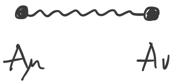
II. QED & Renormalization

The QED Feynman Rules

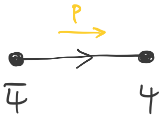
$$\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}\,(i\not{\partial} - Q e \not{A} - m)\,\psi \; - \frac{1}{2\xi}(\partial \cdot A)^2$$

let's take this for granted ... ⇝ part III

- propagators:

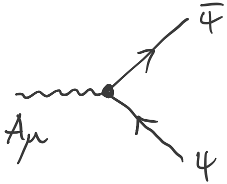


$$= \frac{-i}{k^2 + i0}\left[g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2} \right]$$



$$= \frac{i}{\not{k} - m + i0}$$

- vertex:



$$= -ieQ_f\,\gamma^\mu$$

- external legs:

incoming $\xrightarrow{k}$

$= \varepsilon_\mu(k,\lambda)$

$= u(k,\sigma)$

$= \bar{v}(k,\sigma)$

outgoing $\xrightarrow{k}$

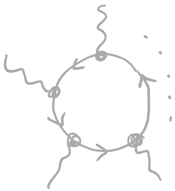
$= \varepsilon^*_\mu(k,\lambda)$

$= \bar{u}(k,\sigma)$

$= v(k,\sigma)$

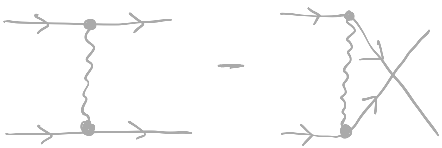
More Rules ...

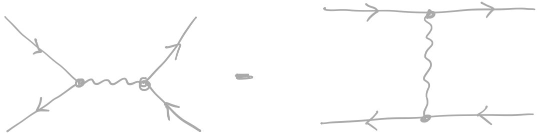
- momentum conservation @ each vertex
- undetermined (loop) momenta $\rightsquigarrow \int \frac{d^4p}{(2\pi)^4}$
- traverse fermion lines in opposite direction to the arrow
- symmetry factors $1/S_G$



- (-1) for each closed fermion loop
- relative (-1) between diagrams related by exchange of external fermion legs

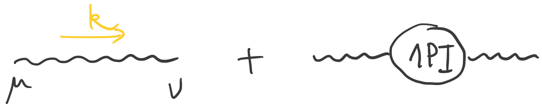
$e^- e^- \to e^- e^-$ or Bhabha





QED Vertex Functions

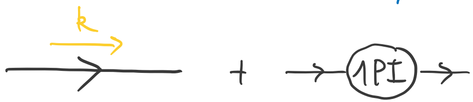
- photon 2-pt :



$$\Gamma^{AA}_{\mu\nu}(k,-k) = -\left[G^{AA}_{\mu\nu}(k,-k)\right]^{-1} = -i\left[g_{\mu\nu}k^2 - \left(1-\frac{1}{\xi}\right)k_\mu k_\nu\right] - i\,\Sigma^{AA}_{\mu\nu}(k)$$

"self energies"

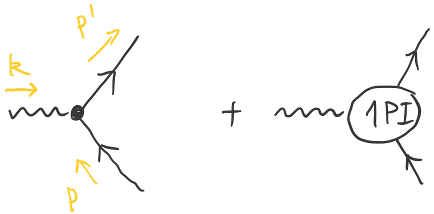
- electron 2-pt :



$$\Gamma^{\bar\psi\psi}(-k,k) = -\left[G^{\psi\bar\psi}(k,-k)\right]^{-1} = i(\not k - m) + i\,\Sigma^{\bar\psi\psi}(k)$$

- electron-photon vertex :

"vertex corrections"



$$\Gamma^{A\bar\psi\psi}_{\mu}(k,-p',p) = -ieQ\gamma_\mu - ieQ\,\Lambda_\mu(p',p)$$

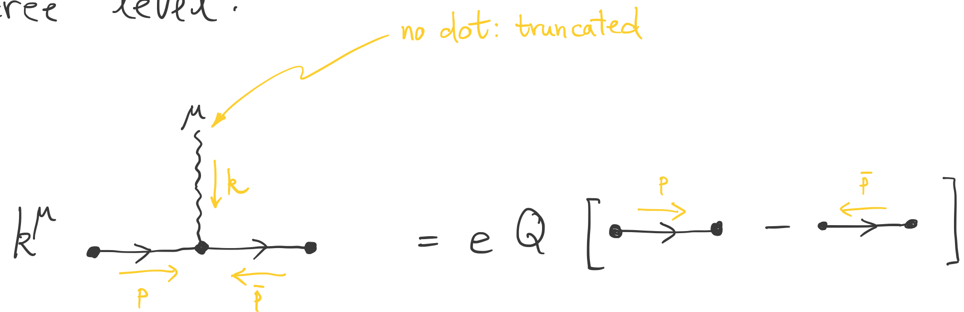
The Ward-Takahashi Identity

- identities between correlation functions due to electromagnetic $U(1)$ symmetry

- Ward identity for the electron-photon vertex:

$$k^\mu \; G^{\psi\bar\psi}(\bar p, -\bar p)\; \Gamma_\mu^{A\bar\psi\psi}(k,\bar p, p)\; G^{\psi\bar\psi}(-p,p) = eQ\left[G^{\psi\bar\psi}(-p,p) - G^{\psi\bar\psi}(\bar p,-\bar p)\right]$$

$\hookrightarrow$ tree level:



EX 7

$\Big($ not using the vertex $\Gamma_\mu^{A\bar\psi\psi}$ but the Green's function

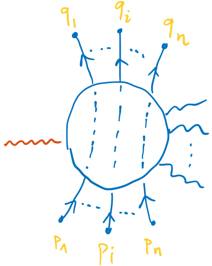
$$\frac{i}{\xi} k^2 k^\mu\, G_\mu^{A\psi\bar\psi}(k,\bar p,p) = eQ\left[G^{\psi\bar\psi}(-p,p) - G^{\psi\bar\psi}(\bar p,-\bar p)\right]$$

$\Big)$

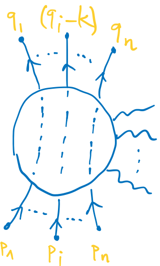
The Ward-Takahashi Identity

- general amplitude with an external photon:

$$\mathcal{A} = \varepsilon_\mu^{(*)}(k)\,\mathcal{A}^\mu =$$



$$\Rightarrow k_\mu \mathcal{A}^\mu = eQ \sum_{i=1}^{n} \Big[$$



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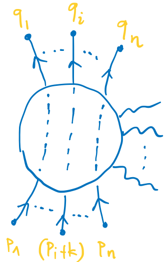
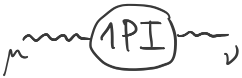


$$\Big]$$

- for scattering amplitudes (LSZ $\longleftrightarrow$ truncation / on-shell residue)

$$\mathcal{M} = \varepsilon_\mu^{(*)}(k)\,\mathcal{M}^\mu \quad \Rightarrow \quad k_\mu \mathcal{M}^\mu = 0$$

1-Loop Corrections: Photon Self Energy

$$-i\Sigma^{AA}_{\mu\nu}(k) =$$  $$=$$  $$+ \dots$$

$$\Rightarrow \Sigma^{AA}_{\mu\nu}(k) = i(-1)\int \frac{d^4q}{(2\pi)^4} \operatorname{Tr}\left\{(-ieQ\gamma_\mu)\frac{i}{\not{q}-m}(-ieQ\gamma_\nu)\frac{i}{\not{q}+\not{k}-m}\right\}$$

$$= -ie^2Q^2 \int \frac{d^4q}{(2\pi)^4} \frac{\operatorname{Tr}[\gamma_\mu(\not{q}+m)\gamma_\nu(\not{q}+\not{k}+m)]}{(q^2-m^2)((q+k)^2-m^2)}$$

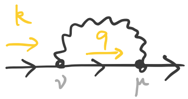
$\hookrightarrow$ UV behaviour $(q \to \infty)$

$$\int d^4q \sim \int dq\, q^3 \;; \quad (q^2-m^2),\ ((q+k)^2-m^2) \sim q^2$$

$$\Rightarrow \Sigma^{AA}_{\mu\nu} \sim \int dq \left\{\frac{1}{q}, 1, q\right\} \Rightarrow$$ quadratically divergent!

