$.

d) Consider some small full subcategory of T-algebras $T \hookrightarrow \mathcal{C} \hookrightarrow \mathbf{Alg_T}^{\mathbf{op}}$. Refine Isbell duality to adjunction $\mathcal{O} \dashv \operatorname{Spec}$ such that $\operatorname{Spec}(A)(B) = \operatorname{Hom}_{\mathbf{Alg_T}}(A, B)$.

$$\mathbf{Alg_T}^{\mathbf{op}} \underset{\operatorname{Spec}}{\overset{\mathcal{O}}{\longleftrightarrow}} [\mathcal{C}^{op}, \mathbf{Ab}]_+$$

146 Perfect groups*

A group is perfect if its equal to its commutator.

a) Show that every element of a perfect group is a product of commutators.

b) Show that the category $\mathcal{P}$ of perfect groups is cocomplete.

c) Show that $\mathcal{P} \hookrightarrow \mathbf{Grp}$ is coreflective

d) Show that a homomorphism in $\mathcal{P}$ is mono iff has abelian kernel

e*) Find non-injective monomorphism in $\mathcal{P}$

147 Metric spaces

a) Show that complete metric spaces with uniformly continuous maps is reflective subcategory of metric spaces.

b) Let U be forgetful functor $\mathbf{CompMet}_u \to \mathbf{Met}_u$ from complete metric spaces with uniformly continuous maps to metric spaces with uniformaly continuous maps and $c_M : 1 \to \mathbf{Met}_u$ be contant functor with value M. Show that initial object in the comma category $(c_M \downarrow U)$ can be identified with the completion of M.

148 The zoo of graded algebras

a) Show that graded $\Bbbk$-algebra is equivalent to monoid in the monoidal category of vector spaces with tensor product. Show that group algebras are graded.

b) Construct tensor product of augmented graded algebras ϵGA.

c) A differential graded algebra (DGA) is a graded algebra with derivation, linear endomorphism of degree ± 1 satisfying

$$d(ab) = d(a)b + (-1)^{|a|}ad(b)$$

Show that differential graded algebras are monoids in the category of (co)chain complexes of vector spaces. Extend this construction to graded commutative differential algebras (CDGA).

d) Show that tensor algebra is a free graded algebra of a raded vector space, and that free graded commutative algebra is

$$\bigvee V^{odd} \oplus \mathrm{Sym}\, V^{even}$$

e) (Word length filtration) Show that images F^pA of iterated multiplications of augmented differential graded algebra

$$\mu^p : \bigotimes^p A \to A$$

form a filtration on A extending to a functor $Q : \epsilon\mathbf{DGA} \to DGVS$.

f) Show that differential forms on smooth manifold form a CDGA.

g) Show that polynomial differential forms on a simplex, defined as a graded algebra

$$\Omega^\bullet_{poly}(\Delta^n) := \frac{\mathbb{R}[x_0, \cdots, x_n]}{(x_0 + \cdots + x_n - 1)} \otimes \frac{\bigvee_{\mathbb{R}}[dx_0, \cdots, dx_n]}{(dx_0 + \cdots + dx_n)}$$

has unique structure of $CDGA$ making the inclusion

$$\Omega^\bullet_{poly}(\Delta^n) \hookrightarrow \Omega^\bullet_{poly}(\Delta^n) \otimes C^\infty(\Delta^n) \otimes_{\mathbb{R}} \Omega^\bullet_{poly}(\Delta^n) \simeq \Omega^\bullet(\Delta^n)$$

a morphism of CDGA's.

h*) Formulate axioms of differential graded coalgebras - comonoids in chain complexes. Show that linear maps from dg-coalgebra to dg-algebra under products and coproducts has a structure of dg-algebra called convolution algebra. Show that this construction is functorial in both variables

$$[-,-] : \mathbf{DGcoAlg}^{op} \times \mathbf{DGAlg} \to \mathbf{DGAlg}$$

i*) Find its two adjoints functors, called Swidler products and Swidler hom:

$$\lhd : \mathbf{DGcoAlg} \times \mathbf{DGAlg} \to \mathbf{DGAlg}$$

$$\{-,-\} : \mathbf{DGAlg}^{op} \times \mathbf{DGAlg} \to \mathbf{DGcoAlg}$$

149 Weil algebras*

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$ (tangent space to identity with the Lie bracket of vector fields) generated by e_i with duals α_i

a) (feel free to skip this if it feels to tedious; the computations are not very relevant to homological applications) Define the Weil algebra of $\mathfrak{g}$ as free CGA on $\mathfrak{g}^\vee$ with generators θ^α, u_α of degree 1 and 2 indexed by α_i.

$$W\mathfrak{g} = \bigvee g^\vee \otimes \operatorname{Sym} \mathfrak{g}^\vee$$

Show that if the exterior derivative of left-invariant forms, under previously chosen basis has a form

$$d(\theta^\alpha) = -\frac{1}{2} \sum c^\alpha_{\beta\gamma} \theta^\beta \theta^\gamma$$

then there is a derivation

$$D\theta^\alpha = d\theta^\alpha - u_\alpha$$
$$Du_\alpha = \sum c^\alpha_{\beta\gamma} u_\beta \theta^\gamma$$

making $W\mathfrak{g}$ CDGA.

b) Find the Weil algebra of $G = (S^1)^n$.

c) Show that $W\mathfrak{g}$ is D-acyclic - i.e. $H^0_D(W\mathfrak{g}) \simeq \mathbb{R}$ and $H^{>0}_D(W\mathfrak{g}) = 0$.

Homological flavour

150 Examples

Verify whether categories are additive and abelian

a) $S - R$-bimodules

b) finitely generated abelian groups

c) finitely generated $\mathbb{Z}[[x]]$-modules

d) countably generated projective modules

e) topological abelian groups

f) vector spaces with involution

g) finite abelian p-groups

h) finite abelian groups

i) graded modules of finite length

151 0-th ring homology

Let R be a commutative ring. Let $H_0(R)$ be a ring of continuous functions $\operatorname{Spec} R \to \mathbb{Z}$. (For reader not familiar with spectra: alternatively, every nontrivial idempotent decomposes ring R as a product $eR \times (1-e)R$. Fix maximal set of idempotent decomposing R into product of indecomposable rings $R = \prod_{i \in I} R_i$. Then continuous function $\operatorname{Spec} R \to \mathbb{Z}$ can be identified with ordinary function $I \to \mathbb{Z}$)

a) Show that $H_0(R) \simeq H_0(R[t]) \simeq H_0(R[t, t^{-1}])$

b*) Let $G = \mathbb{Z}/n$ be a cyclic group and $n = p_1 \cdots p_n$ is a prime decomposition of n. Consider subring $R = \prod \mathbb{Z}[\mathbb{Z}/p_i] \subset \prod \mathbb{Q}[\mathbb{Z}/p_i] \simeq \mathbb{Q}[G]$. Show that for some $r \in \mathbb{Z}$ there is an exact sequence:

$$0 \to \mathbb{Z} \to H_0(R) \oplus H_0((\mathbb{Z}/n)[G]) \to H_0(R/n) \to \mathbb{Z}^r \to 0$$

152 Unimodular row

A projective R-module P is called stably-free (of rank $n - m$) if there is an isomorphism $P \oplus R^m \simeq R^n$.

a) Show that infinitely generated stably free module is free.

b) Prove that P is stably free iff it is a kernel of surjection $R^n \to R^m$.

In case $m = 1$, matrix of such map can be identified with vector $\sigma = (r_i)$ called unimodular row.

c) Show that the vector $\sigma = (r_i)$ is unimodular iff it induce split surjection iff there are $s_i \in R$ such that $1 = \sum r_i s_i$.

d) Show that stably free module associated to unimodular row σ is free iff σ can be completed to invertible matrix.

e) Show that if R is commutative, every unimodular row of rank 2 can be completed to invertible matrix, so kernel of every surjection $R^2 \to R$ is free.

f) Identify tangent bundle of S^2 with a projective module over $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$ associated to unimodular row. Using hairy ball theorem deduce that it cannot be completed to invertible matrix.

g) Let $\mathbb{H}$ be quaternion algebra. Show that $(x + i, y + j) \in \mathbb{H}[x, y]$ is an unimodular row of rank 2 with non-free kernel, so the commutativity condition in e) is essential.

153 Kleisli rectification

Denote by $P(R)$ the category of finitely generated projective modules over R.

a) Show that if $R = \operatorname{colim} R_i$, $i \mapsto P(R_i)$ is not generally functorial.

b) Construct a category $\mathbb{P}_i$ of idempotent matrices over R_i and show that it is equivalent to $P(R_i)$.

c) Find a functor $i \mapsto \mathbb{P}_i$ such that $P(R) \simeq \varinjlim \mathbb{P}_i$

d) A big projective module over R is a family of projective S-modules P_f indexed by homomorphisms $f : R \to S$ with an isomorphism $P_f \otimes_S T \simeq P_{gf}$ for any $g : S \to T$ over S. Show that the forgetful functor from the category of big projective modules to ordinary ones $\mathbb{P}(R) \to P(R)$ is an equivalence.

e) Show that unlike in case of $P(R_i)$, there is natural functor $i \mapsto \mathbb{P}(R_i)$ with colimit $\mathbb{P}(R)$.

f) Show that $\mathbb{P}$ is a contravariant functor from rings to exact categories and deduce that the equivalence $\mathbb{P}(R) \to P(R)$ is additive

154 N functor

Let $F : \mathbf{Ring} \to \mathbf{Ab}$ be any functor and $NF(R)$ be the cokernel of natural morphism $F(R) \to F(R[t])$.

a) Show that NF is a functor $\mathbf{Ring} \to \mathbf{Ab}$.

b) Show that evaluation $t = 1 : R[t] \to R$ induce a natural splitting $F(R[t]) \simeq F(R) \oplus NF(R)$.

c) A ring is called F-regular if $F(R) \simeq F(R[t_1, \cdots, t_n])$ for all n. Show that R is F-regular iff $N^i F(R) = 0$ for all $i > 0$.

d) Let $R = \bigoplus R_i$ be a graded ring. Show that if $NF(R)$, then $F(R) \simeq F(R_0)$.

