III. Yang-Mills Theory & QCD

QED from Gauge Symmetry

- the free Dirac theory $\mathcal{L}_\psi(\psi, \partial_\mu\psi)$ is invariant under global $U(1)$

$$\psi(x) \longrightarrow \psi'(x) = e^{-ieQ\theta}\psi(x)$$

- let's try to impose it as a local $(\theta = \theta(x))$ symmetry:

<u>problem</u>: $\partial_\mu \psi \longrightarrow \partial_\mu \psi' = \partial_\mu\left(e^{-ieQ\theta}\psi\right) = e^{-ieQ\theta}\left(\partial_\mu - ieQ\,(\partial_\mu\theta)\right)\psi \neq e^{-ieQ\theta}\partial_\mu\psi$

- solution: introduce the covariant derivative $D_\mu + ieQA_\mu(x)$ (minimal substitution)

$$\Rightarrow \quad D_\mu\psi \longrightarrow D'_\mu\psi' = e^{-ieQ\theta} D_\mu\psi \,, \qquad A_\mu \longrightarrow A'_\mu = A_\mu + (\partial_\mu\theta)$$

$\Rightarrow$ we generate interactions $\sim A_\mu \bar{\psi}\gamma^\mu\psi$ by imposing <u>local</u> $U(1)$!

- make the gauge field a dynamic quantity:

$$\mathcal{L}_{gauge} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \left(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu\right)$$

$\Rightarrow$ principle of minimal gauge-inv. coupling to a dynamic gauge field

$$\mathcal{L}_{QED} = \mathcal{L}_\psi(\psi, D_\mu\psi) + \mathcal{L}_{gauge}$$

Yang – Mills Theory

- apply the principle of gauge symmetry to (non-abelian) symmetries

① field multiplet $\vec{\Phi} = (\Phi_1, \ldots, \Phi_n)^T$ & Lagrangian $\mathcal{L}_\Phi(\vec{\Phi}, \partial_\mu \vec{\Phi})$, invariant under global transformations of group G:

$$\vec{\Phi} \longrightarrow \vec{\Phi}' = U(\underline{\theta})\, \vec{\Phi} \,, \qquad U(\underline{\theta}) = \exp\left\{-i g T^a \theta^a\right\}$$

generators of the group, $a = 1, \ldots, \dim G$

arbitrary gauge coupling

structure constants: $[T^a, T^b] = i\, c^{abc}\, T^c$

normalization: $\mathrm{Tr}(T^a T^b) = \frac{1}{2} \delta^{ab}$

② make the symmetry local

$$D_\mu = \partial_\mu + i g \underbrace{A^a_\mu T^a}_{} =: \partial_\mu + i g \mathcal{A}_\mu$$

with property

$$D_\mu \vec{\Phi} \to D'_\mu \vec{\Phi}' = U(\underline{\theta})\, D_\mu \vec{\Phi}$$

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Yang–Mills Theory

③ make the gauge field dynamic

↪ field-strength tensor: $\mathcal{F}_{\mu\nu} = -\frac{i}{g}[D_\mu, D_\nu] = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + ig[\mathcal{A}_\mu, \mathcal{A}_\nu]$ transforms gauge co-variantly

$$\mathcal{F}_{\mu\nu} = F^a_{\mu\nu} T^a \rightsquigarrow F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g\, c^{abc} A^b_\mu A^c_\nu$$

$$\Rightarrow \mathcal{L}_A = -\frac{1}{2} \mathrm{Tr}[\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}] = -\frac{1}{4} F^a_{\mu\nu} F^{a,\mu\nu}$$

↪ Lorentz- & gauge-invariant

↪ non-abelian: contains $(\partial A)A^2$, A^4 terms

⇒ gauge-boson self-interaction

↪ naive mass term $M^2(A^a_\mu A^{a,\mu})$ not allowed

⇒ from $\mathcal{L}_\Phi(\vec{\Phi}, \partial_\mu \vec{\Phi})$ (global symm.) $\rightsquigarrow$ gauge theory: $\mathcal{L}_\Phi(\vec{\Phi}, D_\mu \vec{\Phi}) + \mathcal{L}_A$

QCD

- gauge symmetry: $SU(3)_c$ — fundamental representation
- matter fields: quark fields

$$\psi_q(x) = \begin{pmatrix} q_r(x) \\ q_g(x) \\ q_b(x) \end{pmatrix} \longleftrightarrow \text{colour-triplett}$$