1-Loop Corrections : Elektron Self Energy

$i\Sigma_1^{\bar\psi\psi}(k) =$ 

$=$ 

$+ \dots$ (Feynman gauge: $\xi = 1$)

$$\Rightarrow \Sigma_1^{\bar\psi\psi}(k) = (-i)\int \frac{d^4q}{(2\pi)^4}\,(-ieQ\gamma_\mu)\,\frac{i}{\not{q}-m}\,(-ieQ\gamma_\nu)\cdot\frac{-i\,g^{\mu\nu}}{(q-k)^2-\lambda^2}$$

$$= i e^2 Q^2 \int \frac{d^4q}{(2\pi)^4}\,\frac{\gamma_\mu(\not{q}+m)\gamma^\mu}{(q^2-m^2)\left((q-k)^2-\lambda^2\right)}$$

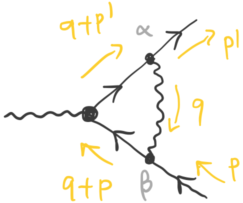
↪ UV behaviour ($q \rightarrow \infty$)

$$\Rightarrow \Sigma_1^{\bar\psi\psi} \sim \int dq\,\left\{\frac{1}{q}, 1\right\}$$ $\Rightarrow$ linearly divergent!

1- Loop Corrections: Electron-Photon Vertex

(Feynman gauge: $\xi = 1$)

$$-ieQ\,\Lambda_\mu(p',p) =$$ 

$=$ 

$+ \ldots$

$$\Rightarrow \Lambda_\mu(p',p) = \frac{i}{eQ} \int \frac{d^4q}{(2\pi)^4} \frac{-i g^{\alpha\beta}}{q^2-\lambda^2} (-ieQ\gamma_\alpha) \frac{i}{\not{q}+\not{p}'-m} (-ieQ\gamma_\mu) \frac{i}{\not{q}+\not{p}-m} (-ieQ\gamma_\beta)$$

$$= -ie^2Q^2 \int \frac{d^4q}{(2\pi)^4} \frac{\gamma_\alpha (\not{q}+\not{p}'+m)\gamma_\mu(\not{q}+\not{p}-m)\gamma^\alpha}{(q^2-\lambda^2)\left((q+p')^2-m^2\right)\left((q+p)^2-m^2\right)}$$

$\hookrightarrow$ UV behaviour ($q \to \infty$)

EX 8

$$\Rightarrow \Lambda_\mu(p',p) \sim \int dq \left\{ \frac{1}{q^3}, \frac{1}{q^2}, \frac{1}{q} \right\} \Rightarrow \text{logarithmically divergent!}$$

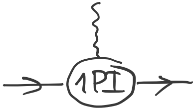
Superficially Divergent Vertex Functions

- EX 8 $\Rightarrow$ $\omega(G) = 4 - \frac{3}{2} E_4 - E_A$

$\omega = 3$ $\longrightarrow$ $= 0$ (quant. number of vacuum)

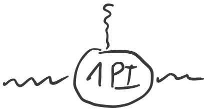
$\omega = 2$

$\omega = 1$

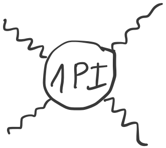


$\omega = 0$

$\left.\begin{matrix} \omega = 2 \\ \omega = 1 \\ \omega = 0 \end{matrix}\right\}$ actually divergent ones



$\omega = 1$ $\longrightarrow$ $= 0$ (Furry's theorem)



$\omega = 0$ $\longrightarrow$ $=$ finite (Ward Id.) $\rightsquigarrow$ Appendix

(gauge symmetry essential in renormalizability!)

The General Idea

Problem: radiative corrections give infinities (UV)

1. regularization: tame the infinities by modifying the theory such that calculations become formally well-defined.
   $\leftrightarrow$ regularization parameter δ for $\delta \to \delta_0$: recover div. expressions

2. renormalization: how do we know what λ_0 (bare param) is?
   $\hookrightarrow$ we have to <u>measure</u> it
   $\hookrightarrow$ prescription / observable how to measure it: scheme
   $\hookrightarrow$ once measurable quantities are re-expressed in terms of other measurable quantities $\Rightarrow$ δ-dependence cancels!
   (independent of how it was regularized)

Regularization Schemes

- momentum cut-off: $\int d^4q \longrightarrow \int\limits_{|q^0|,|\vec{q}|<\Lambda} d^4q$ $\quad (\delta = \Lambda\,,\ \delta_0 = \infty)$
- "Pauli-Villars": $\dfrac{1}{q^2-m^2} \longrightarrow \dfrac{1}{q^2-m^2} - \dfrac{1}{q^2-M^2}$ $\quad (\delta = M,\ \delta_0 = \infty)$
- "Momentum subtraction": ex. $\Gamma(p_1^2, \dots) \longrightarrow \Gamma(p_1^2, \dots) - \Gamma(p_1^2 = q_1^2, \dots)$
- put it on the lattice: $\int d^4x \longrightarrow \sum\limits_{x_i} \Delta_i$ $\quad (\delta = a\,,\ \delta_0 = 0)$
  - $\Leftarrow$ lattice spacing
- dimensional regularization: $\delta = D\,,\ \delta_0 = 4$
  - $\oplus$ Lorentz- & gauge-invariance, IR regularization, simple

Dimensional Regularization (Dim Reg)

$$\int \frac{d^4q}{(2\pi)^4} \longrightarrow \int \frac{d^Dq}{(2\pi)^D} \qquad (D = 4-2\epsilon)$$

- defined as a set of axioms that integrals should satisfy:
  - ↪ shifts of loop momenta <u>always</u> allowed:

$$\int d^Dq\, f(k+p) \equiv \int d^Dq\, f(q)$$

  - ↪ analog of D-dim. rotational symmetry / Lorentz transformations

$$\int d^Dq\, f(\Lambda q) \equiv \int d^Dq\, f(q)$$

scale-less integrals vanish: $\int d^Dq\, (q^2)^\alpha \equiv 0$

  - ↪ scaling property $\int d^Dq\, f(a\cdot q) \equiv a^{-D} \int d^Dq\, f(q)$
  - ↪ linearity: $\int d^Dq\, [f(q)+g(q)] \equiv \int d^Dq\, f(q) + \int d^Dq\, g(q)$
  - ↪ differentiation & integration commute:

$$\frac{\partial}{\partial p_\mu} \int d^Dq\, f(p,q) \equiv \int d^Dq\, \frac{\partial}{\partial p_\mu} f(p,q)$$

More on Dim Reg

- metric $\quad \delta^\mu_\mu = g^\mu_\mu = D$
- Dirac matrices : $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \Rightarrow \gamma^\mu \gamma_\mu = g^\mu_\mu = D$

  $\gamma^\mu \gamma_\nu \gamma_\mu = (2-D)\gamma_\nu \quad , \ldots$

  can define $\mathrm{Tr}(\mathbb{1}) = 4$
- no natural extension of γ^5 in D dimensions

  $\mathrm{Tr}[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5] \sim \varepsilon^{\mu\nu\rho\sigma}$ $\rightsquigarrow$ need to adapt γ^5 algebra
- field / coupling dimensions $\quad ([S]=0 \Rightarrow [\mathcal{L}]=D)$

  $[\psi] = \frac{D-1}{2} \quad , \quad [\phi] = [A_\mu] = \frac{D-2}{2}$

  scale of Dim Reg !

  $\Rightarrow \quad [e] = \frac{4-D}{2} \quad ; \quad$ keep $[e]=0 \Rightarrow e \longrightarrow \mu^{\frac{4-D}{2}} e$ $\quad$ (often absorbed into loop integration)
- $\omega(G)$ in D-dim $\rightsquigarrow$ EX 8

$$\omega(G) = D - \frac{D-1}{2} E_\psi - \frac{D-2}{2} E_A - V\left(\frac{D-4}{2}\right)$$

1-Loop Integrals — The Roadmap

generally confronted with

tensor integrals $T^N_{\mu_1 \dots \mu_M}$

$$\int d^D q \, \frac{q_{\mu_1} \cdots q_{\mu_M}}{\prod_{i=1}^{N} D_i}$$

$$\left(D_i = (q+p_i)^2 - m_i^2 \right)$$

merge propagators

to isolate loop integral

$$\int \left(\prod_{i=1}^{N} dx_i \right) \frac{\delta(1-\sum x_i)}{\left[\sum_i D_i x_i\right]^N}$$

