Sets

$\{1, 7, 3, 5\}$ & $\{$John, red, 1$\}$

↳ Samma reglar som proggy & matte 5

$\in$ tillhör : $7 \in \{1, 7, 3\}$

$\{1, 1, 3, 7\}$ är 3 element.

Oändliga sett: $\mathbb{N} = \{1, 2, 3, \ldots\}$

$\subseteq$ delmängd (subset) till superset.

$\{1, 2\} \subseteq \{1, 2, 3\}$

kan ha set : set

tomma mängden $\emptyset = \{\ \}$

Note $\{\emptyset\} \neq \emptyset$

Note $4/2 \in \{1, 2, 3\}$? $\Rightarrow$ yes

Övn: $\{1, 2, 3\} = \{3, 2, 1\}$

$\{9, 5\} = \{9, 5, 9\}$

$\{0, 2, 8\} = \{|\sqrt{4}|, 0/5, 2^3\}$

$\{7\} \neq 7$

$\{8\} \neq \{\{8\}\}$

$\{$"London", "Beijing"$\} \neq \{$"London", "peking"$\}$

$\{+\} \neq \{'+'\}$

Def $A \subseteq B$ iff (if and only if) every element of A is an element of B.

$\forall$ (for all) if $x \in A$ then $x \in B$

$\nexists x$ (must exist) such that $x \in A$ and $x \notin B$

$\Rightarrow$ Samma som ovan.

skillnad if & iff.

↳ if, om $(A \Rightarrow B)$

↳ $A \Rightarrow B$, $A \Leftarrow B$ (iff)

$\emptyset \subseteq \{1, 2\}$? yes

$\forall x$ if $x \in A$ then $x \in B \Rightarrow$ true

$\Rightarrow$ se $\emptyset$ som ett set.

$\emptyset = \{\}$

$A \subseteq B$, $B \supseteq A$.

Strict subset: $A \subset B \Rightarrow A \neq B$

$A \subsetneq B \Rightarrow$ Samma betydelse.

Witness $\{1\} \subsetneq \{1, 2\} \Rightarrow 2$.

2 motbevisande Witness: motbevisar $\subseteq$: $\subset$ $\Rightarrow$ Oklart.

Setbuilder n.

$\{x \in \mathbb{N}^+ \mid x \text{ is even}\}$

$= \{x \mid x \in \mathbb{N}^+ \text{ and } x \text{ is even}\}$

$= \{2x : x \in \mathbb{N}^+\}$

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ex: $B = \{2, 3, 5, 7\}$

$= \{x \in \mathbb{N} : x < 11 \text{ and } x \text{ is prime}\}$

$4 \in \{x \in \mathbb{N}^+ : \exists y \in \mathbb{N} \text{ and } x = 2y\}$

kolla om för en. M till höll $\mathbb{N}^+$? Finns att y där x är 2y.

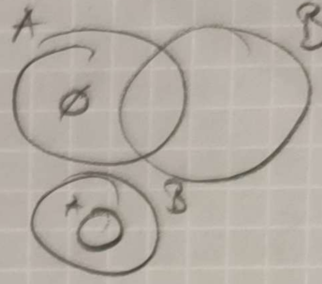
(dvs $\exists n : y = 2 \Rightarrow x = 4$)

 union $A \cup B \Rightarrow x \in A \cup B$ iff $x \in A$ or $x \in B$.

 intersection (meet) $A \cap B \Rightarrow x \in A \cap B$ iff $x \in A$ and $x \in B$.

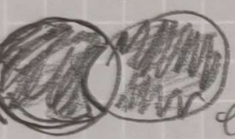
 Diffrence $A \setminus B \Rightarrow x \in A \setminus B$ iff $x \in A$ and $x \notin B$.
$(A - B)$.

kan även användas generellt: $A \subseteq B \Rightarrow$



Symmetric (symetrisk?) : $A \Delta B = A + B = A \oplus B$

$(x \in A \Delta B$ iff $x \in A \setminus B$ or $x \in B \setminus A)$

 eller $\left(A \Delta B = (A \setminus B) \cup (B \setminus A)\right)$

Att bevisa: - "Unpack the defenition"
↳ kolla om def gäller eller v.v.

