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Types of Two-Dimensional Pendulums and Their Uses in Education

ROBERT J. WHITAKER
Department of Physics, Astronomy, and Materials Science, Southwest Missouri State University, Springfield, MO 65804, U.S.A.

Abstract. Pendulums which swing in two dimensions simultaneously and are designed to leave a record of their motion are termed 'harmonographs'. The curves which they draw are known, alternatively, as 'Bowditch curves' or 'Lissajous curves'. A variety of designs of harmonographs have been invented over the years. These may be a 'Y-suspended' 'simple' pendulum, or they may be a complex 'physical' pendulum system. Harmonographs have been built as demonstration apparatus in physics (or mathematics) or as 'art' machines for enjoying the aesthetics of the curves produced.

1. Introduction

The pendulum and its motion are frequently topics in a beginning physics course. A 'simple' pendulum may be considered as a point mass fastened to the end of a massless cord. It is then allowed to swing in a plane, its motion being constrained to one dimension. The history and the educational importance of such a device have recently been discussed by Matthews (2000, 2001).

If a pendulum swings with a small amplitude, its motion is often referred to as 'simple harmonic motion' (SHM). This term was first introduced by Sir William Thomson (later, Lord Kelvin) and P.G. Tait in their classic texts on Natural Philosophy (Thomson & Tait 1873, p. 19). In addition they demonstrated that the projection, on the diameter of the circle, of a point that is moving with uniform circular motion also executes simple harmonic motion. This can easily be observed if one sights, edge on, a rod mounted near the circumference of a large turntable (such as a merry-go-round); as the turntable rotates at a constant angular speed, the rod's horizontal motion is SHM (French 1971, pp. 7–9).

In 1815 James Dean, Professor of Mathematics and Natural Philosophy at the University of Vermont, published an analysis of the motion of the moon about the Earth (Dean 1815; Crowell 1981). Instead of viewing the moon from the Earth, Dean placed an observer on the equator of the moon and observed the Earth. Following his analysis he noted that this motion 'may be easily imitated by a pendulum' that is hung with a Y-suspension (Figure 1) (Dean 1815, p. 245). The pendulum's mass is free to swing with two independent lengths – first, the length from the top of the 'Y' suspension; second, the length from the junction of the three

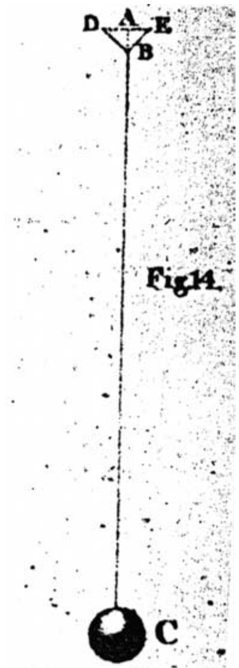

Figure 1. James Dean's 'Y-suspended' pendulum (Dean 1815, Plate I).

cords forming the 'Y'. Each of these, then, was free to swing in two independent, vertical planes.

Nathaniel Bowditch, in the same year, extended Dean's idea for a Y-suspended pendulum and provided a detailed mathematical analysis of its motion (Bowditch 1815). The curves, described by the equations of the motion of a (dimensionless) pendulum mass swinging with a small amplitude in two dimensions, are sometimes referred to as 'Bowditch curves'. While Bowditch was the first to derive and demonstrate these, one may more often find these same curves referred to as 'Lissajous curves', after Jules Antoine Lissajous who produced these curves with a pair of vibrating tuning forks and described his work in a series of papers beginning in 1855 (Lovering 1881, p. 298; Crowell 1981, p. 454; Whitaker 2001a, pp. 169–170).

The Y-suspended pendulum was again introduced around 1844 by Hugh Blackburn while a student at Cambridge. Blackburn was a classmate and friend of William Thomson (who would later be better known as Lord Kelvin). Curiously, Blackburn does not seem to have left any description of this pendulum himself. We learn of it from the secondary literature, where it is referred to as the 'Blackburn pendulum'. The difference in this design is that a permanent (or semi-permanent) record of its motion was made. Several alternative methods are mentioned in the literature. A heavy, hollow funnel is filled with fine sand (or ink) which falls through a small hole leaving a trace as the pendulum swings. Other pendulums have a sharp point which trace the motion on a plate of smoked glass or in a layer of sand. Later versions used electrical sparks to mark a sheet of paper placed on a conducting plate (Whitaker 1991, 2001a, pp. 163–164). Worland and Moelter (2000) have recently described the use of a spreadsheet to analyze the data produced by a modern spark generator of a Y-suspended pendulum in a student laboratory.