- generators: $T^a = \frac{\lambda^a}{2}$ (λ^a: Gell-Mann matrices, $a = 1, \dots, 8$)

$\leftrightarrow$ structure constants: $[T^a, T^b] = i f^{abc} T^c$

- eight gauge fields: G_μ^a ($a = 1 \dots 8$) $\rightsquigarrow$ gluons

$$\Rightarrow \mathcal{L}_{QCD} = \sum_q \bar{\psi}_q (i\not{D} - m_q)\psi_q - \frac{1}{4} F_{\mu\nu}^a F^{a,\mu\nu}$$

$\mathcal{L}_{free} \longrightarrow$ $$= -\frac{1}{4}(\partial_\mu G_\nu^a - \partial_\nu G_\mu^a)(\partial^\mu G^{a,\nu} - \partial^\nu G^{a,\mu}) + \sum_q \bar{\psi}_q (i\not{\partial} - m)\psi_q$$

$g\bar{q}q \longrightarrow$ $$- g_s G_\mu^a \bar{\psi}_q \gamma^\mu T^a \psi_q$$

g^3 & $g^4 \longrightarrow$ $$+ \frac{g_s^2}{2} f^{abc} (\partial_\mu G_\nu^a - \partial_\nu G_\mu^a) G^{b,\mu} G^{c,\nu} - \frac{g_s^2}{4} f^{abc} f^{ade} G_\mu^b G_\nu^c G^{d,\mu} G^{e,\nu}$$

Quantization of Gauge Theories

- the free part of the gluon

$$\mathcal{L}_{G,0} = -\frac{1}{4}\left(\partial_\mu G^a_\nu - \partial_\nu G^a_\mu\right)\left(\partial^\nu G^{a,\mu} - \partial^\mu G^{a,\nu}\right)$$

- from part I. $\rightsquigarrow$ problem with the generating functional:

$$Z_{G,0}[J^a_\mu] = \frac{1}{N}\int \mathcal{D}[G_\mu] \exp\left\{ i\int d^4x \left[\frac{1}{2} G^a_\mu \underbrace{\left(g^{\mu\nu}\Box - \partial^\mu\partial^\nu\right)}_{\text{has no inverse}} G^a_\nu + J^a_\mu G^{a,\mu} \right]\right\}$$

$\lightning$ deriving Green's function

- origin: gauge invariance!

$\mathcal{D}[G_\mu]$ includes physically equivalent field configurations

$$G_\mu \xrightarrow{U(\underline{\theta})} {}^{\underline{\theta}}G_\mu \qquad \left(U(\underline{\theta}) = \exp\left\{-i g_s T^a \theta^a(x)\right\}\right)$$

$\Rightarrow$ solution: restrict $\mathcal{D}[G_\mu]$ to only in-equivalent configurations

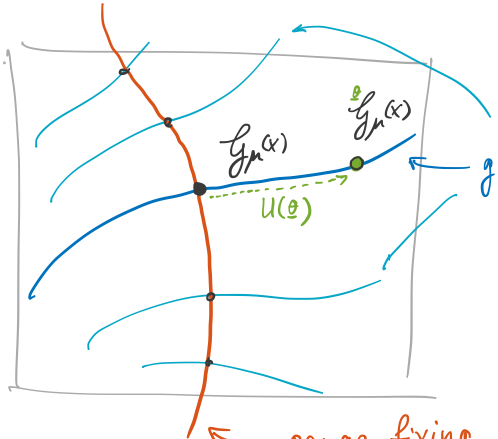
Faddeev-Popov Procedure

① define gauge-fixing conditions

$$f^a[\overset{\theta}{\mathcal{A}}_\mu(x)] \overset{!}{=} 0 \qquad a = 1, \dots 8$$

(QED in covariant gauge:
$f[A_\mu(x)] = \partial^\mu A_\mu$)

$\Rightarrow$ selects one unique $\theta^a(x)$ from gauge orbit



gauge-fixing condition:

$$f^a[\overset{\theta}{\mathcal{A}}(x)] = 0$$

Faddeev–Popov Procedure

② insert a clever one:

$$1 = \Delta[\mathcal{G}_\mu] \int \mathcal{D}[U(\theta)] \; \delta(f[\mathcal{G}^{\theta}])$$

Faddeev–Popov determinant:

Jacobi functional determinant

$$\underbrace{\Delta[\mathcal{G}_\mu]}_{\text{gauge invariant}} \propto \operatorname{Det}\left(\frac{\delta f^a[\mathcal{G}_\mu(x)]}{\delta\theta^b(y)}\right)\Bigg|_{f^a[\mathcal{G}_\mu]=0}$$