Antal element $= \#(A) = |A|$

Note $\#(\{1, 1, 2\}) = 2$
$\#(\emptyset) = 0$
$\#(\{\emptyset\}) = 1$

$A_i = \{x \in \mathbb{N}, x \geq i\}$

$\bigcup_{i \in \mathbb{N}} A_i = \mathbb{N}$ $\left(\sum_{i=1}^{n} 2^i\right)$ Generalised Union

Power set: $\mathcal{P}(A) = 2^A = \{B : B \subseteq A\}$

Dvs $A = \{1, 2\}$

$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Inverse (converse) av funktion

$f^{-1} = \{(b, a) \in B \times A : f(a) = b\}$

Composition $f: B \to C$ $\quad g: A \to B$

$f \circ g(x) = f(g(x))$

$f \circ g = \{(a, c) \in A \times C : f(g(a)) = c\}$

<u>Closure of f on $X \subseteq A$</u>

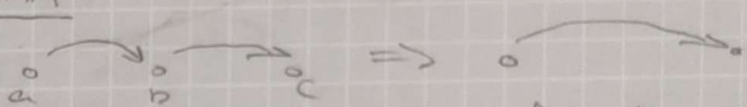
$X \subseteq f[X]$

$f(x) \subseteq f[x] \forall \quad y \subseteq f[x]$

Binary relations (Note, $a \to b$ betyder $(a, b) \in R$)

↳ <u>Regarding $R \subseteq A \times A$</u>

① <u>Transitivity</u> – $a \to b \to c \Rightarrow$



R is transitive over some $X \subseteq A$, if for all $a, b, c \in X$ it holds that if $(a, b) \in R$ & $(b, c) \in R$ then $(a, c) \in R$ (ex $>$)

② <u>Intransitive</u>

↳ $(a, b) \in R$ & $(b, c) \in R \Rightarrow (a, c) \notin R$

↳ (Ex mother of)

③ <u>Symmetri</u>

Ex sibling

	a	b	c
a	0	1	0
b	1	1	0
c	0	0	0

– R is symmetric over A iff for $a, b \in X$

↳ $(a, b) \in R \Rightarrow (b, a) \in R$

④ <u>Assymmetric</u>

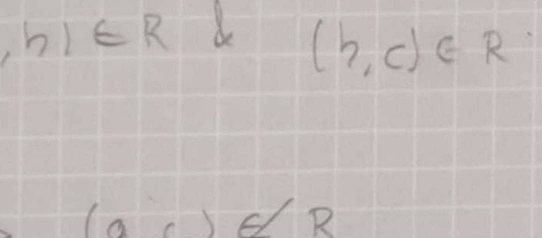
↳ $(a, b) \in R \Rightarrow (b, a) \notin R$

↳ Ex $>$

Symmetrisk:

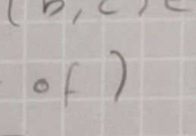
nollor för annars måste def för (i, i)

Assymmetric inte sense



⑤ <u>Antisymmetric</u>

Antisym



↳ R is ~~...~~ over X iff: $(a, b) \in R$ & $(b, a) \in R \Rightarrow a = b$

(6) Reflexive $\subseteq A$

R is reflexive over X iff holds:

↳ if $\forall a \in X \Rightarrow (a,a) \in R$

↳ Ex $\geq$, $=$

(7) Irreflexive

R irreflexive iff $\forall a \in X \Rightarrow (a,a) \notin R$

↳ Ex $>$, sibling of.