Passarino-Veltman reduction

Feynman parameters

tadpole

generic integral

perform $\int d^D q$

$$I_N(A) := \int d^D q \, \left[q^2 - A + i0 \right]^{-N}$$

scalar integrals S^N

$$\int d^D q \, \frac{1}{\prod_{i=1}^{N} D_i}$$

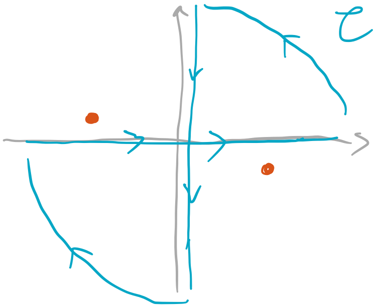
Our First Integral

$$I_n(A) := \int d^D q \, \frac{1}{(q^2 - A + i0)^n}$$

$(D < 2n \,,\; A > 0)$
convergence

① Wick-rotation in the complex q^0 plane

$\hookrightarrow$ poles @ $q^0 = \pm\sqrt{\vec{q}^{\,2} + A^2 - i0} = \pm\sqrt{\vec{q}^{\,2} + A^2} \mp i0$



$$\oint_{\mathcal{C}} dq^0 \, (q^2 - A^2 + i0) = 0$$

Eucledian coordinates: $q_E^0 = i\, q^0$

$$\Rightarrow \int_{-\infty}^{+\infty} dq^0 (\ldots) = \int_{-i\infty}^{i\infty} dq^0 (\ldots) = i \int_{-\infty}^{\infty} dq_E^0 (\ldots)$$

$$\left(q^2 = (q^0)^2 - \vec{q}^{\,2} = -(q_E^0)^2 - \vec{q}^{\,2} = -\, q_E^2 \right)$$

② Integration in polar coordinates:

$$\int d^D q_E = \int d\Omega_D \int_0^\infty dq_E \, (q_E)^{D-1} = \int d\Omega_D \int_0^\infty dq_E^2 \, \frac{1}{2} \left(q_E^2\right)^{\frac{D}{2}-1}$$

Ex 9

Result for $I_n(A)$

$$I_n(A) \overset{(y=q_E^2)}{=} i(-1)^n \frac{\pi^{D/2}}{\Gamma(D/2)} (A-i0)^{\frac{D}{2}-n} \int_0^\infty dy\, y^{\frac{D}{2}-1} (1+y)^{-n}$$

$B(\frac{D}{2}, n-\frac{D}{2})$

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

$$= i(-1)^n \pi^{D/2} \frac{\Gamma(n-D/2)}{\Gamma(n)} (A-i0)^{\frac{D}{2}-n}$$

$\hookrightarrow$ analytic continuation in D & A possible!

- properties of the Gamma function

$\hookrightarrow$ $\Gamma(z)$ is meromorphic: Poles @ $z=-n$, $n \in \mathbb{N}_0$

$\hookrightarrow$ $\Gamma(z+1) = z\,\Gamma(z)$

$\hookrightarrow$ $\Gamma(n+1) = n!$ $\forall n \in \mathbb{N}$

$\hookrightarrow$ $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$ with $\gamma_E = 0.57\ldots$ (Euler const)

UV divergences as $\frac{1}{\epsilon}$ poles

1-Loop Integrals — The Roadmap

generally confronted with

tensor integrals $T^N_{\mu_1 \dots \mu_M}$

$$\int d^D q \, \frac{q_{\mu_1} \cdots q_{\mu_M}}{\prod_{i=1}^{N} D_i}$$

$$\left(D_i = (q+p_i)^2 - m_i^2 \right)$$

Passarino-Veltman reduction

scalar integrals S^N

$$\int d^D q \, \frac{1}{\prod_{i=1}^{N} D_i}$$

Feynman parameters

merge propagators

to isolate loop integral

$$\int \left(\prod_{i=1}^{N} dx_i \right) \frac{\delta(1 - \sum x_i)}{\left[\sum_i D_i x_i \right]^N}$$

tadpole

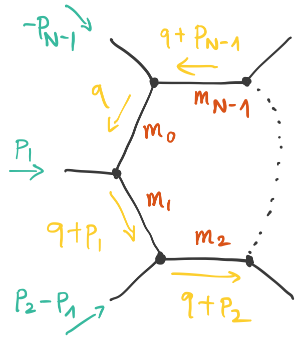
generic integral

perform $\int d^D q$

$$I_N(A) := \int d^D q \, \left[q^2 - A + i0 \right]^{-N}$$

Scalar Integrals

$$S^N(p_1, \dots, p_{N-1}, m_0, \dots, m_{N-1}) := \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dq \prod_{n=0}^{N-1} \frac{1}{(q+p_n)^2 - m_n^2 + i0} \qquad (p_0 = 0)$$



Notation:

$S^1 = A_0$ (tadpole)

$S^2 = B_0$ (bubble)

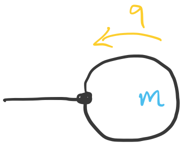
$S^3 = C_0$ (triangle)

$\vdots$ $\vdots$

- all scalar integrals can be brought into the form of $I_n(A)$

$\hookrightarrow$ D-dimensional momentum integration ✓

1-pt Function

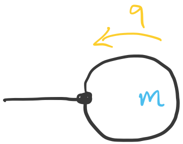


$$A_0(m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \underbrace{\int d^D q \, (q^2 - m^2 + i0)^{-1}}_{I_1(m^2)}$$

$$= -m^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} \Gamma\left(\frac{2-D}{2}\right)$$

EX10

1-pt Function



$$A_0(m) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \underbrace{\int d^D q \left(q^2 - m^2 + i0 \right)^{-1}}_{I_1(m^2)}$$

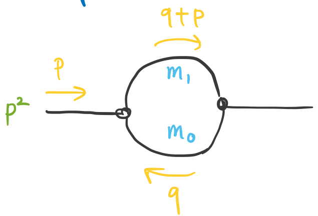
$$= -m^2 \left(\frac{m^2}{4\pi\mu^2} \right)^{\frac{D-4}{2}} \Gamma\left(\frac{2-D}{2} \right)$$

$$= m^2 \Big[\underbrace{\frac{1}{\epsilon} - \gamma_E + \ln(4\pi)}_{=: \Delta \ \text{(divergent constant)}} - \ln\left(\frac{m^2}{\mu^2} \right) + 1 \Big] + \mathcal{O}(\epsilon)$$

$\rightsquigarrow$ $\overline{\text{MS}}$ scheme

- note: $A_0(m)\big|_{\text{div}} = \dfrac{m^2}{\epsilon}$

2-pt Function



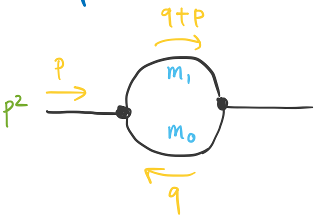
$$B_0(p, m_0, m_1) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dq \; \underbrace{(q^2 - m_0^2 + i0)^{-1} \left((q+p)^2 - m_1^2 + i0\right)^{-1}}_{\frac{1}{a} \cdot \frac{1}{b}}$$

- to reduce S^N to $I_n(A)$ we use Feynman parameters

$$\frac{1}{A_1 \cdot \ldots \cdot A_N} = \Gamma(N) \int_0^1 \prod_{i=1}^{N} dx_i \; \frac{\delta(1 - \sum_{i=1}^{N} x_i)}{\left[\sum_{i=1}^{N} x_i A_i\right]^N}$$

EX 11

2-pt Function



$$B_0(p, m_0, m_1) = \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dq \; \underbrace{(q^2 - m_0^2 + i0)^{-1}\left((q+p)^2 - m_1^2 + i0\right)^{-1}}$$

(Feynman param)

$$= \int_0^1 dx \left\{ (q^2 - m_0^2 + i0)(1-x) + \left[(q+p)^2 - m_1^2 + i0\right] x \right\}^{-2}$$

$$= \int_0^1 dx \left\{ \underbrace{(q+xp)^2}_{q'} \underbrace{- x^2p^2 + x(p^2 - m_1^2 + m_0^2) - m_0^2}_{=: -A} + i0 \right\}^{-2}$$

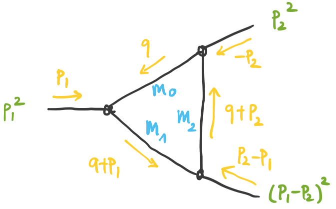
$$\Rightarrow B_0(p, m_0, m_1) = \frac{(2\pi\mu)^{4-D}}{i\pi} \int_0^1 dx \, I_2(A) = (4\pi\mu^2)^{\frac{4-D}{2}} \, \Gamma\left(\frac{4-D}{2}\right) \int_0^1 dx \left[x^2p^2 + x(p^2 - m_1^2 + m_0^2) + m_0^2 - i0\right]^{\frac{D-4}{2}}$$

$$= \Delta - \int_0^1 dx \, \ln\left[\frac{x^2p^2 - x(p^2 - m_1^2 + m_0^2) + m_0^2 - i0}{\mu^2}\right] + \mathcal{O}(\epsilon)$$

- note: $B_0(p, m_0, m_1)\big|_{div} = \frac{1}{\epsilon}$
- alternative notation: $B_0(p^2, m_0, m_1)$
- symmetry: $B_0(p^2, m_0, m_1) = B_0(p^2, m_1, m_0)$