An alternative version, built by John Dobson around 1877, was described by J.G. Hagen (Hagen 1879, pp. 287, 297–299; Rigge 1926, pp. 68–71). Dobson's version had the advantage of having bifilar suspensions so that twisting was eliminated as the mass swung. A pen recorded its motion on paper.

2. The 'Physical' Pendulum

The Y-suspended pendulum was basically a simple pendulum which was free to swing in two independent directions simultaneously. These usually had large masses and long supporting cords. This limited the location where this pendulum could be demonstrated. If one removes the restrictions of a 'small' mass and a 'massless' support, one has a 'physical pendulum'. A common example of this in introductory physics is a rigid rod (such as a meter stick), fitted with a knife edge clamp, which may swing on an appropriate support. The knife edge is adjusted along different positions of the stick, and the period of this pendulum is found to be dependent on the position of the knife edge. Because of the uniformity of mass distribution along a meter stick, the period of the pendulum, as a function of mass distribution, may be readily obtained (Stephenson 1969, pp. 210–218; Halliday et al. 1993, pp. 390–393).

An early example of a physical pendulum for precision time measurements was introduced in 1817 by Captain Henry Kater (Kater 1818, pp. 33–102). Known now as 'Kater's reversible (or convertible) pendulum', Kater's pendulum

> consisted of a brass rod to which were attached a flat circular bob of brass and two adjustable weights, the smaller of which was adjusted by a screw. The convertibility of the pendulum was constituted by the provision of two knife edges turned inwards on opposite sides of the center of gravity. The pendulum was swung on each knife edge, and the adjustable weights were moved until the times of swing were the same about each knife edge. When the times were judged to be the same, the distance between the knife edges was inferred to be the length of the equivalent simple pendulum, ... (Lenzen & Multhauf 1965, p. 314)

Appropriate corrections for error (such as buoyancy of the air) were made, and this equivalent seconds pendulum made possible improved accuracy for measurements of gravitational acceleration at different locations to aid in determining the 'figure of the Earth'. While studies of this kind were a part of 'physics' in the Nineteenth Century, one must look for them today in references on 'geodesy' or 'geophysics' as these have evolved as separate disciplines. Kater's pendulum has served as a model for subsequent modifications and refinements in the measurement of gravitational acceleration or differences in acceleration (Garland 1965, pp. 6–27).

3. 'Harmonographs'

The preceding discussion is indicative of the use made of the physical pendulum as part of the 'mainstream' physics research in the 19th century. The pendulums used also oscillated in one dimension. We now return to the use of those physical pendulums which, like the Y-suspended pendulum, were designed to oscillate in two dimensions. In 1873 S.C. Tisley reported on an apparatus made of two vertical rods, fitted with knife edges, which could swing at right angles to each other. This is illustrated in Figure 2 (*Engineering* 1874, p. 101). A ball-and-socket joint was attached to the top of each rod; a wire arm was attached to the joint and perpendicular to the rods. A pen, mounted at the intersection of the two wires, traces the curves resulting from the motion of the pendulums. The period of each pendulum could be adjusted by means of weights attached to each rod (Tisley 1873, p. 48). Tisley's pendulum apparatus was soon offered for sale by the firm, 'Tisley and Spiller, Opticians, etc.'. By May 1877 the apparatus was advertised as 'Tisley's Harmonograph. For drawing Lissajous' and Melde's figures . . . '. This seems to be the first use of the term, 'harmonograph', in the literature (Whitaker 2001a, p. 171). The term is subsequently applied to a variety of different designs of curve drawing apparatus.

A simplified modification of Tisley's design was offered by Newton & Co. and described by Herbert Newton in 1909 (Goold et al. 1909, pp. 4–8). This is shown in Figure 3. The drawing table is mounted on the top of one of the pendulums, and the pen is attached, by means of a long rod, to the top of the second pendulum. Each pendulum is mounted on a knife edge and can be set in motion in any direction.

