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Effect of the mass of the cord on the period of a simple pendulum

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It is well known that, when a weight hanging on a spring oscillates up and down, the spring, owing to its mass, has some effect on the motion and, in particular, on the period. Both exact and approximate solutions to that problem have been given. Likewise, the cord of a simple pendulum, because of its mass, has some effect on the motion and on the period. The solution to the problem is investigated here. It leads to a rather complicated equation involving Bessel functions of both kinds. However, it is possible to find simpler approximations which are likely to be legitimate in practice. Also, the results obtained are compared with some other arrangements, which seem physically to have some resemblance to the cord and pendulum bob, and which can be solved fairly readily.

When a weight, hanging on a spring, is let oscillate up and down, it is quite apparent that the mass of the spring will have some effect, In fact, an over-simplified solution to this problem is often stated without qualification. A complete solution, however, was given in this journal some years ago.[1]

Likewise, the mass of the cord of a simple pendulum has some effect on the motion. It is likely that the problem has been solved previously. If so, however, the solution is evidently not well known; and there comes a time when it is easier to produce a new solution.

Consider, then, an arrangement as shown in Fig. 1. The bob of the pendulum, of mass M, hangs on the cord, of length l and mass m. The radius of the bob is negligible compared with l. The coordinates are as shown; and it is assumed that the motion is small. Then the motion of the bob can be considered to be along a straight line, the y axis.

Consider a section of the cord, of length dx, at position x. The tension there is $g[M + (mx/l)]$. The component of that tension in the y direction, in the approximation of small motion, is, at any point, $g[M + (mx/l)](\partial y/\partial x)$. Hence the net force, in the y direction, on that section of cord, is $(g\, dx)(\partial/\partial x)\{[M + (mx/l)](\partial y/\partial x)\}$. Since this force causes the acceleration, it must be equal to $(m/l) \times (\partial^2 y/\partial x^2)\, dx$. When the indicated differentiations are carried out this gives

$$g\left(M+\frac{m}{l}x\right)\frac{\partial^2 y}{\partial x^2}+\frac{gm}{l}\frac{\partial y}{\partial x}=\frac{m}{l}\frac{\partial^2 y}{\partial t^2}. \qquad (1)$$

It may be supposed also that the motion at any point is simple harmonic, at an angular frequency ω, so that differentiation twice with respect to time may be replaced with multiplication by $-\omega^2$. So that gives

$$g\left(M+\frac{m}{l}\right)\frac{d^2 y}{dx^2}+\frac{gm}{l}\frac{dy}{dx}+\frac{m\omega^2}{l}y=0. \qquad (2)$$

Dividing through by gm/l gives the slightly simpler form

$$\left(\frac{M}{m}l+x\right)\frac{d^2 y}{dx^2}+\frac{dy}{dx}+\frac{\omega^2}{g}y=0. \qquad (3)$$

To get this into a workable form, change the independent variable into $z = (Ml/m) + x$, and divide through by z, to get

$$\frac{d^2 y}{dz^2}+\frac{1}{z}\frac{dy}{dz}+\frac{\omega^2}{gz}y=0. \qquad (4)$$

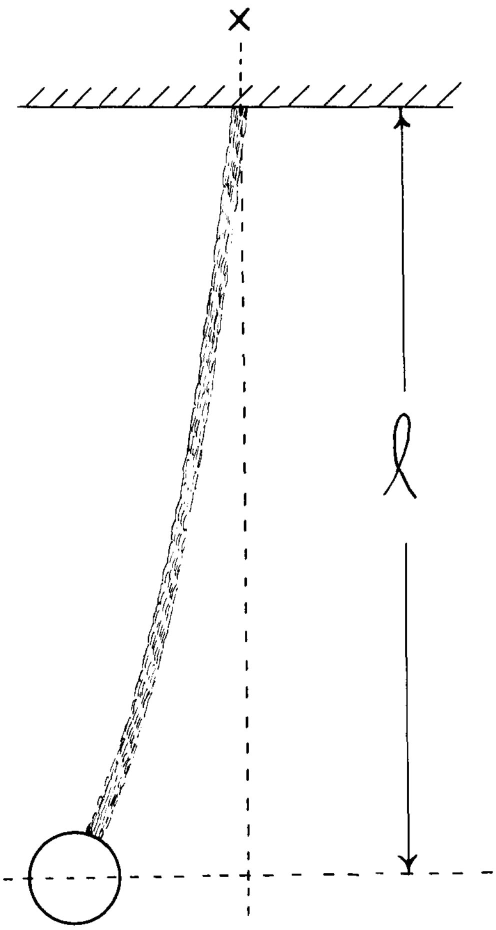

Fig. 1. Pendulum, the mass of whose cord is not negligible, discussed in the text. The system of coordinates used in the discussion is also shown.

