Radiative Corrections & Renormalization

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slides: github.com/AlexanderHuss/MariaLaach

The Plan

Ø Motivation & Conventions

I. QFT in a Nutshell

II. QED & Renormalization

III. Yang–Mills Theory & QCD

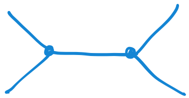
IV. Infrared Singularities

∅. Motivation & Conventions

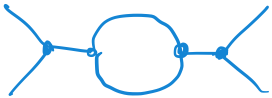
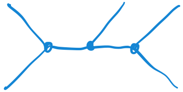
Motivation

- radiative corrections ⟷ perturbative evaluation of a QFT

  ↪ diagrammatic representation: Feynman diagrams

  Born-level  , ...

  "radiative corrections"

  higher order  ,  , ...

  ↪ ultraviolet (UV) divergences

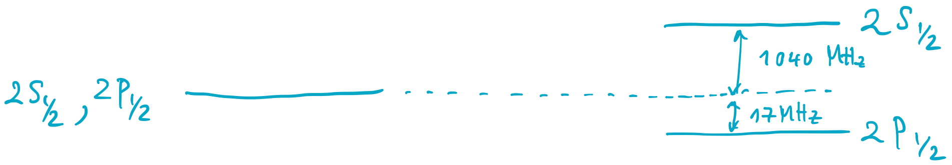
  ⇒ renormalization

  ↪ infrared (IR) divergences

  ⇒ compensation between virtual & real-emission

Motivation: Measurable effects

- Lamb-Shift: (H - atom)



(QED corrections)

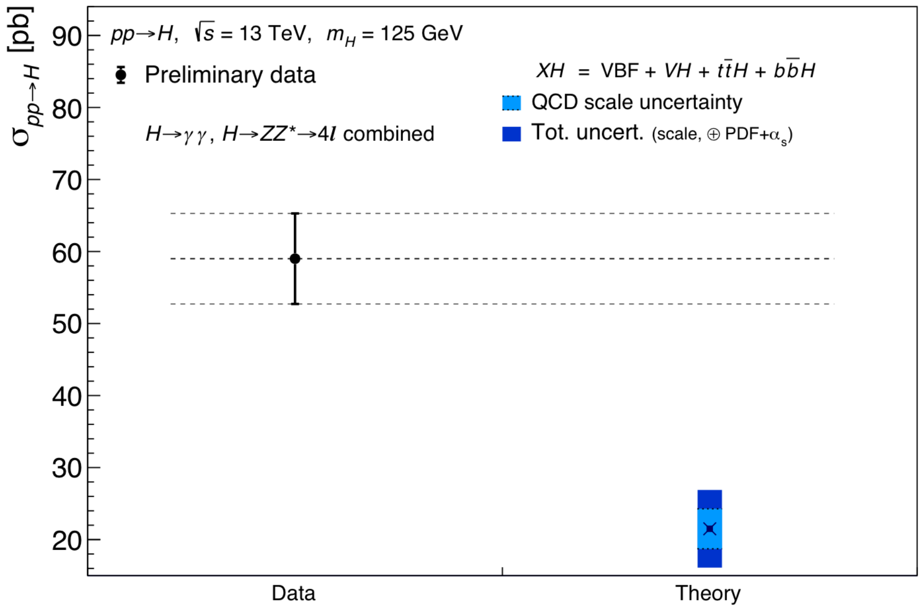
- anomalous magnetic moment (e, μ)

$$g_\mu = \underset{\substack{\uparrow \\ \text{Dirac}}}{2} \left[1 + \underset{\substack{\uparrow \\ \text{Schwinger}}}{\frac{\alpha}{2\pi}} + \dots \right]$$

- electroweak precision tests Input parameters $\rightsquigarrow$ observables
  - $\hookrightarrow$ fit Standard Model (SM) to Data: good agreement
  - $\hookrightarrow$ predict m_t, constrain M_H

Predict & Compare

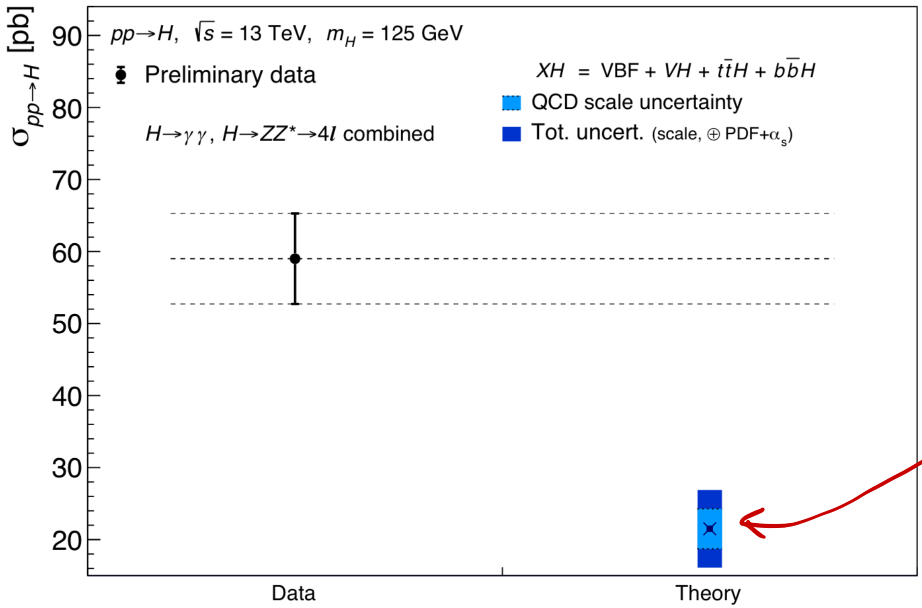
- Higgs production @ LHC:



- 3.8 σ deviation !!!
- Standard Model fails?!

