Part 1

Given $x_1 = 0$, $x_2 = \Delta x$, $x_3 = 2\Delta x$

Now

$$\tilde{f}(x) = \sum_{i=1}^{3} \prod_{j \neq i}^{3} \frac{x - x_j}{x_i - x_j} f_i$$

$$= \frac{(x-\Delta x)(x-2\Delta x)}{(-\Delta x)(-2\Delta x)} f_1 + \frac{x(x-2\Delta x)}{\Delta x(\Delta x - 2\Delta x)} f_2 + \frac{x(x-\Delta x)}{2\Delta x(2\Delta x - \Delta x)} f_3$$

$$= \frac{x^2 - 3x\Delta x + 2(\Delta x)^2}{2(\Delta x)^2} f_1 + \frac{x^2 - 2x\Delta x}{-(\Delta x)^2} f_2 + \frac{x^2 - x\Delta x}{2(\Delta x)^2} f_3$$

Taylor Expanding f_1, f_2, f_3, we get

$$f_1 = f(x_1) = f(0)$$
$$f_2 = f(x_2) = f(0) + f'(0)\Delta x + (1/2) f''(0)(\Delta x)^2 + \mathcal{O}((\Delta x)^3)$$
$$f_3 = f(0) + f'(0)2\Delta x + f''(0)2(\Delta x)^2 + \mathcal{O}((\Delta x)^3)$$

Replacing f_1, f_2, f_3 in $\tilde{f}(x)$ gives

$$\tilde{f}(x) = \left\{ \frac{x^2 - 3x\Delta x + 2(\Delta x)^2}{2(\Delta x)^2} + \frac{x^2 - 2x\Delta x}{-(\Delta x)^2} + \frac{x^2 - x\Delta x}{2(\Delta x)^2} \right\} f(0)$$
$$+ \left\{ \frac{-x(x-2\Delta x)}{\Delta x} + \frac{x^2 - x\Delta x}{\Delta x} \right\} f'(0) + \left\{ -\frac{x(x-2\Delta x)}{2} + x^2 - x\Delta x \right\} f''(0)$$
$$+ \underbrace{\left\{ \frac{x^2 - 2x\Delta x}{-(\Delta x)^2} + \frac{x^2 - x\Delta x}{2(\Delta x)^2} \right\}}_{\text{Of order } \sim 1 \text{, so can drop for approximation}} \mathcal{O}((\Delta x)^3)$$

Of order ~ 1, so can drop for approximation

~~$\tilde{f}(\Delta x)$~~ $=$

$$\tilde{f}(x) = \left\{ \frac{\cancel{x^2} - \cancel{3x\Delta x} + \cancel{2(\Delta x)^2} - \cancel{2x^2} + \cancel{4x\Delta x} + \cancel{x^2} - \cancel{x\Delta x}}{\cancel{2(\Delta x)^2}} \right\} f(0)$$

$$+ \left\{ \frac{\cancel{-x^2} + 2x\cancel{\Delta x} + \cancel{x^2} - \cancel{x\Delta x}}{\cancel{\Delta x}} \right\} f'(0) + \left\{ \frac{\cancel{-x^2} + \cancel{2x\Delta x} + 2x^2 - \cancel{2x\Delta x}}{2} \right\} f''(0) + \mathcal{O}(\Delta x^3)$$

$$= f(0) + f'(0)\, x + \frac{1}{2} f''(0)\, x^2 + \mathcal{O}((\Delta x)^3)$$

Thus $\hat{f} - f = \mathcal{O}((\delta x)^3)$ where $\delta x = \max(x, \Delta x)$,

since that term will dominate at higher powers