Note that in Eqs. (2)–(4), derivatives have been written as ordinary, not partial, since only one independent variable is involved.

Equation (4) is in a standard form, as shown for instance by Jahnke and Emde,[2] whose notation will be used here. The solution is

$$y = Z_0\left[(4\omega^2/g)(Ml/m + x)\right]^{1/2}. \tag{5}$$

Z_0 indicates a Bessel function of order zero, either of the first or the second kind or a linear combination of the two. Functions of the second kind are not excluded here, since the argument is never zero.

There are two conditions to be met. When $x = l$, it must be that $y = 0$. Moreover, when $x = 0$, at the bottom, since the tension in the cord is there Mg, the force in the y direction on the bob will be $Mg(\partial y/\partial x)$. And again, since this force causes the acceleration, it must be equated to $-M\omega^2 y$. The working out of these conditions is straightforward but tedious. The eventual result is

$$2\omega\left(\frac{lM}{gm}\right)^{1/2}\frac{J_0[2\omega(lM/gm)^{1/2}]N_0[2\omega(l/g)^{1/2}(M/m+1)^{1/2}] - N_0[2\omega(lM/gm)^{1/2}]J_0[2\omega(l/g)^{1/2}(M/m+1)^{1/2}]}{J_1[2\omega(lM/gm)^{1/2}]N_0[2\omega(l/g)^{1/2}(M/m+1)^{1/2}] - N_1[2\omega(lM/gm)^{1/2}]J_0[2\omega(l/g)^{1/2}(M/m+1)^{1/2}]} = 2. \tag{6}$$

In principle, M, m, and l being given, and g known, this could be solved for ω. The solution, presumably, would be by something like Newton's method. With the help of modern computing devices, the work might even not be too bad.

In fact, though, it is time to take into account what is realistic and what is not. Rarely indeed would m be even comparable with M. Usually m would be small, but not quite negligible. And for that condition it is possible to get an approximate general solution.

When m is small the arguments of the Bessel functions are large. Then the asymptotic forms of the Bessel functions, which are, or involve, trigonometric functions, may be substituted.[3] Then, with the help of standard trigonometric identities, the relation takes the form

$$\omega(lm/gM)^{1/2}\tan[\omega(lm/gM)^{1/2}] = m/M. \tag{7}$$

Since m is now in the numerator of the argument of the tangent, use the first two terms of the series $\tan x = x + x^3/3 + \cdots$, to get, as an approximation,

$$\omega^2(lm/gM)[1 + \tfrac{1}{3}\omega^2(lm/gM)] = m/M. \tag{8}$$

Since the deviation of the behavior from that of an ideal simple pendulum is small, $\omega^2 = g/l$ may be substituted in the second term, to give, in turn,

$$\omega^2(l/g)(1 + m/3M) = 1. \tag{9}$$

Thus this shows, to the approximation used, the effect of the mass of the cord on the angular frequency; as for the period T, of course,

$$T = 2\pi\left(\frac{l}{g}\right)^{1/2}\left(1 + \frac{m}{3M}\right)^{1/2} \approx 2\pi\left(\frac{l}{g}\right)^{1/2}\left(1 + \frac{m}{6M}\right). \tag{10}$$

It may be of interest to compare the case in which the cord is replaced by a uniform rigid rod of the same length and mass. The behavior is then readily calculated as that of a compound pendulum; and the period is found to be

$$T = 2\pi\left(\frac{l}{g}\right)^{1/2}\left(\frac{1 + m/3M}{1 + m/2M}\right)^{1/2} \approx 2\pi\left(\frac{l}{g}\right)^{1/2}\left(1 - \frac{m}{12M}\right). \tag{11}$$

It seems curious that while the mass of a rigid fastening acts to reduce the period, that of a flexible one acts to increase it.

Some light may be thrown onto this matter by consideration of the case in which a hanging cord, with nothing on the end, swings back and forth. That could be represented, of course, by setting $M = 0$. Equation (6) would still apply; but it would need much interpretation. Since the argument of the Bessel functions would now extend to zero, the functions of the second kind need, in effect, to drop out. The condition becomes that $y = 0$ when $x = l$. That is done by having $2\omega(l/g)^{1/2}$ be a zero of the Bessel function of the first kind. These zeros are tabulated, in Jahnke and Emde, for instance[4]; and the first will be wanted here, the others applying to higher modes of oscillation. The magnitude of the first zero of J_0, in fact, is 2.4048. So this gives

$$2\omega(l/g)^{1/2} = 2\cdot 4048. \tag{12}$$

The period, of course, is then

$$T = (2\pi/1\cdot 2024)(l/g)^{1/2}. \tag{13}$$

So for the cord swinging without anything on the end, the period is less than that of a simple pendulum of length l, but the effect of the cord on the pendulum is to increase the period.