Predict & Compare

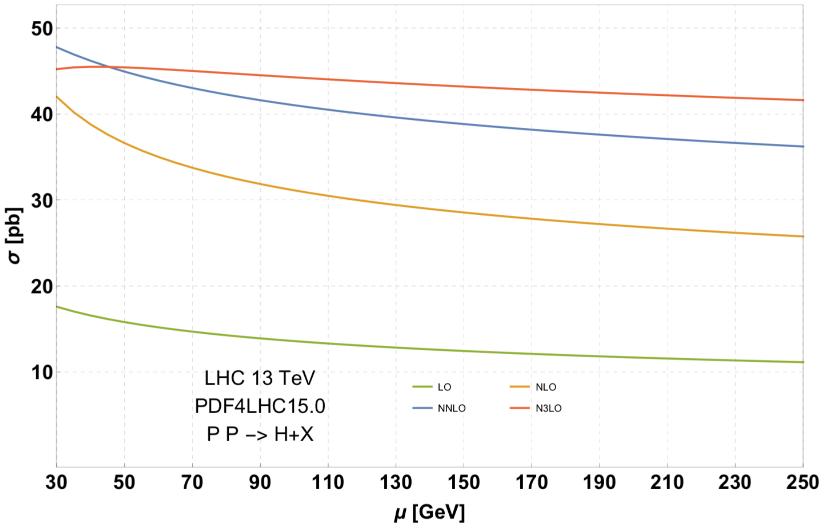
- Higgs production @ LHC:



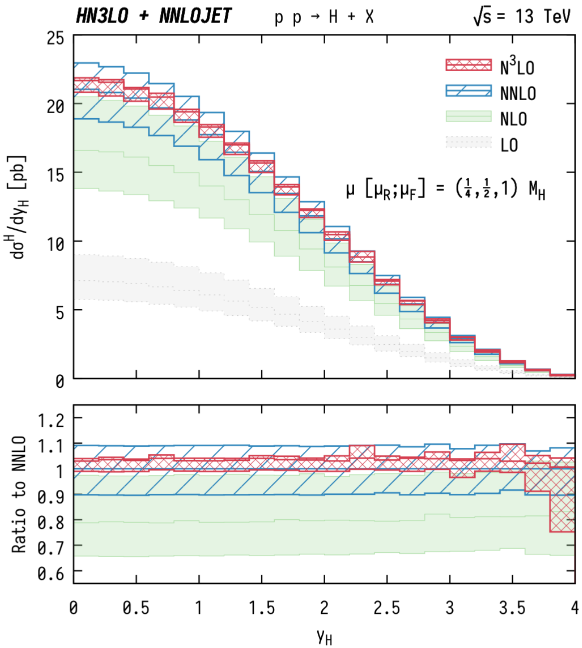
- ~~3.8 σ deviation !!!~~
- ~~Standard Model fails?!~~

LEADING ORDER

Higgs Production @ N³LO



[Mistlberger '18]



[Cieri, Chen, Gehrmann, Glover, AH '18]

Conventions & Notation

- natural units: $\hbar = c = 1$ — one should avoid this...
- metric tensor $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$

  contravariant vector: $x^\mu = (x^0, \vec{x}) = (t, x, y, z)$

  covariant vector: $x_\mu = g_{\mu\nu} x^\nu = (x^0, -\vec{x})$

  derivative: $\partial_\mu \equiv \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \vec{\nabla}\right)$

  scalar product: $x \cdot y = x^\mu y_\mu = x^\mu y^\nu g_{\mu\nu} = x^0 y^0 - \vec{x} \cdot \vec{y}$

  $$\Box \equiv \partial^2 = \partial^\mu \partial_\mu = \partial_t^2 - \Delta$$

  Levi-Civita: $\varepsilon^{\mu\nu\rho\sigma} = -\varepsilon_{\mu\nu\rho\sigma} = \begin{cases} +1, & \{\mu,\nu,\rho,\sigma\} \text{ even perm } \{0,1,2,3\} \\ -1, & \text{—"— odd —"—} \\ 0, & \text{else} \end{cases}$

Conventions & Notation

- Dirac algebra (4 - dimensional)

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu = 2 g^{\mu\nu} \mathbb{1} \qquad (\mathrm{Tr}(\mathbb{1}) = 4)$$

$$\gamma^5 = -i\gamma_0\gamma_1\gamma_2\gamma_3 = +i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

$$\{\gamma^5, \gamma^\mu\} = 0, \quad (\gamma^5)^2 = \mathbb{1}$$

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] = -\sigma^{\nu\mu}$$

Dirac - slash: $\not{a} = \gamma^\mu a_\mu$

- Dirac spinors: $(\bar{\psi} = \psi^\dagger \gamma^0)$

Ex 1

$$(\not{p} - m)\, u(p) = 0, \qquad (\not{p} + m)\, v(p) = 0$$

$$\bar{u}(p)\, (\not{p} - m) = 0, \qquad \bar{v}(p)\, (\not{p} + m) = 0$$

I. QFT in a Nutshell

Overview

- starting point: (local) Lagrange density $\mathcal{L}$ $\leftrightarrow$ defines our theory

  $\hookrightarrow$ ϕ^4-theory: $\mathcal{L}_{\phi^4} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!}\phi^4$

  $\hookrightarrow$ QED: $\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - eQ A_\mu \bar{\psi}\gamma^\mu \psi$

  $e > 0$

  $\hookrightarrow$ construction via symmetry considerations: $SU(3)_c \times SU(2)_L \times U(1)_Y$

  $\Rightarrow$ equation of motion (e.o.m)

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \Phi)} - \frac{\delta \mathcal{L}}{\delta \Phi} = 0$$

