CS350 : Lab 4

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1. **Recurrence Relation Practice:** Solve the following recurrence relations. Do not use the Master method. Show all of your work.

   (a) $T(n) = 2T(n-1) + c2^n$

   Solution: let n = 4 in:

$$\begin{aligned}
T(4) &= 2T(3) + c2^4 = 2^4k + 2^3c2^1 + 2^2c2^2 + 2c2^3 + c2^4 \\
T(3) &= 2T(2) + c2^3 = 2^3k + 2^2c2^1 + 2c2^2 + c2^3 \\
T(2) &= 2T(1) + c2^2 = 2^2k + 2c2^1 + c2^2 \\
T(1) &= 2T(0) + c2^1 = 2k + c2^1 \\
T(0) &= k
\end{aligned}$$

   From the emerging pattern we can derive a solution for the recurrence relation:

$$\begin{aligned}
T(n) &= 2^nk + 2^{n-1}c2^1 + 2^{n-2}c2^2 + \ldots + 2^1c2^{n-1} + 2^0c2^n \\
&= 2^nk + c[2^{n-1}2^1 + 2^{n-2}2^2 + \ldots + 2^12^{n-1} + 2^02^n] \\
&= 2^nk + c \cdot \sum_{i=1}^{n} 2^n \\
&= 2^nk + cn2^n
\end{aligned}$$

   On inspection we can conclude that $T(n) \in O(n2^n)$

   (b) $T(n) = 7T(n/2) + cn^2$

   Solution: let $n = 2^4$ in:

$$\begin{aligned}
T(2^4) &= 7T(2^3) + c(2^4)^2 = 7^4k + 7^3c(2^1)^2 + 7^2c(2^2)^2 + 7c(2^3)^2 + c(2^4)^2 \\
T(2^3) &= 7T(2^2) + c(2^3)^2 = 7^3k + 7^2c(2^1)^2 + 7c(2^2)^2 + c(2^3)^2 \\
T(2^2) &= 7T(2^1) + c(2^2)^2 = 7^2k + 7c(2^1)^2 + c(2^2)^2 \\
T(2^1) &= 7T(2^0) + c(2^1)^2 = 7k + c(2^1)^2 \\
T(2^0) &= k
\end{aligned}$$

   More generally, if we let $n = 2^m$ and follow the pattern we can see:

$$\begin{aligned}
T(2^m) &= 7^mk + 7^{m-1}c4^1 + \ldots + 7^1c4^{m-1} + 7^0c4^m \\
&= 7^mk + c[7^{m-1}4^1 + 7^{m-2}4^2 + \ldots + 7^14m - 1 + 7^04^m] \\
&= 7^mk + cS \quad \text{, where S} = 7^{m-1}4^1 + 7^{m-2}4^2 + \ldots + 7^14m - 1 + 7^04^m
\end{aligned}$$

If we multiply S by the common ratio $\frac{4}{7}$ and subtract it from S we get:

$$\begin{aligned}
S - \frac{4}{7}S &= (7^{m-1}4^1 + 7^{m-2}4^2 + \ldots + 7^1 4m-1 + 7^0 4^m) - (7^{m-2}4^2 + 7^{m-3}4^3 + \ldots + 7^0 4m + 7^{-1}4^{m+1}) \\
\frac{3}{7}S &= 7^{m-1}4 - 7^{-1}4^{m+1} \\
3S &= 7^m 4 - 4^{m+1} \\
S &= \frac{4}{3}(7^m - 4^m)
\end{aligned}$$

Substituting n back in for $m = lg(n)$ and the closed form solution for S we get:

$$\begin{aligned}
T(n) &= 7^{lg(n)}k + \frac{4}{3}c(7^{lg(n)} - n^2) \quad \textit{, note: } 4^m = (2^2)^m = (2^m)^2 = n^2 \\
&= n^{lg(7)}k + \frac{4}{3}cn^{lg(n)} - \frac{4}{3}cn^2
\end{aligned}$$

Since $2 < lg(7) < 3$ we can conclude $T(n) \in O(n^{lg(7)})$

(c) $T(n) = 2T(n/2) + n^2$

Solution: let $n = 2^4$ in:

$$\begin{aligned}
T(2^4) &= 2T(2^3) + (2^4)^2 = 2^4k + (2^2)^4 + (2^2)^4 + (2^2)^4 + (2^2)^4 \\
T(2^3) &= 2T(2^2) + (2^3)^2 = 2^3k + (2^2)^3 + (2^2)^3 + (2^2)^3 \\
T(2^2) &= 2T(2^1) + (2^2)^2 = 2^2k + (2^2)^2 + (2^2)^2 \\
T(2^1) &= 2T(2^0) + (2^1)^2 = 2k + (2^2)^1 \\
T(2^0) &= k
\end{aligned}$$

More generally, if we let $n = 2^m$ and follow the pattern we can see:

$$\begin{aligned}
T(2^m) &= 2^m k + \sum_{i=1}^{m}(2^m)^2 \\
&= 2^m k + m(2^m)^2
\end{aligned}$$

Substituting n back in for $m = lg(n)$ we get:

$$T(n) = nk + n^2 lg(n)$$

On inspection we can conclude that $T(n) \in O(n^2 lg(n))$

(d) $T(n) = 5T(n/4) + \sqrt{n}$

Solution: let $n = 4^3$ in:

$$\begin{aligned}
T(4^3) &= 5T(4^2) + \sqrt{4^3} = 5^3k + 5^2\sqrt{4}^1 + 5\sqrt{4}^2 + \sqrt{4}^3 \\
T(4^2) &= 5T(4^1) + \sqrt{4^2} = 5^2k = 5\sqrt{4} + \sqrt{4}^2 \\
T(4^1) &= 5T(4^0) + \sqrt{4} = 5k + \sqrt{4} \\
T(4^0) &= k
\end{aligned}$$

More generally, if we let $n = 4^m$ and follow the pattern we can see:

$$\begin{aligned}
T(4^m) &= 5^m k + 5^{m-1}\sqrt{4} + 5^{m-2}\sqrt{4}^2 + \ldots + 5^1\sqrt{4}^m - 1 + 5^0\sqrt{4}^m \\
&= 5^m k + S \quad \text{, where } S = 5^{m-1}2 + 5^{m-2}2^2 + \ldots + 5^1 2^{m-1} + 5^0 2^m
\end{aligned}$$

If we multiply S by the common ratio $\frac{2}{5}$ and subtract it from S we get:

$$\begin{aligned}
S - \frac{2}{5}S &= (5^{m-1}2 + 5^{m-2}2^2 + \ldots + 5^1 2^m - 1 + 5^0 2^m) - (5^{m-2}2^2 + 5^{m-3}2^3 + \ldots + 5^0 2^m + 5^{-1}2^{m+1}) \\
\frac{3}{5}S &= 5^{m-1}2 - 5^{-1}2^{m+1} \\
S &= \frac{2}{3}(5^m - 2^m)
\end{aligned}$$

Substituting n back in for $m = log_4(n)$ and the closed form solution for S we get:

$$\begin{aligned}
T(n) &= 5^{log_4(n)}k + \frac{2}{3}5^{log_4(n)} - \frac{2}{3}2^{log_4(n)} \\
&= n^{log_4(5)}k + \frac{2}{3}n^{log_4(5)} - \frac{2}{3}n^{log_4(2)}
\end{aligned}$$

Since $log_4(5) > log_4(2)$ we can conclude that $T(n) \in O(n^{log_4(5)})$