Note: $\emptyset$ är transitive, intransitive etc

Equivilent Class

$a \equiv b \Rightarrow$ a like mod b för att funktion = tillämpad.

$a = b \pmod{p}$

↳ $a \% p = b \% p$

$2 = 4 \pmod{2}$

1. ↳ Transsitive

↳ $a \equiv_d b$ & $b \equiv_d c \Rightarrow a \equiv_d c$

2. ↳ Symmetric

↳ $a \equiv_d b \Rightarrow b \equiv_d a$

3. ↳ Reflexive

↳ $a \equiv_d a$

Equivilent Classes satisfy all 3 properties

Bild

$\equiv_2 \subseteq \mathbb{N} \times \mathbb{N}$

{0 2 4 6 8 ..} {1 3 5 7 ...}

EC Delar upp setet i tvi klasser (delar) där alla element är lika

uppdelning

Partition of A is set of sets $\{B_i, i \in I\}$ where $\bigcup B_i = A$ and $B_i \cap B_j = \emptyset$ for $i \neq j$ $\Rightarrow B_{i \in N}$ innehåller alla, men i j är disjoint

↳ Direct correspondance between partition of A and equivilent class over A.

↳ DVS: Ex X är Jul, $\sim$ är "är jämna"

↳ Delar i tvi klasser, där alla är lika i klassen.

$X/\sim$ är de alla jämna tal.

$B_0 = \{ N\%=0 \}$
$B_1 = \{ \% = 1 \}$
$B_1 \cup B_0 = A$
$B_1 \cap B_0 = \emptyset$
$\{B_0, B_1\}$ is part of $\mathbb{N}$

Order

- $\leq$ ⇒ reflexive $a \leq a$
  ⇒ antisymmetric : $a \leq b$ & $b \leq a$ ⇒ $a = b$
  ⇒ transitive : $a \leq b$ & $b \leq c$ ⇒ $a \leq c$

- (Partial) Order iff satisfies ref, antisym, transitive

- A partial order $\preceq$ ← relation (generic name for (partial order)?) is a total order iff $\forall\, a, b \in A$
  $a \preceq b$ or $b \preceq a$

$(A, \preceq)$ partially orderd set (poset)

$\preceq \subseteq A \times A$ is partially ordered set.

$(P(A), \subseteq)$ is poset

$(\mathbb{N}, \leq)$ is poset

~~[illegible]~~ A strict (partial) order

R is strict partial order if it is transitive, irreflexive, assymmetric

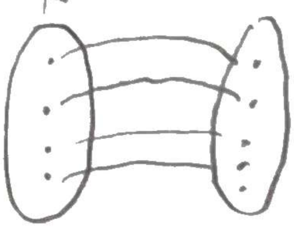
↳ Note en strict partial order är inte en partial order

↳ Ex: $<$, $\subset$

Properties of functions $f : A \to B$

Injective function (one-one func)

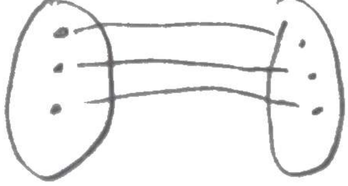
A



⇒ Alla mappar till olika värden av B

↳ implies $\#(B) \geq \#(A)$

Surjective (Onto)



No element left over in B (alla mappas)

↳ $\#B \leq \#A$

Bijective : Both injective & surjective

Injective Def $\Rightarrow \forall\, b \in B \Rightarrow$ at most one $a \in A$ s.t. $f(a) = b$

- $\forall\ a_1, a_2 \in A$ & $a_1 \neq a_2$
  $\Rightarrow f(a_1) \neq f(a_2)$

Surjective def

- $\forall\ b \in B$, exists $a \in A$ s.t. $f(a) = b$.
- $\text{range}(f) = b$.

Bijection

- f^{-1} is a function from B to A
  ↳ Om one-one inte stämmer är $B \to A$ inte funktion
  ↳ Samma om onto inte stämmer.