"matrix" in a & x: $\quad M^{ab}(x,y) = \dfrac{\delta f^a[\mathcal{G}_\mu(x)]}{\delta\theta^b(y)}$

$$\Rightarrow (Mf)^a(x) = \int d^4y \; M^{ab}(x,y)\, f^b(y)$$

$\Rightarrow M(x,y) \sim \delta(x-y)$

(QED: $f[A_\mu(x)] = \partial^\mu A_\mu$, infinitesimal transf. $\delta A_\mu(x) = \partial_\mu \delta\theta(x)$

$$\Rightarrow \frac{\delta f[A_\mu(x)]}{\delta\theta(y)} = \delta(x-y)\,\Box_x \;)$$

Faddeev-Popov Procedure

③ insert into fuctional integral:

$$\int \mathcal{D}[\mathcal{G}_\mu]\, \exp\{i S_0[\mathcal{G}_\mu]\}$$

$$= \int \mathcal{D}[U(\underline{\theta})] \int \mathcal{D}[\mathcal{G}_\mu]\; \Delta[\mathcal{G}_\mu]\; \delta\big(f[{}^{\underline{\theta}}\mathcal{G}_\mu]\big)\, \exp\{i\, S_0[\mathcal{G}_\mu]\}$$

$\downarrow$ g.i. $\qquad \downarrow$ g.i. $\qquad \downarrow$ g.i.

$$= \int \mathcal{D}[U(\underline{\theta})] \int \mathcal{D}[{}^{\underline{\theta}}\mathcal{G}_\mu]\; \Delta[{}^{\underline{\theta}}\mathcal{G}_\mu]\; \delta\big(f[{}^{\underline{\theta}}\mathcal{G}_\mu]\big)\, \exp\{i\, S_0[{}^{\underline{\theta}}\mathcal{G}_\mu]\}$$

$\downarrow$ relabel

$$= \underbrace{\int \mathcal{D}[U(\underline{\theta})]}_{\substack{= \text{const} \\ (\infty \text{ volume}) \\ \rightsquigarrow \mathcal{N}}} \underbrace{\int \mathcal{D}[\mathcal{G}_\mu]\; \Delta[\mathcal{G}_\mu]\; \delta\big(f[\mathcal{G}_\mu]\big)\, \exp\{i\, S_0[\mathcal{G}_\mu]\}}_{\text{independent of } U(\underline{\theta})}$$

Faddeev – Popov Procedure

④ get rid of the $\delta(f[\mathcal{G}_\mu])$ constraint and write it as an extra term in the Lagrangian $\mathcal{L}_{fix}$ instead

↪ modify constraint

$$f^a[\mathcal{G}_\mu] \longrightarrow f^a[\mathcal{G}_\mu] - C^a(x)$$

(phys. obs. independent of $C^a(x)$)

↪ integrate with

$$\int \mathcal{D}[C^a] \ \dots \ \exp\left\{-i \int d^4y \, \frac{1}{2\xi} \, C_a(y)^2\right\}$$

arbitrary gauge param. could also choose ξ^a indiv.

$$\Rightarrow \int \mathcal{D}[C^a] \ \delta(f[\mathcal{G}_\mu] - C) \ \exp\left\{-i \int d^4y \, \frac{1}{2\xi} \, C_a(y)^2\right\} = \exp\left\{i \int d^4x \, \mathcal{L}_{fix}\right\}$$

with $\quad \mathcal{L}_{fix} = -\dfrac{1}{2\xi} \, f_a[\mathcal{G}_\mu]^2$

$\left(\text{QED: } \mathcal{L}_{fix} = -\dfrac{1}{2\xi}(\partial \cdot A)^2\right)$

Faddeev – Popov Procedure

5. rewrite the determinant as a Gaussian integral over Grassmann-valued fields ($\bar{u}^a, u^b$: ghosts)

$$\mathrm{Det}\left(M^{ab}(x,y)\right) \propto \int \mathcal{D}[u^b] \int \mathcal{D}[\bar{u}^a] \exp\left\{-i \int d^4x \int d^4y \ \bar{u}^a(x) M^{ab}(x,y) u^b(y)\right\}$$

$$M^{ab}(x,y) = \frac{\delta f^a[\mathcal{G}_\mu(x)]}{\delta\theta^b(y)} = \delta(x-y) \frac{\delta f^a[\mathcal{G}_\mu(x)]}{\delta\theta^b(x)}$$

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put it as an extra term in the Lagrangian:

$$\mathcal{L}_{FP} = -\bar{u}^a(x) \ M^{ab}(x) \ u^b(x)$$

$u^a, \bar{u}^a$: Grassmann-valued scalars
↪ unphysical!

