On the Adjustment of Kater's Pendulum

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Although a little different from the main object of this note, it may be useful to point out the corresponding arrangements for dealing with the binomial expansion. Let us write

$$X_n = 1 + mx + m_2x^2 + \ldots + m_{n-1}x^{n-1} \quad \ldots\ldots\ldots\ldots\ldots(8)$$

where the coefficients are of the usual form.

Thus $$(1+x)\frac{dX_n}{dx} = mX_n - (m-n+1)m_{n-1}x^{n-1}$$

$$= mX_n - n(m_nx^{n-1}), \quad \ldots\ldots\ldots\ldots\ldots(9)$$

or* $$\frac{d}{dx}\left[\frac{X_n}{(1+x)^m} - 1\right] = -\frac{n(m_nx^{n-1})}{(1+x)^m}. \quad \ldots\ldots\ldots\ldots\ldots(10)$$

Hence, when x is positive, (10) gives

$$\frac{d}{dx}\left|\frac{X_n}{(1+x)^m} - 1\right| \leqq n\,|\,m_n\,|\,x^{n-1}$$

so that $$\left|\frac{X_n}{(1+x)^m} - 1\right| \leqq |\,m_n\,|\,x^n,$$

or $$|\,X_n - (1+x)^m\,| \leqq |\,m_n\,|\,.\,x^n(1+x)^m. \quad \ldots\ldots\ldots\ldots(11_1)$$

But, if x is negative and greater than $-a$, where a is a positive proper fraction, we get from (10)

$$\left|\frac{X_n}{(1+x)^m} - 1\right| \leqq \frac{|\,m_n\,|\,a^n}{(1-a)^m},$$

or $$|\,X_n - (1+x)^m\,| \leqq |\,m_n\,|\frac{a^n}{(1-a)^m}. \quad \ldots\ldots\ldots\ldots\ldots(11_2)$$

T. J. I'A. BROMWICH.

ON THE ADJUSTMENT OF KATER'S PENDULUM.

THE measurement of the acceleration due to gravity by means of the compound pendulum is a very common laboratory experiment; but there are certain simple principles involved in the theory of it which are not, as it seems to me, sufficiently emphasized by teachers, and which are not stated (at all events explicitly) in any text-book known to me. It may be of use to some to enunciate them here, and to shew their importance both to the instrument-maker and the student.

1. One question that sometimes arises is, What will be the effect of shifting an adjustable load to different places along the bar of a compound pendulum?

*I suppose that the rule for differentiating x^m has been proved *independently of the binomial theorem*; the proofs which depend on the binomial (for indices which are not positive integers) appear to me objectionable in every way, at least for beginners.

It is easy to predict the effect of *adding* a particle; but what will be the effect of moving a particle already present? The first case may be at once dismissed by saying that if a particle be added above the centre of oscillation, it tends to swing as a simple pendulum of shorter period, and therefore accelerates the pendulum; while if added below, it retards it. The second case requires further consideration. Let us suppose the particle to be moved along a straight line, called the axis of the pendulum, passing through the centre of suspension and the centre of gravity. Let the particle be of mass m, the mass of the rest of the pendulum being M; let the positions of m and the C.G. of M respectively be denoted by co-ordinates x, h, reckoned along the axis from the centre of suspension. Let the original and altered co-ordinates of m be x_1, x_2, respectively; let the moment of inertia of M about the knife-edge be I; let t_1, t_2 be the periods corresponding to x_1, x_2, and let l_1, l_2 be the corresponding simple equivalent pendulums. Then $t_2 > = < t_1$ according as $l_2 > = < l_1$, or as $\dfrac{I+mx_2^2}{Mh+mx_2} > = < \dfrac{I+mx_1^2}{Mh+mx_1}$. Now the alteration in the numerator is $m(x_2^2-x_1^2)$, and this expressed as a fraction of the original numerator is $\dfrac{m(x_2^2-x_1^2)}{I+mx_1^2}$; similarly the fractional increase in the denominator is $\dfrac{m(x_2-x_1)}{Mh+mx_1}$. The above condition amounts then to $\dfrac{m(x_2^2-x_1^2)}{I+mx_1^2} > = < \dfrac{m(x_2-x_1)}{Mh+mx_1}$, *i.e.* to $(x_2+x_1) > = < l_1$, when $x_2 > x_1$; if $x_2 < x_1$, the signs of inequality must be reversed. This condition may be put into a very simple form. Let us call the point which lies midway between the centres of suspension and oscillation for the *first* position of m the "middle point"; let us call points on the axis situated symmetrically with respect to the middle point "symmetric points." Then if a particle is initially above the middle point, raising it will retard the pendulum, lowering it to any point short of the symmetric point will accelerate it, lowering it beyond the symmetric point will retard it. If it is initially below the middle point, lowering it will increase the period, raising it to any position below the symmetric point will decrease it, raising it still further will increase it. The disadvantage of this method of expressing the results is that we utilise l_1, which is not a constant. Let us now proceed rather differently, taking L as the simple pendulum equivalent to the compound pendulum *without the particle*; thus L is constant.

