Bessel's Corrections to the Keter Pendulum
Determining Gravitation Acceleration on Earth

A. Knight, N. Anna*
USM Department of Physics
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We attempted to experimentally determine the standard gravitational acceleration g by means of a Bessel pendulum, which is a volume symmetric, mass asymmetric reversible pendulum that can be treated as a simple pendulum without consideration to the moment of inertia or most air resistance when the period of each orientation is equivalent. Our experiment consisted of changing the center of mass of the pendulum to equalize the periods. From this period, we then calculated $g = (9.781 \pm 0.017)\frac{m}{s^2}$, which gave us an experimental error of 0.262%. Despite a small experimental error, our uncertainty failed to encompass the actual value of $g = 9.80665\frac{m}{s^2}$.

1. PROBLEM AND RELEVANT THEORY

Pendulums are a prominent tool in classical and modern physics. The oscillatory motion of the pendulum is well known and easily modeled with mathematics. They can be manufactured in order to customize and emphasize many aspects of the pendulum itself.

Pendulums were often used to measure the gravitational force due to their sensitivity and gravity's weak attraction. Today, we have more modern methods using lasers and vacuum chambers, but the old methods are still important and informative.

In 1817 Henry Kater designed and tested a reversible pendulum that was meant to measure the gravitational acceleration of the Earth. From this, Kater showed that the center of gravity and moment of inertia did not have to be calculated. By adjusting small weights on the pendulum, he would change the center of mass, and so change its period. He could then adjust the pendulum until the period was the same for both pivot points. From here, it could be treated as a simple pendulum, where $T^2 = 4\pi^2\frac{l}{g}$. This method was mostly effective, but it failed to take into account several factors that changed the period between the two pivot points, such as air resistance, buoyancy, and the added mass of the air dragged behind.

These problems reduced the accuracy of the reversible pendulum until in 1826 Friedrich Bessel designed a pendulum that was symmetric in volume, but asymmetric in mass. This negated the need to determine most air resistance. Additionally, by balancing the pendulum on a knife blade to find the center of mass, he then could determine the pendulum's period. This design was used for years to make fine measurements of the acceleration of gravity.

*Electronic address: alexander.knighr@maine.edu, nicholas.anna@maine.edu

2. EXPERIMENTAL SKETCH AND SALIENT DETAILS

Kater's pendulum design uses the symmetrical behavior of the period around the center of mass of an object. Following the logic and thought of Candela[2], we start with the equation for small oscillations of a physical pendulum,

$$\omega^2 = \frac{gml}{I} \tag{1}$$

where ω is the angular frequency, m is the mass of the pendulum, l is the length, and I is the moment of inertia. Draw a line through your center of mass, and pick an arbitrary origin such that $x_{cm} \geq 0$, then let your pendulum pivot at any point on this line (x). By the Parallel axis theorem, our equation of motion is

$$\frac{\omega^2}{g} = \frac{m(x - x_{cm})}{I_{cm} + m(x - x_{cm})^2} \tag{2}$$

From here, we can solve for the effective length of our pendulum ($x - x_{cm}$), which gives us a quadratic equation that yields

$$x - x_{cm} = \pm\frac{1}{2F} \pm \sqrt{\frac{1}{4F^2} - \frac{I_o}{m}} \tag{3}$$

where $F = \frac{m(x - x_{cm})}{(I_o + m(x - x_c m)^2)}$. This equation defines four points symmetric around the center of mass at which the angular frequency is equal. By choosing two points such that they are asymmetric and opposite around the center of mass, the radical drops out of the equation, and the moment of inertia does not have to be taken into account. This then give

$$\Delta x = \frac{g}{\omega_o^2} \tag{4}$$

which is expected for a simple pendulum.

Our apparatus (Fig 1) consists of a pair of symmetrical bobs about a meter apart on a metal pole, one wood and the other brass. Mounted below each bob is a knife edge

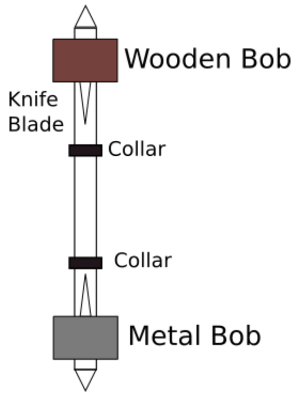

FIG. 1: Bessel Pendulum Diagram: Our pendulum consisted of three main parts. On either end of a solid metal were two bobs of equal volume and shape, one of wood and the other of brass, shown in brown and gray. Two sets of knife blades extended from the bobs to act as pivot points for the far side. Two movable collars are mounted between the knife blades, allowing for adjustments to the center of mass. This design allowed us to change the center of mass of the pendulum while keeping it symmetric.

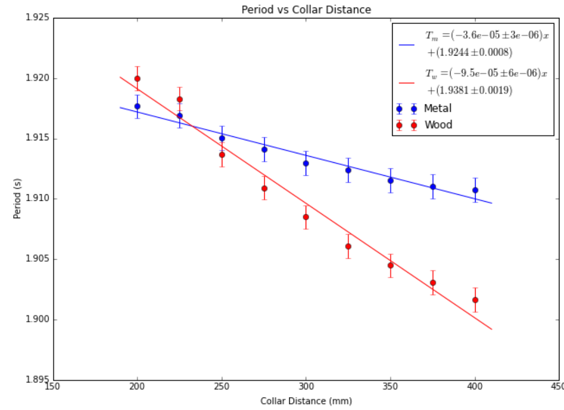

FIG. 2: Period versus collar distance: The collars on our pendulum were moved in set increments and the period of each side was taken. The point where the two lines cross show the approximate location at which the periods should be equal, and the difference in air resistance is negligible. We found this value to be around $232.7mm$.

that rests on a glass plate that the far bob uses as a pivot point. Two collars between the knife blades can be moved to change the center of mass. The pendulum is volume symmetric about the center line, but it is mass asymmetric due to the different densities of the bobs.