155 Contracted functors

Let $F : \mathbf{Ring} \to \mathcal{C}$ be a functor to abelian category $\mathcal{C}$.

a) Show that the contraction $LF(R)$ defined as cokernel of $F(R[t]) \oplus F(R[t^{-1}]) \to F(R[t, t^{-1}])$ is a functor $F : \mathbf{Ring} \to \mathcal{C}$

b) Construct a chain complex $\mathrm{Seq}(F, R)$ having a form

$$0 \to F(R) \to F(R[t]) \oplus F(R[t^{-1}]) \to F(R[t, t^{-1}]) \to LF(R) \to 0$$

and extend this construction to functor $\mathrm{Seq}(F, -) : \mathbf{Ring} \to \mathbf{Ch}(\mathbf{Ab})$.

c*) F is called acyclic if $\mathrm{Seq}(F, R)$ is acyclic for every R and contracted if it is acyclic and and the map $F(R[t, t^{-1}]) \to LF(R)$ has a canonical splitting, natural in both t and R. Find examples of contracted, acyclic but not contracted and not acyclic functors.

d) Show that this construction make sense on every subcategory of $\mathbf{Ring}$, such that if $R \in \mathcal{R}$, then also $R[t], R[t, t^{-1}]$ and maps $R[t] \rightrightarrows R[t, t^{-1}]$ are in $\mathcal{R}$.

e) Show that if F is contracted, so is LF and NF. Moreover, if η is a natural transformation $LF \Rightarrow LG$ between contracted functors commuting with splitting, then $\ker(\eta)$ and $\mathrm{coker}(\eta)$ are also contracted functors.

f) Show that if F is contracted, $NLF \simeq LNF$.

g) Show that for any F

$$F(R[t_1, \cdots, t_n]) \simeq (1 + N)^n F(R)$$

h) Show that for any contracted F

$$F(R[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]) \simeq (1 + 2N + L)^n F(R)$$

i) Show that if $L^2 F = 0$, for any F-regular R

$$(1 + 2N + L)^n F(R) \simeq F(R) \oplus n \cdot LF(R)$$

156 Prüfer groups

Let p be a prime.

a) The Prüfer p-group $\mathbb{Z}(p^\infty)$ is the direct limit of natural inclusions

$$\mathbb{Z}/p \hookrightarrow \mathbb{Z}/p^2 \hookrightarrow \mathbb{Z}/p^3 \hookrightarrow \cdots$$

Show that $\mathbb{Z}(p^\infty) \simeq \mathbb{Z}[\frac{1}{p}]\mathbb{Z}$ and that

$$\mathbb{Q}/\mathbb{Z} \simeq \bigoplus_{p \ prime} \mathbb{Z}(p^\infty)$$

b) Find free resolution of $\mathbb{Z}(p^\infty)$

c) Compute $\mathrm{Ext}^1(\mathbb{Z}(p^\infty), \mathbb{Z})$

d) Compute $\mathrm{Ext}^1(\mathbb{Z}[\frac{1}{p}], \mathbb{Z})$

e) Use these results to compute $\mathrm{Ext}^1(\mathbb{Q}/\mathbb{Z}, \mathbb{Z})$ and $\mathrm{Ext}^1(\mathbb{Q}, \mathbb{Z})$

f) Show that $\mathbb{Z}(p^\infty)$ is an Artinian but not Noetherian $\mathbb{Z}$-module (contrary to the case of rings, where being Artinian implies being Noetherian).

157 Torsion subgroups and Ext

Let A be an abelian group and p a prime. Let A_p be the p-torsion subgroup of A.

a) Show that there is an exact sequence

$$\mathrm{Hom}(pA, \mathbb{Z}) \to \mathrm{Ext}^1(A_p, \mathbb{Z}) \to \mathrm{Ext}^1(A, \mathbb{Z}) \to \mathrm{Ext}^1(A, \mathbb{Z}) \to \mathrm{Ext}^1(A/p, \mathbb{Z}) \to 0$$

b) Show that $\mathrm{Ext}^1(A, \mathbb{Z})$ is injective iff A is torsion-free.

c) Show that if A is injective, $\mathrm{Ext}^1(A, \mathbb{Z})$ is torsion-free.

d) Show that if $\mathrm{Ext}^1(A, \mathbb{Z})$ is torsion-free and $\mathrm{Hom}(A, \mathbb{Z}) = 0$, then A is injective.

158 Divisible groups

a) Show that the category $\mathcal{C}$ of injective abelian groups is additive

b) Show that products and coproducts in $\mathcal{C}$ coincide with those in **Ab**

c) Show that $\mathcal{C}$ has all kernels and cokernels

c) Show that $\mathcal{C}$ is not abelian (hint: find homomorphism that is epi and mono but not iso)

d) More generally express sheaf of solutions to any holomorphic differential equation on X as a kernel of endomorphism $\mathcal{O}_X \to \mathcal{O}_X$.

159 Classifying extensions

Let $\mathcal{L}_1, \mathcal{L}_2$ be complex line bundles on connected space. We say that their extensions are (weakly) isomorphic if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{L}_1 \arrow[r] \arrow[d, "\simeq"] & V \arrow[r] \arrow[d, "\simeq"] & \mathcal{L}_2 \arrow[r] \arrow[d, "\simeq"] & 0 \\ 0 \arrow[r] & \mathcal{L}_1 \arrow[r] & W \arrow[r] & \mathcal{L}_2 \arrow[r] & 0 \end{tikzcd}$$

a) Show that $\mathrm{Aut}(\mathcal{L}_1) \simeq \mathrm{Aut}(\mathcal{L}_2) \simeq \mathbb{C}^\times$.

b) Show that automorphisms of line bundles give an action of $\mathbb{C}^\times \times \mathbb{C}^\times$ on $\mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1)$.

c) Show that the diagonal subgroup acts trivially on $\mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1)$.

d) Show that isomorphism classes of non-trivial extensions are in bijection with $\mathbb{P}(\mathrm{Ext}^1(\mathcal{L}_2, \mathcal{L}_1))$

160 Ext groups in regular local rings

Let R be a regular local ring and I be ideal generated by two relatively prime elements. Show that:

a) $\mathrm{Hom}(I, R) \simeq R$

b) $\mathrm{Hom}(I, I) \simeq R$

c) $\mathrm{Ext}^1(I, R) \simeq R/I$

d) $\mathrm{Ext}^i(I, R) = 0$ for $i > 1$

e) $\mathrm{Ext}^2(R/I, R) \simeq R/I$

161 Scissor relations

Suppose $\mathcal{C}$ is (some) category of topological spaces with functors (called motivic) $h^i : \mathcal{C}^{op} \to \mathbb{Q}^{fd}$ such that inclusion of closed subset $Z \hookrightarrow X$ induce long exact sequence

$$\cdots \to h^i(X \setminus Z) \to h^i(X) \to h^i(Z) \to h^{i+1}(X \setminus Z) \to \cdots$$

Denote as usual $\chi(A^\bullet)$ the alternating sum of dimensions and by $K_0(\mathcal{C})$ the free abelian group on objects modulo scissors relations $[X] - [Z] - [X \setminus Z]$ (in case of category of spaces) or $[B] - [A] - [C]$ induces from exact sequence (in case of abelian categories).

a) Show that if $h^\bullet(\mathcal{C})$ is finite dimensional, then

$$\chi(h^\bullet(X)) = \chi(h^\bullet(Z)) + \chi(h^\bullet(X \setminus Z))$$

b) Show that cohomology in the category of compact orientable smooth manifolds of even dimension is motivic.

c) Show that in case of manifolds of odd dimension, the cohomology is no longer motivic, however the relations are satisfied mod 2.

d) Show Euler characteristic $\chi(h^\bullet(-))$ is a group homomorphism

$$K_0(\mathcal{C}) \to K_0($$

$\mathbb{Q}^{fd}) \simeq \mathbb{Z}$

e) Show that if on some subcategory $\mathcal{D} \hookrightarrow \mathcal{C}$ values of h^i lifts to some abelian category $\mathcal{A}$, then $\chi(h^\bullet(-))$ has a refinement factoring through morphism induced by forgetful functor $\mathcal{A} \to$

$\mathbb{Q}^{fd}$ (a most interesting example is the Hodge structure on cohomology of projective complex manifolds inducing the Hodge characteristic with values in $\mathbb{Z}[u, v] \simeq K_0(\mathbf{HodgeStruct})$)

$$\chi(h^\bullet(-)) \,:\, K_0(\mathcal{D}) \to K_0(\mathcal{A}) \to K_0($$

$\mathbb{Q}^{fd}) \simeq \mathbb{Z}$

162 Smash-nilpotence

Let $\mathcal{C}$ be an monoidal additive category (i.e. with tensor products commuting at hom groups with tensor product in **Ab**). A morphism f is called smash nilpotent or order n if $f^{\otimes n} = 0$. f is smash nilpotent such n exists.

a) Show that the exterior product is an endofunctor $X \mapsto X^{\otimes n}$ that is monoidal (commutes with tensors), but not additive.

b) Show that smash nilpotent morphisms form a subgroup of $\mathrm{Hom}(X, Y)$.

c) Show that smash nilpotent morphism of order n is nilpotent of order n, i.e. $f^{\circ n} = 0$.

d) Let $f \,:\, XY$ be smash nilpotent of order n and $g_1, \cdots, g_{n-1} \in \mathrm{Hom}(Y, X)$. Show that

$$f \circ g_1 \circ f \circ \otimes \circ f \circ g_{n-1} \circ f = 0$$

e) Let $\mathcal{C}$ be a category of representations of group G (finite or linear algebraic). Show that forgetful functor takes smash-nilpotent morphisms to zero.

f) Show that more generally any monoidal functor (preserving tensors) $\mathcal{C} \to \mathbf{Vect}$ takes smash-nilpotent morphisms to zero.

g) Show that if category $\mathcal{C}$ is rigid, i.e. it has internal hom and evaluation map induce isomorphism with double dual $X \simeq X^{\vee\vee}$, then $\mathcal{C}$ has no non-zero smash nilpotent morphisms.

h) Find some nilpotent morphism that is not smash-nilpotent in $\mathbb{Q}$-linear category.

i) Let $\mathcal{C}$ be rigid and $f \in \mathrm{End}(X)$. Construct adjoint morphism $\mathrm{ad}(f) \,:\, 1 \to X \otimes X^\vee$. Show that is $\mathrm{ad}(f)$ is smash-nilpotent, then f is nilpotent element in ring $\mathrm{End}(X)$.

j*) Show that if $\mathrm{Sym}^{n+1} A = \bigwedge^{m+1} B = 0$, then every morphism $f \,:\, A \to B$ is smash-nilpotent of order $nm + 1$.

k*) Let consider canonical decomposition to invariant subobjects under canonical action of S_n indexed by partitions (Schur functors)

$$V^{\otimes n} = \bigoplus_{\lambda \in \mathrm{part(n)}} S_\lambda(V)$$

Show that a morphism $f : A \to B$ such that for all λ $S_\lambda(f) = 0$ is smash nilpotent of order n.