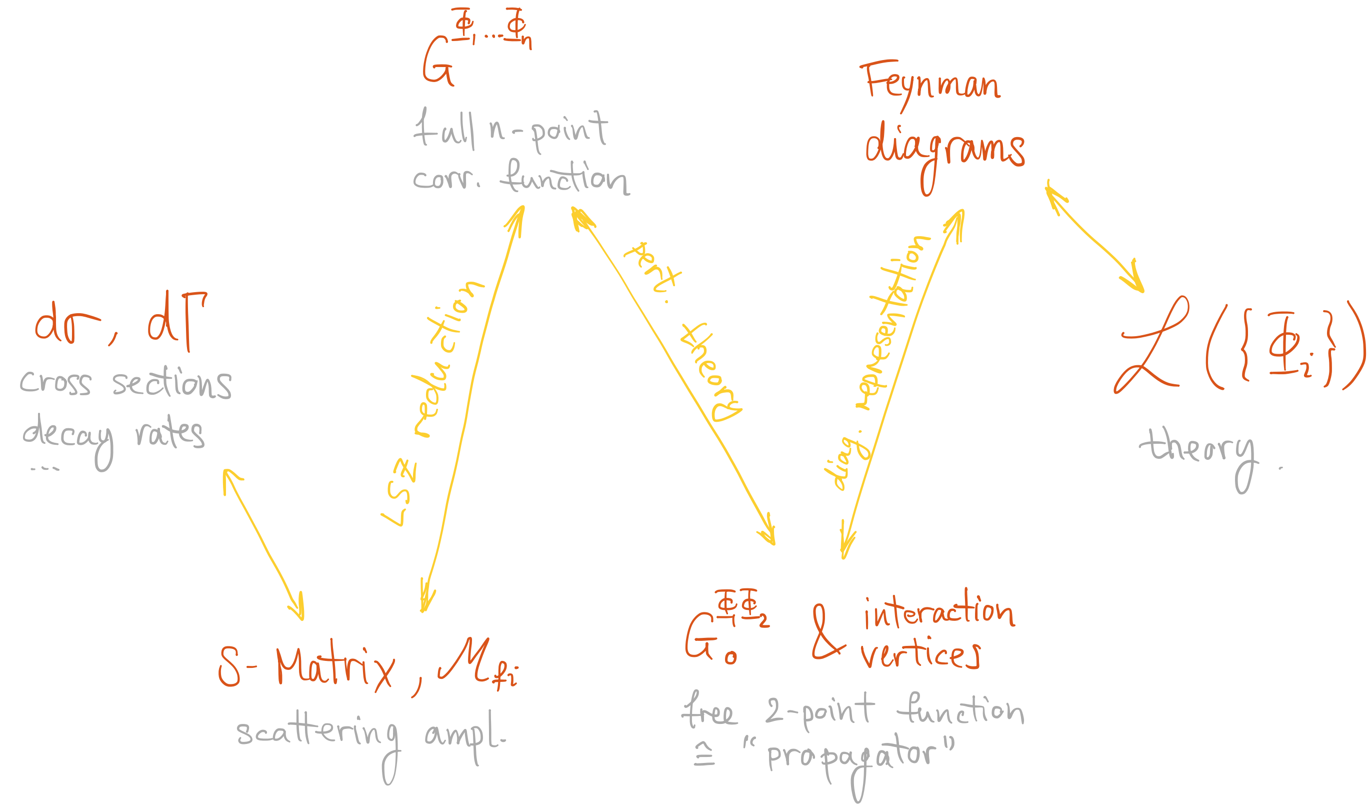
I write: $\boxed{\Phi \in \{\phi, \psi, A^\mu, \ldots\}}$

- fundamental objects: n-point correlation functions

$$G^{\Phi_1 \cdots \Phi_n}(x_1, \ldots, x_n)$$

Ex 2

The Road Map



$G^{\Phi_1 \dots \Phi_n}$
full n-point corr. function
Feynman diagrams
$d\sigma$, $d\Gamma$
cross sections
decay rates
...
LSZ reduction
pert. theory
diag. representation
$\mathcal{L}(\{\Phi_i\})$
theory
S-Matrix, $\mathcal{M}_{fi}$
scattering ampl.
$G_0^{\Phi_1 \Phi_2}$ & interaction vertices
free 2-point function
≙ "propagator"

Free Field theory

- "free" part of $\mathcal{L} \leftrightarrow$ bilinear in the fields:

$\hookrightarrow \quad \mathcal{L}_{\phi,0} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2} m^2 \phi^2$

$\hookrightarrow \quad \mathcal{L}_{\psi,0} = \bar{\psi}(i\not{\partial} - m)\psi$

$\hookrightarrow \quad \mathcal{L}_{A,0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad ; \; F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

- n-point correlation function

time ordering: $\Phi_{i_1}(x_{i_1}) \ldots \Phi_{i_n}(x_{i_n})$ for $x^0_{i_1} > \ldots > x^0_{i_n}$

$$G_0^{\Phi_1 \ldots \Phi_n}(x_1, \ldots, x_n) = \langle 0 | T \, \Phi_1(x_1) \ldots \Phi_n(x_n) | 0 \rangle \quad \text{(canonical quant.)}$$

$$= \frac{1}{N_0} \int \mathcal{D}[\{\Phi\}] \, \Phi_1(x_1) \ldots \Phi_n(x_n) \, \exp\left(i \, S_0[\{\Phi\}]\right)$$

(path integral quant.)

Free Scalar Theory

- $\mathcal{L}_{\phi,0} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 \quad \Rightarrow$ e.o.m: $(\Box + m^2)\phi = 0$
- Green's function: $(\Box + m^2)\Delta_F(x) = -\delta(x)$

$$\Delta_F(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{1}{p^2 - m^2 + i0}$$

(Feynman prop.)

time ordering.

- n-point function:

$$G_0^{\phi\cdots\phi}(x_1, \dots x_n) = \langle 0 | T\, \phi(x_1)\dots\phi(x_n) | 0 \rangle = \frac{1}{N_0}\int \mathcal{D}[\phi]\, \phi(x_1)\dots\phi(x_n)\, e^{iS_0[\phi]}$$

- generating functional:

$$Z_{\phi,0}[J] = \frac{1}{N_0}\int \mathcal{D}[\phi]\, \exp\left\{ i \int d^4x \left[\mathcal{L}_{\phi,0} + J(x)\phi(x) \right] \right\}$$

$$\Rightarrow G_0^{\phi\cdots\phi}(x_1, \dots, x_n) = \frac{\delta}{i\,\delta J(x_1)} \cdots \frac{\delta}{i\,\delta J(x_n)} Z_{\phi,0}[J] \Big|_{J=0}$$

Free Scalar Theory

- generating functional:

P.I.

$$Z_{\phi,0}[J] = \frac{1}{N_0}\int \mathcal{D}[\phi]\, \exp\left\{ i \int d^4x \left[-\frac{1}{2}\phi(\Box + m^2)\phi + J\phi \right]\right\}$$

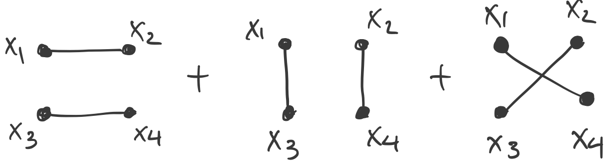
... "complete the square" $\left(\phi = \phi' - \int d^4y\, \Delta_F(x-y) J(y) \right)$

