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Pendulum on a massive cord

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The oscillation frequency of a pendulum supported by a massive string, discussed previously by Armstrong, is analyzed both analytically and using a variational method. The variational approach displays its characteristic benefits in providing both an efficient computational procedure and also useful insight into the physical behavior of the system.

I. INTRODUCTION

In a previous article in this Journal,[1] Armstrong has discussed the frequency of simple harmonic motion of a pendulum on a cord of appreciable mass. This paper provides a more compact expression for the equation determining the frequency when the cord is of uniform density, from which are obtained the frequencies when the mass of the cord is much greater, and much less, than the mass of the bob. It is found that Armstrong's physical argument for the limiting behavior when the bob is light gives the correct result. For a light string, however, Armstrong's results are in error. In particular, the frequency is always reduced for a massive cord.

In addition, a variational formulation of the problem is presented, which is applicable for an arbitrary distribution of mass along the cord. When applied to the case of a uniform cord, quite accurate values of the frequency for all bob-to-cord mass ratios can be computed easily.

To establish our notation, which duplicates that of Armstrong where possible, we repeat an abbreviated derivation of the fundamental equations. We denote by $y(x)$ the horizontal displacement (assumed small) of the cord at a height x above the bottom end of the cord of length l. The horizontal component of the tension in the cord at a height x is

$$\mathcal{M}(x) g \frac{dy}{dx},$$

where $\mathcal{M}(x)$ is the total mass of the bob and the cord below x. In the harmonic approximation, the equation of motion for an element of the cord becomes

$$\frac{d}{dx}\left(\mathcal{M}(x) g \frac{dy}{dx}\right) = -\omega^2 y \frac{d\mathcal{M}}{dx}. \tag{1}$$

The boundary conditions are that at the fixed upper end of the cord

$$y(l) = 0 \tag{2}$$

and that at the bottom, where $x = 0$,

$$\mathcal{M}(0)\left(\frac{dy}{dx} + \frac{\omega^2}{g} y\right) = 0. \tag{3}$$

II. UNIFORM CORD

For a uniform cord of mass m carrying a bob of mass M,

$$\mathcal{M}(x) = M + (m/l)x \tag{4}$$

so that the differential equation becomes

$$\left(M + \frac{m}{l}x\right) g \frac{d^2y}{dx^2} + \frac{m}{l} g \frac{dy}{dx} + \omega^2 \frac{m}{l} y = 0, \tag{5}$$

which (except for a typographical error) is the same as Armstrong's equation (2).

The solution is the zeroth-order Bessel function[2,3]

$$y = Z_0(\zeta), \tag{6}$$

where

$$\zeta^2 = 4(\omega^2 l/g)(M/m + x/l). \tag{7}$$

We denote by ζ_0 and ζ_l the values of ζ when $x = 0$ and l, respectively. The boundary condition for $x = l$ is then

$$Z_0(\zeta_l) = 0. \tag{8}$$

The other boundary condition is that where $\zeta = \zeta_0$

$$\frac{2}{\zeta}\frac{dZ_0}{d\zeta} + Z_0 = 0. \tag{9}$$

But since $Z_0' = -Z_1$ and $Z_{p-1} + Z_{p+1} = (2p/\zeta)Z_p$, the latter condition may be reexpressed concisely as

$$Z_2(\zeta_0) = 0. \tag{10}$$

It is convenient to write the Bessel function explicitly in the form

$$Z_p(\zeta) = \cos\alpha J_p(\zeta) - \sin\alpha N_p(\zeta). \tag{11}$$

The condition for the existence of simultaneous solutions of Eqs. (8) and (10) is that

$$J_0(\zeta_l)/N_0(\zeta_l) = J_2(\zeta_0)/N_2(\zeta_0). \tag{12}$$

This result is equivalent to Armstrong's Eq. (6), but is probably more convenient for solution using standard tables of Bessel functions. Pairs of values ζ_l and ζ_0 obeying Eq. (12) are related to the oscillation frequency and masses by the equations

$$\zeta_l^2 - \zeta_0^2 = 4\omega^2 l/g \tag{13}$$

and

$$(\zeta_l/\zeta_0)^2 - 1 = m/M. \tag{14}$$

We need "adjacent" values of ζ_l and ζ_0—for example, when ζ_0 is the nth zero of J_2 then ζ_l is the $(n + 1)$st zero of J_0.

The limiting behavior of the frequency for extreme values of the ratio m/M can be obtained from the known properties of Bessel functions. For $m \gg M$ we see that ζ_0 is very small and ζ_l is near the first zero of J_0. Expanding J_2 and N_2 for small arguments[4] we find

$$\frac{J_2}{N_2} \sim \frac{\pi}{2}\left(\frac{\zeta_0}{2}\right)^4 = \frac{\pi}{2}(\omega^2 l/g)^2(M/m)^2. \tag{15}$$

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Thus the distance of ζ_l from the first zero of J_0 is of order $(M/m)^2$, so that

$$2\omega(l/g)^{1/2}(1 + M/m)^{1/2} = 2.4048 + O(M^2/m^2). \quad (16)$$

This confirms Armstrong's argument leading to his Eq. (14).