163 Rigid categories

A right rigid category is a monoidal category with right dual objects, i.e. equipped with evaluation and coevaluation $A \otimes A^\vee \Rightarrow 1, 1 \Rightarrow A^\vee \otimes A$ satisfying triangle identities (the same as unit and counits of adjunctions). Dually we may define left rigid categories and rigid categories being both left and right rigid.

a) Show that in the category of endofunctors, left and right duals of F are its left and right adjoints.

b) Show that left rigid categories can be canonically embedded in rigid one.

c) Find right rigid but not rigid category.

d) An indecomposable object is an object that is not a direct sum of non-trivial subobjects. Show that in rigid abelian category the unit is a direct sum of indecomposable objects. Find abelian category where it is not the case.

e) Suppose $\mathcal{C}$ is additive and rigid. Show that if 1 has non-trivial subobject and $\mathrm{End}(1) \simeq \mathbb{Q}$, then $\mathcal{C}$ cannot be abelian.

164 Quivers

A quiver is a directed multigraph $Q = (I, A)$ with vertices I and arrows encoded by source and target functions $s, t : A \rightrightarrows I$. A representation of quiver is a commutative diagram in of vector spaces $(\{V_i\}, \{f_a : V_{s(a)} \to V_{t(a)}\})$ of shape Q. Dimension of representation is the vector $(\dim V_i) \in \mathbb{Z}^I$. Show that:

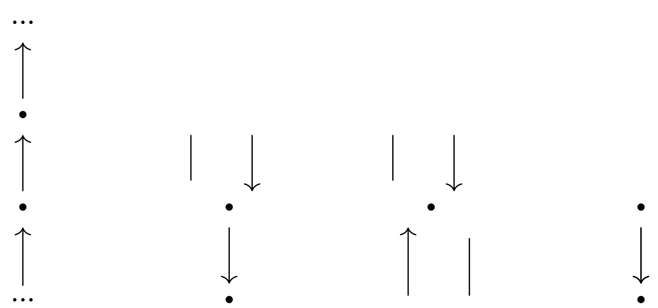
a) representations of quiver Q form abelian category

b) Dimension is additive under short exact sequences

c) Automorphism group of representation of dimension (d_i) is $\prod \mathrm{GL}_{d_i}(\mathbb{k})$.

d) Category of representations $Rep(Q)$ is equivalent to modules over some $\mathbb{C}$-algebra $\mathbb{C}Q$ called path algebra. Find its presentation.

e) Calculate path algebras of following quivers:



A_∞ J L_2 K_2

f) For every vertex v in Q construct simple module $S(v)$ from a trivial path and projective $P(v)$ from paths starting at v.

g) Show that the module $P_+(v) = \bigoplus_{a:v\to u} aP(u)$ yields a canonical projective resolution

$$0 \to P_+(v) \to P(v) \to S(v) \to 0$$

h) Show that every arrow $a : u \to v$ has associated short exact sequence

$$0 \to S(v) \to aP(u)_v \to S(u) \to 0$$

i) Show that number of arrows from u to v is equal to $\dim \operatorname{Ext}^1(S(u), S(v))$.

j) Construct a functor from finitely generated graded modules of $\Bbbk[x, y]$ to $\operatorname{Rep}(K_2)$ with essential image being irreducible representation of dimension $(1, 1)$.

k) Let $\theta \in \mathbb{Z}^I$ and V be a representation of dimension α. The slope of V is

$$\phi_\theta(V) = \frac{\langle \theta, \alpha \rangle}{\sum \alpha_i}$$

V is called θ-semistable if $\phi(V) = 0$ and $\phi(W) \leq 0$ for any subrepresentation $W \subseteq V$. Show that θ-semistable representations form an abelian subcategory of (Q).

165 Flatness and length

We say that the A-module M has finite length $l_A(M)$ if there is a finite filtration of length $M = M_0 \subset \cdots M_l = 0$ such that $M_{i-1}/M_i \simeq A/\mathfrak{m}$.

a) Show that the length is independent of chosen filtration

b) Show that length is additive on short exact sequences. In particular, whenever two modules in a sequence have finite length, so does the third one.

c) Let $A \to B$ be a flat morphism between Artinian local rings. Show that the length of B can be expressed as

$$l_B(B) = l_A(A) \cdot l_B(B/\mathfrak{m})$$

d) Suppose M has a finite resolution by modules of finite length M_i. Show that

$$\sum (-1)^i l_A(M_i) = 0$$

166 Herbrand quotient

Let $\varphi : M \to M$ be an endomorphism with kernel and cokernel of finite lengths. Then we define $e(\phi) = l(\operatorname{coker} \phi) - l(\ker \phi)$.

a) Show that $e(\phi) = 0$ if M is of finite length.

b) Show that for a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, "f"] & B \arrow[r] \arrow[d, "g"] & C \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0
\end{tikzcd}$$

$$e(g) = e(f) + e(h)$$

c) Show that either $e(\varphi) = e(\det \varphi)$ or neither of them is well defined.

d) Show that if M is finitely generated and with cokernel of finite length, $e(\varphi)$ is well defined and non-negative.

e) Show that e can be generalise to arbitrary homomorphism $\varphi : M \to N$ and that if M, N are both projective of rank n, $e(\varphi) = e(\bigwedge^n \varphi)$.

167 Cohomology of inverse limit

Let X_i be a filtration of CW complex X by subcomplexes.

a) Show that $H_n(X) \simeq \varinjlim H_n(X_i)$.

b) Show that there is a short exact sequence

$$0 \to \varinjlim^1 H^{n-1}(X_i) \to H^n(X) \to \varinjlim H^n(X_i) \to 0$$

c) Show that if $H_n(X)$ is not finitely generated abelian group, then either $H^n(X)$ or $H^{n+1}(X)$ is uncountable.

d) Compute singular homology and cohomology of the solenoid - inverse limit of system of p-fold covers of the circle

$$S^1 \xrightarrow{p} S^1 \xrightarrow{p} \cdots$$

168 Split exact categories

We can consider exact sequences in categories that are only additive, but embedded in abelian category where exactness makes sense. An exact category is a full additive subcategory $\mathcal{C}$ of abelian category $\mathcal{A}$ with class $\mathcal{E}$ of sequences from $\mathcal{C}$ exact in $\mathbb{A}$ and moreover, $\mathcal{C}$ is closed under extensions - for any sequence in $\mathbb{A}$ such that $A, C \in \mathcal{C}$ we require B to be isomorphic with object from $\mathcal{C}$ as well. A category is split exact if every sequence in $\mathcal{E}$ splits.

a) Show that complex vector bundles over any space form exact category. Over paracompact space, this category is split, while category of holomorphic vector bundles over compact complex manifold is almost never split.

b) Show that the category of modules with projective resolution by finitely generated modules is exact. Show that free modules form an additive subcategory that is not exact unless all projective modules are free.

c) Show that category of projective modules with endomorphisms $\mathbf{End}_R$ is exact, as well its subcategory $\mathbf{Nil}_R$ where endomorphisms are nilpotent. Construct a biexact functor

$$\mathbf{End}_R \times \mathbf{Nil}_R \to \mathbf{Nil}_R$$

d) Show that categories of modules admitting finite projective resolution and projective resolution of length at most n are exact subcategories of $\mathbf{Mod}$-R, as well as their subcategories consisting only S-torsion modules when S is a multiplicatively closed set of central nonzerodivisors.

e) Show that the category of finite abelian p-groups has an additive subcategory of groups with cyclic summands of order p^{2i} that is not exact.

f) Show that $\mathbf{Nil}_R$ is equivalent to t-torsion $R[t]$-modules admitting projective resolution by finitely generated modules of length 1.

169 Pure Hodge structures

A pure weight m Hodge structure on complex vector space H consists of

- structure of real vector space $H_\mathbb{R}$
- bigradation $H = \bigoplus_{p+q=m} H^{p,q}$ such that $H^{p,q} = \overline{H}^{q,p}$ (complex conjugation)
- finitely generated abelian group (playing a role of a lattice, despite it can have torsion) $H_\mathbb{Z}$ with isomorphism $H_\mathbb{Z} \otimes \mathbb{R} \simeq H_\mathbb{R}$

We say that such structure is of rank r if $H_\mathbb{Z} \simeq \mathbb{Z}^r$

a) Show that Hodge structures form a category HS with maps preserving grading and lattices

b) Show that there are no rank 1 Hodge structures of pure odd weights

c) Show that the lattice $(2\pi i)^k\mathbb{Z} \hookrightarrow \mathbb{C} = H^{-k,-k}$ is unique up to isomorphism rank 1 structure of pure weight $-2k$

d) Show that HS has direct sums of objects of the same weight and tensor product additive on weights

e) Show that the Tate twist $H \mapsto H(i) := H \otimes \mathbb{Z}(i)$ is an endofunctor

f) Construct a dualising functor $H \mapsto H^\vee$ such that

$$\mathrm{Hom}(\mathbb{Z}(0), H^\vee \otimes G) \simeq \mathrm{Hom}(H, G)$$

g) Show that HS is abelian.