4. Wheatstone's Kaleidophone

While not a pendulum, *per se*, the motion of the 'kaleidophone' discussed by Charles Wheatstone (Wheatstone 1827) is closely related. The kaleidophone consists of a thin rod, clamped at one end. This could be set in vibration with a small hammer or violin bow. A small silvered glass bead, mounted on the end, reflects a beam of light incident upon it. Persistence of vision permits one to observe the path of the bead. William Sang, in 1832, produced a mathematical analysis of the kaleidophone in which there is asymetry in the vibration of the rod in two directions. The motion of the bead is described by equations identical to those of Bowditch (Greenslade 1992; Whitaker 1993). A common automobile radio antenna will exhibit the same effects under appropriate conditions (Annet 1979; Merivuori & Sands 1984; Newburgh & Newburgh 2000).

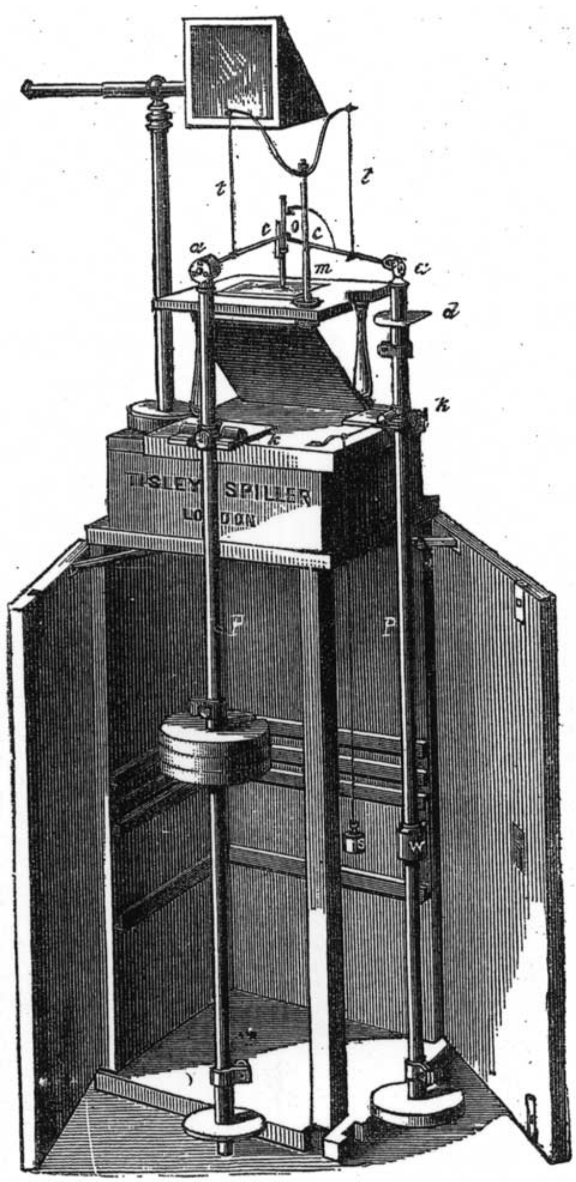

Figure 2. Tisley's compound pendulum (*Engineering* 1874, p. 101).

Figure 3. Newton's harmonograph (Goold et al. 1909, p. 5).

5. 'Bowditch/Lissajous' Curves

When the pendulums are set to vibrate perpendicular to one another, their motion traces the same system of curves described by Bowditch. Since the periods of each pendulum can be adjusted independently, a nearly infinite number of different curves of varying complexity may be drawn. Figure 4 shows two simple examples (Poynting & Thomson 1900, p. 74). Of particular interest are those curves produced when the periods of the two pendulums are in integer ratios of one another. A range of ideal curves (without damping) are shown in the plate reproduced as Figure 5.

The plate itself is of interest; a number of authors have illustrated their works with this identical plate. Zahm (1892, p. 416) and Tyndall (1894, p. 418), for example, used it to demonstrate the curves produced by properly adjusted tuning forks based on Lissajous' work. Poynting and Thomson (1900, p. 76) reproduced the same plate as an example of kaleidophone figures. Interestingly, this practice was not that uncommon. In his study of scientific illustration Knight has noted: 'Plates were expensive to make, and wherever possible were reused; otherwise,

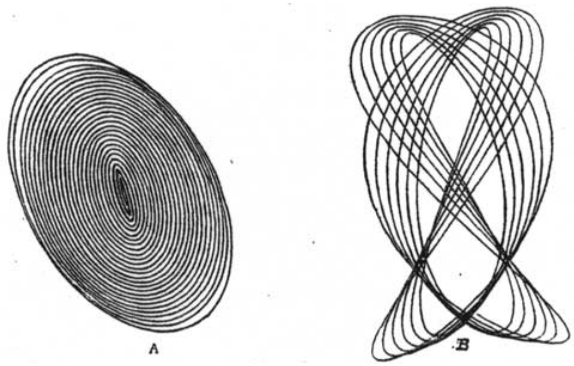

Figure 4. Damped harmonic curves (Poynting & Thomson 1900, p. 74).

they were often copied (sometimes reversed in the process) – we keep meeting recycled illustrations' (Knight, 1998, p. 249). This writer has made no effort to determine the first use of this plate, nor to list its (probably) many other locations.

