Assignment #2

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1 Phys 581 Winter 2019

2 Assignment #2: Geomagnetic Bottle

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```
In [1]: #Import useful libraries
        import numpy as np
        import matplotlib.pyplot as plt
        from mpl_toolkits.mplot3d import Axes3D
        import scipy.integrate
        %matplotlib inline
```

2.1.1 Introduction

The Earth's magnetic field, also known as the geomagnetic field, is wildly complex and is of great interest to space physicists. In particular, understanding how charged particles interact with the geomagnetic field is key to understanding how the Earth interacts with the solar wind.

To first order, the geomagnetic field can be approximated as a magnetic dipole. The magnetic field of a dipole in standard spherical coordinates is given by:

$$\vec{B}\left(\vec{r}\right) = \frac{\mu_0}{4\pi}\frac{M_E}{r^3}\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right)$$

where $M_E = 8.05 \times 10^{22}$ Am^2 is the magnetic dipole moment of the earth, and $\mu_0 = 4\pi \times 10^{-7}$ Tm/A is the permeability of free space. We will measure all distances in units of the radius of earth $R_E = 6371$ km. Define a base unit of magnetic flux density

$$B_0 := \frac{\mu_0}{4\pi}\frac{M_E}{R_E^3} = 3.113 \times 10^{-5}\ \mathrm{T}$$

then we can rewrite the Earth's magnetic field as

$$\vec{B} = B_0\left(\frac{R_E}{r}\right)^3\left(2\cos\theta\hat{r} + \sin\theta\hat{\theta}\right).$$

Using the standard spherical to cartesian transformation, we have

$$r = \sqrt{x^2 + y^2 + z^2}, \qquad \cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}, \qquad \sin\theta = \sqrt{\frac{x^2 + y^2}{x^2 + y^2 + z^2}} \tag{1}$$

$$\hat{r} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left(x\hat{x} + y\hat{y} + z\hat{z} \right), \qquad \hat{\theta} = \frac{1}{\sqrt{x^2 + y^2 + z^2}} \left(x\hat{x} + y\hat{y} + z\hat{z} \right) \tag{2}$$

$$\tag{3}$$

so the dipole magnetic field in cartesian coordinates is

$$\vec{B} = B_0 \frac{R_E^3}{(x^2 + y^2 + z^2)^{5/2}} \left[3xz\hat{x} + 3yz\hat{y} + \left(2z^2 - x^2 - y^2 \right) \hat{z} \right].$$

A particle of charge q will then experience a force governed by the Lorentz force law

$$\vec{F} = q\vec{v} \times \vec{B}$$

In this notebook, we will consider the motion of a proton ($q = e$) through the Earth's magnetic field. In this case, the equations of motion are given by

$$\vec{a} = \frac{e}{m} \vec{v} \times \vec{B}.$$

The physical parameters of interest are therefore the radius of the earth R_E, the magnetic flux density B_0, and the charge and mass of a proton $e = 1.60217662 \times 10^{-19}$ C, $m = 1.6726219 \times 10^{-27}$ kg. Since the physical parameters at play vary greatly in order of magnitude, it is a good idea to make the equations of motion dimensionless. These dimensionless equations are readily obtained by noting that

$$t_0 := \frac{m}{eB_0} = 3.353577 \times 10^{-4} \text{ s}$$

has units of time, so I will take this to be the characteristic time scale. Defining a dimensionless time $\tau := t/t_0$, position $\vec{\rho} := \vec{r}/R_E$, magnetic field $\vec{\beta} := \vec{B}/B_0$, and velocity

$$\vec{\nu} = \frac{d\vec{\rho}}{d\tau} = \frac{t_0}{R_E} \vec{v}$$

the equations of motion simplify to

$$\frac{d\vec{\nu}}{d\tau} = \vec{\nu} \times \vec{\beta}$$

where it is understood that lengths are now measured in units of R_E, and time in units of t_0.

Task: Read https://www2.mps.mpg.de/solar-system-school/lectures/space_plasma_physics_2007/Lectur and then numerically integrate the motion of a $T = 1$ keV proton starting at the surface of the Earth ($r = 6371$km) with latitude 66° moving antiparallel to the local $\vec{B}$.

We begin by defining the parameters relevant to this problem. Given an initial kinetic energy of T, the initial speed will be $v_0 = \sqrt{2T/m}$, so the initial dimensionless speed will be

$$\nu_0 = \frac{t_0}{R_E} v_0 = \frac{t_0}{R_E} \sqrt{2T/m} = 2.3039 \times 10^{-5}$$