170 Mixed Hodge structure

A mixed Hodge structure is a real (or rational) vector space H with two filtrations - increasing weight filtration W on H and decreasing Hodge filtration on $H_\mathbb{C}$, such that F induce a pure Hodge structure of weight k on each associated graded $Gr_k^W := W_k/W_{k-1}$. Morphisms of mixed Hodge structures are linear maps preserving both filtrations.

a) Show that given two filtration on vector space, each one induce a filtration on associated graded of the other and describe these filtrations induced on a mixed Hodge structure.

b) Show that category of mixed Hodge structures has direct sums and tensor products adjoint to internal homs.

c) (Deligne decomposition) Let $I^{p,q} = Gr_F^p Gr_{p+q}^W$. Show that

$$H_{\mathbb{C}} \simeq \bigoplus_{p\in\mathbb{Z}} \bigoplus_{q\in\mathbb{Z}} I^{p,q}$$

$$W_k = \bigoplus_{i=0}^{k} \bigoplus_{p+q=i} I^{p,q}$$

$$F^k = \bigoplus_{p=k}^{\infty} \bigoplus_{q\in\mathbb{Z}} I^{p,q}$$

d*) Let $f : H_1 \to H_2$ be a morphism of mixed Hodge structures. Show that f is strict for both filtration, i.e.

$$f(W_k H_1) = W_k H_2 \cap f(H_1)$$

$$f(F^k H_1) = F^k H_2 \cap f(H_1)$$

e) Show that f is an isomorphism of mixed Hodge structures iff it is isomorphism of vector spaces.

f) Deduce that the category of mixed Hodge structures is abelian.

171 Trivial extensions

In this exercise fix associative finite dimensional $\Bbbk$-algebra A and its bimodule A. The trivial extension of A by M, denoted $A \rtimes M$, is the module $A \oplus M$ equipped with algebra structure with multiplication

$$(a, m)(b, n) = (ab, an + mb)$$

a) Show that there is a decomposition of A as right module over itself into

$$A \simeq \bigoplus^{n} e_i P_i$$

Where e_i are idempotent and P_i idecomposable and projective

b) Show that M is a square-zero two-sided ideal of $A \rtimes M$

c) Show that following sequence of $A \rtimes M$-bimodules is exact

$$0 \to M \to A \rtimes M \to A \to 0$$

d) Show that there is an indecomposable projective $A \rtimes M$-bimodule P_i^A fitting into short exact sequence of $A \rtimes M$-bimodules

$$0 \to e_i M \to P_i^A \to P_i \to 0$$

172 Nakayama functor

Let A be a finite-dimensional $\Bbbk$-algebra. We can consider two types of dual modules of $A : DM = \mathrm{Hom}(M, \Bbbk)$ and $M\vee = \mathrm{Hom}(M, A)$. The Nakayama functor combines these two into an endofunctor $\nu(M) := (DM)^\vee$.

a) Show that ν takes projective modules to injective modules

b) Show that $\bar{\nu}(M) = \mathrm{Hom}(DA^{op}, M)$ takes injective modules to projective modules

c) Show that ν restricts to equivalence of categories **ProjMod**-$A \to$ **InjMod**-A^{op} with quasi-inverse $\bar{\nu}(M)$.

d) Show that ν is right exact, but not left exact in general.

e*) Let $\mathcal{C} = [\mathbf{n}, \mathbf{Vect}_{\mathbb{C}}]$ where $\mathbf{n}$ is a total order on n elements. Show that for $M, N \in \mathcal{C}$

$$\mathrm{Ext}^1(M, N) = D\,\mathrm{Hom}(N, L^1\nu(M))$$

173 Self-injective algebras

An $\Bbbk$-algebra A is called self-injective if it is projective and injective right module over itself.

a) Show that A is self-injective iff A-modules are projective iff they are injective

b) Show that for any finite-dimensional $\Bbbk$-algebra A, $A \rtimes A^\vee$ is self-injective

c*) The i-th syzygy $\Omega^i M$ of M is the kernel of i-th map in (some) projective resolution of M. Show that for self-injective algebras we have

$$L^1\nu(M) \simeq \Omega^2\nu(M)$$

174 Symmetric algebras

A finite-dimensional $\Bbbk$-algebra is called symmetric if $A \simeq DA = \mathrm{Hom}(A, \Bbbk)$ as A-bimodules. Fix such symmetric algebra A.

a) Show A is self-injective

b) Show that $\nu(M) \simeq M$ for any M-bimodule

c) Show that $L^1\nu(M) \simeq \Omega^2(M)$

d) Show that $A \rtimes DA$ is symmetric

175 The pseudo-abelian hull

An additive category is called pseudo-abelian if every idempotent endomorphism has a kernel and a canonical isomorphism $A \simeq \ker p \oplus \ker(1 - p)$

a) Find exact category that is not pseudo-abelian and pseudo-abelian that is not abelian. Find additive, not pseudo-abelian category where all kernels of idempotent endomorphisms exist.

b) The pseudo-abelian hull $\mathcal{C}^\sharp$ of additive category $\mathcal{C}$ has objects (A, p) for $A \in \mathcal{C}$ and p idempotent endomorphism of A and morphism groups

$$\mathrm{Hom}_{\mathcal{C}^\sharp}((A,p),(B,q)) = q \circ \mathrm{Hom}(A,B) \circ p$$

Show that $\mathcal{C}^\sharp$ is pseudo-abelian.

d) Show that p is identity element of $\mathrm{End}((A,p))$

e) Let $M = (A,p), N = (B,q) \in \mathcal{C}^\sharp$. Show that given retraction $N \xrightarrow{\alpha} M \xrightarrow{\beta} N$, $\alpha \circ \beta$ is an idempotent of M, so can be identified with idempotent $\bar{p}$ of A. Prove that there are isomorphisms

$$N \simeq (A, \bar{p})$$

$$M \simeq N \oplus (A, p - \bar{p})$$

f) Realise $\mathrm{Hom}_{\mathcal{C}^\sharp}((A,p),(B,q))$ as quotient of subgroup of $\mathrm{Hom}_{\mathcal{C}}(A,B)$. Show that in such quotient we have

$$[f] = [q \circ f \circ p] = [q \circ f] = [f \circ p]$$

g) Show that $(A,p) \simeq (B,q)$ iff there are morphisms $A \xrightarrow{f} B \xrightarrow{g} A$ such that

$$\begin{cases} p \otimes g \otimes q \otimes f \otimes p = p \\ q \otimes f \otimes p \otimes g \otimes q = q \end{cases}$$

h) Justify that $\mathcal{C}^\sharp$ can be obtained by formally adding kernels of idempotent endomorphisms.

i) Construct pseudo-abelianisation $\Psi_{\mathcal{C}} : \mathcal{C} \to \mathcal{C}^\sharp$: additive, fully faithful functor with universal property that for every additive functor $F : \mathcal{C} \to \mathcal{D}$ to pseudo-abelian $\mathcal{D}$ there exists an additive functor $F^\sharp : \mathcal{C}^\sharp \to \mathcal{D}$ and natural isomorphism $F \simeq F^\sharp \circ \Psi_{\mathcal{C}}$.

j) Describe pseudo-abelian hull of your favourite additive but not pseudo-abelian category.

176 Invariants

Let $X \in \mathcal{C}$ be an object of abelian category and G a finite group acting on X. Assume moreover that $|G| \cdot 1_X$ is an automorphism of X.

a) Show that $X^G = \bigcap_{g \in G} \ker(1 - g)$ is a direct summand of X.

b) Show $\frac{1}{|G|} \sum_G g$ is an idempotent endomorphism with image X^G.

c) Can X^G be constructed if $\mathcal{C}$ is only pseudo-abelian?

d*) A polynomial functor $\mathcal{C} \to \mathcal{D}$ between abelian linear categories is a functor such that $\mathrm{Hom}(X,Y) \to \mathrm{Hom}(F(X), F(Y))$ is a polynomial function between vector spaces. Show that every polynomial functor homogeneous of degree n can be expressed as

$$F(X) = G(X, \cdots, X)^{S_n}$$

For some additive functor $G : \mathcal{C}^n \to \mathcal{D}$ acted on by symmetric group.

177 Correspondences

Recall that singular cohomology with rational coefficients due to Poincare duality has a covariant pushforward map defined for maps between smooth orientable compact manifolds, satisfying the projection formula $f_\bullet(f^\bullet(y) \cdot x) = y \cdot f_\bullet(x)$. This makes possible to extend the functor $H^\bullet$ to bigger class of generalised maps constructed as spans. A correspondence between x and Y is an element of $H^\bullet(X \times Y)$.

a) Show that correspondences form a category **Cor** with smooth orientable manifolds as objects and correspondences as maps.

b) Show that every correspondence $u \in \mathrm{Hom}_{\mathbf{Cor}}(X, Y)$ induces group homomorphisms $u^\bullet : H^\bullet(Y) \to H^\bullet(X)$ and $u_\bullet : H^\bullet(X) \to H^\bullet(Y)$

c) Construct a functor $h : \mathbf{SmoothOrMan} \to \mathbf{Cor}$ such that $h(f)^\bullet = f^\bullet, h(f)_\bullet = f_\bullet$ for any $f : X \to Y$.

d) Show that **Cor** is additive

e) Show that **Cor** is not abelian (hint: consider the Poincare dual of $[\{\bullet\} \times \mathbb{C}P^1]$ as idempotent endomorphism of $\mathbb{C}P^1$)

f*) Show that Lefschetz fixed point theorem can be generalised to correspondences in a following way: for X of dimension n and correspondence $u \in H^n(X \times X)$ the "virtual fixed points" of u satisfy the equation that for a graph of function coincide with Lefschetz trace formula:

$$\sum(-1)^i \mathrm{tr}_i(u^\bullet) = \langle \Delta, u \rangle_{X \times X}$$

178 Filtrations

Show that if $\mathcal{C}$ is abelian, a category of objects from $\mathcal{C}$ with finite filtrations and homomorphisms preserving filtrations ($f(F^i(X)) \subseteq F^i(Y)$) is additive, but not always abelian.