(QED: only one ghost with $\mathcal{L}_{FP} = -\bar{u}(x) \ \Box \ u(x)$
↪ no interactions ⇒ in abelian theories: ghosts decouple!)

Faddeev-Popov Procedure

- when the dust settles:

the effective Lagrangian

$$\mathcal{L}_{G,0} \longrightarrow \mathcal{L}_{G,0} + \mathcal{L}_{fix} + \mathcal{L}_{FP} = \mathcal{L}_{eff}$$

$$\Rightarrow Z[J^a_\mu, j^a, \bar{j}^a] = \frac{1}{N} \int \mathcal{D}[G^a_\mu] \int \mathcal{D}[u^a] \int \mathcal{D}[\bar{u}^a] \exp\Big\{ i \int d^4x \Big[\mathcal{L}_{G,0} + \mathcal{L}_{fix} + \mathcal{L}_{FP} + J^a_\mu G^{a,\mu} + \bar{j}^a u^a - \bar{u}^a j^a \Big] \Big\}$$

- QCD in covariant gauge: $f^a[G_\mu] = \partial^\mu G^a_\mu$

$$\Rightarrow \frac{\delta f^a[G_\mu(x)]}{\delta \theta^b(y)} = \delta(x-y) \left[\delta^{ab} \Box_x + g\, c^{abc} (\partial \cdot G^c) + g\, c^{abc} G^c_\mu \partial^\mu_x \right]$$

$$\hookrightarrow \mathcal{L}_{fix} = -\frac{1}{2\xi} (\partial \cdot G^a)^2$$

ghosts only interact with the gluons.

$$\hookrightarrow \mathcal{L}_{FP} = -\bar{u}^a \Box u^a - g\, c^{abc} \bar{u}^a \partial^\mu (G^c_\mu u^b)$$

BRS Symmetry

- Faddeev-Popov: $\mathcal{L}_{ferm} + \mathcal{L}_{gauge} \xrightarrow{FP} \underbrace{\mathcal{L}_{ferm} + \mathcal{L}_{gauge}}_{\text{gauge-inv.}} \underbrace{+ \mathcal{L}_{fix} + \mathcal{L}_{FP}}_{\text{not gauge inv.}}$ (the whole point)

  $\hookrightarrow$ phys. obs. gauge inv., manifest gauge inv. in $\mathcal{L}$ gone!

- what happened?

  $\hookrightarrow$ $\mathcal{L}_{fix}$: allows unphysical d.o.f. to propagate

  invertible diff. op $\Rightarrow$ compute Green's fn. ($\hat{=}$ propagator)

  $\hookrightarrow$ $\mathcal{L}_{FP}$: ghosts are there to compensate for them $\longleftrightarrow$ cancel unphysical d.o.f introduced by $\mathcal{L}_{fix}$

- what is the symmetry that corresponds to gauge symmetry after FP?

  Becchi-Rouet-Stora (BRS) symmetry

  $\hookrightarrow$ gauge transformation with $\delta\theta^a(x) = \delta\lambda\, u^a(x)$

  $\left(U(\underline{\theta}) = \exp\{-ig_s T^a \theta^a\} \right)$

  Grassmann!

BRS Symmetry

- gauge transformation with $\delta\theta^a = \delta\bar\lambda\, u^a$ $\quad (U(\underline{\theta}) = \exp\{-ig_s T^a \theta^a\})$

$\hookrightarrow$ $\delta_{BRS}\, G^a_\mu = \delta\bar\lambda\, [g_s f^{abc} u^b G^c_\mu + \partial_\mu u^a]$

$\hookrightarrow$ $\delta_{BRS}\, \psi = \delta\bar\lambda\, [-ig_s T^a u^a \psi]$

- clearly, both $\mathcal{L}_{ferm}$ & $\mathcal{L}_{gauge}$ are invariant under δ_{BRS}
- we need $\delta_{BRS}(\mathcal{L}_{fix} + \mathcal{L}_{FP}) = 0$ for $\mathcal{L}_{eff}$ to be invariant

$\hookrightarrow$ $\delta_{BRS}\, \bar u^a \equiv \delta\bar\lambda\, \left[-\frac{1}{\xi} f_a[\mathcal{G}_\mu]\right] = \delta\bar\lambda\, \left[-\frac{1}{\xi}\partial^\mu G^a_\mu\right]$