EX 12

3-pt Function



finite in D=4 dimensions!

$$C_0(p_1,p_2,m_0,m_1,m_2) = \frac{1}{i\pi^2}\int d^4q \, \underbrace{[q^2-m_0^2+i0]^{-1}\,[(q+p_1)^2-m_1^2+i0]^{-1}\,[(q+p_2)^2-m_2^2+i0]^{-1}}$$

(Feynman param) $\displaystyle \frac{1}{abc} = 2\int_0^1 dx \int_0^{1-x} dy \, [a(1-x-y)+bx+cy]^{-3}$

... complete the square in q, shift momentum $q \to q + \dots$, identify $I_3(A)$ ...

$$\Rightarrow C_0(p_1,p_2,m_0,m_2,m_3) = -\int_0^1 dx \int_0^{1-x} dy \, \left[x^2p_1^2 + y^2p_2^2 + 2xy\,p_1p_2 - x(p_1 - m_1^2 + m_0^2) - y(p_2^2 - m_2^2 + m_0^2) + m_0^2 + i0\right]^{-1}$$

- steps:
  - linearize in x or y: $x \to x+\alpha$ to kill all y^2 terms $\rightsquigarrow$ y-integration
  - decompose quad. forms into lin. factors $\rightsquigarrow$ $\displaystyle\int dx \frac{\ln(ax+b)}{cx+d}$ $\rightsquigarrow$ $\ln$, $\ln^2$, Li_2
- note: $C_0(p_1,p_2,m_0,m_1,m_2)\Big|_{\mathrm{div}} = 0$
- alternative notation: $C_0(p_1^2, (p_1-p_2)^2, p_2^2, m_0, m_1, m_2)$

1-Loop Integrals — The Roadmap

generally confronted with

tensor integrals $T^N_{\mu_1 \dots \mu_M}$

$$\int d^D q \, \frac{q_{\mu_1} \cdots q_{\mu_M}}{\prod_{i=1}^{N} D_i}$$

$$\left(D_i = (q+p_i)^2 - m_i^2 \right)$$

Passarino-Veltman reduction

scalar integrals S^N

$$\int d^D q \, \frac{1}{\prod_{i=1}^{N} D_i}$$

Feynman parameters

merge propagators

to isolate loop integral

$$\int \left(\prod_{i=1}^{N} dx_i \right) \frac{\delta(1 - \sum x_i)}{\left[\sum_i D_i x_i \right]^N}$$

tadpole

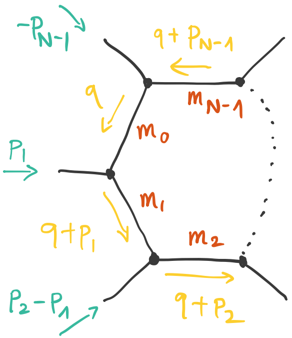
generic integral

perform $\int d^D q$

$$I_N(A) := \int d^D q \, \left[q^2 - A + i0 \right]^{-N}$$

Tensor Integrals

$$T^N_{\mu_1\cdots\mu_M}(p_1,\dots,p_{N-1},m_0,\dots,m_{N-1}) := \frac{(2\pi\mu)^{4-D}}{i\pi^2}\int d^Dq\; q_{\mu_1}\cdots q_{\mu_M}\prod_{n=0}^{N-1}\frac{1}{(q+p_n)^2-m_n^2+i0} \qquad (p_0=0)$$



Notation:

$T^1 = S^1 = A_0$

$T^2 = S^2 = B_0\,,\quad T^2_\mu = B_\mu\,,\quad T^2_{\mu\nu} = B_{\mu\nu}$

$T^3 = S^3 = C_0\,,\quad T^3_\mu = C_\mu\,,\quad T^3_{\mu\nu} = C_{\mu\nu}$

$\vdots \qquad \vdots$

- symmetric in $\mu_1\dots\mu_M$, Lorentz-covariance

$\Rightarrow$ decomposition into symmetric tensors of degree M

example:

scalar!

$$C_{\mu\nu}(p_1,p_2,m_0,m_1,m_2) = g_{\mu\nu}C_{00} + p_{1,\mu}p_{1,\nu}C_{11} + p_{2,\mu}p_{2,\nu}C_{22} + (p_{1,\mu}p_{2,\nu}+p_{2,\mu}p_{1,\nu})C_{12}$$

Passarino – Veltman Reduction

- algebraic reduction of tensor coefficients $B_1, B_{11}, B_{00}, C_1, \dots$ to scalar integrals $A_0, B_0, C_0, \dots$
- recursive algorithm: $(T^N_{\mu_1 \dots \mu_M})$

① contraction of integral representation & tensor decomposition with external momenta p_i^μ and the metric $g^{\mu\mu}$

↪ in the integrand:

- $$p_i^\mu q_\mu = \frac{1}{2}\left[(q+p_i)^2 - q^2 - p_i^2\right] = \underbrace{\frac{1}{2}\overbrace{\left[(q+p_i)-m_i^2\right]}^{\text{kill prop. } i} - \frac{1}{2}\overbrace{\left[q^2-m_0^2\right]}^{\text{kill prop. } 0}}_{(N-1)\text{-pt integrals}} - \underbrace{\frac{1}{2}(p_i^2-m_i^2+m_0^2)}_{\text{rank }(M-1)}$$

- $$g^{\mu\nu} q_\mu q_\nu = \underbrace{\overbrace{[q^2-m_0^2]}^{\text{kill prop. } 0}}_{(N-1)\text{-pt}} + \underbrace{m_0^2}_{\text{rank }(M-2)}$$

↪ in the tensor decomposition ⇝ linear combination of tensor coefficients

② solve the system of linear equations for $T^N_{ij\dots}$

2-pt Function

- short-hand notation $\quad \left\langle \ldots \right\rangle_q \equiv \frac{(2\pi\mu)^{D-4}}{i\pi^2} \int d^Dq \, (\ldots)$

- rank 1: $\quad B_\mu(p, m_0, m_1) = \left\langle \frac{q_\mu}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q = p_\mu B_1(p^2, m_0, m_1)$

$\hookrightarrow$ contraction with p^μ

$$\Rightarrow p^2 B_1 = \left\langle \frac{p\cdot q}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q$$

$$= \left\langle \frac{\frac{1}{2}\left[(q+p)^2-m_1^2\right] - \frac{1}{2}\left[q^2-m_0^2\right] - \frac{1}{2}(p^2-m_1^2+m_0^2)}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q$$

$$= \frac{1}{2}\left\langle \frac{1}{q^2-m_0^2} \right\rangle_q - \frac{1}{2}\left\langle \frac{1}{(q+p)^2-m_1^2} \right\rangle_q - \frac{1}{2}\left\langle \frac{p^2-m_1^2+m_0^2}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q$$

$$= \frac{1}{2} A_0(m_0) - \frac{1}{2} A_0(m_1) - \frac{1}{2}(p^2-m_1^2+m_0^2) B_0$$

$$\Rightarrow B_1 = \frac{1}{2p^2}\left[A_0(m_0) - A_0(m_1) - (p^2-m_1^2+m_0^2) B_0\right]$$

$\hookrightarrow$ note: $\quad B_1(p^2, m_0, m_1)\Big|_{div} = \frac{1}{2p^2}\left[\frac{m_0^2}{\epsilon} - \frac{m_1^2}{\epsilon} - (p^2-m_1^2+m_0^2)\frac{1}{\epsilon}\right] = -\frac{1}{2\epsilon}$

2-pt Function

- rank 2 : $$B_{\mu\nu}(p, m_0, m_1) = \left\langle \frac{q_\mu q_\nu}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q = g_{\mu\nu} B_{00} + p_\mu p_\nu B_{11}$$

↪ contraction with $g^{\mu\nu}$

$$\Rightarrow D\, B_{00} + p^2 B_{11} = \left\langle \frac{q^2-m_0^2+m_0^2}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q = A_0(m_1) + m_0^2\, B_0$$

↪ contraction with p^μ

$$\Rightarrow p_\nu\left(B_{00} + p^2 B_{11}\right) = \left\langle \frac{q_\nu \left\{ \frac{1}{2}\left[(q+p)^2-m_1^2\right] - \frac{1}{2}\left[q^2-m_0^2\right] - \frac{1}{2}\left(p^2-m_1^2+m_0^2\right)\right\}}{(q^2-m_0^2)\left[(q+p)^2-m_1^2\right]} \right\rangle_q$$

$$= \frac{1}{2}\underbrace{\left\langle \frac{q_\nu}{q^2-m_0^2} \right\rangle_q}_{=0} - \frac{1}{2}\underbrace{\left\langle \frac{q_\nu}{(q+p)^2-m_1^2} \right\rangle_q}_{\left\langle \frac{q'_\nu - p_\nu}{(q')^2-m_1^2} \right\rangle_{q'} = -p_\nu A_0(m_1)} - \frac{1}{2}\left(p^2-m_1^2+m_0^2\right)\underbrace{B_\nu}_{p_\nu B_1}$$

$$= p_\nu\left[\frac{1}{2} A_0(m_1) - \frac{1}{2}\left(p^2-m_1^2+m_0^2\right) B_1\right]$$

