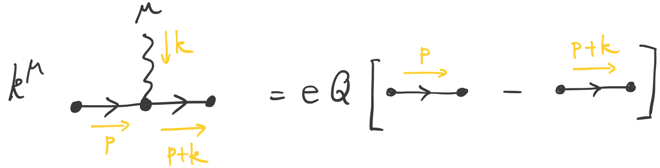
Exercise 7

- using the tree-level identity



show the transversality of the photon self-energy

k^μ  $= k^\mu \; \Sigma^{AA}_{1\mu\nu}(k,-k) = 0$

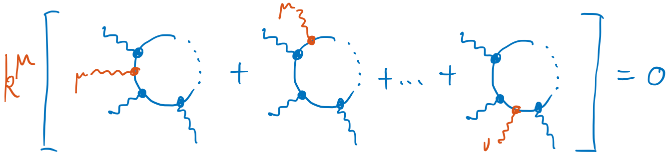
at 1-loop and 2-loop order.

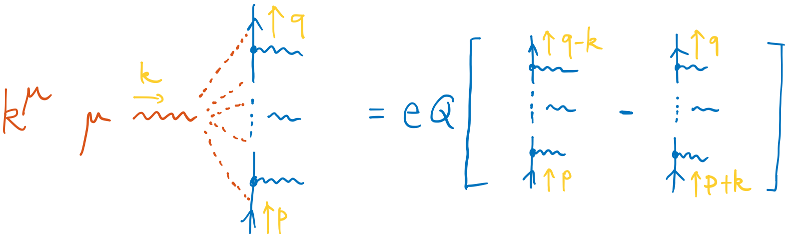
$\Rightarrow$ Ward Identity for the photon propagator:

$$-\frac{1}{\xi} k^2 k^\mu G^{AA}_{\mu\nu}(k,-k) = i k_\nu$$

(no h.o. corrections to the longitudinal part)

- similarly, one can show





Exercise 8

(a) determine a formula for the superficial degree of divergence

$$\omega(G) = \begin{cases} 0 \rightarrow \text{log. div} \\ 1 \rightarrow \text{lin. div.} \\ 2 \rightarrow \text{quad. div.} \\ \vdots \end{cases}$$

graph

(we're interested in 1PI graphs)

① write $\omega(G)$ in terms of L (# loops), I_A (# internal photon prop.), I_ψ (# internal fermion prop.).

② "Euler's loop equation": count the # of momentum-consv. constraints to write L in terms of I_A, I_ψ, and V (# vert.)

③ inspect the vertex ($A_\mu \bar{\psi}\psi$) and find relations

$$V = X(I_\psi, E_\psi) \quad \& \quad V = Y(I_A, E_A)$$

with $E_{\psi/A}$: # of external ψ/A fields

④ show that $\omega(G)$ only depends on $E_{\psi/A}$

cont.

Exercise 8

(b) find all superficially divergent vertex functions in QED

Exercise 9

- compute the D-dimensional solid angle $\Omega_D = \int d\Omega_D$

$\hookrightarrow$ the 1-dim Gaussian integral:

$$\sqrt{\pi} = \int_{-\infty}^{+\infty} dx \; e^{-x^2}$$

$\hookrightarrow$ Gamma function:

$$\Gamma(z) = \int_0^{\infty} t^{z-1} \, e^{-t} \, dt$$

Exercise 10

- expand $$A_0(m) = -m^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} \Gamma\left(\frac{2-D}{2}\right)$$ $(D = 4-2\epsilon)$
$$= -m^2 \left(\frac{m^2}{4\pi\mu^2}\right)^{-\epsilon} \Gamma(-1+\epsilon)$$

  in ϵ $(D \to 4)$ up to finite terms

- Γ has poles for $z = 0, -1, -2, \ldots$
- $\Gamma(\epsilon) = \frac{1}{\epsilon} - \gamma_E + \mathcal{O}(\epsilon)$
- $\Gamma(z+1) = z\,\Gamma(z)$