179 Semisimple categories

Let $\mathcal{C}$ be an additive category. Objects $A \in \mathcal{C}$ is simple if it has no subobjects except for 0 and A and semisimple if is a direct sum of simple objects.

a) Show that morphism between simple objects is either zero or an isomorphism

b) An objects is called indecomposable if is not a direct sum on two non-zero subobjects. Show that in pseudo-abelian category object is indecomposable iff has no non-trivial idempotents

c) Show that $\mathbf{Vect}_{\mathbf{fd}}$ is semi-simple, but **Vect** is not

d) Find indecomposable but not semi-simple object in $[I, \mathbf{Vect}_{\mathbf{fd}}]$ (where I is a poset $0 \leq 1 \leq 2$)

e) Show that **Mod**-$M_3(\mathbb{H})$ is a semisimple category (where $\mathbb{H}$ is the quaternion algebra) but **Mod**-$M_3(\mathbb{Z})$ is not

180 A_3 quiver

Consider representations of the A_3 quiver, i.e. the category $[A_3, \mathbf{Vect}_{\mathbb{C}}^{\mathbf{fd}}]$ where A_3 is the poset $0 \leq 1 \leq 2$.

a) Find objects M, N such that $\mathrm{Hom}(M, N) = 0$, but $\mathrm{Hom}(N, M) \simeq \mathbb{C}$. Deduce that this category is not semisimple.

b) Find non-split short exact sequence in $\mathcal{C}$

c) Describe simple and projective objects in $\mathcal{C}$

d) Find projective resolution of $\mathbb{C} \xrightarrow{1} \mathbb{C} \xrightarrow{1} \mathcal{C}$

e) Let M, N be the diagrams $0 \to \mathbb{C} \xrightarrow{1} \mathbb{C}$, $\mathbb{C} \xrightarrow{1} \mathbb{C} \to 0$. Compute $\mathrm{Ext}^1(M, N)$.

181 Schur functors

Let $\mathcal{C}_R$ be a category of finitely generated modules over commutative $\mathbb{k}$-algebra R. A Schur functor is an endofunctor of $\mathcal{C}_R$ commuting with tensor products, direct sums and arbitrary change of rings (or more precisely, any symmetric monoidal additive functor).

a) Show that $M^{\otimes n}, \mathrm{Sym}^n M, \bigwedge^n M$ are Schur functors.

b) Show that direct sum, tensor product and composition of Schur functors are also Schur functors.

c*) Find a Schur functor that cannot be constructed as sum, tensor or composition of $M^{\otimes n}, \mathrm{Sym}^n M, \bigwedge^n M$.

182 Structure matters

Find a morphism of ringed spaces that is an identity on underlying topological spaces but is not an isomorphism.

183 Enough

a) Find a category where not every SES splits, but objects are injective iff they are projective

b) Show that finitely generated abelian groups have not enough injectives

c) Show that torsion abelian groups have enough injectives

d) Show that chain complexes over $\mathcal{C}$ have enough projective/injectives iff $\mathcal{C}$ has

e) Show that finite abelian groups have only one projective object. What about torsion abelian groups?

184 Not projective G-modules

Find finitely generated, not projective S_3-module over an algebraically closed field.

185 Hereditary categories

Let $\mathcal{C}$ be abelian category with enough injectives and projectives. We call such $\mathcal{C}$ hereditary if the functors $\operatorname{Ext}^i(-,-)$ are zero for $i > 1$.

a) Show that $\mathcal{C}$ is hereditary if all subobjects of projective objects are projective.

b) Dualise the characterisation for injective objects

c) Show that representations of quivers form hereditary category.

d*) Let $Q = (I, A)$ be a quiver with n vertices and no loops. Show that the Euler form $\langle V, W\rangle = \dim \operatorname{Hom}(V, W) - \dim \operatorname{Ext}^1(V, W)$ depends only on the dimension of representation and form a bilinear form $\mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{Z}$ given by

$$\langle V, W\rangle = \sum_{i\in I} v_i w_i - \sum_{a\in A} v_{s(a)} w_{t(a)}$$

for $w_i = \dim W_i, v_i = \dim V_i$.

e*) Show that the Euler form depends on the orientation of the edges, but the Tits form

$$q(V) = \langle V, V\rangle$$

depends only on the underlying graph.

186 BGK reflection functors*

Let $Q = (I, A)$ be a quiver. A vertex i is called a sink if there are no edges $j \to i$. For any sink i, let Q_i^+ be a quiver with edges $i \to j$ reversed. The BGK (Boris-Gelfand-Ponomarev) reflection functor $\Phi_i^+ : \mathbf{Rep}(Q) \to \mathbf{Rep}(Q_i^+)$ on vertices is defined as

$$\Phi_i^+(V)_j = \begin{cases} V_j & i \neq j \\ \ker\left(\bigoplus_{a:k\to i} V_k \to V_i\right) & i = j \end{cases}$$

and on arrows in the obvious way.

a) Show that Φ_i^+ is additive, left exact functor.

b) Show that $R^i\Phi_i^+ = 0$ for $i > 1$

c) Show that $R^1\Phi_i^+(V) = 0$ iff $\bigoplus_{a:k\to i} V_k \to V_i$ is surjective

d) Find functor Φ_i^- left adjoint to Φ_i^+. Find Φ_i^--acyclic objects.

e) Show that Φ_i^+ and Φ_i^- restrict to equivalence of categories on acyclic objects.

f) Show that the functor $R\Phi_i^+ : \mathbf{Rep}(Q) \to \mathbf{Ch}(\mathbf{Rep}(Q_i^+))$ given by

$$R\Phi_i^+(V)_j = \begin{cases} 0 \to 0 \to V_j \to 0 & i \neq j \\ 0 \to \bigoplus_{a:k\to i} V_k \to V_i \to 0 & i = j \end{cases}$$

is exact

g) Show that for any representation V and some $n \geq 0$ there is a short exact sequence

$$0 \to \Phi_i^+\Phi_i^-(V) \to V \to S(i)^n \to 0$$

h) Show that for any non-zero indecomposable representation V

$$\Phi_i^- \Phi_i^+(V) \simeq \begin{cases} 0 & V \simeq S(i) \\ V & V \neq\simeq S(i) \end{cases}$$

187 Representation orbits*

Let $Q = (I, A)$ be a quiver. Its representation space in dimension $\alpha \in \mathbb{N}^I$ we define as

$$R(\alpha) = \bigoplus_{a \in A} \mathrm{Hom}(V_{s(a)}, V_{t(a)})$$

a) Show that the group

$$GL(\alpha) := \prod GL_{\alpha(i)}(\mathbb{k}) \subset \oplus \mathrm{End}(\mathbb{k}^{\alpha(i)}) =: \mathrm{End}(\alpha)$$

acts freely on $R(\alpha)$ by conjugation.

b) Show that isomorphism classes of α-dimensional representations are in bijection with $GL(\alpha)$-orbits in $R(\alpha)$.

For $x \in R(\alpha)$ denote V_x the corresponding representation and O_x or O_{V_x} - its orbit.

c) Show that any two representations V, W induce an exact sequence

$$0 \to \mathrm{Hom}(V, W) \to \bigoplus_I \mathrm{Hom}(V_i, W_i) \to \bigoplus_{a : i \to j} \mathrm{Hom}(V_i, W_j) \to \mathrm{Ext}^1(V, W) \to 0$$

Deduce that for any $x \in R(\alpha)$ there is an exact sequence

$$0 \to \mathrm{End}(V_x) \to \mathrm{End}(\alpha) \to R(\alpha) \to \mathrm{Ext}^1(V_x, V_x) \to 0$$

d) Show that the orbit O_x is open iff $\mathrm{Ext}^1(V_x, V_x) = 0$

e) Show that the orbit O_x is closed iff all the maps in V_x are zero.

f) Show that if O_V has maximal dimension, V can be decomposed as $\bigoplus I_i$, where I_k are indecomposable and for $i \neq j$ $\mathrm{Ext}^1(I_i, I_j) = 0$

g) Assuming each orbit O_x is a smooth manifold, identify the orbit and its tangent and normal spaces at point x as

$$O_x = GL(\alpha) / \mathrm{stab}(x)$$
$$T_x O_x \simeq \mathrm{End}(\alpha) / \mathrm{End}(V_x)$$
$$N_x O_x \simeq \mathrm{Ext}^1(V_x, V_x)$$

188 Pontryagin duality

a) Show that the category of compact Hausdorff abelian groups is abelian.

b*) Show that it is equivalent to $\mathbf{Ab}^{\mathrm{op}}$ (hint: consider maps to the circle group and the Fourier transform)

c) Show that the category of locally compact abelian groups is self-dual

d) Show that categories of topological abelian groups, Hausdorff abelian groups and locally compact abelian groups are additive, have all kernels and cokernels but are not abelian.

189 Hyperderived functors

Let $F : \mathcal{C} \to \mathcal{D}$ be left exact functor between abelian categories, $\mathcal{C}$ with enough injectives.

a) Show that every complex $C^\bullet \in C^+(\mathcal{C})$ is quasi-isomorphic to complex with injective objects $\mathcal{I}^\bullet$.

b) Define the hyperderived functor of F by $\mathcal{H}^i(A^\bullet) = H^i(F(I^\bullet)))$. Show that it is independent on complex $I^\bullet$ and the quasi-isomorphism up to unique isomorphism.

c) Show that $\mathbb{H}^i F \simeq R^i(F \circ H^0) \simeq R^i(H^0 \circ F)$.

190 Hypercohomology

Let $F : \mathcal{C} \to \mathcal{D}$ be left exact functor between abelian categories, $\mathcal{C}$ with enough injectives. Denote the category of cochain complexes bounded on the left as C^+ and sheaves of abelian groups on X by $\mathbf{Ab(X)}$. The hypercohomology is the hyperderived functor of global sections $\mathbb{H}^i : C^+(\mathbf{Ab(X)}) \to \mathbf{Ab}$

a) Describe the sheaves of cohomology, functors $\mathcal{H}^i : C^+(\mathbf{Ab(X)}) \to \mathbf{Ab(X)}$

b) Show that for any sheaf $\mathcal{F}$

$$\mathbb{H}^i(\mathcal{F}[n]) \simeq H^{i+n}(\mathcal{F})$$

c) Show that for bounded complex of acyclic sheaves

$$\mathbb{H}^i(\mathcal{F}^\bullet) \simeq H^i(H^0(\mathcal{F}^\bullet))$$

d) Now consider sheaves of vector spaces and denote $h^i(\mathcal{F}^\bullet) = \dim \mathbb{H}^i(\mathcal{F}^\bullet), h^{i,j}(\mathcal{F}^\bullet) = H^i(\mathcal{F}^j)$. Show that for bounded complex

$$h^n(\mathcal{F}^\bullet) \leq \sum_{i+j=n} h^{i,j}(\mathcal{F}^\bullet)$$

e) Show that if $\sum_{i+j=n} h^{i,j}(\mathcal{F}^\bullet)$ is finite,

$$\sum(-1)^n h^n(\mathcal{F}^\bullet) = \sum(-1)^{i+j} h^{i,j}(\mathcal{F}^\bullet)$$

f*) Realise de Rham cohomology as hypercohomology of the complex of sheaves of differential forms.