↳ Obs. Bra att räkna: Om f är bijective, och du vet $\#A$ så vet $\#B = \#A$.

A set A is infinite iff there exists $B \subset A$ s.t. is a bijection f between A and B.

Directed Graph

pair $G: (V, E)$
↳ V is set of vertices/notes and $E \subseteq V \times V$ is a relation, called a set of directed edges/arcs.

↳ Directed graph (har directions)

Undirected Graph (Tänk den som dubbel directed)

↳ a pair $G: (V, E)$
↳ G is a graph
↳ $E \subseteq V \times V$ is a symmetric relation, also called a set of undirected ~~eges~~ edges/ar
↳ Note: alltid symmetrisk
↳ Ex: Collaboration graph
Ex om undirected graph som:
↳ $E = \{(a, b), (b, c), (e, d)\}$
Menar dom egentligen $E \cup E^{-1}$ (så den blir symmetr

Symmetric closure of a relation $R \subseteq A \times A$

$R^{\leftrightarrow} = R \cup R^{-1}$
↳ a relation $\subseteq A \times A$.
↳ kalles closure för det är den minsta symmetriska relati av R.

Adjacensy matrix

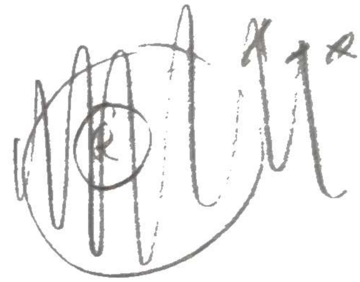
$R \subseteq A \times A$; $A = \{1, 2, 3, 4\}$

	1	2	3	4
1	0	1	1	1
2	0	0	0	1
3	1	1	0	1
4	1	0	0	1

=> representerar R.

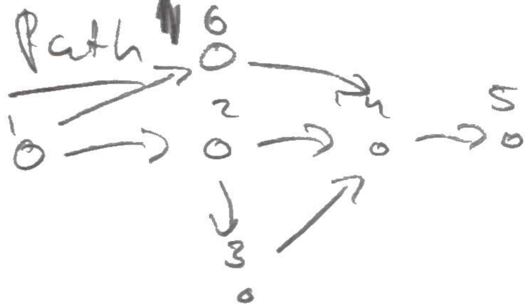
$(3,1) \in R$

$(1,1) \notin R$



Self loops: Pekar på sig själv.

Path



Given Graph $G = (V, E)$

$(1, 6, 4, 5)$ är en path.

Path, sequence of vertices:

$(v_0, v_1, v_2, \dots v_n)$, where $v_i \in V$ for All $i \in \{0, 1, 2, \dots, n\}$ and $(v_i, v_{i+1}) \in E$ for $i \in \{0, 1, \dots$

Dvs. $\Leftarrow$

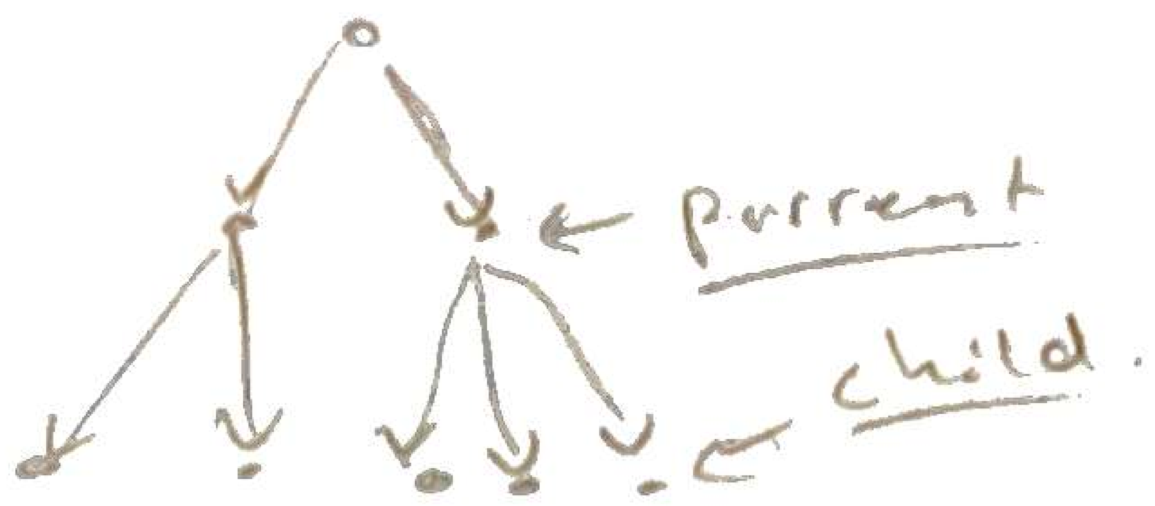
Du kan gå den vägen enligt grafen.