$\hookrightarrow$ $\delta_{BRS}\, u^a = \delta\bar\lambda\, [-\frac{1}{2} g_s f^{abc} u^b u^c]$

$\Rightarrow$ Derive relations between n-pt functions from BRS symmetry

$\rightsquigarrow$ Slavnov-Taylor identities

Slavnov-Taylor Identities

- let us consider the transformation $\Phi_i \to \Phi_i + \delta_{BRS}\Phi_i$ in the P.I.

$$\hookrightarrow \int \mathcal{D}[\Phi_i + \delta_{BRS}\Phi_i] = \int \mathcal{D}[\Phi_i] \,, \qquad \mathcal{L}_{eff}(\Phi_i + \delta_{BRS}\Phi_i) = \mathcal{L}_{eff}(\Phi_i)$$

$$\Rightarrow 0 = Z[J_i]\Big|_{\Phi_i + \delta_{BRS}\Phi_i} - Z[J_i] \qquad \text{only linear because } (\delta\bar{\lambda})^2 = 0$$

$$= \frac{1}{N}\int \mathcal{D}[\Phi_i] \exp\left\{ i\int d^4x \left[\mathcal{L}_{eff} + J_i\Phi_i\right]\right\} \cdot \overbrace{i\int d^4y\, J_i(y)\left(\delta_{BRS}\Phi_i(y)\right)}$$

- functional differentiation $\Rightarrow$ Relations between n-pt functions

- Slavnov-Taylor identities:

$$0 \overset{!}{=} \delta_{BRS}\left\langle T\, \Phi_1(x_1)\dots\Phi_n(x_n)\right\rangle$$

$$= \left\langle T\left(\delta_{BRS}\Phi_1(x_1)\right)\Phi_2(x_2)\dots\Phi_n(x_n)\right\rangle + \dots + \left\langle T\,\Phi_1(x_1)\dots\Phi_{n-1}(x_{n-1})\left(\delta_{BRS}\Phi_n(x_n)\right)\right\rangle$$

Slavnov–Taylor Identity: Gluon Propagator

$$0 \stackrel{!}{=} \delta_{BRS} \langle T\, \bar{u}^a(x)\, G^b_\mu(y) \rangle$$

$$= \langle T \underbrace{(\delta_{BRS} \bar{u}^a(x))}_{\delta\bar{\lambda}\left[-\frac{1}{\xi}\partial^\nu G^a_\nu(x)\right]} G^b_\mu(y) \rangle + \langle T\, \bar{u}^a(x) \underbrace{(\delta_{BRS} G^b_\mu(y))}_{\delta\bar{\lambda}\left[g_s f^{bcd} u^c(y) G^d_\mu(y) + \partial_\mu u^b(y)\right]} \rangle$$

Grassmann!

- bring $\delta\bar{\lambda}$ to the front & cancel
- differentiate w.r.t. y: $\partial^\nu_y(\dots)$
- use the e.o.m. for the Green's function

$$\langle T\, \bar{u}^a(x)\, M^{bc}(y)\, u^c(y) |0\rangle = i\, \delta^{ab}\, \delta(x-y)$$

- $\leadsto$ momentum space: $\int d^4x \int d^4y \; e^{-ik(x-y)}$

$$\Rightarrow \quad k^\mu k^\nu\, G^{g^a g^b}_{\mu\nu}(k,-k) = -i\, \xi\, \delta^{ab}$$

$\hookrightarrow$ no higher-order corrections to longitudinal component of the gluon!

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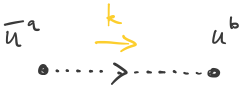
QCD Feynman Rules

- propagators

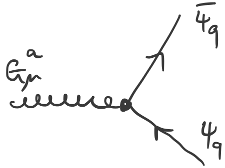
$$\frac{-i\,\delta^{ab}}{k^2+i0}\left[g_{\mu\nu}-(1-\xi)\frac{k_\mu k_\nu}{k^2}\right]$$