```
In [2]: v0 = 2.3039e-5 #Initial dimensionless speed
        Lat = 66 #Initial latitude in degrees
```

Next, several functions are defined to compute the initial conditions in spherical coordinates, as well as to define the system of differential equations to be integrated. Although the equations of motion as written are second order in position, they can be readily converted to a system of twice as many first order equations by rewriting them as

$$\frac{d\vec{v}}{d\tau} = \vec{v} \times \vec{\beta} \tag{4}$$

$$\frac{d\vec{\rho}}{d\tau} = \vec{v}. \tag{5}$$

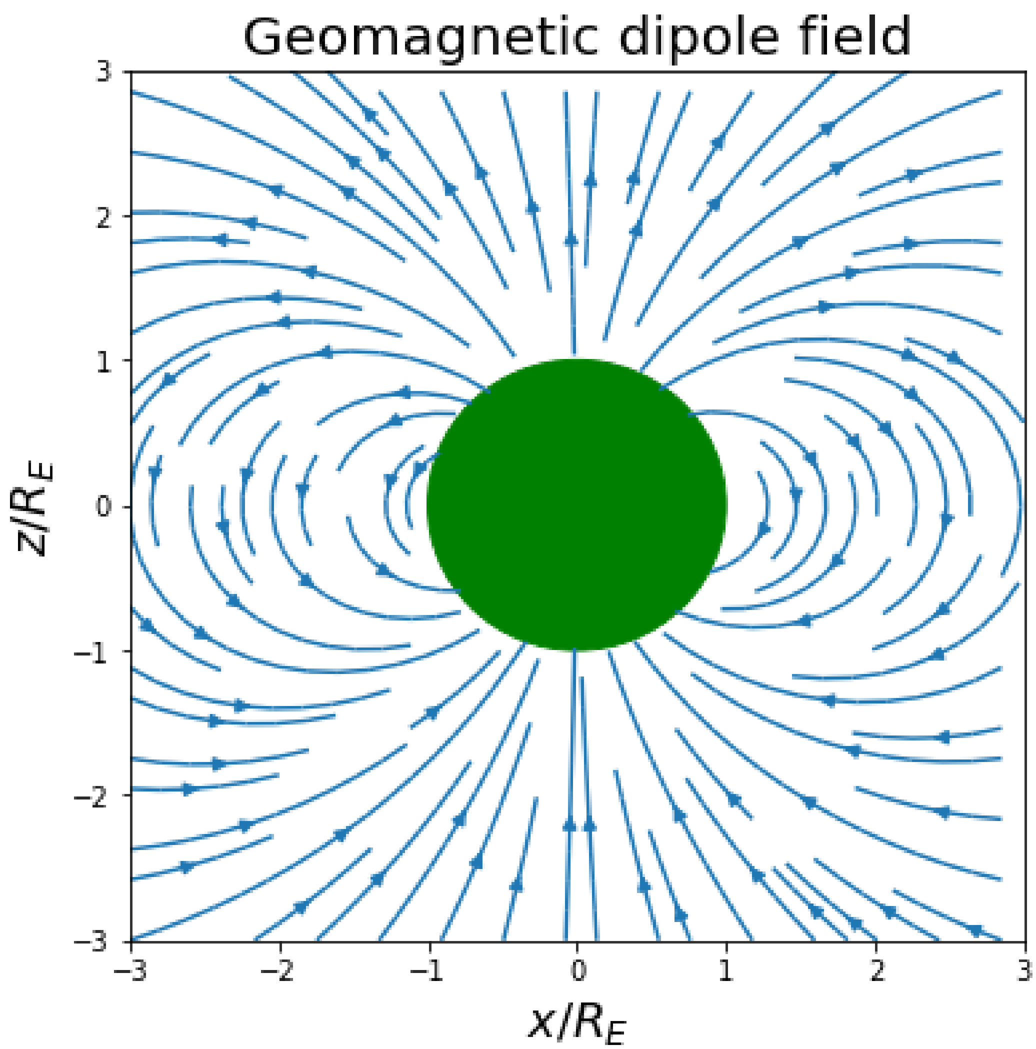
```
In [3]: def B(x,y,z):
            '''
            This function defines the geomagnetic field vector in cartesian
            coordinates using the dipole approximation.