... absorb $\int \mathcal{D}[\phi']$ into normalisation $\left(Z_{\phi,0}[0] \overset{!}{=} 1 \right)$

$$= \exp\left\{ \int d^4x \int d^4y\, \frac{1}{2}\, iJ(x)\, i\Delta_F(x-y)\, iJ(y) \right\}$$

$$\Rightarrow \quad G_0^{\phi\phi}(x_1,x_2) = \frac{\delta}{i\delta J(x_1)}\frac{\delta}{i\delta J(x_2)} Z_{\phi,0}[J]\Big|_{J=0} = i\Delta_F(x_1-x_2) =$$

$$G_0^{\phi\phi\phi\phi}(x_1,x_2,x_3,x_4) = \dots =$$



$\left(\text{no } G^{\overbrace{\phi\dots\phi}^{\text{uneven}}} \right)$

Free Dirac Theory

- $\mathcal{L}_{\psi,0} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ $\Rightarrow$ e.o.m. $(i\partial\!\!\!/ - m)\psi = 0$

- Green's function: $(i\partial\!\!\!/ - m) S_F(x) = \delta(x)$

$$S_F(x) = \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \frac{1}{p\!\!\!/ - m + i0}$$

- generating functional:

Grassmann-valued fields.

$$Z_{\psi,0}[\eta,\bar{\eta}] = \frac{1}{N_0} \int \mathcal{D}[\psi]\, \mathcal{D}[\bar{\psi}]\, \exp\left\{ i \int d^4x \left[\bar{\psi}(i\partial\!\!\!/ - m)\psi - \bar{\psi}\eta + \bar{\eta}\psi \right] \right\}$$

$$= \ldots = \exp\left\{ - \int d^4x \int d^4y \; i\bar{\eta}(x)\; iS_F(x-y)\; i\eta(y) \right\}$$

$$\Rightarrow \quad G_0^{\psi\bar{\psi}}(x_1, x_2) = \langle 0 | T\, \psi(x_1)\, \bar{\psi}(x_2) | 0 \rangle = \begin{cases} t_1 > t_2 : \text{ create } f \text{ @ } x_2 \longrightarrow \text{destroy @ } x_1 \\ t_2 > t_1 : \text{ create } \bar{f} \text{ @ } x_1 \longrightarrow \text{destroy @ } x_2 \end{cases}$$

$=$ 

$\Leftrightarrow$ definite fermion-number flow $\triangleq$ arrow

$$= \frac{\delta}{i\,\delta\bar{\eta}(x_1)} \frac{\delta}{i\,\delta\eta(x_2)} Z_{\psi,0}[\eta,\bar{\eta}]\Big|_{\eta=\bar{\eta}=0}$$

$$= i\, S_F(x_1 - x_2)$$

Free Photon Field

- $\mathcal{L}_{A,0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$ $\left(F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \right)$ $\Rightarrow$ e.o.m. $\left[g^{\mu\nu} \Box - \partial^\mu \partial^\nu \right] A_\nu = 0$
- Green's function: $\left(g^{\mu\nu} \Box - \partial^\mu \partial^\nu \right) \Delta_{F,\nu\rho}^{(x)} = g^\mu{}_\rho \, \delta(x)$ ↯ $\partial_\mu [\dots]$

$\hookrightarrow$ origin gauge symmetry: $A_\mu(x) \longrightarrow A_\mu(x) + \partial_\mu \Lambda(x)$

(unphysical d.o.f)

$$\left(g^{\mu\nu} \Box - \partial^\mu \partial^\nu \right) \partial_\mu \Lambda(x) \equiv 0 \quad \hat{=} \quad \text{eigenvalue zero}$$

$\hookrightarrow$ Green's function $\Delta_{F,\nu\rho}$ is "inverse" of the differential-operator $\left[g^{\mu\nu} \Box - \partial^\mu \partial^\nu \right]$

eigenvalues zero $\Rightarrow$ not invertible $\Rightarrow$ $\nexists \; \Delta_{F,\nu\rho}$

Ex 3

Free Photon Field (with covariant gauge fixing)

- $\mathcal{L}_{A,0} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial \cdot A)^2 \quad \Rightarrow$ e.o.m. $\left[g^{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] A_\nu = 0$

- Green's function: $\left[g^{\mu\nu} \Box - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right] \Delta_{F,\nu\rho}(x) = g^\mu{}_\rho \, \delta(x)$

$$\Delta_F^{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4} \, e^{-ipx} \, \frac{-1}{p^2} \left[g^{\mu\nu} - (1-\xi) \frac{p^\mu p^\nu}{p^2} \right]$$

- generating functional:

$$Z_{A,0}[J_\mu] = \exp\left\{ \frac{1}{2} \int d^4x \int d^4y \; iJ_\mu(x) \; i\Delta_F^{\mu\nu}(x-y) \; iJ_\nu(x) \right\}$$

$$\Rightarrow G^{AA}_{\mu\nu,0}(x_1, x_2) = \frac{\delta}{i\delta J^\mu(x_1)} \frac{\delta}{i\delta J^\nu(x_2)} Z_{A,0}[J^\mu] \Big|_{J^\mu = 0}$$

= 

$= i\Delta_{F,\mu\nu}(x_1 - x_2)$

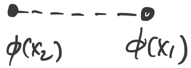
Summary : Free Field Theory

- exact solutions for $Z_0[J]$ $\Rightarrow$ arbitrary n-point functions
- general procedure: $\mathcal{L}_0 = \Phi^{(+)} \mathbb{D} \Phi$ $\Rightarrow$ e.o.m. $\mathbb{D}\Phi = 0$

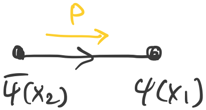
  ↳ solve using Green's functions: $\mathbb{D} \Delta_F(x) \sim \delta(x)$

  ↳ perform path integral: $Z_0[J] = \exp\left\{ \pm \int d^4x \int d^4y \; iJ(x) \, i\Delta_F(x-y) \, iJ(y) \right\}$

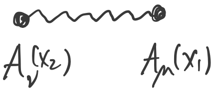
  $\Rightarrow$ basic building block: 2-point functions $G^{\Phi\Phi}(x_1, x_2) = i\Delta_F(x-y)$
  ("PROPAGATORS")