To determine the center of mass for which the period is equal for both ends of the pendulum, the collars were moved symmetrically away from the knife edge in set increments, and the period was measured. Plotting the period versus the collar distance from the blades gave a linear relationship, with each bob having a different slope and intercept. By comparing where these lines crossed, we could then find the position of the collars that would result in an equal period. We then used equation 4 to find g.

The period of the pendulum was measured by means of a photogate. As the pendulum oscillated the bottom tip broke the infrared beam from the photogate, and a timestamp was recorded. By analyzing the timestamps, a period was determined.

3. DATA PRESENTATION AND ERROR ANALYSIS

The initial data collection was to determine roughly the point at which the two periods were equal, so that we could then take finer data in the region of interest.

The collars were initially set $200mm$ away from the blades, and moved inward in $25mm$ increments, until the collars could not be moved closer together. This gave us the graph shown in figure 2. From this graph we could see that our area of interest was approximately at $230mm$. Algebraic calculation of the interception point of the lines gave $232.66mm$.

We then took finer date, from $229mm$ to $231mm$. We stopped taking data after $231mm$ because we noticed that the difference between the period was increasing. We got the following table:

$\Delta X (mm)$	Metal T	Wood T	ΔT	Metal ω	Wood ω	$\Delta\omega$
229.0	1.91500	1.91717	0.0022	3.281	3.277	0.004
229.5	1.91695	1.91757	0.0006	3.278	3.277	0.001
230.0	1.91666	1.91759	0.0009	3.378	3.277	0.002
230.5	1.91629	1.91699	0.0007	3.279	3.278	0.001
230.5	1.91748	1.91713	0.0003	3.277	3.277	0.001
230.5	1.91626	1.91754	0.0013	3.279	3.277	0.002
230.5	1.91687	1.91797	0.0011	3.278	3.276	0.002
230.5	1.91631	1.91736	0.0010	3.279	3.277	0.002
231.0	1.90721	1.91820	0.0110	3.294	3.276	0.019

Here, ΔX is the separation from the blades to closest side of the collars, T is the period of the pendulums in seconds, and ω is the angular frequency. The five measurements at $230.5mm$ was to determine the uncertainty in our period measurements.

We determined the equal period for each side of the pendulum (T_o) to be $1.917s \pm 0.001s$. From this we calculate the angular frequency to be $\omega_o = 3.277\frac{rad}{s} \pm 0.002\frac{rad}{s}$.

We measured the distance between the two pivot points to be $910.887mm \pm 0.5mm$. This relatively large error was due to the difficulty in keeping our measuring device lined up with the pendulum. If it became askew, it would add as much as a millimeter, despite being much more precise.

From here, we can determine g:

$$g = (\Delta x)(\omega_o)^2 \tag{5}$$

$$g = (0.910887m \pm 0.0005m)(3.277\frac{rad}{s} \pm 0.002\frac{rad}{s})^2 \tag{6}$$

$$g = 9.781\frac{m}{s^2} \pm 0.017\frac{m}{s^2} \tag{7}$$

The maximum value inside our uncertainty, $9.798\frac{m}{s^2}$ just comes under the accepted value for g, which is $9.80665\frac{m}{s^2}$. The percent error for our average value is 0.262%.

The primary source of error in our experiment was the period measurement. In the uncertainty equation

$$\Delta g = \sqrt{(\partial_X g * \Delta X)^2 + (\partial_{\omega_o} g * \Delta\Omega_o)^2} \tag{8}$$

our value of $\partial_X g * \Delta X = 0.005369$ and $\partial_{\omega_o} g * \Delta\omega_o = 0.01194$. From this equation we can see that despite a photogate that was capable of measuring to 0.00001 seconds, our uncertainty was signicantly larger.

Since our experimental value with our uncertainty does not meet the expected value, there must be additional, unaccounted sources of error. This could be some shake in the mount of the pendulum, misplacing the knife blade on the pivot points, bumping the table on which the photogate was resting, or disturbing air currents in the room. There may have also been uncertainties from air resistance not canceled out by the volume symmetric pendulum, such as the finite amplitude and the dampening shift of the frequency (Candela [2]).

4. CONCLUSIONS

Our results for $g = (9.781 \pm 0.017)\frac{m}{s^2}$ was very close to the accepted $g = 9.80665\frac{m}{s^2}$ with only a 0.262% error, but our uncertainty was not enough to include the accepted value. This would indicate that either there was some additional source of error that we did not account for, or that we were imprecise in our measurements. As one possible improvement to this experiment, we would devise a consistent and reliable method to set the pendulum on the glass plates to reduce any additional drag. Another possible improvment would be reducing the skew of our measuring device, by manufacturing a alignment mechanism, or taking multiple periods at a given distance. I would say that we were mostly successful in our experiment, though there is room for improvement.

[1] A.O. Allen. On the adjustment of kater's pendulum. *The Mathematical Gazette*, 3(57):307–313, May 1906.

[2] R.V. Krotkov D. Candela, K.M. Martini and K.H. Langley. Bessel's improved kater pendulum in the teaching lab. *American Journal of Physics*, 69:714–720, 2001.

[3] K.E. Jesse and H.J. Born. Notes: Possible sources of error when using the kater pendulum. *The Physics Teacher*, 10: 466, 1972.

[4] Kenneth E. Jesse. Kater pendulum modification. *American Journal of Physics*, 48:785, 1980.

[5] Michele Rossi and Lorenzo Zaninetti. The cubic period-distance relation for the kater reversible pendulum. *Central European Science Journals*, 3:636–659, 2005.

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