            Args:
                x: float, x component of position
                y: float, y component of position
                z: float, z component of position

            Return:
                B: array, Magnetic field components in the form [Bx,By,Bz].
            '''

            #Common factor to each component, no need to compute 3 times
            scale = scale = 1/(x*x+y*y+z*z)**(5/2)

            #Define each component
            Bx = 3.0*x*z
            By = 3.0*y*z
            Bz = 2.0*z*z-x*x-y*y

            return np.array([Bx,By,Bz])*scale


        def sphere_to_cart(rtp):
            '''
            This function converts an array of standard spherical coordinates to
            cartesian coordinates.

            Args:
                rtp: Array of the form [r,theta,phi] with theta the polar angle.

            Return:
                xyz: Array of cartesian coordinates [x,y,z]
```

```
    '''

    xyz = np.zeros(rtp.shape)

    xyz[0] = rtp[0]*np.sin(rtp[1])*np.cos(rtp[2])
    xyz[1] = rtp[0]*np.sin(rtp[1])*np.sin(rtp[2])
    xyz[2] = rtp[0]*np.cos(rtp[1])

    return xyz


def time_evol(t,state):
    '''
    This function defines the time evolution of a proton in the geomagnetic
    dipole field.

    Args:
        t: Time to evaluate system
        state: Input state of the form [vx,vy,vx,x,y,z]

    Return:
        newstate: Time derivative of [vx,vy,vx,x,y,z]
    '''

    #Create output array
    newstate = np.zeros(state.shape)

    #Unpack input state
    x,y,z = state[3:]

    #Define time derivative of each component
    newstate[:3] = np.cross(state[:3],B(x,y,z))
    newstate[3:] = state[:3]

    return newstate
```

The dipole approximation of the geomagnetic field can be nicely visualized using a streamplot.

In [4]:
```
#Compute magnetic field vectors in the xz plane
x, y, z = np.linspace(-3,3,100), 0.0 , np.linspace(-3,3,100)
xx,zz = np.meshgrid(x,z)
Bx, By, Bz = B(xx,y,zz)

#Mask to delete points within the earth
mask = xx**2+zz**2 > 1

#Plot dipole field as streamplot
fig, ax = plt.subplots(figsize = (6,6))
```

```
ax.streamplot(x, z, Bx*mask, Bz*mask)
ax.set_xlabel('$x/R_E$',fontsize=18)
ax.set_ylabel('$z/R_E$',fontsize=18)
ax.set_title('Geomagnetic dipole field',fontsize=20)

#Plot a circle to represent the Earth
circ = plt.Circle((0, 0), radius=1, edgecolor='green', facecolor='green')
ax.add_patch(circ)