Sheaves

191 Fat point

Consider smooth manifolds as locally ringed $\mathbb{R}$-algebras with sheaf of real functions. The fat point is the locally ringed space F with topological space having two points $\bullet, \circ$ such that $\bullet$ is closed and lies in the closure of $\circ$. Its sheaf has sections

$$\Gamma(\{\bullet, \circ\}, F) = \mathbb{R}[x]/(x^2)$$

$$\Gamma(\{\circ\}, F) = \mathbb{R}$$

Describe the functor **SmoothMan** (in terms of well known objects)

$$F(X) = \mathrm{Hom}_{LRS}(F, X)$$

and show that it's not representable.

192 Exponential sequence

Let $X \in$ **Top** and G be a topological abelian group.

a) Show that continuous functions $X \to G$ form a sheaf.

b) Show that if X is locally simply connected there is an exact sequence of sheaves of abelian groups

$$0 \to \underline{\mathbb{Z}} \to \mathrm{Hom}(X, \mathbb{R}) \to \mathrm{Hom}(X, S^1) \to 0$$

c) Show that if X is simply connected and locally simply connected, $H^1(\underline{\mathbb{Z}}_X) = 0$

d) Compute $H^i(S^1, \underline{\mathbb{R}})$ by constructing exact sequence

$$0 \to \underline{\mathbb{R}} \to \mathcal{O} \to \mathcal{O} \to 0$$

193 Logarithmic complex

Let X be a compact complex manifold with sheaf of holomorphic functions $\mathcal{O}_X$. A normal crossing is a closed subset D that can be described in local coordinates $(z_1, \cdots, z_n)$ by the equation $z_1 \cdot \cdots \cdot z_k = 0$. Denote complement of D by U.

a) A rational 1-forms with at most logarithmic pole at D is a differential form in local coordinates $(z_1, \cdots, z_n)$ expressed as

$$\omega = \sum_{i=1}^{k} f_i \frac{dz_i}{z_i} + \sum_{i=k+1}^{n} f_i dz_i \mid f_i \in \mathcal{O}_X(U_z)$$

Show that such forms form a sheaf of $\mathcal{O}_X$-modules $\Omega^1_X(\log D)$

b) As usual we set $\Omega^k_X(\log D) = \bigwedge^k \Omega^1(\log D)$. Construct a cochain complex

$$0 \to \mathcal{O}_X \to \Omega^1(\log D) \to \Omega^2(\log D) \to \cdots$$

194 Čech vs sheaf cohomology*

Consider the plane X with topology where proper closed sets are finite. Construct Y from two copies of X glued at two points. Show that $H^2(\underline{\mathbb{Z}}_Y) = \mathbb{Z}$, but $\check{H}^2(\underline{\mathbb{Z}}_Y) = 0$.

195 Sheaf of solutions to differential equation

Let $U \subset \mathbb{C}$ be open subset and $A \in M_n(\mathcal{O}(U))$ - a matrix of holomorphic functions on U.

a) Show that the solutions to the system of linear differential equations form a sheaf

$$\mathcal{F}(U) = \{f \in \mathcal{O}(U)^n \mid f' = Af\}$$

b) Show that $\mathcal{F}$ is locally constant

c) Find A and U for which $\mathcal{F}$ is not a sheafification of constant presheaf.

196 Local systems

Recall that a fundamental groupoid of a path-connected space X is a groupoid with points of X as objects and homotopy classes of paths as maps.

a) A local system if a functor $\Pi(X) \to \mathbf{Ab}$. Show that for every $x \in X$, there is a canonical action of $\pi_1(X, x)$ on $F(x)$.

b) Show that the category of local systems is equivalent to $\pi_1(X)$-modules

c) Construct a functor associating to every local system a canonical locally constant sheaf of abelian groups

d) Show that the category of local systems is equivalent to locally constant sheaves of abelian groups.

e) Let $E \to X$ be a fiber bundle. Construct a local system taking x to $H^i(E_x)$ in each 3 equivalent categories.

197 Operations on sheaves

Find adjoints or show they don't exist of

a) Inverse image of presheaves of rings

b) Extension by zero of sheaves of abelian groups on open subset $j : U \hookrightarrow X$

$$j_!\mathcal{F}((V) = \begin{cases} \mathcal{F}(V) & V \subseteq U \\ 0 & V \nsubseteq U \end{cases}$$

c) Inclusion of locally ringed spaces in ringed spaces

d) Show that inclusions of open subset $j : U \hookrightarrow X$ and its complement $i : X \setminus U \hookrightarrow X$ for any sheaf of abelian groups induce the short exact sequence

$$0 \to j_! j^{-1\mathcal{F}\to\mathcal{F}\to i_* i^{-1}\mathcal{F}\to 0}$$

198 Pushing finite covers

Let X, Y be locally ringed spaces, $\mathcal{L}$ a line bundle on X and $f : X \to Y$ a morphism.

a) Show that if f is a finite cover, $f_*\mathcal{L}$ is a vector bundle

b) Show that if f is stalkwise flat map with fibers being length d finite algebras, $f_*\mathcal{L}$ is a vector bundle of rank d.

199 Flasque sheaves

a) Show that pushforward of flasque sheaf is flasque

b) Show that direct limit of flasque sheaves is flasque

c) Show that $C^k(\mathbb{R}^n)$ ale flasque for all $k \in [0, \infty]$

d) Show that sheaf of invertible meromorphic functions $\mathcal{K}_X^\times$ on complex manifolds are flasque sheaves of abelian groups.

e) Show that for flasque $\mathcal{F}$, $R^i f_* \mathcal{F} = 0$ for $i > 0$

f) Let X be a topological space that is not an union of two proper closed subsets. Show that locally constant sheaves on X are flasque.

g) Let X be a topological space such every open set is connected. Show that locally constant sheaves on X are flasque.

200 Flabbification

Let $(X, \mathcal{O}_X)$ be locally ringed space and $\mathcal{F}$ be an $\mathcal{O}_X$-module. Define $G(\mathcal{F})$ to be a presheaf with sections $G(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x$.

a) Show that $G(\mathcal{F})$ is a flasque sheaf.

b) Show that G is exact.

c) Show that G is right adjoint to inclusion of the subcategory of flasque sheaves.

d) Show that $G(\mathcal{F}) = p_* p^*$ where p is the canonical map $X_{discrete} \to X$.

e) (Godeman resolution) Put $G_0 = G(\mathcal{F})$ and $d_0 : \mathcal{F} \to G_0(\mathcal{F})$ be the unit of adjunction between p_* and p^*. Show that d_0 is a monomorphism.

f) We define the complex $G_*(\mathcal{F})$ by $G_i = G(\operatorname{coker} d_{i-1}$ and d_i be a composition $G_{i-1} \to \operatorname{coker} d_{i-1} \to G_i$. Show that $G_*(\mathcal{F})$ is the flasque resolution of $\mathcal{F}$.

g*) Show that for all $x \in X$, complexes of $\mathcal{O}_{X,x}$-modules $G_*(\mathcal{F})_x$ and $\mathcal{F}_x[0]$ are homotopic.

201 Flasque shift

Let G be a flabbification defined above and put

$$C^i(\mathcal{F}) = \begin{cases} \mathcal{F} & i = 0 \\ G(\mathcal{F})/\mathcal{F} & i = 1 \\ C^{i-1}(C^1(\mathcal{F})) & i > 1 \end{cases}$$

a) Show that there is short exact sequence

$$0 \to C^n(\mathcal{F}) \to G(C^n(\mathcal{F})) \to C^{n+1}(\mathcal{F}) \to 0$$

b) Show that $H^{i+1}(\mathcal{F}) \simeq H^1(C^i(\mathcal{F}))$

c) Show that every short exact sequence can be extended to diagram with exact row and columns

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{G} \arrow[r] \arrow[d] & H \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & G(\mathcal{F}) \arrow[r] \arrow[d] & G(\mathcal{G}) \arrow[r] \arrow[d] & G(H) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & C^1(\mathcal{F}) \arrow[r] \arrow[d] & C^1(\mathcal{G}) \arrow[r] \arrow[d] & C^1(H) \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0 & \end{tikzcd}$$

202 Extensions with constant sheaf

Let $\mathcal{F}$ be a sheaf of abelian groups and $\underline{\mathbb{Z}}$ be a constant sheaf. Show that $H^1(\mathcal{F}) \simeq \mathrm{Hom}(\mathrm{Ext}^1(\underline{\mathbb{Z}}, \mathcal{F}), H^1(\mathcal{F}))$.

203 Invertible sheaves

Let $(X, \mathcal{O})$ be ringed space. A sheaf of $\mathcal{O}$-modules $\mathcal{L}$ is called invertible if $- \otimes_{\mathcal{O}} \mathcal{L}$ is autoequivalence of category of $\mathcal{O}$-modules.

a) Show that $\mathcal{L}$ is invertible iff for some $\mathcal{E}$ there is isomorphism $\mathcal{E} \otimes \mathcal{L} \simeq \mathcal{O}$

b) Show that locally free sheaf of rank 1 is invertible.

c) Show that pullback of invertible sheaf is invertible

d) Show that the converse is true if X is locally ringed, but not in general.

e) Show that $gr(\mathcal{L}) = \bigoplus_{i=0}^{\infty} H^0(\mathcal{L}^{\otimes i})$ is a graded ring and that for any $\mathcal{F}$ $\bigoplus_{-\infty}^{\infty} H^0(\mathcal{F} \otimes \mathcal{L}^{\otimes i})$ is a graded $gr(\mathcal{L})$-module.

204 Torsors

Let X be a topological space space, $\mathcal{G}$ be a sheaf of groups and $\mathcal{F}$ - sheaf of sets. We say that $\mathcal{F}$ is a $\mathcal{G}$-torsor iff there is an action $\mathcal{F} \times \mathcal{G} \to \mathcal{F}$ that is free and transitive at all sections $\mathcal{F}(U) \times \mathcal{G}(U) \to \mathcal{F}(U)$ and all stalks of $\mathcal{F}$ are non-empty.

a) Show that torsors form a groupoid.

b) Show that torsor is trivial iff it has a global section.

c*) Show that if $\mathcal{G}$ is abelian, isomorphism classes of $\mathcal{G}$-torsors are in bijection with $H^1(X, \mathcal{G})$.