For the case of a light cord, $m \ll M$, the arguments ζ_0 and ζ_l of the Bessel functions are large and nearly equal. We may make use of expressions[5] for the zeros of Bessel functions to see that conditions (8) and (10) are satisfied when for some constant, β, we have

$$\zeta_l = \beta + \frac{1}{8\beta} + \frac{31}{384\beta^3} + \cdots \quad (17)$$

$$\zeta_0 = \beta - \frac{15}{8\beta} - \frac{405}{128\beta^3} - \cdots. \quad (18)$$

Equations (13) and (14) then become

$$\omega^2 l/g = \tfrac{1}{4}(\zeta_l^2 - \zeta_0^2) = 1 + \frac{2}{3\beta^2} + O(\beta^{-4}) \quad (19)$$

$$\frac{m}{M} = \frac{\zeta_l^2 - \zeta_0^2}{\zeta_0^2} = \frac{4}{\beta^2} + O(\beta^{-4}) \quad (20)$$

so that we have

$$\omega^2 l/g = 1 + m/6M + \text{terms of order } (m/M)^2. \quad (21)$$

It can be seen that three terms in the asymptotic expansions are required to obtain the frequency correct to first order in the ratio of masses. Apparently Armstrong's error was due to the use of approximations of lower accuracy. An alternate evaluation of the correct frequency in the light-cord limit has recently been given.[6]

III. VARIATIONAL TREATMENT

The differential equation for the horizontal displacement of a cord with any mass distribution, Eq. (1), may be rewritten

$$\frac{d}{dx}\left(\mathcal{M}\frac{dy}{dx} + \frac{\omega^2}{g}\mathcal{M}y\right) = \frac{\omega^2}{g}\mathcal{M}\frac{dy}{dx}, \quad (22)$$

which may be recognized as having the form of the Euler-Lagrange equation

$$\frac{d}{dx}\left(\frac{\partial \mathcal{L}}{\partial y'}\right) = \frac{\partial \mathcal{L}}{\partial y} \quad (23)$$

for the variation of $\int \mathcal{L}(y,y')\,dx$ with

$$\mathcal{L} = \tfrac{1}{2}\mathcal{M}(x)\left(\frac{dy}{dx}\right)^2 + \frac{\omega^2}{g}\mathcal{M}(x)y\frac{dy}{dx}. \quad (24)$$

An examination of the end-point contributions shows that

$$\delta\int_0^l \mathcal{L}\,dx = \left[\left(y' + \frac{\omega^2}{g}y\right)\mathcal{M}\delta y\right]_0^l. \quad (25)$$

The contribution from $x = 0$ vanishes by virtue of the boundary condition of Eq. (3), and that from $x = l$ will also vanish if we require that the variation of y be zero at the point of support. Thus we can obtain an estimate of the frequency from the equation

$$\frac{2\omega^2}{g} = -\int_0^l \mathcal{M}y'^2\,dx \Big/ \int_0^l \mathcal{M}yy'\,dx, \quad (26)$$

using for $y(x)$ an approximate displacement which makes ω^2 stationary. It can be recognized that a frequency ω found in this way must be an upper limit to the true value, since it is the frequency of an oscillation with the shape of the cord constrained to be proportional to the chosen trial function $y(x)$. But the removal of a constraint can only lower, never raise, the lowest eigenfrequency of oscillation.

For a uniform cord,

$$\mathcal{M}(x) = M + mx/l,$$

choosing $y(x)$ a straight line through the point of support gives, of course, the rigid-rod result,

$$\frac{\omega^2 l}{g} = \frac{3m + 6M}{2m + 6M}. \quad (27)$$

This is actually quite a good approximation, the error in ω^2 being less than 4% even in the extreme case of $M = 0$.

For a better estimate, if one is needed, we may use a parabola as a trial function:

$$y = (x - l) + (\gamma/l)(x - l)^2. \quad (28)$$

The result for ω^2 from Eq. (26) is

$$\frac{\omega^2 l}{g} = \frac{15(m + 2M) + 20\gamma(m + 3M) + 10\gamma^2(m + 4M)}{10(m + 3M) + 15\gamma(m + 4M) + 6\gamma^2(m + 5M)}. \quad (29)$$

This has its minimum value when γ is a solution of the quadratic equation

$$6\gamma^2[m^2 + 8m + (20)^2] + 4\gamma[m^2 + 7m + (30)^2] - 5(m^2 + 6m) = 0. \quad (30)$$

Again the greatest error occurs when $M = 0$, in which case the estimate is $\omega^2 l/g = 2(8 - \sqrt{34})/3 \simeq 1.4460$, giving the frequency (or period) correctly to better than one part in 12 000.

IV. CONCLUSIONS

Besides confirming the results of an analytical treatment for the case of a uniform string, the variational approach displays several characteristic advantages. Among the foremost of these must be counted the physical insight gained in recognizing the approximate eigenfrequencies as corresponding to constrained motions of the system. This clarifies the rigid-rod approximation, for example, and provides both qualitative and quantitative conclusions about the exact results as well as the validity of various approximations.

Moreover, the variational approach is suitable for an arbitrary distribution of mass along the cord, and permits the efficient computation of highly accurate results without reliance on either analytical properties or tabulated values of special functions. In fact this problem provides a rather neat example of the conceptual and practical features which are provided by variational calculations, and may be a useful addition to more standard examples in the discussion of these features.

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[4]Reference 2, pp. 128, 132; Ref. 3, p. 360.
[5]Reference 2, p. 143; Ref. 3, pp. 370–371.
[6]Saul T. Epstein and M. G. Olsson, Am. J. Phys. **45**, 671 (1977).