- scalar 

$= i\Delta_F(x_1 - x_2)$

- fermion 

$= i S_F(x_1 - x_2)$

- photon 

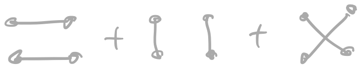
$= i\Delta_F^{\mu\nu}(x_1 - x_2)$

momentum space $\longrightarrow$

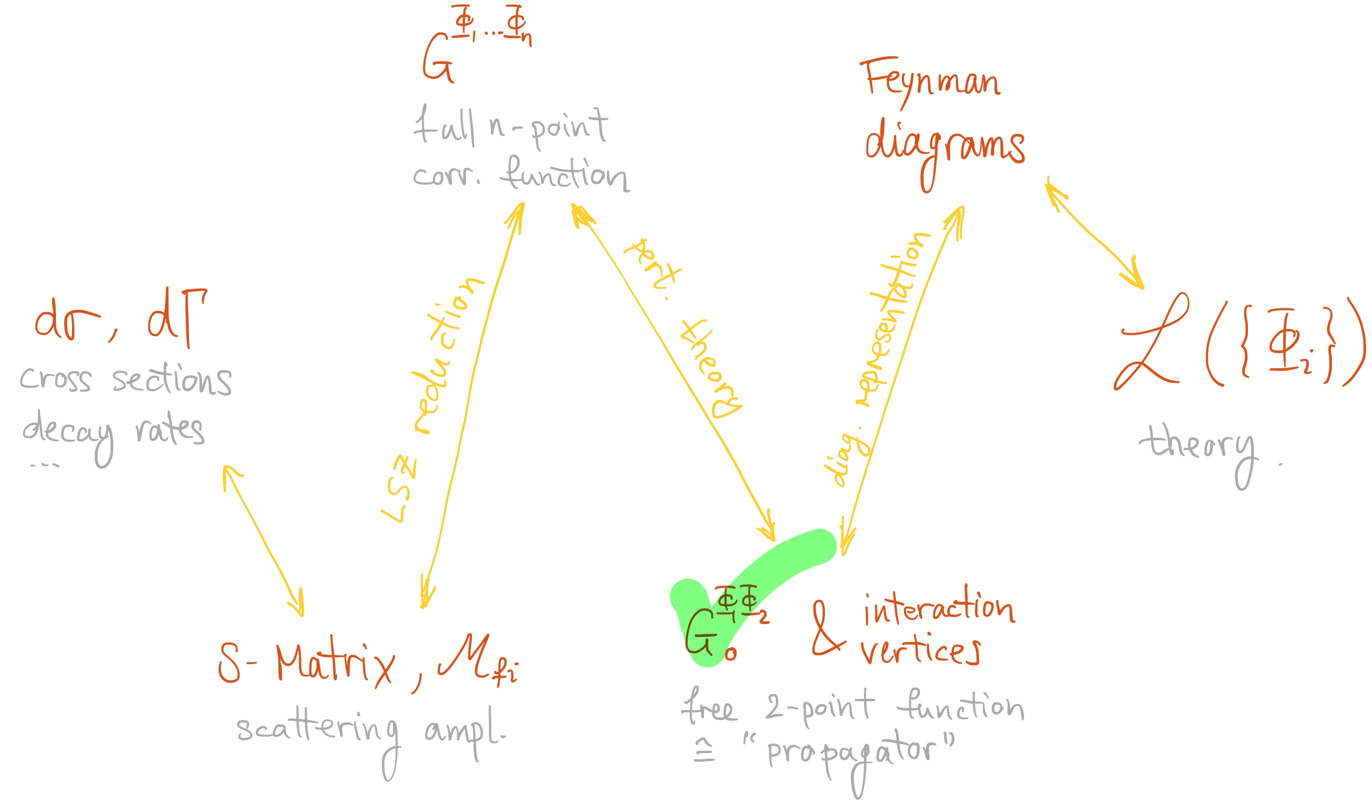
$$\frac{i}{p^2 - m^2}$$

$$\frac{i}{\not{p} - m}$$

$$\frac{-i}{p^2}\left[g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2} \right]$$

- so far: not very exciting: $G^{\phi\phi\phi\phi} =$ 

The Road Map


$G^{\Phi_1 \ldots \Phi_n}$
full n-point corr. function
Feynman diagrams
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decay rates
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theory
S-Matrix, $\mathcal{M}_{fi}$
scattering ampl.
$G_0^{\Phi_1 \Phi_2}$ & interaction vertices
free 2-point function
≙ "propagator"
LSZ reduction
pert. theory
diag. representation

Interacting Field Theory

- separate interactions from free part: $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$

  $\hookrightarrow \mathcal{L}_{\phi^4} = -\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 - \frac{\lambda^4}{4!}\phi^4$

  $\hookrightarrow \mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - eQA_\mu\bar{\psi}\gamma^\mu\psi$

- n-point function

vacuum of the interacting theory $(\neq |0\rangle)$

$$G^{\Phi_1\dots\Phi_n}(x_1,\dots,x_n) = \langle\Omega|T\,\Phi_1(x_1)\dots\Phi_n(x_n)|\Omega\rangle \quad \text{(canonical quant.)}$$

$$= \frac{1}{N}\int\mathcal{D}[\{\Phi\}]\,\Phi_1(x_1)\dots\Phi_n(x_n)\exp\left\{i\,S[\{\Phi\}]\right\} \quad \text{(path-integral quant.)}$$

Perturbation Theory

- generating functional for n-point functions

$$Z[\{J\}] = \frac{1}{N'}\int \mathcal{D}[\{\Phi\}]\, \exp\left\{ i\int d^4x \left[\mathcal{L}(\{\Phi(x)\}) + \sum_i J_i(x)\,\Phi_i(x) \right] \right\}$$

$$= \frac{1}{N'}\int \mathcal{D}[\{\Phi\}]\, \exp\left\{ i\int d^4y\, \mathcal{L}_{int}(\{\Phi_i(y)\}) \right\} \exp\left\{ i\int d^4x \left[\mathcal{L}_0(\{\Phi_i(x)\}) + \sum_i J_i(x)\,\Phi_i(x) \right] \right\}$$

$$= \frac{1}{N}\, \underbrace{\exp\left\{ i\int d^4y\, \mathcal{L}_{int}\left(\left\{ \Phi_i(y) \longrightarrow \frac{\delta}{i\,\delta J_i(y)} \right\}\right) \right\}}_{\text{series expansion } \hat{=} \text{ perturbative expansion in } \mathcal{L}_{int}}\; Z_0[\{J_i\}]$$

series expansion $\hat{=}$ perturbative expansion in $\mathcal{L}_{int}$

- $Z[\{J=0\}] \overset{!}{=} 1$

$\longleftrightarrow$ no "vacuum bubbles" 

