Exercise 21

- determine the transformation property of $\mathcal{A}_\mu := A_\mu^a T^a$ from

$$D_\mu \longrightarrow D_\mu' = U D_\mu U^\dagger \quad ; \quad D_\mu = \partial_\mu + ig A_\mu^a T^a = \partial_\mu + ig \mathcal{A}_\mu$$

$$U = U(\underline{\theta}) = \exp\{-ig\,\theta^a T^a\}$$

- show that for infinitesimal gauge transformations $\delta\underline{\theta}$ the gauge fields transform as

$$\delta A_\mu^a = g\, c^{abc}\, \delta\theta^b A_\mu^c + \partial_\mu(\delta\theta^a)$$

$[T^a, T^b] = i\, c^{abc}\, T^c$

Exercise 22

- Let $y_1, \dots y_N, y_1^*, \dots y_N^*$ be the 2N generators of a Grassmann alg.

$\hookrightarrow$ $\{y_i, y_j\} = \{y_i, y_j^*\} = \{y_i^*, y_j^*\} = 0$

$\Rightarrow (y_i)^2 = (y_i^*)^2 = 0 \quad \Rightarrow$ only function $f(y) = a + b\,y$ $\quad \in \mathbb{C}$

$\hookrightarrow$ derivatives:

$$\frac{\partial}{\partial y_i} y_k = \delta_{ik} = \frac{\partial}{\partial y_i^*} y_k^* \;, \quad \frac{\partial}{\partial y_i} y_k^* = 0 = \frac{\partial}{\partial y_i^*} y_k$$

$\hookrightarrow$ "integration" equiv. to derivative

$$\int dy_i = 0 = \int dy_k^* \;, \quad \int dy_k\, y_i = \delta_{ik} = \int dy_k^*\, y_i \;, \quad \int dy_i\, y_k^* = \int dy_k^*\, y_i = 0$$

- show the substitution rule: (A = $N \times N$ regular matrix)

$$z_i = A_{ij} y_j \quad \Rightarrow \quad dz_i = (A^{-1})_{ji}\, dz_j$$

- show $\displaystyle \int dz_1 \dots dz_N\, f(\underline{z}) = \int dy_1 \dots dy_N\, f(z(y))\, [\det A]^{-1}$

- Finally, show: $\displaystyle \int dy_1 \dots \int dy_N \int dy_N^* \dots \int dy_1^* \; \exp\{y_i^* A_{ij} y_j\} = \det A$

Exercise 23

- derive the Ward identity for $k^\mu G_\mu^{A\psi\bar\psi}(k,p,p')$ using BRS symmetry

$$\delta_{BRS} A_\mu = \delta\bar\lambda\ [\partial_\mu u]$$

$$\delta_{BRS}\,\psi = \delta\bar\lambda\ [-ieQu\psi]\ , \qquad \delta_{BRS}\,\bar\psi = \delta\bar\lambda\ [+ieQu\bar\psi]$$

$$\delta_{BRS}\,\bar u = \delta\bar\lambda\ [-\tfrac{1}{\xi}\partial^\mu A_\mu]\ , \qquad \delta_{BRS}\, u = 0$$

① consider $\quad 0 \overset{!}{=} \delta_{BRS}\langle T\ \bar u(x)\,\psi(y)\,\bar\psi(z)\rangle$

② exploit the fact that the FP ghosts decouple:

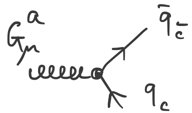
$$\langle T\ u(y_1)\,\bar u(y_2)\ \Phi_1(x_1)\ldots\Phi_n(x_n)\rangle = \underbrace{\langle T\,u(y_1)\,\bar u(y_2)\rangle}_{=i\Delta_F(y_1-y_2) = \int\frac{d^4k}{(2\pi)^4}\, e^{-ik(y_1-y_2)}\frac{i}{k^2}}\langle T\,\Phi_1(x_1)\ldots\Phi_n(x_n)\rangle$$

③ go into momentum space using

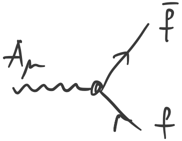
$$\int d^4x\, d^4y\, d^4z\ \ e^{-ikx-ipy-ip'z}\ \ (\ \ldots\ )$$

Exercise 24

- compare the vertices in QCD vs QED



$-ig_s T^a_{\bar{c}c} \gamma_\mu$ vs.

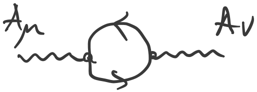


$-ie Q_f \gamma_\mu$

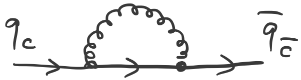
to derive conversion rules (looking at the colour structure) between:

(a)

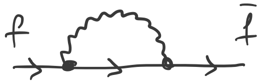
vs.



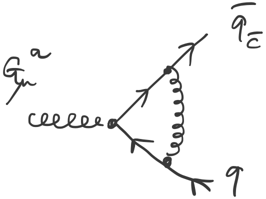
(b)



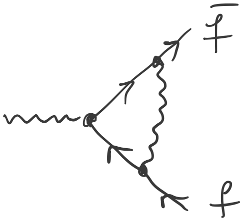
vs.



(c)



vs.



Exercise 25

- calculate the 1-loop beta function

① consider $g_{s,0} = \mu^{\epsilon} g_s Z_g$ with $Z_g = 1 + \delta Z_g$, $\delta Z_g = \frac{\alpha_s}{4\pi} \Delta \underbrace{\left(\frac{2}{3} N_f T_F - \frac{11}{6} C_A\right)}_{=: -\beta_0}$

and we obtain

$$\alpha_{s,0} = \mu^{2\epsilon} \alpha_s \left[1 - \frac{\alpha_s}{2\pi} \Delta \beta_0 + \dots\right]$$

② use the fact that the bare coupling is scale-indep $\left(\mu \frac{d\alpha_{s,0}}{d\mu} = 0\right)$ to obtain an equation for

$$\mu \frac{d\alpha_s}{d\mu} = \dots$$

③ take the limit $D \to 4$ $(\epsilon \to 0)$