Derivation of 2nd Order Runge Kutta

With $\partial_t u(x,t) = f(u,x,t)$, we take a Taylor expansion around $u(x,t+h) = u(t+h)$, (Ignoring the spacial component)

$$u(t+h) = u(t) + h\dot{u}(t) + (h^2/2)\ddot{u}(t) + O(h^3)$$
$$= u(t) + h f(u,t) + (h^2/2)\ddot{u}(t) + O(h^3)$$

Now $\ddot{u} = \dfrac{d f(u,t)}{dt} = \dfrac{\partial f}{\partial t} + \dfrac{\partial f}{\partial u}\dfrac{du}{dt} = \dfrac{\partial f}{\partial t} + f\dfrac{\partial f}{\partial u}$

So

$$u(t+h) = u(t) + hf + \frac{h^2}{2}\left[\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial u}\right](u,t) + O(h^3) \qquad (1)$$

We note that

$$f(u+k, t+h) = f(u,t) + k\partial_u f(u,t) + h\partial_t f(u,t) + \dots$$

and we can see that the bracket part of eq 1 can be seen as

$$f(u+hf, t+h) = f(u,t) + h\partial_t f(u,t) + hf(u,t)\partial_u f(u,t)$$

So rearranging eq 1

$$u(t+h) = u(t) + hf(u,t) + \left(\frac{h}{2}\right)h\partial_t f(u,t) + \left(\frac{h}{2}\right)hf(u,t)\partial_u f(u,t) + O(h^3)$$
$$= u(t) + \frac{hf(u,t)}{2} + \frac{h}{2}\left(f(u,t) + h\partial_t f(u,t) + hf(u,t)\partial_u f(u,t)\right) + O(h^3)$$

$$u(t+h) = u(t) + \frac{h}{2}f(u,t) + \frac{h}{2}f(u+hf, t+h) + O(h^3) = u(t) + h\left[\tfrac{1}{2}k_1 + \tfrac{1}{2}k_2\right]$$

with $k_1 = f(u,t)$
$k_2 = f(u+hk_1, t+h)$