plt.show()
```



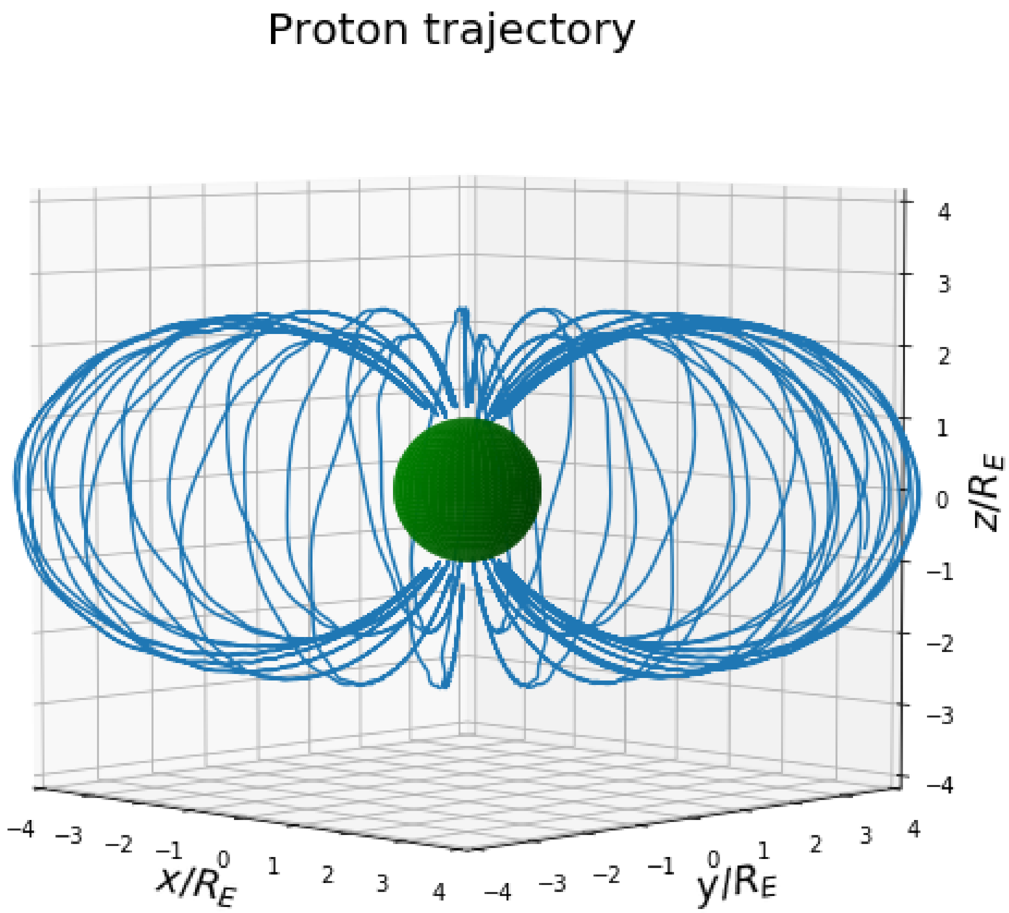
We see that the magnetic field function reproduces what we would expect of a perfect dipole. The starting point of the proton is given as being at the surface of the earth with a latitude of 66°, which corresponds to a polar angle of $\theta = 90° - 66° = 24°$ in standard spherical coordinates. Additionally, we know the initial velocity is antiparallel to the local magnetic field, so $\vec{v} = -v_0\vec{B}/|\vec{B}|$.

In [5]:
```python
#Initialize state vector
state0 = np.zeros(6)

#Convert initial conditions to cartesian coordinates (phi is arbitrary)
r,theta,phi = 1.0, (90-Lat)*np.pi/180, 0.0
x0,y0,z0 = sphere_to_cart(np.array([r,theta,phi]))

#Compute local magnetic field and initial velocity
B0vec = B(x0,y0,z0)
v0vec = -v0*B0vec/np.linalg.norm(B0vec)

#Define inital state
state0[:3], state0[3:] = v0vec, [x0, y0, z0]
```

Next we simply integrate the equations of motion to find the trajectory. I will use the scipy.integrate.solve_ivp function, which by default uses the RK45 method to dynamically compute time steps to stay within some tolerance.

In [6]:
```python
#Define time interval and integrate
t_span = (0,5e5)
dat = scipy.integrate.solve_ivp(time_evol,t_span,state0,rtol=1e-13)

#Extract time, position and velocity arrays
time = dat.t
[x,y,z] = dat.y[3:]
[vx,vy,vz] = dat.y[:3]
```

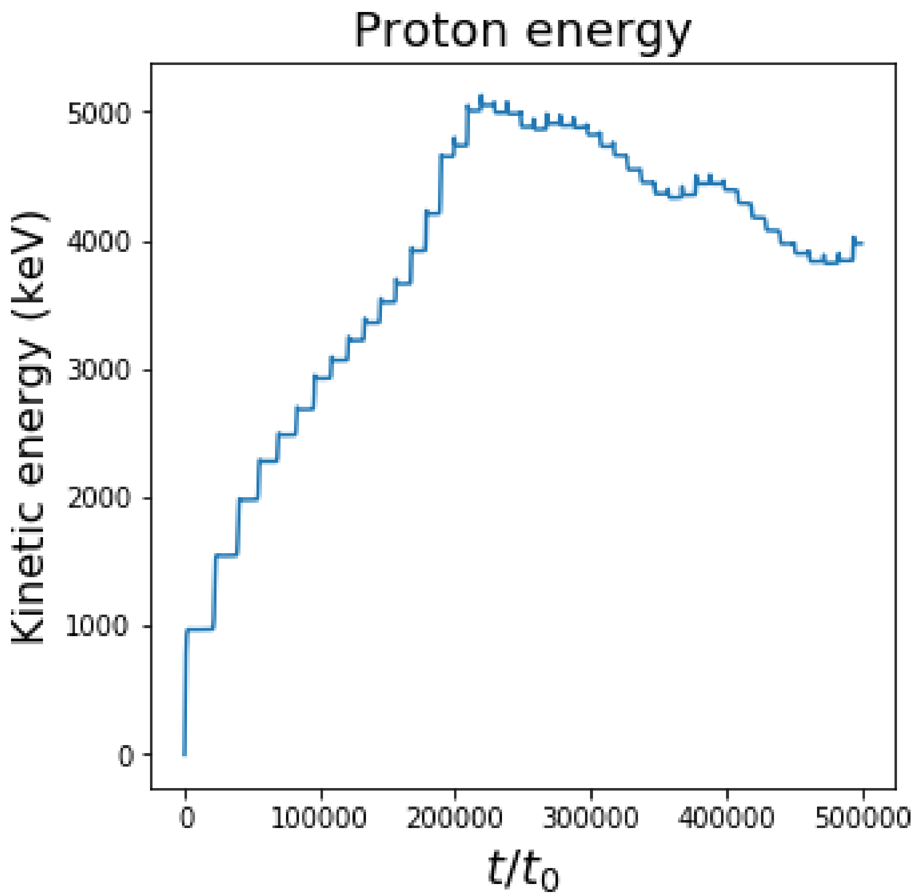
In [7]:
```python
#Plot trajectory in 3D
fig = plt.figure(figsize=(9,9))
ax = fig.gca(projection='3d')
ax.plot3D(x,y,z)