We wish to discover a value x_2 which will give to the expression $\dfrac{I+mx^2}{Mh+mx}$ the value $\dfrac{I+mx_1^2}{Mh+mx_1}$. This leads to a quadratic, of which of course one root is x_1; the other is $\frac{I-Mhx_1}{Mh+mx_1}$. Putting $L=\frac{I}{Mh}$, this reduces to $x_2=\frac{L-x_1}{1+\frac{mx_1}{Mh}}$. Moving the particle within the limits x_1 and x_2 as given by this expression will accelerate the pendulum; transferring it to a point outside these limits will retard it. Moving it from the upper limit to any position short of the lower limit increases the restoring couple more than the moment of inertia; moving it beyond the lower limit increases the moment of inertia more than the couple; raising it from the upper limit to any position short of the knife-edge decreases the couple more than the moment of inertia; raising it still higher will increase the moment of inertia while still decreasing the couple. Of course there must be positions of m which will make the period a minimum; these correspond to the cases when

$$x_2=x_1=-\frac{Mh}{m}+\sqrt{\left(\frac{Mh}{m}\right)^2+\frac{Mh}{m}.L}$$

and when

$$x_2=x_1=-\frac{Mh}{m}-\sqrt{\left(\frac{Mh}{m}\right)^2+\frac{Mh}{m}.L};$$

when m is a small fraction of M, the first will be very little below the middle point of L, while the second will be far above the knife-edge and need not be considered.

2. If we suppose next that the mass m is not a particle but has an appreciable moment of inertia about its own C.G., allowance can readily be made for this. In the first case, that of addition of load, the altered period corresponds to a simple pendulum $\frac{I+i+mx^2}{Mh+mx}$, where i denotes the moment of inertia of m about its own C.G. The question is then how this expression compares with $\frac{I}{Mh}$, *i.e.* with L; and there is no difference between them provided the C.G. of m is placed where

$$x=\frac{L}{2}\pm\frac{1}{2}\sqrt{L^2-4.\frac{i}{m}}.$$

When the radius of gyration of m about its C.G. is small compared with L, these two positions are respectively just below the knife-edge and just above the centre of oscillation of the pendulum unloaded. If m be attached at any intermediate position it decreases the period; if outside these limits it has a retarding effect.

So far as transferring m from one position to another is concerned, the only change in the previous investigation is that

$I+i$ replaces I, so that L must now be taken to mean the simple pendulum equivalent to the compound one with m so fixed that its C.G. is at the knife-edge.

3. The whole of the above discussion may be represented in a convenient graphical manner as follows. Let the axis of the pendulum be taken as x-axis, and at each point erect an ordinate equal to the length of the simple pendulum equivalent to the compound one when m is fixed at the point in question. The equation of the graph so obtained is $y=\dfrac{I+i+mx^2}{Mh+mx}$, which is of the form $x^2-xy=$ a constant, and represents a hyperbola. One of its asymptotes is $y=x-\dfrac{Mh}{m}$, the other is $x=-\dfrac{Mh}{m}$. These lines cross the x-axis at the two points where m would have the same moment as M about the knife-edge; the latter of the two corresponds to period infinity, as is obviously right since the C.G. of the whole pendulum is at the knife-edge. If m be placed beyond the point where $x=-\dfrac{Mh}{m}$, we obtain the other branch of the hyperbola, of which the ordinates are negative. This of course is a sign that under the increased influence of m the pendulum is now hanging the other way up. Provided this is borne in mind, the ordinates may be plotted as if they were positive, and the second branch will then be a reflection of the first in the asymptote. The equivalent simple pendulum is a minimum for points given by