This state of affairs may be considered in the light of another approximate calculation. When M is much greater than m, the tension in the cord may be taken, to a good approximation, as Mg. Then the motion of the cord is given by an ordinary wave equation. And that, with the other matters concerned, leads exactly to Eq. (7), as can readily be seen.

It would appear that if the tension is nearly the same everywhere in the cord the effect is to increase the period; if it varies widely along the length, the effect is to reduce it.

Attempts to check this matter from the other end, so to speak, by finding an approximate solution for Eq. (6) valid if M is much less than m led nowhere; the matter just becomes too complicated. However, there is another way of looking at it which should be at least suggestive. The situation for M much less than m, but not quite negligible, should be much the same as if the length were increased from l to $l[1 + (M/m)]$. That substitution into Eq. (13) gives

$$T=\frac{2\pi}{1\cdot 2024}\left(\frac{l}{g}\right)^{1/2}\left(1+\frac{M}{m}\right)^{1/2}$$
$$\approx\frac{2\pi}{1\cdot 2024}\left(\frac{l}{g}\right)^{1/2}\left(1+\frac{M}{2m}\right). \tag{14}$$

So the period is greater than for the cord alone; presumably there is some ratio of m to M for which the period is the same as that for a simple pendulum of length l.

It might be noticed also that a rigid rod, swinging about one end, stands in about the same relation to the cord alone as the pendulum with the rigid rod does to the one with the cord. The rigid rod, of course, is easily solved as a pendulum; and its period is found to be

$$T=2\pi(\tfrac{2}{3})^{1/2}(l/g)^{1/2}\approx(2\pi/1\cdot 22)(l/g)^{1/2}. \tag{15}$$

APPENDIX

It has been suggested that the effects of other things, such as the finite size of the bob, the finite amplitude of the motion, and air resistance, ought to be considered.

It is plain that to attempt to include all of these things in one treatment would lead to intolerable complications. But they can be considered one at a time, and their effects compared.

Hughes treated the effect of the finite size of the bob at some length.[5] The motion really has two normal modes; however, it is likely that the pendulum-like one will prevail, the other, faster mode being damped out sooner. Then the relative increase in the period, due to the finite radius r of the bob, is approximately $r^2/5l^2$, provided r be small. At most, r is likely to be a few percent of l; so the period would be affected by rather less than 1%.

The motion with finite amplitude of a pendulum is discussed in many books on elliptic functions, and elsewhere. A simple argument notices that at any time the acceleration of the bob is proportional to the sine of the angle by which it is displaced, rather than the angle itself. Also, the average acceleration is proportional to the square of the period. Thus the true period would be expected to be between 1 and $(\theta/\sin\theta)^{1/2}$ times the period for infinitesimal motion. As an approximation, take the average of the two factors mentioned; the relative increase in the period is of magnitude approximately $\theta^2/24$. In all of this, θ is the maximum angle with the vertical which the string reaches. (An expansion of elliptic integrals, it may be noted, gives $\theta^2/16$.) Thus an amplitude of 20 deg would cause an increase of less than 1%.

As for air resistance, it occurs in two ways: on the cord and on the bob. When the air resistance on the cord is taken into account in the partial differential equation, it is found that no effect on the period is indicated, to the same stage of approximation as has already been used. It is indicated, of course, that the motion decreases with time.

The treatment of air resistance on the bob is a standard case of damped harmonic motion. It is shown quite readily that if the damping is such that the amplitude of oscillation is reduced to half of its original magnitude in N cycles of the oscillation, the relative change in the period due to the damping is approximately $1/(160N^2)$; or, the period is increased by about $1/(1.6N^2)$ percent. So even if the damping should reduce the amplitude to half of its original magnitude in as few as 10 cycles, the period would be increased by less than 1%.

The treatment above, of the effect of the mass of the cord, showed, in Eq. (10), that the relative increase in the period is of magnitude $m/6M$. Thus if m should be as little as one per cent of M, the effect would be comparable in magnitude with that of the other things mentioned.

[1]H. L. Armstrong, Am. J. Phys. **37**, 447 (1969).
[2]E. Jahnke and F. Emde, *Tables of Functions* (Dover, New York, 1945), pp. 144–147.
[3]Reference 2, p. 138.
[4]Reference 2, p. 166.
[5]J. V. Hughes, Am. J. Phys. **21**, 47 (1953).