ax.set_xlim(-4, 4)
ax.set_ylim(-4, 4)
ax.set_zlim(-4, 4)
ax.set_xlabel('$x/R_E$',fontsize=18)
ax.set_ylabel('$y/R_E$',fontsize=18)
ax.set_zlabel('$z/R_E$',fontsize=18)
ax.set_title('Proton trajectory',fontsize=20)
ax.view_init(5,-45)

#Overlay unit sphere to represent earth
u, v = np.linspace(0, 2 * np.pi, 100), np.linspace(0, np.pi, 100)
xc = np.outer(np.cos(u), np.sin(v))
yc = np.outer(np.sin(u), np.sin(v))
zc = np.outer(np.ones(np.size(u)), np.cos(v))
ax.plot_surface(xc, yc, zc, color='g')

plt.show()
```



Note that the trajectories seem to be spaced unevenly, and some seem to be quite rigid. Additionally, it seems that the distance from the earth at which the bounce occurs (i.e. velocity flip) is non-uniform. To get an idea of what might be happening, let's look at how the energy changes with time.

```
In [8]: #Relative Kinetic energy
        Tlist = (vx**2+vy**2+vz**2)/(vx[0]**2+vy[0]**2+vz[0]**2)

        #Plot
        plt.figure(figsize=(5,5))
        plt.plot(dat.t,Tlist)
        plt.xlabel('$t/t_0$',fontsize = 18)
        plt.ylabel('Kinetic energy (keV)',fontsize = 16)
        plt.title('Proton energy',fontsize = 18)
```