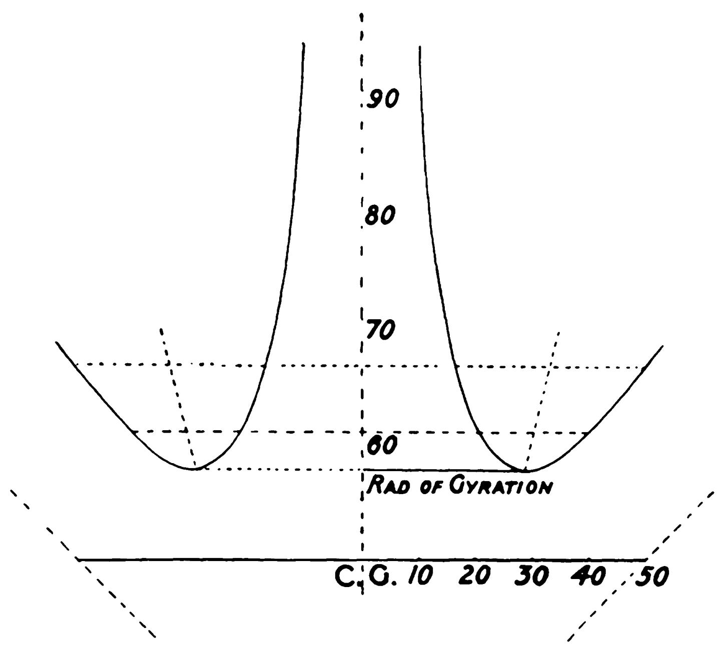
$$2x(mx+Mh)=I+i+mx^2,$$

$$i.e.\text{ where } x=-\frac{Mh}{m}\pm\sqrt{\left(\frac{Mh}{m}\right)^2+\frac{I+i}{m}};$$

when m is in either of these positions, the corresponding simple pendulum is of length $2x$.

4. A common laboratory experiment on the compound pendulum is to swing a bar from a knife-edge whose position can be varied, and then to plot a curve shewing how the time of swing depends on where the knife-edge is, but (so far as I have seen) it is not customary to point out all that may be learned from such an experiment. Let us take x now to denote the height of the knife-edge above the C.G., which is regarded as origin; then the length of the simple equivalent pendulum is $\dfrac{I+Mx^2}{Mx}$, so that the graph in this case is $y=\dfrac{I+Mx^2}{Mx}$; the asymptotes are $x=0$ and $y=x$, of which the former corresponds to infinite period with the knife-edge at the C.G. If the knife-edge be moved still further, the bar hangs with the other end up, and we realise the other branch of the

hyperbola; the two branches may very well be plotted on the same side of the x-axis, and they will then be symmetrical about the y-axis. The minimum value of y will be twice the radius of gyration about the C.G., which is $\sqrt{\frac{I}{M}}$; and this corresponds to positions of the knife-edge at $x = \pm\sqrt{\frac{I}{M}}$. A line drawn across the graph parallel to the x-axis at any altitude exceeding $2\sqrt{\frac{I}{M}}$ will of course give four intersections, symmetrically placed with respect to the y-axis; and the distance between an asymmetric pair will be equal to the ordinate of the four points,



Uniform bar, a metre long.

since it will be the length of the simple pendulum equivalent to the compound pendulum swung from the four corresponding points. If the altitude of the line drawn exceeds $2\sqrt{\frac{I}{M}}$ only slightly, the four intersections will nearly coincide, two and two; and the length of the equivalent simple pendulum might then easily be misjudged owing to the selection of a symmetric instead of an asymmetric pair. This illustrates the point mentioned in Gray's *Text-book of Physics*, Vol. I., where the reader is cautioned against selecting two points equidistant from the C.G.; and it is stated that the outrageously wrong value obtained for g will usually reveal the nature of the mistake. It might be added that with a pendulum reasonably designed such a mistake cannot fail to be revealed; for, as will presently be seen in another way, the very essence of proper adjustment is that the period should be such as to give four well separated intersections on the graph, or (in other words) to place the C.G. at very unequal distances from the centre of suspension and the centre of oscillation. (See § 9.)