Exercise 11

- derive the Feynman parametrisation: $\frac{1}{A_1 A_2} = \int_0^1 dx \, \frac{1}{[A_1(1-x) + A_2 x]^2}$

① convince yourself that $\frac{1}{A} = \int_0^\infty dt \, e^{-At}$ & write

$$\frac{1}{A_1 A_2} = \int_0^\infty dt_1 \, dt_2 \, e^{-A_1 t_1 - A_2 t_2} \qquad \text{(Schwinger parametrisation)}$$

② hint: insert $1 = \int_0^\infty d\lambda \, \delta(\lambda - t_1 - t_2)$

- generalization to higher powers:

$$\frac{1}{A^\nu} = \frac{1}{\Gamma(\nu)} \int_0^\infty dt \, t^{\nu-1} e^{-At} \qquad \text{(by differentiation)}$$

- Cheng-Wu Theorem

only a subset $S \subset \{1, \dots, N\}$ in the delta: $\delta\left(1 - \sum_{i=1}^{N} x_i\right) \rightsquigarrow \delta\left(1 - \sum_{i \in S} x_i\right)$

for $j \notin S$: $\int_0^\infty dx_j$

Exercise 12

imagine: 

- compute $B_0(p^2, 0, m)$ up to $\mathcal{O}(\epsilon)$

$\hookrightarrow$ result: $\Delta - \ln\left(\frac{m^2}{\mu^2}\right) + 2 + \frac{m^2-p^2}{p^2}\ln\left(\frac{m^2-p^2-i0}{m^2}\right) + \mathcal{O}(\epsilon)$

Exercise 13

- solve the system of linear equations for B_{00} & B_{11} in terms of A_0, B_0, B_1

$$D\, B_{00} + p^2 B_{11} = A_0(m_1) + m_0^2\, B_0$$

$$B_{00} + p^2 B_{11} = \frac{1}{2} A_0(m_1) - \frac{1}{2}\left(p^2 - m_1^2 + m_0^2\right) B_1$$

- give the divergent part of B_{00} & B_{11}

$$A_0(m)\Big|_{div} = \frac{m^2}{\epsilon}\,, \quad B_0(p^2, m_0, m_1)\Big|_{div} = \frac{1}{\epsilon}\,, \quad B_1(p^2, m_0, m_1) = -\frac{1}{2\epsilon}$$

Exercise 14

- the photon self energy is given by

$$\Sigma^{AA}_{\mu\nu}(k) = \frac{\alpha}{2\pi} Q^2 \left\{ g_{\mu\nu}\left[k^2 B_0 - 2A_0 + 4B_{00}\right] + 4k_\mu k_\nu\left[B_{11} + B_1\right]\right\}$$

apply the reduction to scalar integrals:

$$B_1(k^2, m, m) = -\frac{1}{2}B_0$$

$$B_{00}(k^2; m, m) = \frac{1}{6}\left[A_0(m) + 2m^2 B_0 + k^2 B_1 + 2m^2 - \frac{k^2}{3}\right] + \mathcal{O}(\epsilon)$$

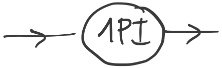
$$B_{11}(k^2; m, m) = \frac{1}{6k^2}\left[2A_0(m) - 2m^2 B_0 - 4k^2 B_1 - 2m^2 + \frac{k^2}{3}\right] + \mathcal{O}(\epsilon)$$

and identify the transversal & longitudinal parts in the decomposition

$$\Sigma^{AA}_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Sigma^{AA}_{T}(k^2) + \frac{k_\mu k_\nu}{k^2}\Sigma^{AA}_{L}(k^2)$$

Exercise 15

- the electron self energy is given by:

$i\Sigma^{\bar\psi\psi}(k) = $  $=$  $+ \ldots$ (Feynman gauge: $\xi = 1$)

$$\Rightarrow \Sigma^{\bar\psi\psi}(k) = -\frac{\alpha}{4\pi} Q^2 \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^Dq \, \frac{\gamma_\mu (\not{q}+m)\gamma^\mu}{(q^2-m^2)(q-k)^2}$$

and can be decomposed into its vector & scalar part as

$$\Sigma^{\bar\psi\psi}(k) = \not{k}\, \Sigma_V^{\bar\psi\psi}(k^2) + m\, \Sigma_S^{\bar\psi\psi}(k^2)$$

determine $\Sigma_V^{\bar\psi\psi}$ & $\Sigma_S^{\bar\psi\psi}$ in terms of scalar integrals.

careful with the sign!