EX13

3-pt Function

- rank 1: $C_\mu(p_1, p_2, m_0, m_1, m_3) = p_{1,\mu}\, C_1 + p_{2,\mu}\, C_2$

↪ contraction with p_1^μ

$$\Rightarrow p_1^2\, C_1 + p_1 \cdot p_2\, C_2 = \frac{1}{2} B_0(p_2^2, m_0, m_2) - \frac{1}{2} B_0\big((p_1-p_2)^2, m_1, m_2\big) - \frac{1}{2}(p_1^2 - m_1^2 + m_0^2)\, C_0$$

↪ contraction with p_2^μ : as above, $1 \leftrightarrow 2$

$f_i = p_i^2 - m_i^2 + m_0$

$$\Rightarrow \begin{pmatrix} p_1^2 & p_1 \cdot p_2 \\ p_2 \cdot p_1 & p_2^2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} B_0(p_2^2, m_0, m_2) - \frac{1}{2} B_0\big((p_1-p_2)^2, m_1, m_2\big) - \frac{1}{2} f_1\, C_0 \\ \frac{1}{2} B_0(p_1^2, m_0, m_1) - \frac{1}{2} B_0\big((p_1-p_2)^2, m_1, m_2\big) - \frac{1}{2} f_2\, C_0 \end{pmatrix}$$

$\Rightarrow \quad C_1, C_2 = \ldots$

↪ note: $\quad C_1\big|_{div} = C_2\big|_{div} = 0$

1-Loop Integrals — The Roadmap

generally confronted with

tensor integrals $T^N_{\mu_1\cdots\mu_M}$

$$\int d^Dq \frac{q_{\mu_1}\cdots q_{\mu_M}}{\prod_{i=1}^N D_i}$$

$\left(D_i = (q+p_i)^2 - m_i^2 \right)$

Passarino-Veltman reduction

scalar integrals S^N

$$\int d^Dq \frac{1}{\prod_{i=1}^N D_i}$$

Feynman parameters

merge propagators

to isolate loop integral

$$\int \left(\prod_{i=1}^N dx_i \right) \frac{\delta(1-\sum x_i)}{\left[\sum_i D_i x_i\right]^N}$$

tadpole

generic integral

perform $\int d^Dq$

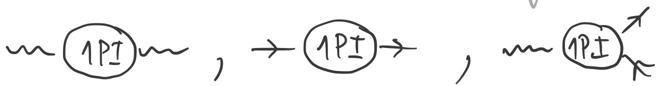
$$I_N(A) := \int d^Dq \left[q^2 - A + i0 \right]^{-N}$$

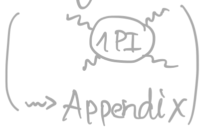
A Bird's Eye View

① QED Ward identities

$k^{\mu}\Sigma^{AA}_{\mu\nu}(k) = 0$ $\qquad$ $k^{\mu}\Lambda_{\mu}(p',p) = \Sigma^{\bar{\psi}\psi}(p') - \Sigma^{\bar{\psi}\psi}(p)$

② Study UV divergent behaviour (superficial degree of divergence)

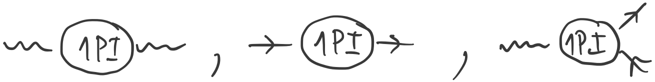
our problem childs: 



REGULARIZATION

③ Let's get a handle on infinities

dimensional regularization & 1-loop integrals

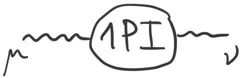
④ compute  in DimReg

RENORMALIZATION

⑤ renormalize the theory

⑥ the on-shell scheme

1-Loop Corrections: Photon Self Energy

$$-i\Sigma^{AA}_{\mu\nu}(k) = $$  $$=$$  $$+ \mathcal{O}(\alpha^2)$$

$$\Rightarrow \Sigma^{AA}_{\mu\nu}(k) = -ie^2 Q^2 \mu^{4-D} \int \frac{d^D q}{(2\pi)^D} \frac{\mathrm{Tr}\left[\gamma_\mu (\not{q}+m)\gamma_\nu(\not{q}+\not{k}+m)\right]}{(q^2-m^2)\left((q+k)^2-m^2\right)}$$

(Dim Reg)

- trace in the numerator:

$$\mathrm{Tr}\left[\gamma_\mu(\not{q}+m)\gamma_\nu(\not{q}+\not{k}+m)\right]$$

(only even # of γ's)

$$= \underbrace{\mathrm{Tr}\left[\gamma_\mu \not{q}\gamma_\nu(\not{q}+\not{k})\right]}_{4\left[-g_{\mu\nu}\, q\cdot(q+k) + 2q_\mu q_\nu + q_\mu k_\nu + k_\mu q_\nu\right]} + m^2 \underbrace{\mathrm{Tr}\left[\gamma_\mu\gamma_\nu\right]}_{4 g_{\mu\nu}}$$

$\left(\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\right] = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)\right)$

$$= -2g_{\mu\nu}\left\{\underbrace{[q^2-m^2]}_{\text{kills a propagator}} + \underbrace{[(q+k)^2-m^2]}_{\text{kills a propagator}} - k^2\right\} + 8q_\mu q_\nu + 4q_\mu k_\nu + 4k_\mu q_\nu$$

1-Loop Corrections: Photon Self Energy

$$\Rightarrow \Sigma^{AA}_{\mu\nu}(k) = \frac{\alpha}{2\pi} Q^2 \left\{ -g_{\mu\nu}\left[A_0(m) + A_0(m) - k^2 B_0\right] + 4 B_{\mu\nu} + 2 k_\mu B_\nu + 2 k_\nu B_\mu \right\}$$

$\left(\alpha = \frac{e^2}{4\pi}\right)$

fine structure constant ~ 1/137

- tensor decomposition: $B_{\mu\nu} = g_{\mu\nu} B_{00} + k_\mu k_\nu B_{11}$, $B_\mu = k_\mu B_1$

$$\Rightarrow \Sigma^{AA}_{\mu\nu}(k) = \frac{\alpha}{2\pi} Q^2 \left\{ g_{\mu\nu}\left[k^2 B_0 - 2A_0 + 4 B_{00}\right] + 4 k_\mu k_\nu \left[B_{11} + B_1\right] \right\}$$

- reduction to scalar integrals:

$$B_1(k^2, m, m) = -\frac{1}{2} B_0$$

$$B_{00}(k^2, m, m) = \frac{1}{6}\left[A_0(m) + 2m^2 B_0 + k^2 B_1 + 2m^2 - \frac{k^2}{3}\right] + \mathcal{O}(\epsilon)$$

$$B_{11}(k^2, m, m) = \frac{1}{6k^2}\left[2A_0(m) - 2m^2 B_0 - 4k^2 B_1 - 2m^2 + \frac{k^2}{3}\right] + \mathcal{O}(\epsilon)$$

$$B_0(0, m, m) = \frac{\partial A_0(m)}{\partial m^2}$$

$$= (1-\epsilon)\frac{A_0(m)}{m^2}$$

$$\Rightarrow A_0(m) = m^2\left[B_0(0, m, m) + 1\right] + \mathcal{O}(\epsilon)$$

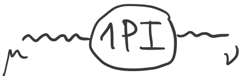
- decomposition into transversal & longitudinal parts:

EX 14

$$\Sigma^{AA}_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Sigma^{AA}_T(k^2) + \frac{k_\mu k_\nu}{k^2}\Sigma^{AA}_L(k^2)$$

Scalars

1-Loop Corrections: Photon Self Energy

$$-i\,\Sigma^{AA}_{\mu\nu}(k) = \;{}_{\mu}$$  $${}_{\nu} \;=$$  $$+\;\mathcal{O}(\alpha^2)$$

$$\Rightarrow \Sigma^{AA}_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Sigma^{AA}_{T}(k^2) + \frac{k_\mu k_\nu}{k^2}\,\Sigma^{AA}_{L}(k^2)$$

$$\hookrightarrow \Sigma^{AA}_{T}(k^2) = \frac{\alpha}{3\pi}\left[k^2 B_0(k^2,m,m) + 2m\left(B_0(k^2,m,m) - B_0(0,m,m)\right) - \frac{k^2}{3}\right]$$

$$\hookrightarrow \Sigma^{AA}_{L}(k^2) = 0$$

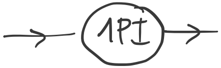
- $\Sigma^{AA}_{L}(k^2) = 0$ to all orders due to Ward Identity ($\rightsquigarrow$ EX7: $k^\mu \Sigma^{AA}_{\mu\nu}(k) = 0$)
- UV divergence: $\Sigma^{AA}_{T}(k^2)\Big|_{div} = \frac{\alpha}{3\pi}k^2\frac{1}{\epsilon}$
- propagator ("vacuum polarization" $\Pi^{AA}(k^2) := \frac{\Sigma^{AA}_{T}(k^2)}{k^2}$)

$$G^{AA}_{\mu\nu}(k,-k) = \frac{-i}{\underbrace{k^2}\left[1+\Pi^{AA}(k^2)\right]}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) - \xi\frac{i}{k^2}\frac{k_\mu k_\nu}{k^2}$$

EX 15

no shift in the pole @ $k^2 = 0$ $\longleftrightarrow$ no photon mass!