5. Another matter worth mentioning in this connection is that as the graph is a conic, the locus of parallel chords is a straight line. Therefore, when the curve representing the observations has been plotted, a series of lines parallel to the x-axis should be drawn across it; if we then confine our attention to one branch at a time, the chords should be bisected and the middle points used in order to construct a "diameter" of the conic; this will intersect the curve at a point corresponding to the shortest period. The same thing having been done with the other branch, the distance between the two minima should be ascertained; the square of half this distance multiplied by the mass of the pendulum gives its moment of inertia about its C.G.

6. The simplicity of these results is often masked by plotting as ordinate the period, instead of the length of the equivalent simple pendulum; the graph has then the equation

$$\frac{y}{2\pi}=+\sqrt{\frac{Mx^2+I}{Mgx}},$$

a cubic whose only asymptote not entirely at infinity is $x=0$. It has no points below the x-axis or on the negative side of the y-axis, and in fact when the knife-edge is taken to the other side of the C.G. the equation requires a change of sign in the term Mgx; it then gives a new curve which is the reflection in the y-axis of the previous one, and corresponds to cases in which the pendulum hangs the other way up. It will be noticed that although in general a straight line cuts a cubic in three points, in this case a line parallel to the x-axis will only cut it in two; yet imaginary points can only come in in pairs. Of course the explanation is that the third point has gone away to infinity. Had there been three intersections at a finite distance, there must have been three points on the same side of the C.G. giving the same period; whereas we know there are only two. Similarly a line parallel to the y-axis meets the curve in one point at infinity, and the y-axis itself meets it in three points at infinity. The experiment is often carried out with a uniform bar drilled with holes along its length; each of these in succession is used as a means of supporting the bar upon a knife-edge. This method is preferable to attaching a knife-edge to the bar and sliding it to different positions. With a uniform bar the cubic character of the curve is not noticeable, for the point of inflexion lies beyond the end of the bar; viz., at a distance from the centre equal to $\sqrt{1+\frac{2}{3}\sqrt{3}}$ times the half-length. However, by loading the bar at the centre the point of inflexion may be brought nearer; *e.g.* if at the C.G. of the bar there be attached a small mass having thrice the mass of the bar, then the point of inflexion is at only half the above distance.

If the experiment is carried out by means of a sliding knife-edge clamped to the bar, the only effect on the expression for the simple equivalent pendulum is to increase the numerator by one constant and the denominator by another, and the character of the graph is unchanged. A. O. ALLEN.

REVIEWS.

Volume and Surface Integrals used in Physics. By J. G. LEATHEM, M.A. (Cambridge University Press, 1905.) Price 2*s.* 6*d.* net.

This is No. 1 of a series of "Tracts" in the course of appearance dealing with mathematics and mathematical physics. So far as we can see, the object of these tracts is to deal with portions of mathematical theory too small to form a book, and either too large to form a paper or of too general application to remain buried in transactions until some specialist unearths them. If this view is correct, the useful purpose served by such tracts might almost justify their being multiplied indefinitely; indeed the "Tract" method of publication has many advantages over the "Transactions" method. The volume and surface integrals discussed in the book have special reference to quantities satisfying the partial differential equations of the first and second degrees occurring in physics. In the introduction the author discusses at some length the applicability of infinitesimal analysis to the properties of bodies whose ultimate structure is molecular, and suggests the term "physical smallness" as designating the order of magnitude of the elements involved in the analysis of physical bodies. Mr. Leathem's "physically small element" is thus the same as the "differential element" of the present reviewer. G. H. BRYAN.

Lectures on the Theory of Functions of Real Variables. By JAMES PIERPONT, Professor of Mathematics in Yale University. Volume I., xii + 560 pages. (Ginn & Co., Boston, [1905].)

In its historical development, the theory of functions of real variables is not always separable from the theory of analytic functions of complex variables. For example, Weierstrass' continuous function with no derivative was discovered by considering the real part of a certain analytic function on its circle of convergence, which is also a line of essential singularities; further, it casts, I think, some light on the theory of functions of a real variable to reflect that the first systematic treatment of this theory was made (by Hankel in 1870) long after the theory of analytic functions, as given by Cauchy and Riemann, had come to be widely known, and that it was made with the express