205 Torsors and invertible sheaves

a) Let $\mathcal{F}$ be an $\mathcal{O}^\times$-torsor. Define the presheaf of sets

$$\mathcal{L}_1(\mathcal{F})(U) = \frac{\mathcal{F}(U) \times \mathcal{O}(U)}{\mathcal{O}^\times(U)}$$

with $\mathcal{O}^\times$-action $g(s,f) = (gs, g^{-1}f)$. Show that this action is well-defined and equip $\mathcal{L}_1(\mathcal{F})$ with the structure of presheaf of $\mathcal{O}$-modules.

b) Show that $\mathcal{L}_1(\mathcal{F})$ need not be a sheaf.

c) Show that sheafification of $\mathcal{L}_1(\mathcal{F})$ is invertible.

d) Let $\mathcal{L}$ be invertible sheaf. Consider the presheaf of sets

$$\mathcal{L}^\star(U) = \{s \in \mathcal{L}(U) \mid s \cdot \mathcal{L}_U \simeq \mathcal{L}_U \simeq \mathcal{O}_U\}$$

Show that $\mathcal{L}^\star$ is an $\mathcal{O}^\times$-torsor.

e) Show that $(\mathcal{L}_1(\mathcal{F})^{sh})^\star \simeq \mathcal{F}$ and deduce that isomorphism classes of invertible sheaves are in natural bijections with isomorphism classes of $\mathcal{O}^\times$-torsors.

206 Hom complex

Let $(X, \mathcal{O}_X)$ be a ringed space and $\mathcal{F}^\bullet, \mathcal{G}^\bullet, H^\bullet$ - complexes of $\mathcal{O}_X$-modules.

a) Construct a complex $\mathrm{Hom}^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$ such that

$$\mathcal{H}om^n(\mathcal{F}^\bullet, \mathcal{G}^\bullet) = \prod_{n=i+j} \mathcal{H}om(\mathcal{F}^{-i}, \mathcal{G}^j)$$

b) Construct natural isomorphism functorial in every variable

$$\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{H}om^\bullet(\mathcal{G}^\bullet, H^\bullet)) \simeq \mathcal{H}om^\bullet(\mathrm{Tot}(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet), H^\bullet)$$

c) Construct morphisms functorial in every variable:

- $\mathrm{Tot}(\mathcal{H}om^\bullet(H^\bullet, \mathcal{G}^\bullet) \otimes \mathcal{H}om^\bullet(\mathcal{F}^\bullet, H^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet)$
- $\mathrm{Tot}(\mathcal{G}^\bullet \otimes \mathcal{H}om^\bullet(\mathcal{F}^\bullet, H^\bullet)) \to \mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathrm{Tot}(\mathcal{H}om^\bullet(\mathcal{G}^\bullet, H^\bullet)))$
- $\mathcal{F}^\bullet \to \mathcal{H}om^\bullet(\mathcal{G}^\bullet, \mathrm{Tot}(\mathcal{H}om(\mathcal{G}^\bullet \otimes \mathcal{F}^\bullet)))$
- $\mathrm{Tot}(\mathcal{H}om^\bullet(\mathcal{F}^\bullet, \mathcal{G}^\bullet) \otimes H^\bullet) \to \mathcal{H}om^\bullet(\mathcal{H}om^\bullet(H^\bullet, \mathcal{F}^\bullet), \mathcal{G}^\bullet)$

207 Affine bundles

A holomorphic fiber bundle $E \to B$ is called affine if the transition functions take values in the group of affine linear transformations $GL_n(\mathbb{C}) \hookrightarrow AGL_n(\mathbb{C})$, i.e. are of the form $(x, v) \mapsto (x, g(x)v + \xi(x))$ for $g_x \in GL_n(\mathbb{C}), \eta(x) \in C^n$.

a) Write cocycle conditions for affine bundles as single equation of block upper-triangular matrices.

b) Show that there is a short exact sequence of vector bundles

$$0 \to \mathcal{F} \to \mathcal{E} \to \mathcal{O} \to 0$$

Such that $E \simeq \mathbb{P}(\mathcal{E}) \setminus \mathbb{P}(\mathcal{F})$

c) Show that E is a vector bundle iff the sequence splits.

d) Show that classes of $\mathbb{G}_m$-bundles, (affine bundles of rank 1) are classified by $\mathrm{Ext}^1(\mathcal{O}_X, \mathcal{O}_X)$.

e) Find some interesting affine bundle

208 Projective sheaves*

Show that sheaves of abelian groups over the interval do not have enough projectives.

209 Line bundles on product

Let X, Y be paracompact spaces and and $(x, y) \in X \times Y$. Recall isomorphism classes of line bundles form a group $\mathrm{Pic}(-)$ under tensor product. Denote by $\mathrm{Pic}^{-}(X \times Y)$ the subgroup of $\mathrm{Pic}(X \times Y)$ containing classes of line bundles trivial under restriction to $X \times \{y\}$ and $\{x\} \times Y$.

a) Show that

$$\mathrm{Pic}(X \times Y) \simeq \mathrm{Pic}(X) \oplus \mathrm{Pic}(Y) \oplus \mathrm{Pic}^{-}(X \times Y)$$

b) Find X, Y with nontrivial $\mathrm{Pic}^{-}(X \times Y)$.

210 Branched double covers

Let X be a smooth/complex compact manifold with sheaf of functions $\mathcal{O}_X$ and submanifold D. Suppose that there is a line bundle $\mathcal{L}$ with total space $p : L \to X$ such that $\mathcal{L} \otimes \mathcal{L}$ has a global section s vanishing exactly on D. Define $Y = \{t \in \mathcal{L}_x \mid t^2 = s(x)\} \subseteq L$.

a) Show that Y is a smooth manifold and the canonical map $f : Y \to X$ is a double cover outside of D restricting to homeomorphism on preimage of D.

b) Describe local sections of $f^{\bullet}\mathcal{L}$ in local coordinates.

c) Show that Y has an involution σ exchanging sheets, fixing subspace isomorphic to D.

d) Show that $f_{\bullet}\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{L}^{\vee}$ and that this decomposition is induced by eigenspaces of of σ acting on fibers.

211 Higher pushforward

a) Let $f : X \to Y$ be a morphism of ringed spaces and $\mathcal{F}$ a vector bundle on T. Prove that

$$F(U) = H^i(f^{-1}(U), \mathcal{F})$$

is a presheaf on Y.

b) Show that it is not a sheaf in general.

c*) Show that sheafification of F is the i-th right derived functor of the pushforward

$$F(U)^{sh} \simeq R^i f_{\bullet}\mathcal{F}$$

212 Fine sheaves

Let X be a smooth manifold. A sheaf on X is called fine if for any locally finite open cover $\{U_\alpha\}$ it has a subordinate sheaf partition of unity: A collections of endomorphisms $\eta_\alpha \in \mathrm{End}(\mathcal{F})$ such that:

- $\mathrm{supp}(\eta_\alpha) \subseteq U_\alpha$
- $\sum \eta_\alpha = 1$

a) Show that vector bundles are fine

b) Show that if sheaf of rings $\mathcal{A}$ is fine, every $\mathcal{A}$-module is fine.

c) Show that fine sheaves are acyclic.

d*) Deduce that the de Rham cohomology is isomorphic to sheaf cohomology of the constant sheaf $\underline{\mathbb{C}}_X$

213 Sheaves of algebras

Let $(X, \mathcal{O}_X)$ be ringed space.

a) Give axioms for sheaves of $\mathcal{O}_X$-algebras refining the notion of $\mathcal{O}_X$-modules.

b) Assume that sheaf of $\mathcal{O}$-algebras $\mathcal{A}$ is also a complex vector bundle as locally free $\mathcal{O}_X$-module. Show that trace of left multiplication gives a map $\mathcal{A} \to \mathcal{O}_X$.

c) Show that composition of multiplication with map from above yields a morphism of vector bundles

$$\mathcal{A} \to \mathcal{A}^\vee$$

d*) Show that discriminant of associated quadric form $\mathcal{A} \otimes \mathcal{A} \because \mathcal{O}_X$ vanish at points x such that $\mathcal{A}_x$ is not isomorphic to the product of fields.

e) Show that this is the support of the kernel of map $\mathcal{A} \to \mathcal{A}^\vee$ constructed in c)

214 Non-abelian cohomology

Let X be a compact smooth manifold.

a) Show that $\mathrm{GL}_n(X)(U) = \mathrm{GL}_n(\mathcal{O}_X(U))$ is a sheaf of groups.

b) Construct the first Čech cohomology $\check{H}^1(X, \mathcal{G})$ for sheaf of (non-abelian) groups as a functor to pointed sets.

c) Show that isomorphism classes of rank r complex vector bundles form functors $\mathbf{Vect}_r$: $\mathbf{SmMan} \to \mathbf{Set}$. lifting to abelian groups for $r = 1$.

d) Show that $\mathbf{Vect}_r(X) \simeq \check{H}^1(X, \mathrm{GL}_r(X))$. In particular, isomorphism classes of line bundles have an abelian group structure $\mathrm{Pic}(X) \simeq \check{H}^1(X, \mathcal{O}_X^\times)$.

e) Use the exponential sequence

$$0 \to \underline{\mathbb{Z}}_X \to \mathcal{O}_X \xrightarrow{exp} \mathcal{O}_X^\times \to 0$$

To show that $\mathrm{Pic}(X) \simeq H^2(X, \underline{\mathbb{Z}}_X)$ (hint: use the previous exersize)

f) Construct a 6-term "exact sequence" in of pointed sets in cohomology associated to short exact sequence of sheaves of groups.