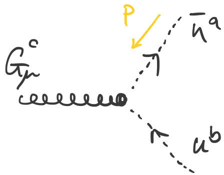
$$\frac{i\,\delta^{ab}}{k^2+i0}$$

$$\frac{i}{\not{k}-m_q+i0}$$

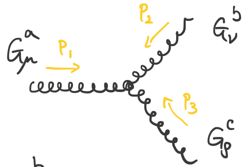
- vertices



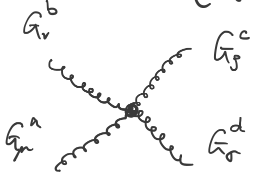
$$-ig_s\,T^a\gamma_\mu$$



$$p_\mu\, g_s f^{abc}$$



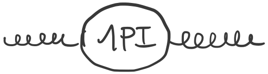
$$-g_s f^{abc}\Big[g^{\mu\nu}(p_1-p_2)^\rho + g^{\nu\rho}(p_2-p_3)^\mu + g^{\rho\mu}(p_3-p_1)^\nu\Big]$$



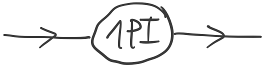
$$-ig_s^2\Big[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}\right) + \text{cycl.}\Big]$$

Superficially Divergent Vertex Functions

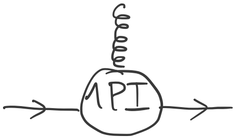
$\omega = 3 \longrightarrow 0$ (quant. number of vacuum)



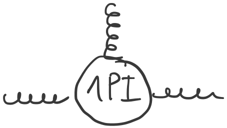
$\omega = 2$



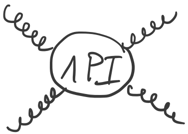
$\omega = 1$



$\omega = 0$



$\omega = 1$



$\omega = 0$

Ex 24

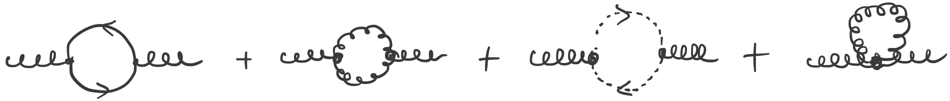
all divergent!

↪ more div. vertex functions than parameters!

↪ gauge symmetry essential

1-Loop Corrections: Gluon Self Energy

$-i\,\Sigma^{G^aG^b}_{\mu\nu}(k) =$



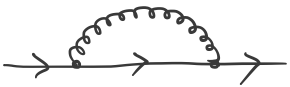
$$\leftrightarrow\ \Gamma^{G^aG^b}_{\mu\nu}(k) = -i\,\delta^{ab}k^2\left[g_{\mu\nu}-\frac{k_\mu k_\nu}{k^2}\right]\left[1+\Pi^{GG}(k^2)\right] - \frac{i}{\xi}\,\delta^{ab}k_\mu k_\nu$$

- gluon vacuum polarization

$$\Pi^{GG}(k^2) = \sum_q \frac{\alpha_s}{3\pi}\,T_F\,\frac{1}{k^2}\left\{(k^2+m_q^2)\,B_0(k^2,m_q,m_q) - \frac{k^2}{3} - 2m_q^2\,B_0(0,m_q,m_q)\right\}$$
$$-\frac{\alpha_s}{12\pi}\,C_A\left\{\frac{13-3\xi}{2}\,B_0(k^2,0,0) + \frac{59}{12} + \frac{9}{2}\xi + \frac{3}{4}\xi^2\right\}$$

- fermion loop: analogue to QED $\quad e^2Q^2 \longrightarrow g_s^2\,T_F\,\delta^{ab}$
- boson loop: $\quad \Sigma_L^{GG} = 0$ only after summation
- Π^{GG} gauge dependent already @ 1-loop (not in QED)

1-Loop Corrections: Quark Self Energy

$i\,\Sigma^{\bar{q}q}(k) =$ 

$\hookrightarrow \quad \Sigma^{\bar{q}q}(k) = \not{k}\,\Sigma_V^{\bar{q}q}(k^2) + m_q\,\Sigma_S^{\bar{q}q}(k^2)$

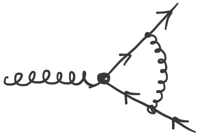
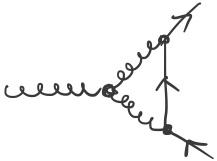
- result $\quad (\xi = 1)$

$$\Sigma_V^{\bar{q}q}(k^2) = -\frac{\alpha_s}{4\pi} C_F \frac{1}{k^2}\left\{ A_0(m_q) - (k^2+m_q^2)\,B_0(k^2, m_q, 0) + k^2 \right\}$$

$$\Sigma_S^{\bar{q}q}(k^2) = -\frac{\alpha_s}{4\pi} C_F \left\{ 4\,B_0(k^2, m_q, 0) - 2 \right\}$$