Example : ϕ^3 Theory

$\mathcal{L}_{int} = \frac{g}{3!}\phi^3$

permutation over vertices $(y_1 \leftrightarrow y_2)$

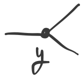
$$Z[J] = \frac{1}{N}\left\{1 + \frac{ig}{3!}\int d^4y \left(\frac{\delta}{i\delta J(y)}\right)^3 + \frac{1}{2}\left(\frac{ig}{3!}\right)^2 \int d^4y_1\, d^4y_2 \left(\frac{\delta}{i\delta J(y_1)}\right)^3 \left(\frac{\delta}{i\delta J(y_2)}\right)^3 + \dots\right\}$$

$$\times \left\{1 + \int d^4x \int d^4x' \, \frac{1}{2}\, iJ(x)\, i\Delta_F(x-x')\, iJ(x') + \dots\right\}$$

compensates:

$$\left(\frac{\delta}{i\delta J(y)}\right)^3 J(x_1)J(x_2)J(x_3) = 3!\,\delta(x_1-y)\,\delta(x_2-y)\,\delta(x_3-y)$$

- rules for $G^{\phi\dots\phi} = \frac{\delta}{i\delta J} \cdots \frac{\delta}{i\delta J} Z[J]\Big|_{J=0}$ at $\mathcal{O}(g^N)$:

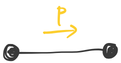
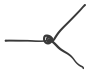
① propagator  $= i\Delta_F(x-x')$, vertex:  $= ig\int d^4y$

② draw all graphs with N vertices and n external legs

(drop all "vacuum bubbles" $\leftrightarrow$ $Z[0] = 1$)

③ multiply by symmetry factor $\frac{1}{S_G}$; S_G: # of permutations leaving graph inv.

- momentum space : (all momenta incoming)

①  $= \frac{i}{p^2 - m^2 + i0}$;  $= ig$

④ momentum cons. @ each vertex ; undetermined $\to \int \frac{d^4p}{(2\pi)^4}$

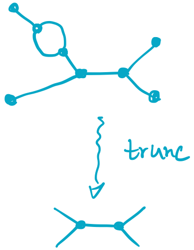
EX4

Vertex function

- we saw $G \longrightarrow G_{con}$ (connected) $\longrightarrow$ EX 4

- truncated ("amputated") correlation function

$G_{trunc} = G_{con} \Big|$ no external propagators / propagator corrections (do not draw ext. field points)



$$G_{trunc}^{\Phi_1 \cdots \Phi_n}(p_1, \dots, p_n) = G_{con}^{\Phi_1 \cdots \Phi_n}(p_1, \dots, p_n) \cdot \prod_{j=1}^{n} \left[G^{\Phi_j \Phi_j^{(+)}}(p_j, -p_j) \right]^{-1}$$

$$\Rightarrow G_{trunc}^{\Phi \Phi^{(+)}} = \left[G^{\Phi \Phi^{(+)}} \right]^{-1}$$

- vertex functions ← elementary building blocks

EX 5 & 6

$$\Gamma^{\Phi_1 \Phi_2}(p, -p) := -G_{trunc}^{\Phi_1 \Phi_2}(p, -p) = -\left[G^{\Phi_1 \Phi_2}(p, -p) \right]^{-1}$$

$$\Gamma^{\Phi_1 \cdots \Phi_n}(p_1, \dots, p_n) := G_{trunc}^{\Phi_1 \cdots \Phi_n}(p_1, \dots p_n) \Big|_{\text{only 1PI}}$$

"1PI": 1-particle irred.
no disconnected subgraphs by cutting <u>one internal</u> line

Källén-Lehmann spectral representation

interaction: ϕ also creates multi-particle states!

$$G^{\phi\phi}(x_1,x_2) = \langle \Omega | T \phi(x_1)\phi(x_2) | \Omega \rangle$$

$$= \ldots = \int_0^\infty \frac{d\mu^2}{2\pi} \rho(\mu^2) \int \frac{d^4p}{(2\pi)^4} \frac{i\, e^{-ip(x_1-x_2)}}{p^2-\mu^2+i0}$$

- momentum space

$$G^{\phi\phi}(p,-p) = \frac{i\, R_\phi}{p^2 - m^2 + i0} + \int_{\sim 4m^2}^{\infty} \frac{d\mu^2}{2\pi} \rho(\mu^2) \frac{i}{p^2-\mu^2+i0}$$

$\hookrightarrow$ R_ϕ : residuum $\hat{=}$ $|\langle p | \phi | \Omega \rangle|^2$

(prob. for 1-particle state)

$\hookrightarrow$ m : physical mass (in general $\neq$ param. in $\mathcal{L}$)