Cycle $\boxed{v_0 = v_n}$

Acycle graphs are graphs that don't have cycles.

Directed acycle graph (DAG)

Tree



Directed tree (aka rooted tree)

↳ Needs to be a DAG.

↳ Needs $\exists$ a root node $r \in V$ for all nodes $v \in V \setminus \{r\}$ there is a unique $u \in V$ s.t $(u,v) \in E$

many trees = forrests

for all $v \in V$ there is at most one $\mu \in V$ s.t $(\mu, v) \in E$.

Properties

- unique root (if $E \neq \emptyset$)
- $\#(v) = n \Rightarrow \#(\text{edges}) = n-1$ (Varje nod förutom root har exakt en förälder)

  vertices

Labeled tree

↳ rooted tree is a labelled tree, if were given a set L of labels, and a labeling funktion.

$\lambda : V \to L$ (vertices to label)

Ex. $V = \{c1, c2, c3\}$

$L = \{4, 5, 7.5\}$

Ex på funktionen $= a + a \cdot b$ dig förelrer.

Ordered tree

↳ A tree (V, E) withan a funktion $\mu: V \setminus \{r\} \to \mathbb{N}^+$

S.t ~~Rn~~ $\forall$ noode with n children, its children are labeked $1, 2, \ldots, n$.

Binary Tree

Given a rooted tree (V, E) with root r, we say it is a binary tree if every node has at most two children and there is a labellin funktion $\beta: V \setminus \{r\} \to \{left, r$

S.t no two children of the same node has same label

Binary Search Tree

Given BST (V, e) with root r, binary labels $B: V \setminus \{r\} \to$ left/ ringtht

a set of labels L, that are totaly ordered and labelling func $\lambda : V \to L$

BST iff $\forall$ roots their label to left i smaller than right.

Note:

- if $a, b \in L \Rightarrow a \leq b \; || \; b \leq a$
- får ha dubbletter : totally ordered?

Unrooted trees

↳ Undirected graph (V, E) that has no cycles

A spanning tree

↳ Given a connected graph $G = (V, E)$

↳ Spanning tree is a tree (V, F) s.t. $F \subseteq E$

↳ A tree that touches all vertices of a graph. (

$G = (V, E)$

A graph is connected iff $\forall\, u, v \in V$ there is a path from u to v.

(Note: A graph G

Closure (endofunction domain = endo codomain)

An endofunction $f: A \to A$ and a set $X \subseteq A$,

the closure of X under f $f[X]$ is defined as the smallest $Y \subseteq A$ s.t. $X \subseteq Y$ $f(Y) \subseteq Y$