- from QED: $\quad e^2 Q^2 \longrightarrow g_s^2\, C_F$

1-Loop Corrections: Quark-Gluon Vertex

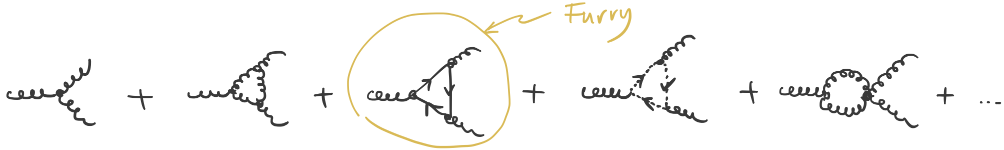
$$-ig_s \Lambda^a_\mu(p',p) =$$  $+$ 

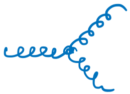
no analogue in QED

- 1st diagram from QED: $e^3Q^3 \longrightarrow -g_s^3\left(C_F - \frac{1}{2}C_A\right)T^a$
- UV divergence:

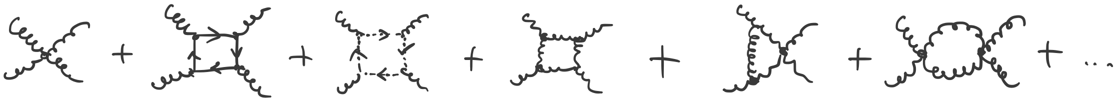
$$\Lambda^a_\mu(p',p)\Big|_{div} = \frac{\alpha_s}{4\pi}\left(C_F + C_A\right)\frac{1}{\epsilon}$$

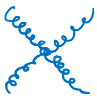
1-Loop Corrections: Gluon Vertices

$$\Gamma^{G^a G^b G^c}_{\mu\nu\rho} =$$ 

$\hookrightarrow$ $\Gamma^{G^a G^b G^c}_{\mu\nu\rho}\Big|_{\mathrm{div}} =$  $\cdot\, C^{GGG}_{\mathrm{div}}$

(QED: zero)

$$\Gamma^{G^a G^b G^c G^d}_{\mu\nu\rho\sigma} =$$ 

$\hookrightarrow$ $\Gamma^{G^a G^b G^c G^d}_{\mu\nu\rho\sigma}\Big|_{\mathrm{div}} =$  C^{GGGG}_{div}

(QED: UV finite)

QCD Renormalization

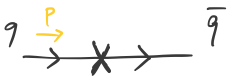
- multiplicative renormalization $\quad$ (1-loop: $Z_x = 1 + \delta Z_x$)

$$G^a_{\mu,0} = \sqrt{Z_G}\, G^a_\mu \,, \qquad \psi_{q,0} = \sqrt{Z_q}\, \psi_q$$

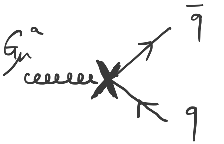
$$g_{s,0} = Z_g\, g_s \,, \qquad m_{q,0} = Z_{m_q} m_q = m_q + \delta m_q \,, \qquad \xi_0 = Z_\xi\, \xi$$

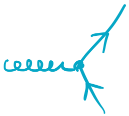
- Counterterm vertices:

$$-i\,\delta^{ab}\,\delta Z_G\, k^2\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) + i\,\delta^{ab}\,\frac{1}{\xi}\left(\delta Z_\xi - \delta Z_A\right) k_\mu k_\nu$$

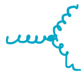


$$i\,\delta Z_q\,(\not{p} - m_q) - i\,\delta m_q$$

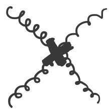


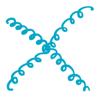


$$\times\left(\delta Z_g + \delta Z_q + \tfrac{1}{2}\,\delta Z_G\right)$$



$$\times\left(\delta Z_g + \tfrac{3}{2}\,\delta Z_G\right)$$





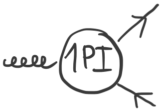
$$\times\left(2\,\delta Z_g + 2\,\delta Z_G\right)$$