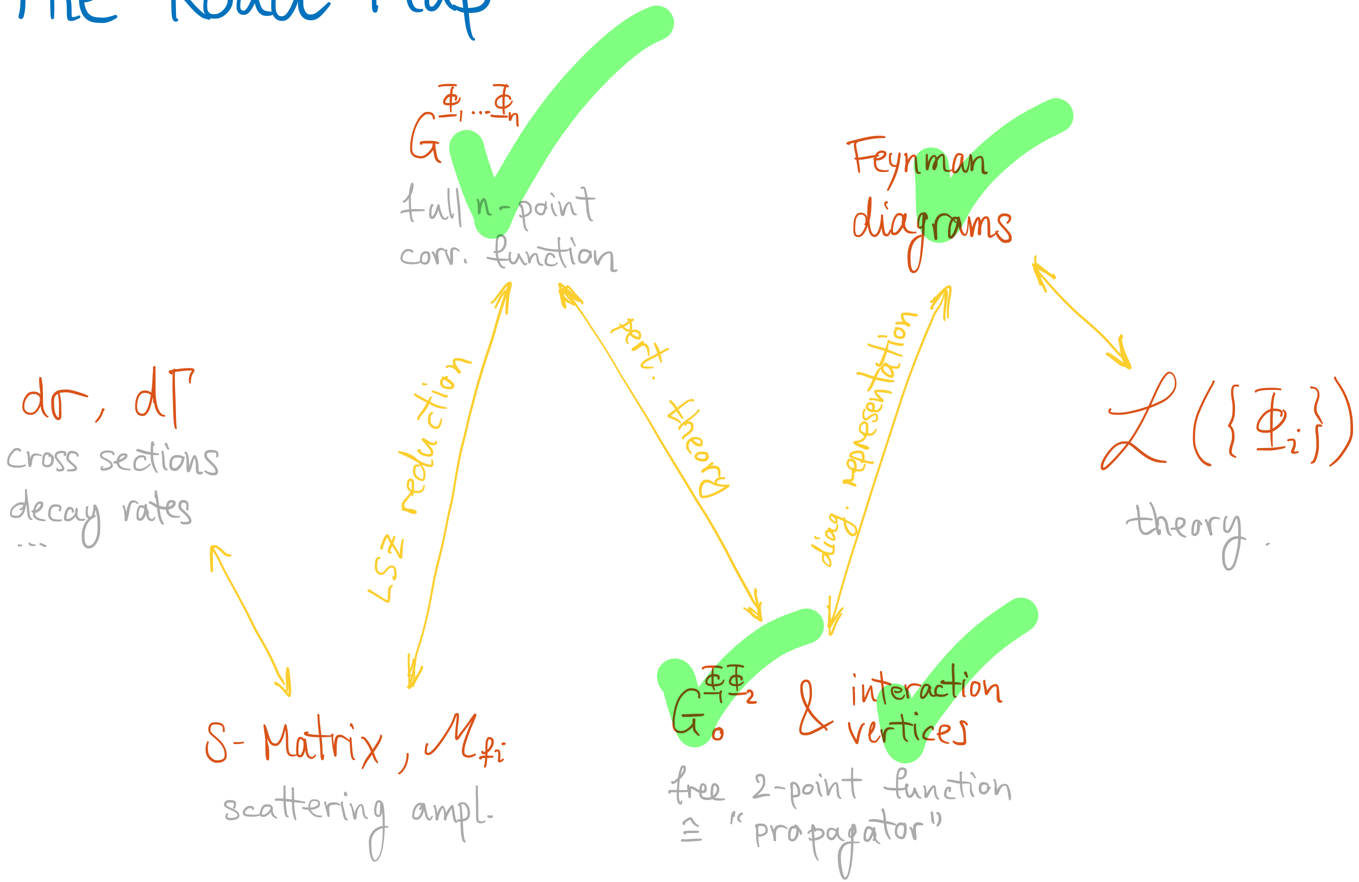
$\rightsquigarrow$ free fields : $R_\phi = 1$, $m = m_0$

Done with n-pt functions

- G : all diags (no vacuum bubbles)
- G_{con} : only connected
- G_{trunc} : "amputate" all external legs
- Γ : only 1PI graphs of G_{trunc} $\left(\Gamma^{\phi\phi} = -[G^{\phi\phi}]^{-1} \right)$

Feynman diagrams & rules to compute them

The Road Map


$G^{\Phi_1 \dots \Phi_n}$
full n-point
corr. function
Feynman
diagrams
$d\sigma$, $d\Gamma$
cross sections
decay rates
...
LSZ reduction
pert. theory
diag. representation
$\mathcal{L}(\{\Phi_i\})$
theory
S-Matrix, $\mathcal{M}_{fi}$
scattering ampl.
$G_0^{\Phi_1\Phi_2}$ & interaction vertices
free 2-point function
≙ "propagator"

The S Matrix & LSZ reduction

- central object in scattering & decays: S matrix
  ↪ time evolution operator from $t \to -\infty$ ("in") to $t \to +\infty$ ("out")

$$S_{fi} = \langle f | S | i \rangle = \langle f | \mathbb{1} + iT | i \rangle$$

$$= \langle f | i \rangle + (2\pi)^4 \delta^4 (p_i - p_f) \, i\mathcal{M}_{fi}$$

scattering amplitude

- Lehmann Symanzik Zimmermann (LSZ) reduction formula:

$$i\mathcal{M}^{n \to m}(p_1, \dots, p_n, p_1', \dots p_m') = \prod_{i=1}^{n} f_{in}^{\Phi_i}(p_i) \sqrt{R_{\Phi_i}} \prod_{j=1}^{m} f^{\Phi_j'}(p_j') \sqrt{R_{\Phi_j'}}$$

$$\times \, G_{trunc}^{\Phi_1 \dots \Phi_n \Phi_1' \dots \Phi_m'}(p_1, \dots p_n, -p_1', \dots, -p_m') \Big|_{\text{on-shell}}$$

↪ truncate ext. legs & replace by 1-particle wave functions

$$f^{\phi}_{in/out} = 1 \, , \quad f^{\psi}_{in/out} = u(p), v(p) \, , \quad f^{\bar\psi}_{in/out} = \bar v(p), \bar u(p) \, , \quad f^{A_\mu}_{in/out} = \varepsilon_\mu^{(*)}(p)$$

↪ go on-shell: $p_i^2 = m_i^2 \, , \quad p_j'^2 = m_j'^2$

What we were after all this time

- differential cross section $\quad p_a + p_b \longrightarrow p_1 + \dots p_n \quad (2 \rightarrow n)$

$$d\sigma = \underbrace{\frac{1}{4\sqrt{(p_a \cdot p_b) - m_a^2 m_b^2}}}_{\text{flux}} \underbrace{|\mathcal{M}_{fi}|^2}_{\text{sq. ME}} \underbrace{\frac{1}{S_{\{n\}}}}_{\text{symmetry factor}} \underbrace{d\Phi_n(p_1, \dots, p_n; p_a + p_b)}_{\text{LIPS}}$$