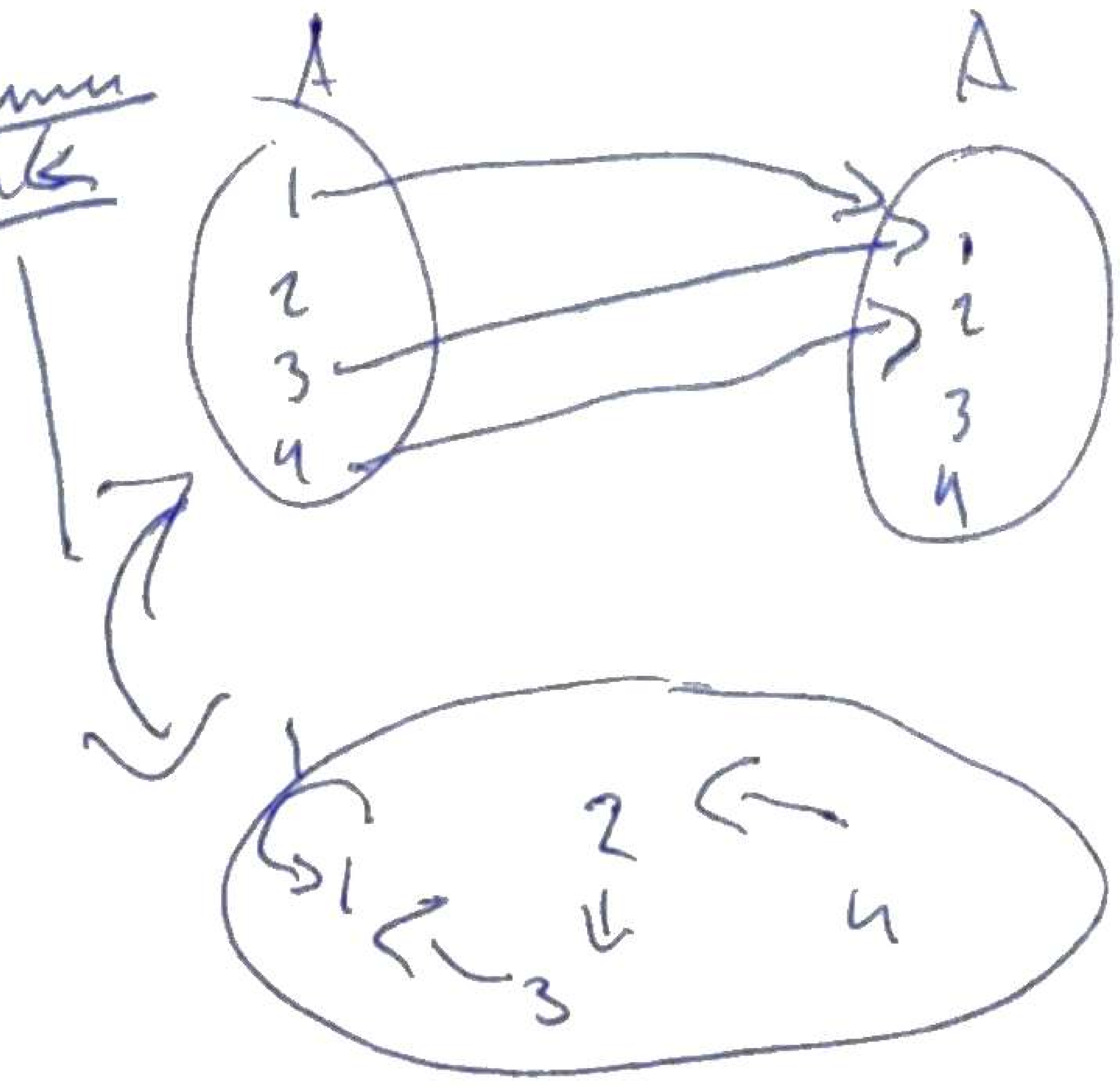
Construction

$Y_0 = X$

$Y_{n+1} = Y_n \cup f(Y_n)$

$\underbrace{f[X] = \bigcup_{i \in \mathbb{N}} Y_i}_{\text{Alla stället}}$

Sammen sats



$f[\{1\}] = 1$

$f[2] = \{1, 2, 3\}$

$f\{4\} = \{2, 3, 1\}$

Rekursion & Induction.