6. Harmonographs in Popular Works

In addition to the 'ideal' curve the gradual reduction of amplitude of each pendulum adds to the complexity (and interest) of the resulting product. Thus, while the harmonograph was used to demonstrate the drawing of these curves, the demonstration was, as often as not, for the purposes of entertainment. As a result, authors of popular works soon began to include descriptions of commercial harmonographs in their books or to describe simplified versions that could be constructed by the home craftsman. J.H. Pepper, in the fourth edition of his *Cyclopaedic Science Simplified*, described Tisley's apparatus and reproduced a picture of it (Pepper 1877, pp. 562–565). Tissandier, in 1883, discussed Tisley's apparatus, but he also provided details for constructing one from simple materials (Tissandier 1883, p. 175; Whitaker 2001a, p. 167).

Cundy and Rollet wrote in 1961 that: 'The harmonograph was a popular diversion in Victorian drawing-rooms, since when it has suffered a decline and is rarely seen today' (Cundy & Rollett, 1961, p. 244). While one might not find the elaborate commercial versions that once were sold, harmonographs are still popular as science projects for students and for hobbyists. Directions for their construction are readily available. Cundy and Rollet, for example, provide detailed directions for the construction of a harmonograph similar to Newton's earlier commercial design. Bulman has provided directions for a similar device, as well as a simple Wheatstone kaleidophone (Bulman 1968a, pp. 86–94, 82—85). In a second book he describes a device for drawing curves that result from the rotation of gears (Bulman 1968b, pp. 12–31). Most recently Ashton has written a little book that describes the construction of a harmonograph similar to Newton's design, as well

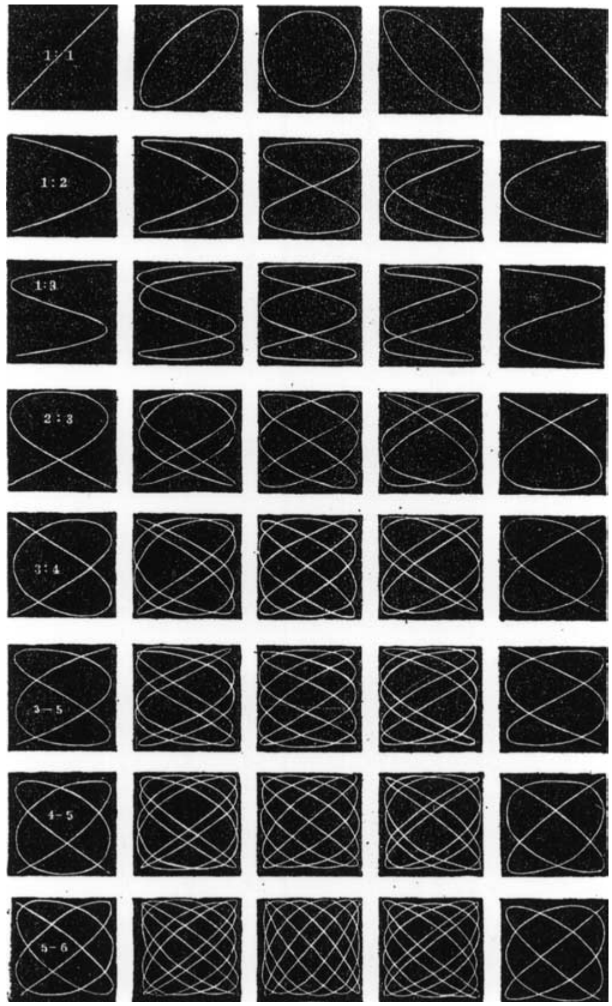

Figure 5. Ideal harmonic curves (Zahm 1892, p. 416; Tyndall 1894, p. 418).

as a three pendulum harmonograph, similar to Tisley's, which has the drawing table mounted on a third pendulum (Ashton 1999, pp. 3, 29). His account is illustrated with drawings produced by his apparatus as well as illustrations from older references. A variety of harmonograph designs, under different names, have been discussed by this author (Whitaker 2001a). These are summarized in Table I.