Renormalized Vertex Functions

$$= -i\,\delta^{ab}\left[g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right]\Big[k^2 + \overbrace{\Sigma_T^{GG}(k^2) + k^2\,\delta Z_G}^{=:\,\hat{\Sigma}_T^{GG}(k^2)}\Big]$$

$$-\,i\,\delta^{ab}\,\frac{1}{\xi}\,\frac{k_\mu k_\nu}{k^2}\Big[k^2 + \underbrace{\cancelto{0}{\Sigma_L^{GG}(k^2)} + k^2\left(\delta Z_G - \delta Z_\xi\right)}_{=:\,\hat{\Sigma}_L^{GG}(k^2)}\Big]$$

$$= i\,\not{p}\Big[1 + \underbrace{\Sigma_V^{\bar{q}q}(p^2) + \delta Z_q}_{=:\,\hat{\Sigma}_V^{\bar{q}q}(p^2)}\Big] + i\,m_q\Big[-1 + \underbrace{\Sigma_S^{\bar{q}q}(p^2) - \delta Z_q - \frac{\delta m_q}{m_q}}_{=:\,\hat{\Sigma}_S^{\bar{q}q}(p^2)}\Big]$$



$$= -i g_s T^a \gamma_\mu - i g_s\Big[\underbrace{\Lambda^a_\mu(p',p) + T^a\gamma_\mu\left(\delta Z_g + \delta Z_q + \tfrac{1}{2}\delta Z_G\right)}_{=:\,\hat{\Lambda}^a_\mu(p',p)}\Big]$$

more:

Renormalization Conditions

- confinement in QCD

  $\hookrightarrow$ α_s definition through elementary qg scattering not accessible

  $\hookrightarrow$ no free propagation of quarks $\rightsquigarrow$ kin pole not accessible

- $\overline{\mathrm{MS}}$ ("modified minimal subtraction") scheme

  δZ_X absorb only the standard divergence

  $$\Delta = \frac{1}{\epsilon} - \gamma_E + \ln(4\pi)$$ (1-loop)

$\Rightarrow$ predictions depend on the reference scale μ (DimReg)

$\hookrightarrow$ running couplings and masses

$\overline{MS}$ Renormalization

- gluon self energy:

$$\delta Z_G = -\left.\frac{\Sigma_T^{GG}(k^2)}{k^2}\right|_\Delta = -\Pi^{GG}(k^2)\Big|_\Delta = -\frac{\alpha_s}{4\pi}\Delta\left[\frac{4}{3}N_f T_F - \frac{5}{3}C_A\right]$$

$$\delta Z_\xi = \delta Z_G \qquad \left(\Rightarrow \quad \hat{\Sigma}_L^{GG}(k^2) = \Sigma_L^{GG}(k^2) = 0\right)$$

- quark self energy:

$$\delta Z_q = -\Sigma_V^{\bar{q}q}(k^2)\Big|_\Delta = -\frac{\alpha_s}{4\pi}\Delta\, C_F$$

$$\frac{\delta m_q}{m_q} = \left[\Sigma_V^{\bar{q}q}(k^2) + \Sigma_S^{\bar{q}q}(k^2)\right]\Big|_\Delta = -\frac{\alpha_s}{4\pi}\Delta\, 3C_F$$

↳ often $m_q = 0$ for light quarks $\Rightarrow \delta m_q = 0$ (chiral symmetry)

- quark-gluon vertex:

$$\Lambda_\mu^a(p', p)\Big|_\Delta + T^a\gamma_\mu\left(\delta Z_g + \delta Z_q + \frac{1}{2}\delta Z_G\right) \stackrel{!}{=} 0 \quad \Rightarrow \quad \delta Z_g = \frac{\alpha_s}{4\pi}\Delta\left[\frac{2}{3}N_f T_F - \frac{11}{6}C_A\right]$$

Renormalization Group

- equations for observables depend on μ
  $\Rightarrow$ extracted valus for g, etc. are μ-dependent $g = g(\mu)$
- in the end: explicit & implicit μ-dependence compensate each other
  $\Rightarrow$ renormalization group equation

$$\mu \frac{d\alpha_s}{d\mu} = -\beta_0 \frac{\alpha_s^2}{\pi} \qquad (1\text{-loop})$$

EX25

$$\beta_0 = \frac{11}{6} C_A - \frac{2}{3} N_f T_F = \frac{11}{6} 3 - \frac{2}{3} N_f \frac{1}{2} > 0 \quad \text{for } N_f < 16.5 \quad (\checkmark)$$

beta function

- solution

$$\alpha_s(\mu_2) = \frac{\alpha_s(\mu_1)}{1 - \frac{\alpha_s(\mu_1)}{2\pi} \beta_0 \ln\left(\frac{\mu_2^2}{\mu_1^2}\right)} \Rightarrow \mu_2 > \mu_1 \Rightarrow \alpha_s(\mu_2) < \alpha_s(\mu_1)$$

asymptotic freedom