f(n)

rekursion definieras med sig själv
↳ har basfall

Induction

Show $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ ⊛

Base case $n=1$: $\sum_{i=1}^{1} = 1 = \frac{1(1+1)}{2}$ ok!

induction hypothesis.
↓ Suppose ⊛ holds for n. Want to prove that it holds for $n+1$. (Induction step)

Want to show $\sum_{i=1}^{n+1} = \frac{(n+1)(n+2)}{2}$

$$\sum_{i=1}^{n+1} = n+1+\sum_{i=1}^{n} \overset{I.H.}{=} n+1+\frac{n(n+1)}{2} = \frac{2(n+1)+n(n+1)}{2} = \frac{(n+2)(n+1)}{2}$$

Shows inductionstep holds.

Stark rekursion

$$f(n) = \begin{cases} 1 & ; n=0 \\ n \cdot f(n-1) & ; n>0 \end{cases} \quad \Bigg\| \quad f.n \mapsto \begin{cases} 1 & ; n=0 \\ n \cdot f(n-1) & ; n>0 \end{cases}$$

factorial som pairs

$\{(0,1), (1,1), (2,2), (3,6)\}$

$\Rightarrow f_0 = \{(0,1)\}$

$f_{n+1} = f_n \cup \{(n+1, (n+1)v) : v \in f_n(n)\}, n \geq 0$

note: set of value of n! Det = image of ~~position~~ n. Där finns n!

$f = \bigcup f_n.$

Well-founded sets $a_1, a_2 \in A \quad a_1 < a_2$?

A poset $(A, \leq)$ is well founded iff any non empty subset of A has a minimal element.

↳ a minimal element m of a set $X \subseteq A$ is s.t

$\nexists\, a \in X \quad a \not\leq m$

Ex: $\emptyset$ är för $P(\{1,2,3\})$

- $\{1\}\{2\}\{3\}$ är minimal för $P(\{1,2,3\}) \setminus \{\emptyset\}$ är inte well founded. för eken.
- eller ~~$\mathbb{R}$~~ är $(\mathbb{R}, <)$ är inte well founded för det finns alltid ett mindre tal.

$(P(n), \subset)$ inte well founded för det finns subsets som inte har ett minimal element.

↳ Ex: $\{\mathbb{N}, \mathbb{N}\setminus\{0\}, \mathbb{N}\setminus\{0,1\}, \dots\}$

↳ har inget minsta.

Note: Rekursion måste användas på wellfounded sets.

minimal element är inte den minsta, utan en endpoint. Det största elementet kan vara minimal.

När dom frågar om det är well-founded: Hur vet man. på vilken basis? Ges i poset def $(A, \leq)$
⇑ mindre än.

Dubbelingparametrar:

$(m, n) \leq (m', n')$ iff $m < m'$

or iff $m = m'$ & $n < n'$

Recap:

Simple induction:

$$\circledast \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \text{for } n \in \mathbb{N}^+$$

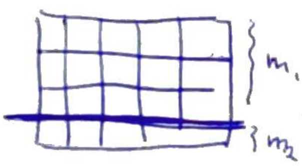
- Base case $n=1$
- Inductive step:
  - ↳ Induction hypothesis: $\circledast$ holds for n. $\| \; \frac{n-1}{n}$:
  - ↳ Prove $\circledast$ holds for $n+1$.

Jobbigare exempel.

Fibonaccitalen. Tänk hur det definieras sen.

Visa $f(n) = f(n-1) + f(n-2)$

Strong induction. (Needs to hold



$m \times n$ unit squares.

How many times do we need to break the bar?

↳ To get unit squares.