Table I. Various pendulum apparatus

Device	Year	Designer	Reference
'Y-suspended Pendulum'	1815	Dean; Bowditch	Crowell 1981, pp. 452-454
'Blackburn Pendulum'	1844	Blackburn	Whitaker 1991, pp. 330–333
'Tisley's Compound Pendulum'	1873	Tisley	Whitaker 2001a, pp. 164–165
'Tisley's Harmonograph'	1877	Tisley & Spiller	Whitaker 2001a, p. 165
'Sympalmograph'	1877	Browning; Benham	Goold et al. 1909, pp. 39–50; Whitaker 2001a, p. 166
'Dobson Duplex Pendulum'	1877	Dobson	Hagen 1879, pp. 287, 297–299; Rigge, 1926, pp. 68–71
'Double Pendulum'	1894	Bryan	Whitaker 2001a, p. 166
'Pendulograph'	1895	Andrew	Whitaker, 2001a, pp. 167–168
'Quadruple Harmonic-Motion Pendulum'	1899	Hoferer	Whitaker, 2001a, p. 168
'Twin-elliptic Pendulum'	1906	Benham	Goold et al. 1909, pp. 61–79
'Benham's Triple Pendulum'	1909	Benham	Goold et al. 1909, pp. 51–61

7. Harmonic Motion in College Textbooks

As with the simple pendulum, the compound pendulum (and Kater's pendulum in particular) has seen reduced discussion in introductory college physics textbooks. However some older textbooks present detailed accounts of the topic. Among several of these are Watson (1902, pp. 132–134) and Poynting & Thomson (1909, pp. 12–27). Pointing and Thomson, particularly, provide a detailed discussion of various pendulum methods and sources of error as well as extensive historical account of the problems involved in determining a standard unit of length and of the figure of the Earth. The textbook by Millikan, Roller, and Watson was noted for the introduction of historical material into the text. The authors' discussion of simple harmonic motion and pendulum motion (including the Kater pendulum) in its historical context is noteworthy (Millikan et al. 1937, pp. 330–345). Similar historical emphasis was provided by Lloyd William Taylor in his book published in 1941 (Taylor 1941, 1959, pp. 182–197). Feather's account is insightful (Feather 1959, pp. 182–194). And, in an intermediate level text, Stephenson includes the Kater pendulum as an example of a physical pendulum (1969, pp. 213–215).

One example of the Kater's pendulum, as an experiment in the laboratory, may be found in Searle (1934, pp. 7–15). Searle provides a thorough discussion of the theory, as well as sample data gathered from an experiment. Apparatus for the study of Kater's pendulum and for the study of the compound pendulum were sold for many years by Central Scientific Company (Cenco), and detailed instructions in their 'Selective Experiments in Physics' series were provided with these to assist in their use (Eaton 1940, 1941). A simplified version may be found in Ingersoll, Martin, and Rouse, who include it as part of a series of experiments on moment of inertia (1953, pp. 54–57). Modifications in the Kater pendulum have been discussed more recently as laboratory experiments in physics (Jesse 1980, pp. 785–786; Peters 1999, pp. 390–393; Candela et al. 2001, pp. 714–720). It is less clear, however, how many institutions may be using these as part of their instruction.

Arnold Sommerfeld, in his classic work on theoretical physics, wrote in the introduction to his chapter on 'oscillation problems':

> 'The investigations that are to follow will teach us nothing new about the principles of mechanics. So great, however, is the significance of oscillation processes for physics and engineering that their separate systematic treatment is deemed essential' (Sommerfeld 1952, p. 87).

Sommerfeld's mathematical analysis of a 'simple' pendulum and of a 'physical' pendulum is based on assumptions regarding the mass distribution of the pendulum as well as the location of its support. Similar restrictions are placed on 'coupled' pendulums, including the 'double pendulum', in which a mass is connected by a cord to a second mass, which is in turn connected to a support. (This is the basis of the 'twin-elliptic' pendulum harmonograph.) Assumptions again arise involving 'point' masses and 'massless' cords. His mathematical analysis of such a problem, which is usually restricted to advanced courses in mechanics, is a model of clarity (Sommerfeld 1952, pp. 87–117).