```
plt.show()
```

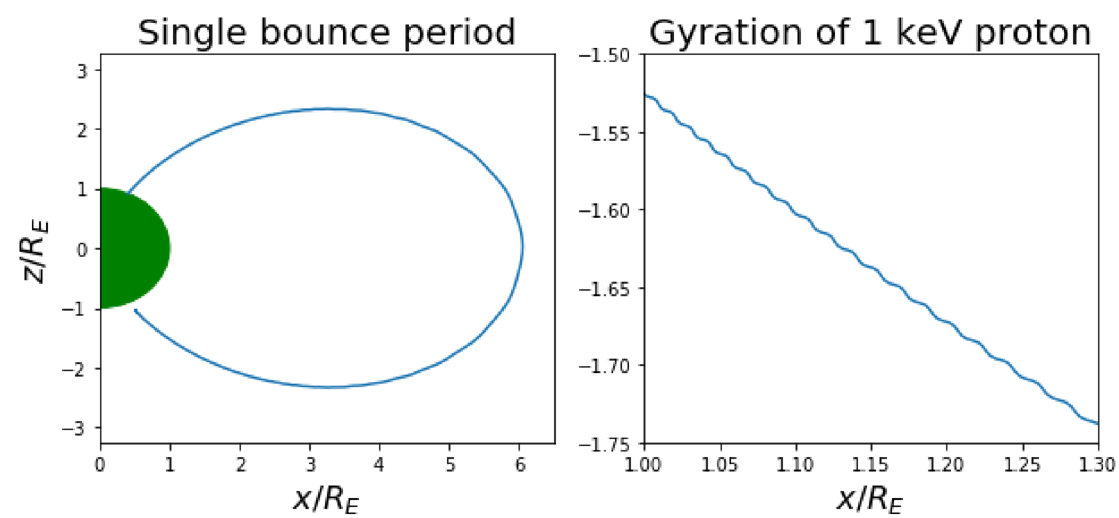


As we see, the kinetic energy fluctuates enormously with time, at some points getting up to 5000 times larger than the initial condition. This is a huge problem, as magnetic forces cannot do any work on a point charge, so the kinetic energy should in principle remain uniform. It seems that the kinetic energy experiences a rapid change each time the particle bounces, and since there is no mechanism within our model that allows for this, it can be attributed precision issues in the numerical integration. I suspect that such an issue would be improved by allowing for arbitrarily small time steps in the numerical integration. For the rest of the notebook, I will consider the motion prior to the first bounce, as this part of the data will best model the true nature of magnetic forces.

Task: Does you result look like the plot in slide#7? If not, what values (eg. energy, pitch angle, latitude) produce a better resemblance?

As expected, we see that the long term trajectory includes the bounce motion, as well as the drift motion. Contrary to the diagram in the resource provided, it is not clear if there are gyrations in the trajectory. To see this, I will zoom in on the trajectory over the first bounce period.

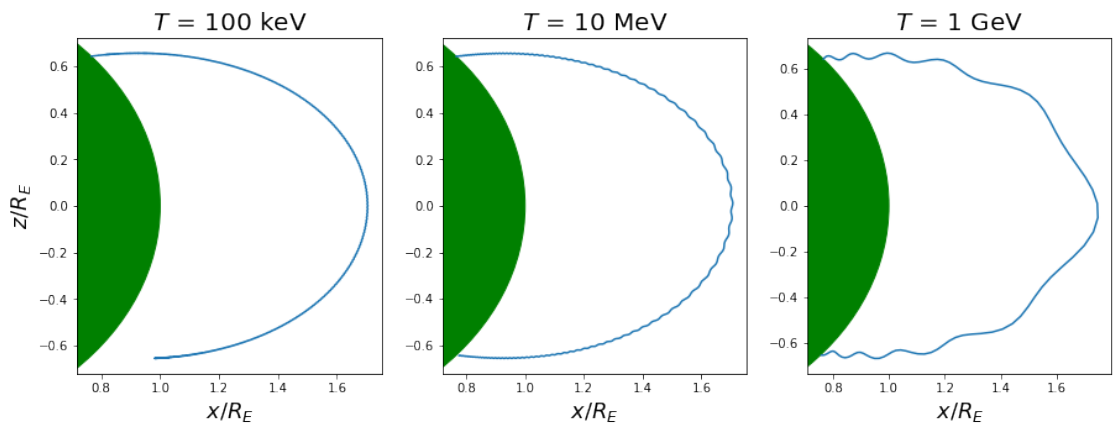
```
In [9]: fig, ax = plt.subplots(1, 2, figsize=(10,4))

        #Plot first bounce period
        ax[0].plot(x[:3900],z[:3900])
        ax[0].set_xlim(0,6.5)
        ax[0].set_ylim(-3.25,3.25)
        ax[0].set_xlabel('$x/R_E$',fontsize=18)
        ax[0].set_ylabel('$z/R_E$',fontsize=18)
        ax[0].set_title('Single bounce period',fontsize=20)
        circ = plt.Circle((0, 0), radius=1, edgecolor='green', facecolor='green')
        ax[0].add_patch(circ)

        #Zoom in to small region of trajectory
        ax[1].plot(x[3000:3150],z[3000:3150])
        ax[1].set_xlim(1,1.3)
        ax[1].set_ylim(-1.75,-1.5)
        ax[1].set_xlabel('$x/R_E$',fontsize=18)
        ax[1].set_title('Gyration of 1 keV proton',fontsize=20)