$\boxed{\text{Prove } f(m,n) = m \cdot n - 1 \text{ times}}$

Base case $m = n = 1 \Rightarrow f(m,n) = 0$

Induction step. $m = m_1 + m_2$

Induction hyp $f(m',n') = m' \cdot n' - 1$.

$$\underset{\text{optimal}}{f(m,n)} = 1 + f(m_1, n) + f(m_2, n)$$

$$\overset{IH}{=} 1 + (m_1 \cdot n - 1) + (m_2 \cdot n - 1) = -1 + (m_1 + m_2)n = m \cdot n - 1$$

Stämmer!

f is optimal breaking

Structural Induction

$R = \{R_1, R_2, R_3\}$

↳ Base case

↳ Inductionstep: Show that it is perserved through all rules.

Det likk för S_1 & S_2. & nästa steg är

"(" S_1 "+" S_2 ")"

antar att
R_2 S_1 har likadan mängd
öppning & slingde parenthesis
↳ tydligt att även nästa
steg har det
↳ Måste gälla för alla regler

prove Inequalities (with induction)

1) $n < 2^n$; $n \in \mathbb{N}$ Basecase $n = 0$ $0 < 2^0$

Inductionstep step: if $n < 2^n$

show $n+1 < 2^{n+1}$

2) Note: att bevisa något starkare kan vara lättare.

typ: lättare bevisa $\sum_{k=1}^{n} \frac{1}{k^2} < 2 - \frac{1}{n}$ än $\sum_{k=1}^{n} \frac{1}{k^2} < 2$.

Direct proves

1. prove that if x is odd, then x^2 is odd
2. Direct prove by cases.

prove that $1+(-1)^n(2n-1)$ is a multiple by four ; $n \in \mathbb{N}^{+}$

↳ lättast att consider n even or n odd.

Proof by contrapositive

Show $A \to B$. $\iff$ show $\neg B \to \neg A$

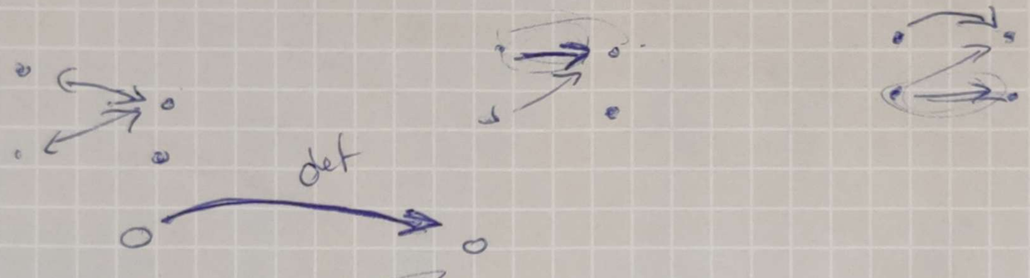
Prove x^2 even $\to$ x even.

$x^2 = 2k \Rightarrow x = \sqrt{2k}$ ... ?

easier:

x odd

if x $\neg$even $\to$ x^2 $\neg$even



$\neg$ o ⟶ o. $\neg$ must come from $\neg$

Proof by contradiction.

prove $\sqrt{2}$ is ~~an~~ irrational.

Suppose $\sqrt{2}$ is rational. $\Rightarrow \underbrace{\sqrt{2} = \frac{a}{b}}$; $a, b \in \mathbb{Z}$

↳ most simplified (no common divisor)

$2 = \frac{a^2}{b^2} \Rightarrow 2b^2 = a^2$ ~~(can't both be even)~~

↳ a^2 is even. $\Rightarrow a$ is even.

i.e $a = 2k$ ⟶ not most simplified

$2b^2 = (2k)^2 = 4k^2 \Rightarrow b^2 = 2k^2 \Rightarrow b = 2m$ → ↳ motsigelse.

want to prove P

Suppose $\neg P$ is true

want to prove Q is true

& Q is false → $\neg P$ is false. hypothesis breaks.