215 Smallest flasque sheaf*

Let M be a smooth manifold. Recall that a set is called nowhere dense if its closure has empty interior.

a) Let C^k_{nd} be a sheaf of classes of real functions on U of class C^k defined outside some nowhere dense set, identified if are equal on common domain of definition. Show that C^k_{nd} is flasque.

b) Show that C^k_{nd} is the smallest flasque sheaf containing C^k in the sense that it has no flasque subsheaves containing C^k as proper subsheaf.

c) Show that in general every sheaf $\mathcal{F}$ we may construct flasque sheaf $\mathcal{F}_{nd}$

$$\mathcal{F}_{nd}(U) = \left(\bigcup_{V \subseteq U open} H^0(U \setminus \partial V) \right) / \sim$$

d) Show that in general $\mathcal{F}$ is not a subsheaf of $\mathcal{F}_{nd}$.

216 Sheaves of nearby cycles*

Let X be a complex manifold, $\Delta \subset \mathbb{C}$ a unit disc and $\Delta^\star = \Delta \setminus \{0\}$. Consider holomorphic map $f : X \to \Delta$ submersive over $\Delta^\star$, denote its fibers over t by X_t. To avoid making this problem about topology, you may assume following known elementary facts

- X_t is a complex manifold for all $t \in \Delta^\star$
- there is a fiber-preserving retration $X \to X_0$ inducing maps $r_t : X_t \to X_0$
- For any $x \in X_0$ and sufficiently small $\varepsilon > 0$ and $0 < \eta < \varepsilon$, for all $t \in B(0, \eta)$ the spaces $B(x, \varepsilon) \cap X_t$ are homeomorphic. Any such representative is called a Milnor fiber $\mathrm{Mil}(f, x)$.

a) Find a family of sheaves of vector spaces $\mathcal{M}^k$ on X_0 such that for all $x \in X_0$

$$\mathcal{M}^k_x \simeq H^k(\mathrm{Mil}(f, x), \mathbb{Q})$$

b) Lift the exponential function to find a canonical action of $\mathbb{Z} \simeq \pi_1(\Delta^\star, t)$ on $H^\bullet(X_t)$.

Simplicial

217 Cubical spaces

A (n-truncated) cubical category $\square$ ($\square_n$) is a poset of finite subsets of $\mathbb{N}$ ($\{0, \cdots, n-1\}$). A (n-)cubical object is a functor $\square^{op} \to \mathcal{C}$ (${\square_n}^{op} \to \mathcal{C}$).

a) An semi-simplicial category $\widehat{\Delta}$ is a subcategory of Δ containing only inclusions. Construct a functor realising $(n+1)$-cubical object as augmented n-semi-simplicial one. Is this an equivalence?

b) Construct a geometric realisation of a cubical manifold, a functor $\square$**Man** $\to$ **Top**.

c) A normal crossing is an (not disjoint) union of k smooth manifolds of dimension n locally homeomorphic to an open subset of intersection of k coordinate hyperplanes in $\mathbb{R}^{n+1}$. Realise normal crossing as realisation of a cubical manifold.

d*) Realise normalisation of a $\Bbbk$-algebra of Krull dimension 1 as a 2-cubical object in the opposite category of regular $\Bbbk$-algebras.

e*) Realise blow-up of complex manifold as 2-cubical complex manifold.

218 Cohomology complex

Show that cohomology groups of a simplicial object in ringed spaces form a double complex.

219 Intertia sets

Let a $\sigma : G \times X \to X$ be an action of group G on set X. The inertia set $I_G(X) = \coprod_G \{g\} \in X^g \subset G \times X$ be a disjoint union of sets stabilised by $g \in G$ with G-action $h(g,x) = (h^{-1}gh, hx)$.

a) Show that $I_G(X)$ is isomorphic to the pullback of diagram

$$\begin{tikzcd} & X \arrow[d, "\Delta"] \\ G \times X \arrow[r, "(\sigma,\pi)"] & X \times X \end{tikzcd}$$

b) Let $X^{(g_1,\cdots,g_n)} := \bigcap X^{g_i}$. Define higher inertia sets $I^n_G(X) = \coprod_{\mathbf{g}\in G^n}\{\mathbf{g}\} \times X^{\mathbf{g}} \subset G^n \times X$. Realise $I^n_G(X)$ as pullback and equip with G-action.

c) Show that if G is abelian, $I^{n+1}_G(X) = I_G(I^n_G(X))$.

d) Show that maps

$$f^i_n(g_1,\cdots,g_n,x) = (g_1,\cdots,g_{i-2},g_{i-1}g_i,\cdots,g_n,x)$$
$$d^i_n(g_1,\cdots,g_n,x) = (g_1,\cdots,g_{i-1},e,g_i,\cdots,g_n,x)$$

Make $I^\bullet_G(X)$ into simplicial G-set.

e) Describe $I^{G^\bullet(\{\bullet\})}$.

f)

220 Standard resolutions

Let Δ be a simplex category. A simplicial object in $\mathcal{C}$ is the functor $\Delta^{op} \to \mathcal{C}$. The category of simplicial objects is denoted by $\Delta(\mathcal{C})$.

a) For a pair of adjoint functors F, G construct simplicial object $X_\bullet$ with $X_n = (FG)^{\circ(n+1)}$.

b) For abelian category $\mathcal{C}$, construct the Morre complex - a functor $s : \Delta(\mathcal{C}) \to \mathbf{Ch}_{\geq 0}(\mathcal{C})$ with $s(X)_n = X_n$.

c) Show that s is exact.

d) Construct canonical free resolution of R-modules using Moore complex.

e*) Construct Godeman resolution as a Moore complex.

f*) Show that homotopy of simplicial objects induce homotopy of their Moore complexes.

221 Normalised complex

Let $A_{\bullet}$ be a simplicial object in abelian category $\mathcal{C}$. Let the normalised chain complex be a sequence of subobjects of A_m with $N(A_0) = A_0$ and

$$N(A_m) = \bigcap_{i=0}^{m-1} \ker d_i^m$$

a) Show that $N(\mathrm{Map}(X, A_n)) \simeq \mathrm{Hom}(X, N(A_n))$

b) For all m Construct isomorphisms natural in A

$$\bigoplus_{[n]\twoheadrightarrow[m]} N(A_m) \to A_m$$

c) Show that restriction of $(-1)^n d_n^n$ gives $N(A_{\bullet})$ a structure of chain complex in a functorial way.

d) Let $D(A_n)$ be the image of map

$$\bigoplus_{[n]\twoheadrightarrow[i], i<n} N(A_i) \to A_n$$

Show that $s(A) \simeq N(A) \oplus D(A)$

e) Show that N and D are exact functors.

f) Show that $D(A)$ is acyclic and $s(A), N(A)$ quasi-isomorphic

222 Chain homotopies

Let $\mathcal{C}$ be abelian category and A a chain complex in $\mathcal{C}$.

a) Show that $H_A(B) = \{(f, g, H) : f, g : A \to B, H : \text{ homotopy between } f, g\}$ is a covariant functor $Ch_+\mathcal{C} \to \mathbf{Ab}$

b) Show that H_A is representable, i.e. there is complex $\diamond A$ such that $H_A(-) \simeq \mathrm{Hom}(\diamond A, -)$

c) Show that $\diamond$ is an endofunctor

d) Construct an exact sequence

$$0 \to A \oplus A \to \diamond A \to A[-1] \to 0$$

e*) Let $X, Y \in \Delta(\mathcal{C})$, $a, b : X \to Y$ and h be a homotopy between chain maps $N(a), N(b) : N(X) \to N(Y)$. Show that every such homotopy comes from simplicial homotopy $X \times \Delta[1] \to Y$ by constructing maps

$$\diamond N(X) \to N(Y)$$
$$N(X \times \Delta[1]) \to \diamond N(Y)$$
$$N(X) \to N(X \times \Delta[1])$$
$$N(X) \to \diamond N(X)$$

f*) Deduce that every chain homotopy comes from morphism $X \times \Delta[1] \to Y$ of simplicial objects.

223 Eilenberg-MacLane objects

A simplicial object is Eilenberg-MacLane $K(A,k)$ iff

$$H_i(s(K(A,k))) = \begin{cases} A & k = i \\ 0 & k \neq i \end{cases}$$

a) Construct $K(A,0)$. Compute the complex $s(K(A,0))$.

b) From any object of $\mathcal{C}$ construct in a functorial way the simplicial object $E(A)$ with

$$E(A)_k = \bigoplus_{[n]\twoheadrightarrow[k+1]} \bigoplus_{[n]\twoheadrightarrow[k]\hookrightarrow[k+1]} A$$

c) Show that $s(E(A))$ is acyclic

d) Construct inductively higher Eilenberg-MacLane spaces using exact sequence

$$0 \to K(A,n) \to E(A) \to K(A,n+1) \to 0$$

224 Truncations and skeletons

Let Δ_n be a subcategory of Δ of ordered sets with at most n elements. A category of n-truncated simplicial objects is $\Delta_n(\mathcal{C}) := [{\Delta_n}^{op}, \mathcal{C}]$. The skeleton functor $sk_n : \Delta(\mathcal{C}) \to \Delta_n(\mathcal{C})$ is induced by restriction. As usual we denote the over category $(\Delta \downarrow [n])$ as Δ^n, i.e. by Yoneda we have then $X_n = \mathrm{Nat}(\Delta^n, X)$.

a) Find right adjoint cosk_n of sk_n.

b) Find left adjoint i_n of sk_n.

b) Construct the functor $X(n) : (\Delta^n_m)^{op} \to \mathcal{C}$ from any n-truncated object $X : {\Delta_m}^{op} \to \mathcal{C}$.

c) Show that

$$(\mathrm{cosk}_m X)_n = \lim_{(\Delta^n_m)^{op}} U(n)$$

d) From object $X \in \mathcal{C}$ construct truncated object $X[n] \in \Delta_n(\mathcal{C})$ with $X[n]_i = 0$ for $i < n$ and $X[n]_n = X$. Describe $\mathrm{Hom}(A, X[n])$ and $\mathrm{Hom}(X[n], A)$ for any A.

e) Describe $\mathrm{cosk}_n X[n]$ and $i_n X[n]$. Show that canonical map $i_n X[n] \to \mathrm{cosk}_n X[n]$ is injective.

f) Show that

$$X = \mathrm{colim}\, i_k \mathrm{sk}_k X$$

g) Show that $K(A,n) \simeq i_n A[n]$

h) Show that $\Delta^n \simeq \mathrm{cosk}_1 \mathrm{sk}_1 \Delta^n$