It should not be surprising, then, that none of the discussions of harmonographs includes a mathematical analysis of the motion of the pendulums. We find that the period of each is adjusted by trial and error. Small differences in adjustment may produce large variations in the curves produced. Under 'proper' adjustment the recording point of the pendulum traces the same curve but with a continually decreasing amplitude. These differences seem to be one of the fascinations with harmonographs. Rigge, for example, devoted a full chapter in his book on the beauty of curves (Rigge 1926, pp. 122–132).

A second class of harmonographs should be mentioned in passing. These make use of gears or pulleys in their operation. While they do not operate under pen-

Table II. Various 'gear' apparatus

Device	Year	Designer	Reference
'Geometric Chuck'	1833	Ibbetson	Whitaker 2001b, p. 174
'Lissajous' Apparatus	1869	Pickering	Whitaker 2001b, p. 174
'Harmonic Curve Apparatus	1873	Donkin	Whitaker 2001b, pp. 174–175
'Cycloidotrope'	1883	Pumphery	Rigge 1926, pp. 74–75
'Campylograph'	1900	Dechevrens	Rigge 1926, pp. 78–81
'Wondergraph'	1913	Tuck	Tuck 1913, pp. 436–439;
	1931	Collins	Collins 1931, pp. 71–74
'Cyclo-harmonograph'	1916	Moritz	Whitaker 2001b, p. 176
'Creighton Compound Harmonic Motion Machine'	1924	Rigge	Rigge 1926, pp. 81–91; Whitaker 2001b, p. 178
'Kukulograph'	1933	Hoferer	Whitaker 2001b, p. 177
'Spirograph®'	1967	Kenner Products, Co.	Whitaker 1988; Whitaker 2001b, pp. 177, 179–180
'Schemagraph'	1968	Bulman	Bulman 1968b, pp. 12–31
'Turntable Oscillators'	1971	Project Physics	Whitaker 2001b, p. 177

dulum motion, they are related to an important problem in astronomy – the effort to describe the motion of the heavenly bodies with a system of circular motions. An appropriate example of one of these is the popular toy, Spirograph®, which came on the market in 1967. It is designed to produce that class of curves known as 'trochoids' (Whitaker 1988b, 2001b, pp. 179–180). A number of elaborate devices, however, have also been invented and described over the years; these are summarized in Table II (Whitaker 2001b).

8. Previous Surveys of Curve Drawing Apparatus

Several surveys of curve drawing apparatus have been published. Among the earlier was Hagen (1879). Lovering, in surveying the role of Bowditch, also summarized those devices of which he was aware. An extensive survey of a variety of devices was provided in Goold et al. (1909), many of which were sold by Newton and Co. Rigge (1926) provided an extensive mathematical description of the curves possible with various machines, including the highly complex 'Creighton Compound Harmonic Motion Machine' which he had begun in 1915 and completed in 1924. Greenslade, for a number of years, has encouraged continual interest in harmonographs in his series of articles (Greenslade 1979, 1992, 1993, 1998). This writer has recently provided a survey of the history of various harmonographs (Whitaker 1988a, b, 1991, 1993, 2001a, b).

9. Conclusions

This article has attempted to summarize those pendulums, constructed to swing in two dimensions, which produce records of those curves known as 'Bowditch curves' or 'Lissajous curves'. These 'harmonographs' were designed as demonstration devices to illustrate vibrations in mechanics as well as in the physics of sound and of harmonies in music. They were also designed to entertain through the ingenuity of the machine or the variety of the curves it produced. While the equations for these various curves may be programed into a computer, and the curves changed almost instantly, there is still a fascination in watching them being drawn by a mechanical device. Tolansky, in introducing his two pendulum harmonographs, has noted: '... it so happens that sophisticated electronics systems cannot create patterns which even remotely compare either in interest or in aesthetic appeal with those that can be formed by quite crude mechanical pendulum devices ... ' (Tolansky 1969, p. 267). Romer, similarly, has described a 'corridor apparatus' for student interaction and has observed that '... this apparatus can be used to produce a wide variety of designs which seem to have considerable aesthetic appeal to many people. We have found it very valuable, both in stimulating an interest in the simple physics on which it is based and perhaps, more honestly, simply as an "art machine" ' (Romer 1970, p. 1116).

Today one may find large scale harmonographs as interactive displays in science museums. While we may no longer find elaborate harmonographs listed in the catalogs of apparatus companies, the continual discussion of these in the literature is indicative of their inherent interest to students, teachers, and the general public.

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