1-Loop Corrections : Elektron Self Energy

$i\Sigma_1^{\bar{4}4}(k) =$ 

$=$ 

$+ \mathcal{O}(\alpha^2)$ (Feynman gauge: $\xi = 1$)

$$\Rightarrow \Sigma^{\bar{4}4}(k) = \not{k}\, \Sigma_V^{\bar{4}4}(k^2) + m\, \Sigma_S^{\bar{4}4}(k^2)$$

$$\longrightarrow \Sigma_V^{\bar{4}4}(k^2) = -\frac{\alpha}{4\pi} Q^2 \frac{1}{k^2}\left[A_0(m) - (k^2+m^2)\, B_0(k^2; m, 0) + k^2\right]$$

$$\hookrightarrow \Sigma_S^{\bar{4}4}(k^2) = -\frac{\alpha}{4\pi} Q^2 \left[4 B_0(k^2; m, 0) - 2\right]$$

- propagator

$$G^{4\bar{4}}(-p, p) = \frac{i}{\not{p} - m + \Sigma^{\bar{4}4}(p)} = i\, \frac{\not{p}\left[1 + \Sigma_V^{\bar{4}4}(p^2)\right] + m\left[1 - \Sigma_S^{\bar{4}4}(p^2)\right]}{\underbrace{p^2 - m^2 \left[\frac{1 - \Sigma_S^{\bar{4}4}(p^2)}{1 + \Sigma_V^{\bar{4}4}(p^2)}\right]}_{\text{radiative corrections shift the pole @ } p^2 = m^2}} \left[1 + \Sigma_V^{\bar{4}4}(p^2)\right]^{-2}$$

- UV divergence :

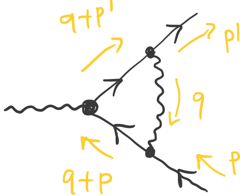
$$\Sigma_V^{\bar{4}4}(k^2)\Big|_{div} = -\frac{\alpha}{4\pi} Q^2 \frac{1}{k^2}\left[\frac{m^2}{\epsilon} - \frac{(k^2+m^2)}{\epsilon}\right] = \frac{\alpha}{4\pi} Q^2 \frac{1}{\epsilon}$$

$$\Sigma_S^{\bar{4}4}(k^2)\Big|_{div} = -\frac{\alpha}{4\pi} Q^2\, 4\, \frac{1}{\epsilon} = -\frac{\alpha}{\pi} Q^2 \frac{1}{\epsilon}$$

1-Loop Corrections: Electron-Photon Vertex

(Feynman gauge: $\xi = 1$)

$-ieQ\Lambda_\mu(p',p) =$ 

$=$ 

$+ \mathcal{O}(\alpha^2)$

$$\Rightarrow \Lambda_\mu(p',p) = \frac{\alpha}{4\pi} Q^2 \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dq \frac{\gamma_\alpha (\not{q}+\not{p}'+m)\gamma_\mu(\not{q}+\not{p}-m)\gamma^\alpha}{(q^2-\lambda^2)((q+p')^2-m^2)((q+p)^2-m^2)}$$

- write in terms of $C_0, C_\mu, C_{\mu\nu}$ & reduce to A_0, B_0, C_0
  $\hookrightarrow$ lengthly & quite complicated result $\rightsquigarrow$ e.g. use computer algebra
- Ward identity: $k^\mu \Lambda_\mu(p',p) = \Sigma_1^{44}(p') - \Sigma_1^{44}(p) \Rightarrow \underbrace{\Lambda_\mu(p,p)}_{\text{photon momentum } k\to 0} = \frac{\partial}{\partial p^\mu}\Sigma_1^{44}(p)$
- UV divergence:

$$\Lambda_\mu(p',p)\Big|_{div} = \Lambda_\mu(p,p)\Big|_{div} = \frac{\partial\Sigma_1^{44}(p)}{\partial p^\mu}\Big|_{div} = \frac{\partial}{\partial p^\mu}\left\{\not{p}\frac{\alpha}{4\pi}\frac{1}{\epsilon} - m\frac{\alpha}{\pi}\frac{1}{\epsilon}\right\}$$

$$= \gamma_\mu \frac{\alpha}{4\pi}\frac{1}{\epsilon} \leftarrow \text{proportional to lowest order!}$$

EX 16

Renormalization: Introduction

- from here on: add suffixes "$_0$" to indicate "bare" quantities

  $\hookrightarrow$ $\mathcal{L}(\psi_0, A_0^\mu, m_0, e_0)$: Do m_0 & e_0 correspond to the physical mass & charge?

- the electron mass

  $\hookrightarrow$ tree level: $G_0^{\psi\bar\psi}(-p,p) = \frac{i}{\not p - m_0} = \frac{i(\not p + m_0)}{p^2 - m_0^2}$ $\rightsquigarrow$ m_{pole} @ m_0

  $\hookrightarrow$ rad. corr: $G^{\psi\bar\psi}(-p,p) = \frac{i}{\not p - m_0 + \Sigma^{\bar\psi\psi}(p)}$ $\rightsquigarrow$ m_{pole} @ $m_0 \left[\frac{1-\Sigma_S^{\bar\psi\psi}}{1+\Sigma_V^{\bar\psi\psi}}\right]$

- similarly for the charge

Conclusion: radiative corrections have an impact on the physical interpretation of the bare quantities!

$\Rightarrow$ redefinition, "renormalization" necessary!

Renormalization: General Idea

- compute n physical observables: $\text{Obs}_i^{\text{th.}}(m_0, e_0)$ divergent expressions in terms of m_0, e_0

- choose two (independent) observables to express m_0, e_0 in terms the measured quantities ($\leftrightsquigarrow$ input parameter scheme)

$$\left.\begin{aligned} \underbrace{\text{Obs}_1^{\text{exp.}}}_{\text{finite, obviously}} &\overset{!}{=} \text{Obs}_1^{\text{th.}}(m_0, e_0) \\ \text{Obs}_2^{\text{exp.}} &\overset{!}{=} \text{Obs}_2^{\text{th.}}(m_0, e_0) \end{aligned}\right\} \Longrightarrow \begin{aligned} &m_0\left(\text{Obs}_1^{\text{exp.}}, \text{Obs}_2^{\text{exp.}}\right) \\ &e_0\left(\text{Obs}_1^{\text{exp.}}, \text{Obs}_2^{\text{exp.}}\right) \end{aligned}$$

divergent expressions in terms of $\text{Obs}_{1/2}^{\text{exp.}}$

- <u>predictions</u> for remaining $(n-2)$ observables

input parameters

$$\text{Obs}_i^{\text{th.}}\left(m_0\left(\text{Obs}_1^{\text{exp.}}, \text{Obs}_2^{\text{exp.}}\right), e_0\left(\text{Obs}_1^{\text{exp.}}, \text{Obs}_2^{\text{exp.}}\right)\right) = \text{Obs}_i^{\text{th.}}\left(\text{Obs}_1^{\text{exp.}}, \text{Obs}_2^{\text{exp.}}\right)$$

$\Rightarrow$ $\text{Obs}_i^{\text{th.}}$ in terms of $\text{Obs}_{1/2}^{\text{exp.}}$: FINITE!