① simplify the Dirac structure in the numerator

② perform a tensor reduction of $B_\mu(p, m_0, m_1) = p_\mu B_1(p, m_0, m_1)$

$$B_1(p^2, m_0, m_1) = \frac{1}{2p^2}\left[A_0(m_0) - A_0(m_1) - (p^2 - m_1^2 + m_0^2)\, B_0(p^2, m_0, m_1)\right]$$

③ get all finite terms using $A_0(m)\big|_{div} = \frac{m^2}{\epsilon}$, $B_0\big|_{div} = \frac{1}{\epsilon}$

Exercise 16

- explicitly calculate the divergent part of the vertex correction.

① for the UV behaviour ($q \to \infty$) external momenta are irrelevant

also neglect masses

$$\Lambda_\mu(p',p)\Big|_{\substack{uv\\div}} = \Lambda_\mu(0,0)\Big|_{\substack{uv\\div}} = \frac{\alpha}{4\pi} Q^2 \frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \, \frac{\gamma_\alpha \not{q} \gamma_\mu \not{q} \gamma^\alpha}{[q^2]^3}\Bigg|_{\substack{uv\\div}}$$

② what is the possible Lorentz-structure of $\Lambda_\mu(0,0)$?

⟶ how can we extract the scalar coefficient of this decomposition?

③ determine the UV-divergent part

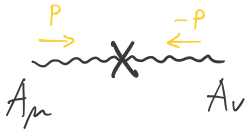
$$\frac{(2\pi\mu)^{4-D}}{i\pi^2} \int d^D q \, \frac{1}{(q^2)^2}\Bigg|_{\substack{uv\\div}} = B_0(0,0,0)\Big|_{\substack{uv\\div}} = \frac{1}{\epsilon}$$

?

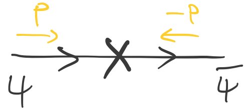
Exercise 17

- derive the counterterm Feynman rules for

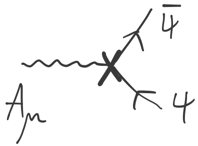
$$\left.\begin{aligned}\mathcal{L}_{ct} = \; & -\delta Z_A \tfrac{1}{2} (\partial_\mu A_\nu)(\partial^\mu A^\nu) \\ & + \left[\delta Z_A\left(1-\tfrac{1}{\xi}\right) + \delta Z_\xi \tfrac{1}{\xi}\right] \tfrac{1}{2} (\partial_\mu A_\nu)(\partial^\nu A^\mu)\end{aligned}\right\} \Rightarrow$$



$$+ \delta Z_\psi \, \bar{\psi}(i\not{\partial} - m)\psi - \delta m \, \bar{\psi}\psi \quad \Rightarrow$$



$$- \left(\delta Z_e + \delta Z_\psi + \tfrac{1}{2}\delta Z_A\right) e Q \, \bar{\psi} \not{A} \psi \quad \Rightarrow$$



Exercise 18

- field strength renormalization ($\Phi_i^0 = \sqrt{Z_{\Phi_i}}\, \Phi_i$) is not needed in S-Matrix elements / physical observables.
  They are useful to make n-pt functions finite.
  how are the n-pt functions with & without Z_{Φ_i} related?

$$G^{\Phi_1^0 \dots \Phi_n^0} \quad \text{vs.} \quad G^{\Phi_1 \dots \Phi_n}$$

- what about truncated n-pt functions?

$$G_{trunc}^{\Phi_1^0 \dots \Phi_n^0} \quad \text{vs.} \quad G_{trunc}^{\Phi_1 \dots \Phi_n}$$

cont.

