

CS350 : Lab 4

Completed by Alexander DuPree

1. **Recurrence Relation Practice:** Solve the following recurrence relations. Do not use the Master method. Show all of your work.

(a) $T(n) = 2T(n-1) + c2^n$

Solution: let $n = 4$ in:

$$T(4) = 2T(3) + c2^4 = 2^4k + 2^3c2^1 + 2^2c2^2 + 2c2^3 + c2^4$$

$$T(3) = 2T(2) + c2^3 = 2^3k + 2^2c2^1 + 2c2^2 + c2^3$$

$$T(2) = 2T(1) + c2^2 = 2^2k + 2c2^1 + c2^2$$

$$T(1) = 2T(0) + c2^1 = 2k + c2^1$$

$$T(0) = k$$

From the emerging pattern we can derive a solution for the recurrence relation:

$$\begin{aligned} T(n) &= 2^n k + 2^{n-1} c 2^1 + 2^{n-2} c 2^2 + \dots + 2^1 c 2^{n-1} + 2^0 c 2^n \\ &= 2^n k + c[2^{n-1} 2^1 + 2^{n-2} 2^2 + \dots + 2^1 2^{n-1} + 2^0 2^n] \\ &= 2^n k + c \cdot \sum_{i=1}^n 2^n \\ &= 2^n k + cn 2^n \end{aligned}$$

On inspection we can conclude that $T(n) \in O(n2^n)$

(b) $T(n) = 7T(n/2) + cn^2$

Solution: let $n = 2^4$ in:

$$T(2^4) = 7T(2^3) + c(2^4)^2 = 7^4k + 7^3c(2^1)^2 + 7^2c(2^2)^2 + 7c(2^3)^2 + c(2^4)^2$$

$$T(2^3) = 7T(2^2) + c(2^3)^2 = 7^3k + 7^2c(2^1)^2 + 7c(2^2)^2 + c(2^3)^2$$

$$T(2^2) = 7T(2^1) + c(2^2)^2 = 7^2k + 7c(2^1)^2 + c(2^2)^2$$

$$T(2^1) = 7T(2^0) + c(2^1)^2 = 7k + c(2^1)^2$$

$$T(2^0) = k$$

More generally, if we let $n = 2^m$ and follow the pattern we can see:

$$\begin{aligned} T(2^m) &= 7^m k + 7^{m-1} c 4^1 + \dots + 7^1 c 4^{m-1} + 7^0 c 4^m \\ &= 7^m k + c[7^{m-1} 4^1 + 7^{m-2} 4^2 + \dots + 7^1 4m - 1 + 7^0 4^m] \\ &= 7^m k + cS \quad , \text{ where } S = 7^{m-1} 4^1 + 7^{m-2} 4^2 + \dots + 7^1 4m - 1 + 7^0 4^m \end{aligned}$$

If we multiply S by the common ratio $\frac{4}{7}$ and subtract it from S we get:

$$S - \frac{4}{7}S = (7^{m-1}4^1 + 7^{m-2}4^2 + \dots + 7^1 4m - 1 + 7^0 4^m) - (7^{m-2}4^2 + 7^{m-3}4^3 + \dots + 7^0 4m + 7^{-1}4^{m+1})$$

$$\frac{3}{7}S = 7^{m-1}4 - 7^{-1}4^{m+1}$$

$$3S = 7^m 4 - 4^{m+1}$$

$$S = \frac{4}{3}(7^m - 4^m)$$

Substituting n back in for $m = \lg(n)$ and the closed form solution for S we get:

$$\begin{aligned} T(n) &= 7^{\lg(n)}k + \frac{4}{3}c(7^{\lg(n)} - n^2) \quad , \text{ note: } 4^m = (2^2)^m = (2^m)^2 = n^2 \\ &= n^{\lg(7)}k + \frac{4}{3}cn^{\lg(n)} - \frac{4}{3}cn^2 \end{aligned}$$

Since $2 < \lg(7) < 3$ we can conclude $T(n) \in O(n^{\lg(7)})$

(c) $T(n) = 2T(n/2) + n^2$

Solution: let $n = 2^4$ in:

$$T(2^4) = 2T(2^3) + (2^4)^2 = 2^4k + (2^2)^4 + (2^2)^4 + (2^2)^4 + (2^2)^4$$

$$T(2^3) = 2T(2^2) + (2^3)^2 = 2^3k + (2^2)^3 + (2^2)^3 + (2^2)^3$$

$$T(2^2) = 2T(2^1) + (2^2)^2 = 2^2k + (2^2)^2 + (2^2)^2$$

$$T(2^1) = 2T(2^0) + (2^1)^2 = 2k + (2^2)^1$$

$$T(2^0) = k$$

More generally, if we let $n = 2^m$ and follow the pattern we can see:

$$\begin{aligned} T(2^m) &= 2^m k + \sum_{i=1}^m (2^i)^2 \\ &= 2^m k + m(2^m)^2 \end{aligned}$$

Substituting n back in for $m = \lg(n)$ we get:

$$T(n) = nk + n^2 \lg(n)$$

On inspection we can conclude that $T(n) \in O(n^2 \lg(n))$

(d) $T(n) = 5T(n/4) + \sqrt{n}$

Solution: let $n = 4^3$ in:

$$T(4^3) = 5T(4^2) + \sqrt{4^3} = 5^3k + 5^2\sqrt{4^1} + 5\sqrt{4^2} + \sqrt{4^3}$$

$$T(4^2) = 5T(4^1) + \sqrt{4^2} = 5^2k + 5\sqrt{4} + \sqrt{4^2}$$

$$T(4^1) = 5T(4^0) + \sqrt{4} = 5k + \sqrt{4}$$

$$T(4^0) = k$$

More generally, if we let $n = 4^m$ and follow the pattern we can see:

$$\begin{aligned} T(4^m) &= 5^m k + 5^{m-1}\sqrt{4} + 5^{m-2}\sqrt{4^2} + \dots + 5^1\sqrt{4^m} - 1 + 5^0\sqrt{4^m} \\ &= 5^m k + S \quad , \text{ where } S = 5^{m-1}2 + 5^{m-2}2^2 + \dots + 5^12^{m-1} + 5^02^m \end{aligned}$$

If we multiply S by the common ratio $\frac{2}{5}$ and subtract it from S we get:

$$\begin{aligned} S - \frac{2}{5}S &= (5^{m-1}2 + 5^{m-2}2^2 + \dots + 5^12^m - 1 + 5^02^m) - (5^{m-2}2^2 + 5^{m-3}2^3 + \dots + 5^02^m + 5^{-1}2^{m+1}) \\ \frac{3}{5}S &= 5^{m-1}2 - 5^{-1}2^{m+1} \\ S &= \frac{2}{3}(5^m - 2^m) \end{aligned}$$

Substituting n back in for $m = \log_4(n)$ and the closed form solution for S we get:

$$\begin{aligned} T(n) &= 5^{\log_4(n)}k + \frac{2}{3}5^{\log_4(n)} - \frac{2}{3}2^{\log_4(n)} \\ &= n^{\log_4(5)}k + \frac{2}{3}n^{\log_4(5)} - \frac{2}{3}n^{\log_4(2)} \end{aligned}$$

Since $\log_4(5) > \log_4(2)$ we can conclude that $T(n) \in O(n^{\log_4(5)})$