Multiplicative Renormalization

- let's make the procedure from the previous slide more systematic
- write (divergent) bare quantities $\psi_0, A_0^\mu, m_0, e_0, \xi$ in terms of (finite) renormalized quantities times (divergent) renormalization constants

$$\psi_0 = \sqrt{Z_\psi}\,\psi\,, \qquad e_0 = Z_e\, e\,, \qquad \xi_0 = Z_\xi\, \xi$$

$$A_0^\mu = \sqrt{Z_A}\, A^\mu, \qquad m_0 = Z_m\, m\,,$$

- in perturbation theory:

$$Z_X = 1 + \alpha\Big(\underbrace{\frac{C_{-1}^X}{\epsilon}}_{\text{divergent part}} + \overbrace{C_0^X}^{\text{finite part}}\Big) + \mathcal{O}(\alpha^2)$$

finite part
"free" choice:
fixed by renormalization conditions

divergent part:
uniquely determined

$$\Rightarrow \sqrt{Z_\psi}\,\psi = \left(1 + \tfrac{1}{2}\,\delta Z_\psi\right)\psi\,, \qquad Z_e\, e = \left(1 + \delta Z_e\right) e\,, \qquad Z_\xi\, \xi = \left(1 + \delta Z_\xi\right)\xi$$

$$\sqrt{Z_A}\, A_\mu = \left(1 + \tfrac{1}{2}\,\delta Z_A\right) A_\mu\,, \qquad Z_m\, m = \left(1 + \delta Z_m\right) m \equiv m + \delta m$$

Counterterms

- We rewrite the Lagrangian in terms of renormalized quantities and "counterterms" (δZ_X):

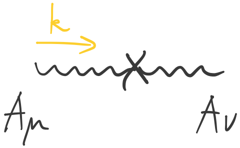
$$\mathcal{L}(\psi_0, A_0^\mu, m_0, e_0) = -\frac{1}{4} F_{0,\mu\nu} F_0^{\mu\nu} + \bar{\psi}_0 (i\partial\!\!\!/ - Q e_0 A\!\!\!/_0 - m_0)\psi_0 - \frac{1}{2\xi_0}(\partial A_0)^2$$

$$\begin{aligned}
= \underbrace{\begin{aligned} & -\tfrac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) \\ & + \tfrac{1}{2}\left(1-\tfrac{1}{\xi}\right)(\partial_\mu A^\mu)(\partial_\nu A^\nu) \\ & + \bar{\psi}(i\partial\!\!\!/ - m)\psi \\ & - eQ\bar{\psi} A\!\!\!/ \psi \end{aligned}}_{\mathcal{L}(\psi, A^\mu, m, e)}
\quad
\underbrace{\begin{aligned} & - \delta Z_A \tfrac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^\nu) \\ & + \left[\delta Z_A\left(1-\tfrac{1}{\xi}\right) + \delta Z_\xi \tfrac{1}{\xi}\right]\tfrac{1}{2}(\partial_\mu A_\nu)(\partial^\nu A^\mu) \\ & + \delta Z_\psi \bar{\psi}(i\partial\!\!\!/ - m)\psi - \delta m\, \bar{\psi}\psi \\ & - \left(\delta Z_e + \delta Z_\psi + \tfrac{1}{2}\delta Z_A\right)\cdot eQ\bar{\psi} A\!\!\!/ \psi \\ & + \mathcal{O}(\delta Z^2) \end{aligned}}_{\text{"counter terms"} \quad \mathcal{L}_{ct}}
\end{aligned}$$

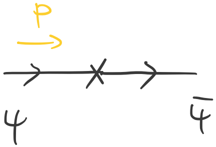
- note: CT's are not "added": still the original theory (re-shuffling)
- Feynman rules: ($m = \mathcal{O}(1)$, $\delta m = \mathcal{O}(\alpha)$)
  1. "old" ones with renormalized quantities
  2. "new" CT vertices

EX 17

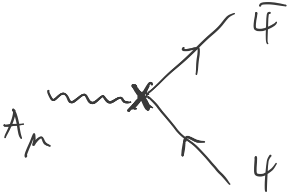
Counterterm Feynman Rules



$$= -i\,\delta Z_A \left[k^2 g_{\mu\nu} - \left(1-\frac{1}{\xi}\right) k_\mu k_\nu\right] + i\,\delta Z_\xi \frac{1}{\xi} k_\mu k_\nu$$



$$= i\,\delta Z_\psi (\not{p} - m) - i\,\delta m$$

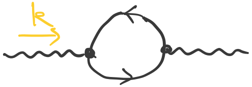


$$= -ieQ\,\gamma_\mu \left(\delta Z_e + \delta Z_\psi + \frac{1}{2}\delta Z_A\right)$$

EX 18

Renormalized Photon Self Energy

"hat" $\hat{=}$ renormalized.

$$-i\,\hat{\Sigma}^{AA}_{\mu\nu}(k) = $$  $$+$$  $$+\ \mathcal{O}(\alpha^2)$$

$$\Rightarrow \hat{\Sigma}^{AA}_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\hat{\Sigma}^{AA}_{T}(k^2) + \frac{k_\mu k_\nu}{k^2}\,\hat{\Sigma}^{AA}_{L}(k^2)$$

$$\hookrightarrow \hat{\Sigma}^{AA}_{T}(k^2) = \Sigma^{AA}_{T}(k^2) + k^2\,\delta Z_A$$

$$\hookrightarrow \hat{\Sigma}^{AA}_{L}(k^2) = \cancelto{0}{\Sigma^{AA}_{L}(k^2)} + \frac{1}{\xi}k^2\left[\delta Z_A - \delta Z_\xi\right]$$

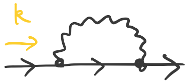
- UV divergences:

$$\delta Z_A\Big|_{div} \overset{!}{=} -\frac{\Sigma^{AA}_T(k^2)}{k^2}\Big|_{div} = -\Pi^{AA}(k^2)\Big|_{div}$$

$$\delta Z_\xi\Big|_{div} \overset{!}{=} \delta Z_A\Big|_{div}$$ gauge-param. renormalization not necessary for phys. obs.

$\hookrightarrow$ we can choose $\delta Z_\xi = \delta Z_A \Leftrightarrow \hat{\Sigma}^{AA}_{L}(k^2) \equiv 0$

Renormalized Elektron Self Energy

$$i\hat{\Sigma}_1^{\bar{4}4}(k) =$$ 

$+$  $+\ \mathcal{O}(\alpha^2)$

$$\Rightarrow \hat{\Sigma}^{\bar{4}4}(k) = \not{k}\,\hat{\Sigma}_V^{\bar{4}4}(k^2) + m\,\hat{\Sigma}_S^{\bar{4}4}(k^2)$$

$$\longleftrightarrow \hat{\Sigma}_V^{\bar{4}4}(k^2) = \Sigma_V^{\bar{4}4}(k^2) + \delta Z_4$$

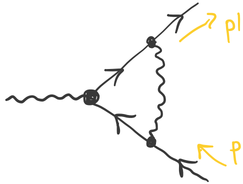
$$\hookrightarrow \hat{\Sigma}_S^{\bar{4}4}(k^2) = \Sigma_S^{\bar{4}4}(k^2) - \delta Z_4 - \frac{\delta m}{m}$$

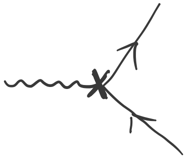
- UV divergences:

$$\delta Z_4\Big|_{div} \stackrel{!}{=} -\Sigma_V^{\bar{4}4}(k^2)\Big|_{div}$$

$$\delta m\Big|_{div} \stackrel{!}{=} m\left[\Sigma_S^{\bar{4}4}(k^2) - \delta Z_4\right]\Big|_{div} = m\left[\Sigma_S^{\bar{4}4}(k^2) + \Sigma_V^{\bar{4}4}\right]\Big|_{div}$$

Renormalized Electron-Photon Vertex

$$-ieQ\hat{\Lambda}_\mu(p',p) =$$ 

$$+$$  $$+ \mathcal{O}(\alpha^2)$$

$$\Rightarrow \hat{\Lambda}_\mu(p',p) = \Lambda_\mu(p',p) + \gamma_\mu\left(\delta Z_e + \delta Z_4 + \tfrac{1}{2}\delta Z_A\right)$$

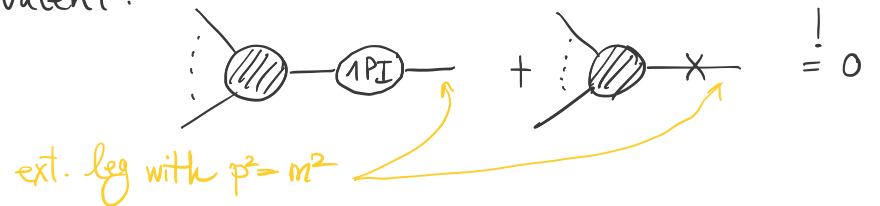
- UV divergences: (remember: $\Lambda_\mu\big|_{div} \sim \gamma_\mu$)

$$\delta Z_e = \left(-\Lambda_{div} - \delta Z_4 - \tfrac{1}{2}\delta Z_A\right)\Big|_{div}$$