Exercise 18:

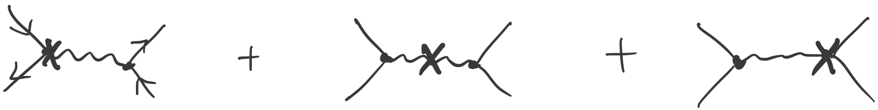
- The LSZ reduction formula is given in terms of bare fields:

$$i\mathcal{M}^{n\to m}(p_1,\dots,p_n,p_1',\dots p_m') = \prod_{i=1}^{n} f_{in}^{\Phi_i}(p_i)\sqrt{R_{\Phi_i}} \prod_{j=1}^{m} f^{\Phi_j'}(p_j')\sqrt{R_{\Phi_j'}}$$

$$\times\, G_{trunc}^{\Phi_1^0\dots\Phi_n^0\,\Phi_1'^0\dots\Phi_m'^0}(p_1,\dots p_n,-p_1',\dots,-p_m')\Big|_{on-shell}$$

what is the formula in terms of $G_{trunc}^{\Phi_1\dots\Phi_m'}$? (convenient choice?)

- Z_{Φ_i} are only relevant for <u>external</u> legs. Show that δZ_A cancels between these diagrams



Exercise 19

- determine the electron WF renormalization from

$$\lim_{p^2 \to m^2} \left(\frac{i}{\not p - m} \hat{\Gamma}^{\bar{4}4}(-p,p) \right) u(p) \stackrel{!}{=} -u(p)$$

$*$ $(\not p - m)\, u(p) = 0$

$*$ $\hat{\Gamma}^{\bar{4}4}(-p,p) = i(\not p - m) + i\hat{\Sigma}^{\bar{4}4}(p)$

$*$ $\hat{\Sigma}^{\bar{4}4}(p) = \not p\, \hat{\Sigma}_V^{\bar{4}4}(p^2) + m\, \hat{\Sigma}_S^{\bar{4}4}(p^2)$

$*$ $\hat{\Sigma}_V^{\bar{4}4}(p^2) = \Sigma_V^{\bar{4}4}(p^2) + \delta Z_4$, $\quad \hat{\Sigma}_S^{\bar{4}4}(p^2) = \Sigma_S^{\bar{4}4}(p^2) - \delta Z_4 - \frac{\delta m}{m}$

$*$ $\frac{\delta m}{m} = \Sigma_V^{\bar{4}4}(m^2) + \Sigma_S^{\bar{4}4}(m^2)$

Exercise 20

- determine the charge renormalization from the condition

$$0 \stackrel{!}{=} \bar{u}(p)\,\hat{\Lambda}_\mu(p,p)\,u(p)$$

① $\hat{\Lambda}_\mu(p',p) = \Lambda_\mu(p',p) + \gamma_\mu\left(\delta Z_e + \delta Z_4 + \frac{1}{2}\delta Z_A\right)$

use the Ward identity and $\Sigma^{\bar{4}4}(p) = \not{p}\,\Sigma_V^{\bar{4}4}(p^2) + m\,\Sigma_S^{\bar{4}4}(p^2)$

$$\Lambda_\mu(p,p) = \frac{\partial}{\partial p^\mu}\Sigma^{\bar{4}4}(p)$$

② Show the special Gordon identity $\bar{u}(p)\gamma_\mu u(p) = \frac{p_\mu}{m}\bar{u}(p)u(p)$ and use it to bracket out $\bar{u}(p)\gamma_\mu u(p)$

③ Use $\delta Z_4 = -\Sigma_V^{\bar{4}4}(m^2) - 2m^2\left[\Sigma_V^{\bar{4}4\,\prime}(m^2) + \Sigma_S^{\bar{4}4\,\prime}(m^2)\right]$ to arrive at $\delta Z_e = -\frac{1}{2}\delta Z_A$