        plt.show()
```



Zooming in, we see that the integrator was precise enough to pick up these gyrations. These gyrations however, are so small compared to the trajectory as a whole that they are not visible when plotting the entire curve. In order to reproduce the plot in slide 7, we need to vary the initial conditions. Recall that for a uniform magnetic field B, a proton will undergo uniform circular motion with a radius $r = mv/eB \sin\Theta$ where Θ is the angle between the magnetic field and the velocity vector. Although the field we're considering is not uniform, this formula gives us some insight into how we can increase the radius of gyration, namely, increase the initial speed. Additionally, although the gyration radius will decrease with an increased pitch angle, this will result in a trajectory which doesn't go as far in the x-direction, thus making the gyrations more pronounced relative to the rest of the path. Finally, decreasing the latitude will have the same effect (i.e. shorter trajectory), so starting closer to the equator should also make the gyrations more pronounced. In this case, I set the latitude to 40° and increase the pitch angle by first setting the velocity to be parallel to the local field, and subsequently increase the z-component by a factor of 5. Note that the kinetic energy is proportional to the square of the speed $T \propto v^2$ so increasing the speed by a factor of k will increase the initial energy of $T = 1$ keV by a factor of k^2.

In [10]:
```
#Rescale initial speed by various factors
scl = [1e1,1e2,1e3]

#Corresponding kinetic energy
lbl = ['100 keV','10 MeV','1 GeV']

#Set varying time bounds
#(more speed = shorter bounce period)
tbnd = [6500,1230,126]

#Initialize state vector
s0 = np.zeros(6)

#Convert initial conditions to cartesian coordinates (phi is arbitrary)
#Latitude set to 40 degrees.
r,theta,phi = 1.0, (90-40)*np.pi/180, 0.0
x0,y0,z0 = sphere_to_cart(np.array([r,theta,phi]))

#Compute local magnetic field, rescale z-component by a factor of 5
Bvec = B(x0,y0,z0)
Bvec[2] = 5*Bvec[2]

#Plot trajectory of various initial speeds
fig, ax = plt.subplots(1,3,figsize=(15,5))
ax[0].set_ylabel('$z/R_E$',fontsize=18)

for j in range(len(scl)):

    #Compute rescaled initial velocity
    vvec = scl[j]*v0*B0vec/np.linalg.norm(B0vec)

    #Define inital state
    s0[:3], s0[3:] = vvec, [x0, y0, z0]

    #Integrate through trajectory
    dat_scl = scipy.integrate.solve_ivp(time_evol,(0,tbnd[j]),s0,rtol=1e-13)
    xs, ys, zs = dat_scl.y[3:]

    ax[j].plot(xs,zs)
    ax[j].set_xlabel('$x/R_E$',fontsize=18)
    ax[j].set_title('$T$ = '+lbl[j],fontsize=20)
    circ = plt.Circle((0, 0), radius=1, edgecolor='green', facecolor='green')
```

```
        ax[j].add_patch(circ)

    plt.show()
```



We see that at a latitude of 40°, an energy of 1 GeV is required to produce gyrations on the order of the ones seen in slide 7. For a proton however, this energy corresponds (classically) to an initial speed of

$$v_0 = \sqrt{\frac{2T}{m}} = \sqrt{\frac{2 \times 1.602 \times 10^{-10}\ \mathrm{J}}{1.67262 \times 10^{-27}\ \mathrm{kg}}} = 4.377 \times 10^8\ \mathrm{m/s} = 1.46c$$

almost 1.5 times larger than the speed of light. These kinds of gyrations are therefore unphysical, and I can conclude that they are greatly exaggerated on slide #7 to illustrate that gyrations are in fact occurring.

Conclusion In this notebook, I examined the trajectory of a proton in the geomagnetic field. The field was modeled using a dipole approximation, and each of the expected motions (i.e. bounce, drift & gyrations) were observed by numerically integrating the Lorentz force law. The path of a 1 keV proton starting at 66° latitude was traced out, and we saw that the numerics led to a noticeable energy increase at each bounce. Additionally, we saw that these initial conditions did not reproduce the well-pronounce gyrations as shown in slide #7 of the reference, meaning that the gyrations are generally much smaller in magnitude than the drift and bounce motions. We saw that an energy of approx. 1 GeV was required to reproduce such large gyrations, but this corresponds to a velocity of almost 1.5c, which is unphysical. At such energies, we would need to treat the proton relativistically in order to get any sort of insight into its true motion.

With more time available, I would be very interested in tracking down the source of rapid energy growth at each of the bounces, and how one could remedy this problem. One way would possibly be to explicitly enforce the speed of the particle to be constant by rescaling the velocity vector at each timestep of the integration process. Furthermore, it wouldn't be too much work to incorporate relativistic effects, which would allow one to properly deal with the high-energy cases.