On-Shell Renormalization

- so far: determined the divergent part of δZ_X
  ↪ sufficient for MS, $\overline{\text{MS}}$ renormalization
- arbitrary finite contributions can be absorbed into the definition of δZ_X ⇝ renormalization scheme. (how we relate to phys. obs.)
  ↪ calculation at $\mathcal{O}(\alpha^n)$ ↔ scheme dependence $\mathcal{O}(\alpha^{n+1})$
- on-shell renormalization
  ↪ $m \overset{!}{=} m_{phys}$ ⇜ location of the propagator pole.
  ↪ normalize fields (Z_Φ) such that residue @ $p^2 = m^2$ is one
  ⇒ no h.o. corrections to the residue

  equivalent:



Renormalization Conditions

(a) photon WF renormalization

$$\Rightarrow \lim_{k^2\to 0}\left(\frac{-i\,g^{\mu\nu}}{k^2}\,\hat{\Gamma}^{AA}_{\nu\rho}(k,-k)\right)\varepsilon^{\rho}(k) \overset{!}{\doteq} -\varepsilon^{\mu}(k) \qquad (k_\mu\varepsilon^\mu(k)=0)$$

$$\Rightarrow \lim_{k^2\to 0}\frac{1}{k^2}\,\hat{\Sigma}^{AA}_{T}(k^2) = \Pi^{AA}(0)+\delta Z_A \overset{!}{=} 0$$

$$\Rightarrow \delta Z_A = -\Pi^{AA}(0)$$

(b) gauge-parameter renormalization:

$$\hat{\Sigma}^{AA}_{L}(k^2) \overset{!}{\doteq} 0 \Rightarrow \delta Z_\xi = \delta Z_A \qquad (Z_\xi = Z_A)$$

$\hookrightarrow$ as ξ, z_ξ has no impact on phys. observables

$\hookrightarrow$ no renormalization for $\mathcal{L}_{fix} = -\frac{1}{2\xi_0}(\partial\cdot A_0)^2 = -\frac{1}{2\xi}(\partial\cdot A)^2$

Renormalization Conditions

(c) electron mass renormalization: $\quad (p^2 = m^2 \hat{=}$ propagator pole$)$

$$\Rightarrow \quad 0 \overset{!}{=} \hat{\Gamma}^{\bar{\psi}\psi}(-p,p)\, u(p) \qquad (\text{Dirac Eq: } (\not{p}-m)\,u(p) = 0)$$

$$= im \Big(\underbrace{\hat{\Sigma}_V^{\bar{\psi}\psi}(m^2) + \hat{\Sigma}_S^{\bar{\psi}\psi}(m^2)}_{\overset{!}{=}0} \Big)$$

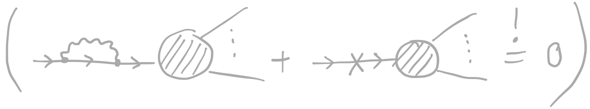
$$\Rightarrow \quad \Sigma_V^{\bar{\psi}\psi}(m^2) + \Sigma_S^{\bar{\psi}\psi}(m^2) - \frac{\delta m}{m} = 0$$

$$\Rightarrow \quad \frac{\delta m}{m} = \Sigma_V^{\bar{\psi}\psi}(m^2) + \Sigma_S^{\bar{\psi}\psi}(m^2)$$

EX 19

Renormalization Conditions

(d) electron WF renormalization



$$\Rightarrow \lim_{p^2 \to m^2} \left(\frac{i}{\not{p} - m} \hat{\Gamma}^{\bar{\psi}\psi}_{(-p,p)} \right) u(p) \overset{!}{=} -u(p)$$

$$\Rightarrow \lim_{p^2 \to m^2} \left(\frac{1}{\not{p} - m} \hat{\Sigma}^{\bar{\psi}\psi}(p) \right) u(p) \overset{!}{=} 0$$

$$\Rightarrow \delta Z_\psi = -\Sigma_V^{\bar{\psi}\psi}(m^2) - 2m^2 \frac{\partial}{\partial p^2} \left[\Sigma_V^{\bar{\psi}\psi}(p^2) + \Sigma_S^{\bar{\psi}\psi}(p^2) \right] \Big|_{p^2 = m^2}$$

EX20

Renormalization Conditions

(e) charge renormalization:

$e \overset{!}{=}$ elementary charge from class. EDyn.

$\hat{=}$ coupling for photon momenta $k \to 0$

$$\Rightarrow \bar{u}(p)\, \hat{\Gamma}^{A\bar{4}4}_{\mu}(k=0,-p,p)\, u(p) \overset{!}{=} -ieQ\, \bar{u}(p)\gamma_\mu u(p)$$

$$\Rightarrow 0 \overset{!}{=} \bar{u}(p)\hat{\Lambda}_\mu(p,p)u(p) = \bar{u}(p)\left[\Lambda_\mu(p,p) + \gamma_\mu\left(\delta Z_e + \delta Z_4 + \tfrac{1}{2}\delta Z_A\right)\right]$$

$$* \quad \Lambda_\mu(p,p) = \frac{\partial}{\partial p^\mu}\Sigma^{\bar{4}4}(p) = \gamma_\mu \Sigma_V^{\bar{4}4}(p^2) + 2p_\mu\left[\not{p}\,\Sigma_V^{\bar{4}4\prime}(p^2) + m\Sigma_S^{\bar{4}4\prime}(p^2)\right]$$

$$* \quad \bar{u}(p)\gamma_\mu u(p) = \frac{p_\mu}{m}\bar{u}(p)u(p)$$

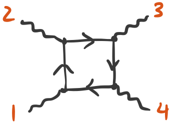
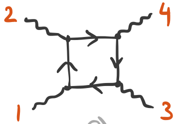
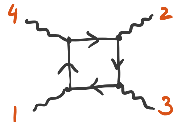
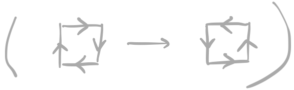
$$\Rightarrow 0 \overset{!}{=} \bar{u}(p)\gamma_\mu u(p)\left[\underbrace{\Sigma_V^{\bar{4}4}(m^2) + 2m^2\left(\Sigma_V^{\bar{4}4\prime}(m^2) + \Sigma_S^{\bar{4}4\prime}(m^2)\right) + \delta Z_4}_{=0} + \delta Z_e + \tfrac{1}{2}\delta Z_A\right]$$

$$\Rightarrow \quad \delta Z_e = -\frac{1}{2}\delta Z_A$$

Appendix: Four-Photon Vertex Function

- we saw that $\Gamma^{AAAA}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4)$ has a superficial degree of divergence $=0$

  $\hookrightarrow$ if the leading UV behaviour is finite, we're ok!

$\Gamma^{AAAA}_{\mu_1\mu_2\mu_3\mu_4} =$  $+$  $+$  $+$ ()

① ② ③

give the same as ①+②+③

- For the leading UV behaviour, we have

$$\Gamma^{①} \overset{\text{div}}{\sim} \int d^4q \; \frac{\mathrm{Tr}\left[\gamma^{\mu_1}\not{q}\,\gamma^{\mu_4}\not{q}\,\gamma^{\mu_3}\not{q}\,\gamma^{\mu_2}\not{q}\right]}{[q^2]^4}$$

we use isotropy: $\int d\Omega \; q^\mu q^\nu q^\rho q^\sigma \sim \left(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right)$

and γ identities (can do this in $D=4$ since $(D-4)\sim\epsilon$ is sub-leading)

$$\left.\begin{aligned}
&\Rightarrow \Gamma^{①} \overset{\text{div}}{\sim} g^{\mu_1\mu_4}g^{\mu_3\mu_2} + g^{\mu_1\mu_2}g^{\mu_3\mu_4} - 2\,g^{\mu_1\mu_3}g^{\mu_2\mu_4} \\
&\quad \Gamma^{②} = \Gamma^{①}\Big|_{\mu_3\leftrightarrow\mu_4} \\
&\quad \Gamma^{③} = \Gamma^{①}\Big|_{\substack{\mu_2\to\mu_4 \\ \mu_3\to\mu_2 \\ \mu_4\to\mu_3}}
\end{aligned}\right\}
\begin{aligned}
&\Rightarrow \Gamma^{①}+\Gamma^{②}+\Gamma^{③} \overset{\text{div}}{\sim} 0 \\
&\Rightarrow \Gamma^{AAAA}_{\mu_1\mu_2\mu_3\mu_4}(k_1,k_2,k_3,k_4)\Big|_{\text{div}} = 0
\end{aligned}$$