$$S_{\{n\}} = n_1! \cdot \dots$$

- decay width $\quad p_a \longrightarrow p_1 + \dots p_n$

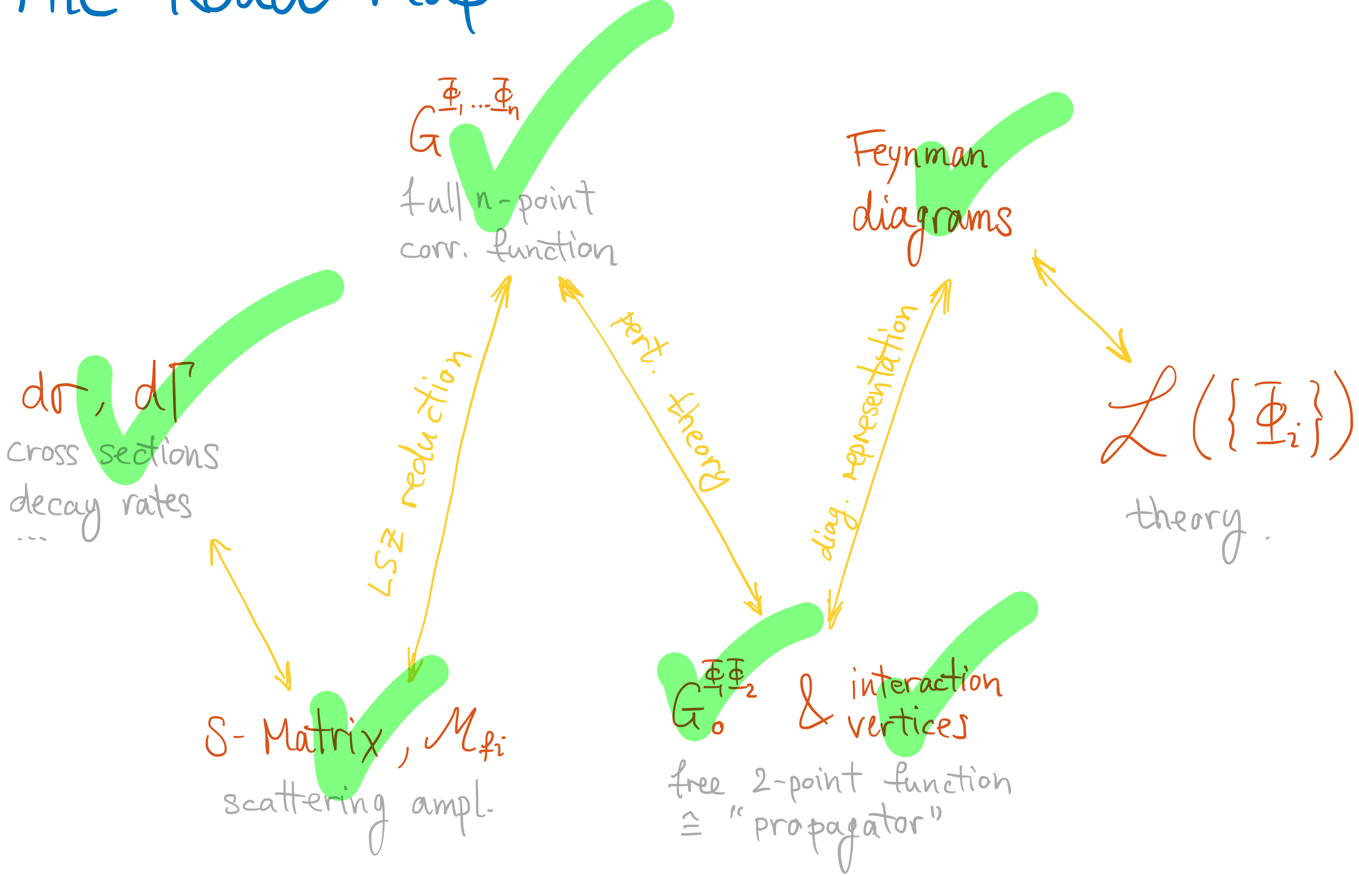
$$d\Gamma = \frac{1}{2E_{p_a}} |\mathcal{M}_{fi}|^2 \frac{1}{S_{\{n\}}} d\Phi_n(p_1, \dots p_n; p_a)$$

- LIPS

$$d\Phi_n(p_1, \dots p_n; Q) = \prod_{i=1}^{n} \left[\frac{d^4 p_i}{(2\pi)^3} \cdot \underbrace{\delta^+(p_i^2 - m_i^2)}_{\delta(p_i^2 - m_i^2)\,\Theta(p^0)} \right] \underbrace{(2\pi)^4 \delta\left(Q - \sum_{i=1}^{n} p_i\right)}_{\text{over-all 4-momentum conservation}}$$

$$= \prod_{i=1}^{n} \left[\frac{d^3 p_i}{(2\pi)^3\, 2E_i} \right] (2\pi)^4 \delta\left(Q - \sum p_i\right)$$

The Road Map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Appendix: Vertex Functions @ Tree-Graph Level

- reintroduce $\hbar$:

$$Z[J] = \frac{1}{N} \int \mathcal{D}[\Phi] \exp\left\{\frac{i}{\hbar}\int d^4x \left[\mathcal{L} + \hbar J(x)\Phi(x)\right]\right\}$$

$$= \frac{1}{N} \exp\left\{\frac{i}{\hbar}\int d^4y\, \mathcal{L}_{int}\left(\frac{\delta}{i\delta J(y)}\right)\right\} \exp\left\{\frac{\hbar}{2}\int d^4x\, d^4x'\; iJ(x)\; i\Delta_F(x-x')\; iJ(x')\right\}$$

$\Rightarrow$ each vertex (V): $\hbar^{-1}$, each propagator (I): $\hbar^{+1}$

$\Leftrightarrow$ for vertex graphs (1PI) $\quad L = I - V + 1$

$\Rightarrow$ $\hbar$ expansion $\hat{=}$ loop expansion : $\Gamma = \sum_L \hbar^L \Gamma^{(L)}$

- method of stationary phase:

$\Rightarrow$ classical $(\hbar \to 0)$ solution Φ_{cl} from e.o.m.: $\quad \frac{\delta\mathcal{L}}{\delta\Phi} = \partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\Phi)}$

0: tree-graph level

$$\Rightarrow Z^{(0)}[J] = \frac{1}{N} \exp\left\{\frac{i}{\hbar}\int d^4x\, \mathcal{L}(\Phi_{cl}, \partial_\mu\Phi_{cl})\right\} \exp\left\{i\int d^4y\, J(y)\,\Phi_{cl}(y)\right\}$$

$$\Rightarrow Z^{(0)}_{con}[J] = \ln\left(Z[J]\right) = \text{const.} + \frac{i}{\hbar}\int d^4x\, \mathcal{L}(\Phi_{cl}, \partial_\mu\Phi_{cl}) + i\int d^4y\, J(y)\,\Phi_{cl}(y)$$

conjugate field: $\quad \frac{\delta Z^{(0)}_{con}[J]}{i\delta J(x)} = \Phi_{cl}(x)$

$$\Rightarrow \text{Legendre-transf:} \quad \Gamma^{(0)}[\Phi_{cl}] = -i\int d^4x\, \Phi_{cl}(x)\, J(x) + Z^{(0)}_{con}[J] = \frac{i}{\hbar}\int d^4x\, \mathcal{L}(\Phi_{cl}, \partial_\mu\Phi_{cl})$$